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GLOBAL PHASE PORTRAITS OF QUADRATIC SYSTEMS WITH A HYPERBOLA AND A STRAIGHT LINE AS INVARIANT ALGEBRAIC CURVES

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Abstract. In this paper we consider the quadratic polynomial differential systems in the plane having a hyperbola and a straight line as invariant algebraic curves, and we classify all its phase portraits. Moreover these systems are integrable and we provide their first integrals.

1. Introduction and statement of the main results

In this paper we consider a planar quadratic differential system

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1}$$

where P and Q are real polynomials such that the maximum of the degree of P and Q is 2. The dot in system (1) denotes derivative with respect to the independent variable t. We introduce some definitions.

Let f is a nonconstant polynomial in the variable x and y. The algebraic curve $f(x,y)=0$ is *an invariant curve* of system (1), if there exists some polynomial K(x,y) such that

$$\mathcal{X}(f) = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf,$$

and $K(x,y)$ is called the *cofactor* of the invariant curve $f(x,y)=0$.

Let $H(x,y)$ be a function defined in a dense and open subset U of $\mathbb{R}^2$. The function $H(x,y)$ is a *first integral* of system (1) if H is constant on the solutions of system (1) contained in U, i.e.

$$\mathcal{X}(H)\,|_U = P\frac{\partial H}{\partial x}(x,y) + Q\frac{\partial H}{\partial y}\bigg|_U = 0.$$

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And a quadratic system is *integrable* in U if it has a first integral H in U.

Up to now several hundred of papers have been published studying differential aspects of quadratic systems, as their integrability, their limit cycles, their global dynamical behavior, and $\cdots$, see for instance the references quoted in the books of Reyn [20] and Ye [24, 25]. But it remains many open problems of these systems. For example, the problem of the maximum number and distribution of limit cycles, or the problem of classifying all the integrable quadratic systems, remain open.

Darboux in [6] introduced the relation between the existence of invariant algebraic curves on a polynomial differential system and its integrability, see for more details in [4, 8].

In [7] Dulac started the studying of the classification of the quadratic centers and their first integrals, see also [2, 3, 10, 11, 16, 21, 26, 27]. In [1] Artés and Llibre studied the Hamiltonian quadratic systems, see also [9, 2]. In [18] Markus studied the class of homogeneous quadratic systems, see also [5, 12, 17, 19, 22, 23].

In this paper we concern about "*Given a class of quadratic systems depending on parameters, how to determine the values of the parameters for which the system has a first integral?*" In [15] and [13] the authors proved the integrability of the class of quadratic systems having an ellipse and a straight line as invariant algebraic curves, or two non-concentric circles as invariant algebraic curves, respectively. Additionally the authors provided all the different topological phase portraits that these classes exhibit in the Poincaré disc.

In this paper we want to study a new class of integrable quadratic systems, the ones having a hyperbola and a straight line as invariant algebraic curves. We prove their integrability and classify all their phase portraits.

Our first result is to provide a normal form for all quadratic polynomial differential systems having a hyperbola and a straight line as invariant algebraic curves.

Theorem 1. *A planar polynomial differential system of degree* 2 *having a hyperbola and a straight line as invariant algebraic curves after an* *affine change of coordinates can be written as either*

$$(2)\qquad \begin{aligned} \dot{x} =& \ c\frac{a^2-b^2}{a^2}y(x-\delta), \\ \dot{y} =& \ C\left(\frac{x^2}{a^2-b^2}-\frac{a^2-b^2}{a^2}y^2-1\right)+\frac{c}{a^2-b^2}x(x-\delta), \end{aligned}$$

or

$$(3)\qquad \begin{aligned} \dot{x} =& \ -cx(x-r), \\ \dot{y} =& \ C(2xy-1)+cy(x-r), \end{aligned}$$

where $a, b, c, C \in \mathbb{R}$ *with* $a \neq 0, a \neq b$ *and* $\delta = \{0, 1\}$.

Theorem 1 is proved in section 2. In the next result we present the first integrals of the polynomial differential system of degree 2 having a hyperbola and a straight line as invariant algebraic curves.

Theorem 2. *The quadratic polynomial differential systems* (2) *have the following first integrals:*

(a) $H = x$ *if* $c = 0$*;*
(b) $H = (x-\delta)^{(-2\frac{a^4+b^4}{(a^2-b^2)^2}\frac{C}{c})}(\frac{x^2}{a^2-b^2}-\frac{a^2-b^2}{a^2}y^2-1)$ *if* $c \neq 0$*;*

and systems (3) *have the following first integrals:*

(c) $H = x$ *if* $c = 0$*;*
(d) $H = (x-r)^{2C/c}(2xy-1)$ *if* $c \neq 0$.

Moreover, the quadratic polynomial differential systems (2) *and* (3) *have no limit cycles.*

Theorem 2 is proved in section 2.

In the next theorem we present the topological classification of all the phase portraits of planar polynomial differential system of degree 2 having a hyperbola and a straight line as invariant algebraic curves in the Poincaré disc.

Theorem 3. *Given a planar polynomial differential system of degree* 2 *having a hyperbola and a straight line as invariant algebraic curves its phase portrait is topological equivalent to one of the* 38 *phase portraits of Figures* 1, 2, 3.

Theorem 3 is proved in sections 3 and 4.

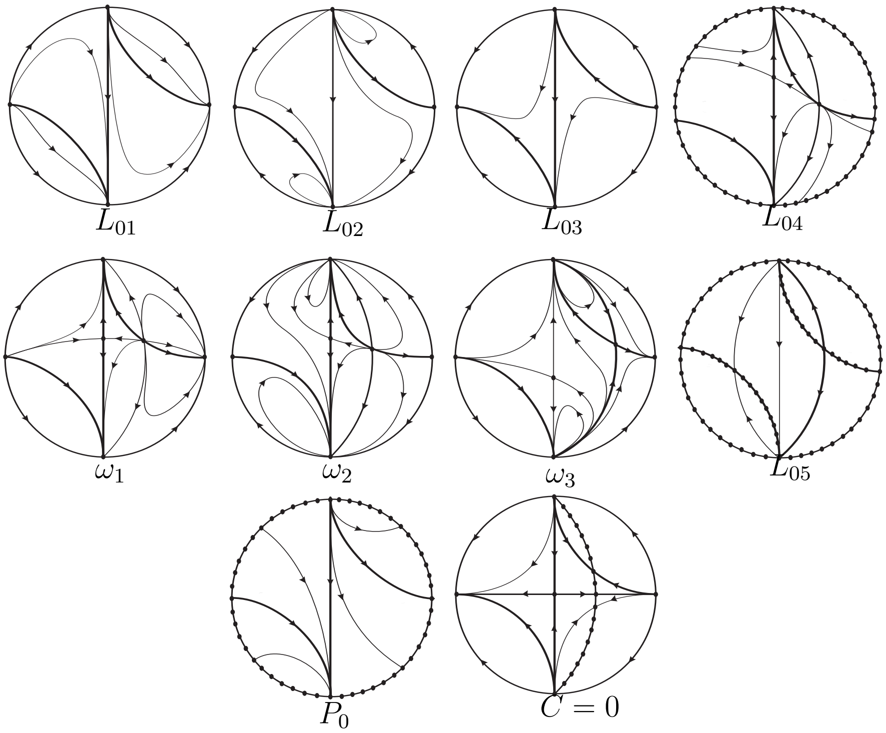

FIGURE 1. The phase portraits of system (3).

2. QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS WITH HYPERBOLA

In this section we consider that system (1) has an invariant hyperbola and an invariant straight line. Then by an affine transformation we can change the hyperbola to the following norm form and to any straight line

(4) $\mathcal{H}: \ f_1(x,y) = x^2 - y^2 - 1 = 0,$ and $\mathcal{L}: \ f_2(x,y) = ax + by - \delta = 0,$

where $\delta = 1$ or 0. Without loss of generality, let $a \geq 0$ for $\mathcal{L}$. According to the properties of the hyperbola, we classify the straight line in the following four cases,

(i) $a = \pm b$; (ii) $0 < a^2 < b^2$, (iii) $a^2 > b^2$, (iv) $a = 0$.

For $a \neq 0, a^2 - b^2 \neq 0$, doing the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{a}{a^2-b^2} & -\dfrac{b}{a} \\ -\dfrac{b}{a^2-b^2} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

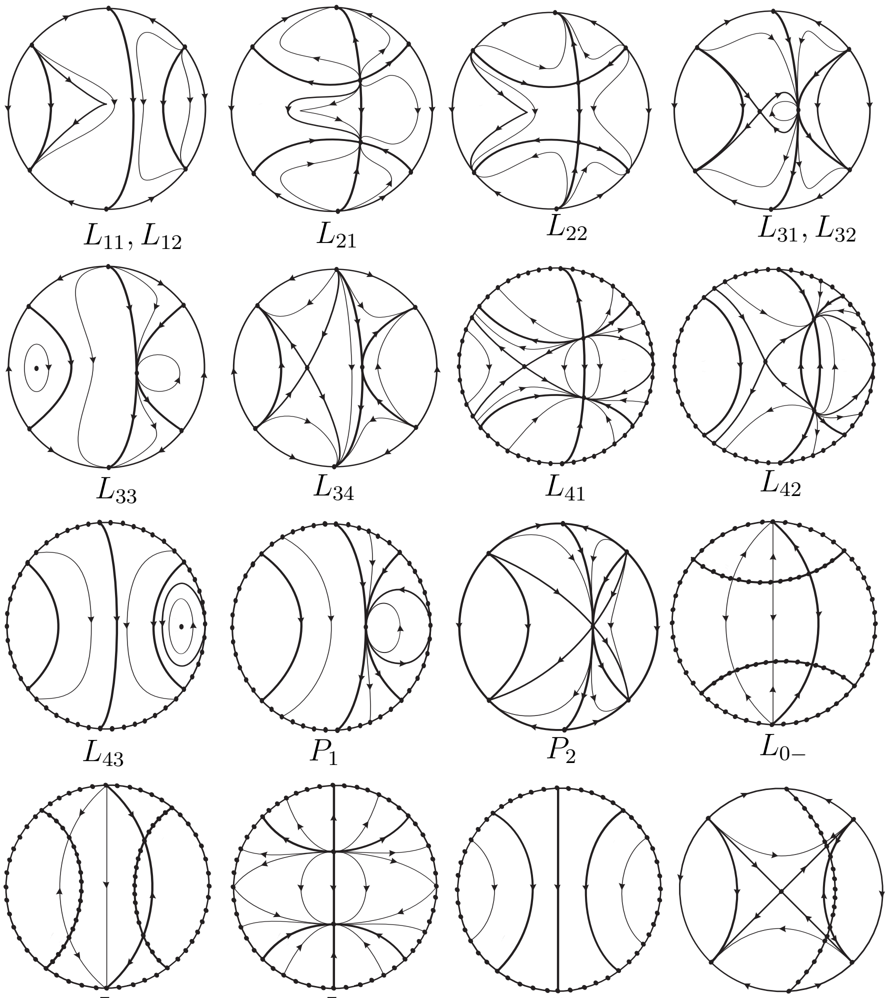

FIGURE 2. The phase portraits I of system (2).

the curves (4) change into

(5)

$$\mathcal{H}: \ f_1(x,y) = \frac{x^2}{a^2-b^2} - \frac{a^2-b^2}{a^2}y^2 - 1 = 0, \text{ and } \mathcal{L}: \ f_2(x,y) = x - \delta = 0,$$

for cases (ii) and (iii), where we rename (u, v) by (x, y). Hence, in the last two cases, without loss of generality, we have that $a > 0, b \geq 0$.

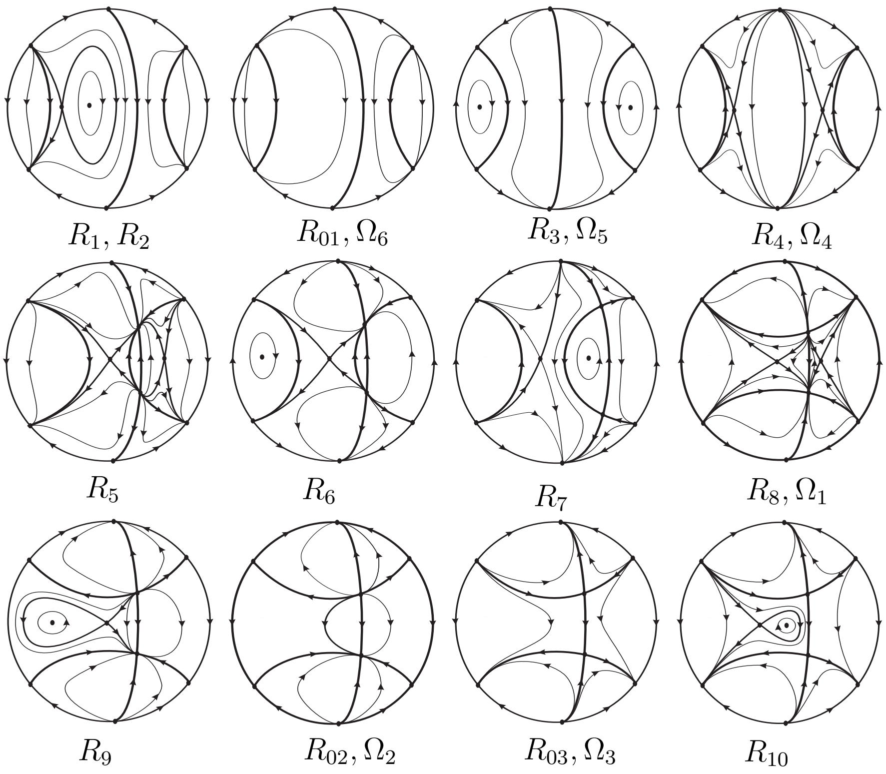

FIGURE 3. The phase portraits II of system (2).

For case (i), doing the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

the invariant hyperbola and the invariant straight line can be written as

$$(6) \qquad \mathcal{H}: \; f_1(x,y) = 2xy - 1 = 0, \; \text{and } \mathcal{L}: \; f_2(x,y) = x - r = 0,$$

where $r = \delta/(\sqrt{2}b)$. Finally, case (iv) pass to case (ii) by the transformation $(x,y) \longrightarrow (y,x)$.

Next we provide a normal form for all quadratic polynomial differential systems having a hyperbola and a straight line as invariant algebraic curves in Theorem 1. We shall need the following result which is a consequence of [14, Corollary 6], which characterizes all rational differential systems having two curves $f_1 = 0$ and $f_2 = 0$ as invariant algebraic curves.

Theorem 4. *Let f_1 and f_2 be polynomials in $\mathbb{R}[x, y]$ such that the Jacobian $\{f_1, f_2\} \not\equiv 0$. Then any planar polynomial differential system which admits $f_1 = 0$ and $f_2 = 0$ as invariant algebraic curves can be written as*

$$\dot{x} = \varphi_1\{x, f_2\} + \varphi_2\{f_1, x\}, \quad \dot{y} = \varphi_1\{y, f_2\} + \varphi_2\{f_1, y\},$$

where $\varphi_1 = \lambda_1 f_1$ and $\varphi_2 = \lambda_2 f_2$, with λ_1 and λ_2 being arbitrary polynomial functions.

Using this theorem we will prove Theorem 1.

Proof of Theorem 1. First for the cases (ii) and (iii) noting that

$$\{x, f_2\} = 0, \ \{y, f_2\} = -1, \ \{f_1, x\} = 2\frac{a^2 - b^2}{a^2}y, \ \{f_1, y\} = \frac{2x}{a^2 - b^2},$$

and applying Theorem 4 we can write systems (1) of degree ≤ 2 having the hyperbola and the straight line given in (5) as invariant algebraic curves into the form

$$\begin{aligned} \dot{x} = & \ 2\lambda_2\frac{a^2 - b^2}{a^2}y(x - \delta), \\ \dot{y} = & \ -\lambda_1\left(\frac{x^2}{a^2 - b^2} - \frac{a^2 - b^2}{a^2}y^2 - 1\right) + \frac{2\lambda_2}{a^2 - b^2}x(x - \delta), \end{aligned}$$

where λ_1, λ_2 are arbitrary constants. Then we have system (2).

Second for the case (i), noting that

$$\{x, f_2\} = 0, \ \{y, f_2\} = -1, \ \{f_1, x\} = -2x, \ \{f_1, y\} = 2y,$$

and applying Theorem 4 we can write systems (1) of degree ≤ 2 having the hyperbola and the straight line given in (6) as invariant algebraic curves into the form

$$\begin{aligned} \dot{x} = & \ -2\lambda_2 x(x - r), \\ \dot{y} = & \ -\lambda_1(2xy - 1) + 2\lambda_2 y(x - r), \end{aligned}$$

where λ_1, λ_2 are arbitrary constants, obtaining system (3). $\square$

Proof of Theorem 2. Statements (a) and (c) follow easily. It is immediate that the function H given in statement (b) or (d) on the orbits of system (2) or (3) satisfies

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0.$$

So H is a first integral of system (2) or (3), and this proves statement (b) and (d).

Since both first integrals are defined in the whole plane except perhaps on the invariant straight line $x = \delta$, or $x = r$, the systems has no limit cycles. This completes the proof of the theorem. □

If $C = 0$ in systems (2) and (3), then it is easy to verify that they are equivalent to a linear differential system with a saddle and the straight line $x = \delta$ or $x = r$ filled of singular points, respectively. Then the phase portraits of systems (2) and (3) are shown in last picture of Figures 1 and 2 with the title $C = 0$, respectively. Assume $C \neq 0$. Doing the rescaling of the time $\tau = Ct$, and renaming $\rho = c/C$ system (2) becomes

$$\begin{aligned} \dot{x} &= \rho\frac{a^2-b^2}{a^2}y(x-\delta), \\ \dot{y} &= \frac{x^2}{a^2-b^2} - \frac{a^2-b^2}{a^2}y^2 - 1 + \frac{\rho}{a^2-b^2}x(x-\delta), \end{aligned} \tag{7}$$

and the quadratic system corresponding to (3) writes as

$$\begin{aligned} \dot{x} &= -\rho x(x-r), \\ \dot{y} &= 2xy - 1 + \rho y(x-r), \end{aligned} \tag{8}$$

with $\rho \in \mathbb{R}$ and $r \geq 0$.

Remark 5. *In system* (8) *we only consider the case $r \geq 0$. If $r < 0$, then it can be changed into the case of $r \geq 0$ by the transformation $(x, y, t) \to (-x, -y, -t)$.*

System (7) *is reversible because it does not change under the transformation $(x, y, t) \to (x, -y, -t)$. Hence we know that the phase portrait of system* (7) *is symmetric with respect to the $x-$axis.*

In the following sections we shall prove our main Theorem 3 for systems (7) and (8).

3. Phase portraits of system (8)

In this section we consider the case of $a^2 - b^2 = 0$, and take the normal form as system (8).

3.1. **The finite singular points.** The finite singular points of system (8) are characterized in the following result.

Proposition 6. *System* (8) *has the following finite singular points.*

(a) *If $\rho = 0$ all the points of the hyperbola $2xy - 1 = 0$.*
(b) *If $\rho \neq 0$ and $r = 0$ there is no singular point.*

(c) *If* $\rho > 0$ *and* $r \neq 0$ *the singular points are* $B_1(0, -1/(\rho r))$ *and* $B_2(r, 1/(2r))$, *and are saddles.*
(d) *If* $\rho < 0$ *and* $r \neq 0$ *the singular points are* $B_1(0, -1/(\rho r))$ *and* $B_2(r, 1/(2r))$, *the first is a saddle and the second a node.*

Proof. It follows easily from (8) that statements (a) and (b) hold. Noting that the Jacobian matrices of system (8) at the points B_1 and B_2 are

$$\begin{pmatrix} \rho r & 0 \\ -\dfrac{\rho+2}{2r} & -\rho r \end{pmatrix}, \quad \begin{pmatrix} -\rho r & 0 \\ \dfrac{\rho+2}{2r} & 2r \end{pmatrix}$$

respectively, it follows the proof of statements (c) and (d). □

3.2. **The infinite singular points.**

Proposition 7. *System* (8) *has the following infinite singular points.*

(a) *If* $\rho = -1$ *the infinity of system* (8) *is filled of singular points.*
(b) *If* $\rho \neq -1$, *system* (8) *has two pairs of infinite singular points. There exits a pair of infinite nodes for* $\rho < -1$ *and* $\rho > 0$, *while a pair of infinite saddles for* $-1 < \rho < 0$. *The other pair of infinite singular points are the union of a parabolic sector and a hyperbolic sector if* $\rho < -1$ *and* $r \geq 0$, *if* $\rho > 0$ *and* $r = 0$, *while they are a pair of infinite singular points each one formed by the union of a parabolic sector and an elliptic sector if* $-1 < \rho < 0$ *and* $r \geq 0$, *or if* $\rho > 0$ *and* $r > 0$.

Proof. Doing the change of variables we take

$$x = \frac{1}{v}, \qquad y = \frac{u}{v}, \tag{9}$$

and the time rescaling $t = v\tau$, system (8) in the coordinates (u, v) is

$$\begin{aligned} \dot{u} &= -v^2 - 2r\rho uv + 2(\rho+1)u, \\ \dot{v} &= \rho v(1 - rv). \end{aligned} \tag{10}$$

Obviously system (10) has a unique singular point $(0, 0)$, which is an unstable node for $\rho > 0$, a saddle if $-1 < \rho < 0$, and a stable node for $\rho < -1$.

Doing the change of variables

$$x = \frac{u}{v}, \qquad y = \frac{1}{v}, \tag{11}$$

and the time rescaling $t = v\tau$, system (8) becomes

$$\begin{aligned} \dot{u} &= -u(2(\rho+1)u - 2r\rho v - v^2), \\ \dot{v} &= v(v^2 - (\rho+2)u + r\rho v). \end{aligned} \tag{12}$$

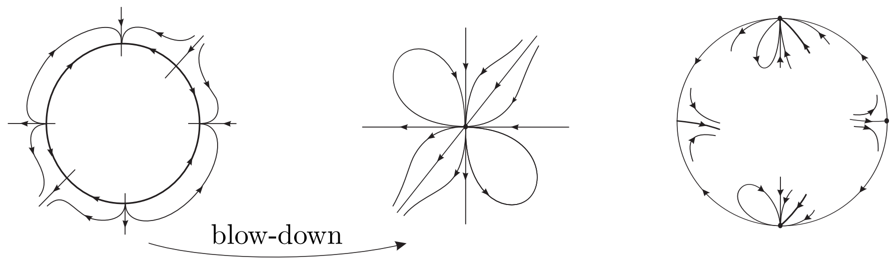

FIGURE 4. Polar blow-down of the singular points of system (13).

Hence $(0,0)$ is a degenerated singular point. Using the blowing-up technique we obtain that it is formed by a pair of parabolic sectors and an elliptic sector if $-1 < \rho < 0$ and $r \geq 0$, or $\rho > 0$ and $r > 0$. And it is formed by a pair of parabolic sectors and a hyperbolic sectors if $\rho < -1$ and $r \geq 0$, or $\rho > 0$ and $r = 0$.

As an example we study the case $-1 < \rho < 0$ and $r > 0$. Considering the degenerated singular point $(0,0)$ of (12), using the polar blowing-up,

$$u = \gamma \cos\theta \text{ and } v = \gamma \sin\theta,$$

we have

$$\begin{aligned} \dot{\gamma} &= ((\cos^2\theta + 1)(r\sin\theta - \cos\theta)\rho - 2\cos\theta)\gamma^2 + \sin^2\theta\gamma^3, \\ \dot{\theta} &= -\rho\cos\theta\sin\theta(r\sin\theta - \cos\theta)\gamma. \end{aligned} \tag{13}$$

System (13) has simple zeroes $\theta = 0, \pi/2, \pi, 3\pi/2$ and $\pm\theta^*$ on $\gamma = 0$, where θ^* satisfies $r\sin\theta - \cos\theta = 0$. It is easy to verify that $\theta = 0, \pi/2$ are stable nodes, $\theta = \pi, 3\pi/2$ are unstable nodes and $\pm\theta^*$ are saddles. Hence doing blow-down we get the phase portrait in a neighborhood of the origin of system (13), shown in Figure 4. Furthermore, taking into account the time scaling transformation $t = v\tau$ and that the infinite singular point of (10) is a saddle, we obtain the phase portraits near the boundary of the Poincaré disk in Figure 4. □

Proof of the phase portraits of Figure 1 for system (8) *in Theorem 3.* We define the following regions in the $(\rho, r)-$ plane:

$$\begin{aligned} \omega_1 = &\ \{(\rho, r)\ :\ \rho < -1,\ r > 0\}, \\ \omega_2 = &\ \{(\rho, r)\ :\ -1 < \rho < 0,\ r > 0\}, \\ \omega_3 = &\ \{(\rho, r)\ :\ 0 < \rho,\ r > 0\}, \end{aligned}$$

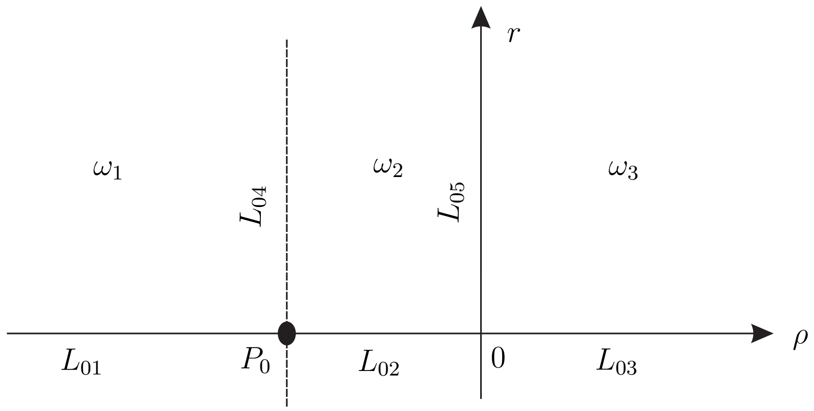

FIGURE 5. The bifurcation diagram of system (8).

the straight lines:

$$
\begin{aligned}
L_{01} = &\quad \{(\rho, r) \ : \ \rho < -1, \ r = 0\}, \\
L_{02} = &\quad \{(\rho, r) \ : \ -1 < \rho < 0, \ r = 0\}, \\
L_{03} = &\quad \{(\rho, r) \ : \ 0 < \rho, \ r = 0\}, \\
L_{04} = &\quad \{(\rho, r) \ : \ \rho = -1, \ r > 0\}, \\
L_{05} = &\quad \{(\rho, r) \ : \ \rho = 0, \ r \geq 0\},
\end{aligned}
$$

and the point $P_0 = (-1, 0)$. In view of Propositions 6 and 7, we show the bifurcation diagram of system (8) with respect to the parameters ρ and r in Figure 5.

From Theorem 2, Propositions 6 and 7, using the invariant straight lines $x = 0$ and $x = r$ with $r \geq 0$, and the invariant hyperbola $2xy = 1$, we obtain the global phase portraits of system (8) in the Poincaré disc described in Figure 1. This completes the proof of Theorem 3. □

4. Phase portraits of system (7)

In this section we study system (7) for $a^2 - b^2 \neq 0$ and $a \neq 0$.

4.1. **The finite singular points.**

The finite singular points of system (7) are characterized in the following result.

Proposition 8. *System* (7) *has the following finite singular points.*

(a) *If $\rho = 0$ all the points of the hperbola* $\dfrac{x^2}{a^2 - b^2} - \dfrac{a^2 - b^2}{a^2} y^2 = 1.$

(b) *For $\delta = 1$, if $\rho \notin \{-1, 0\}$ the singular points are*

$$
(14)\quad
\begin{aligned}
M_{\pm} &= (1, y^*_{\pm}) = \left(1, \pm\frac{a\sqrt{1-(a^2-b^2)}}{a^2-b^2}\right) && \text{if } a^2-b^2 \le 1,\\
N_{\pm} &= (x^*_{\pm}, 0) = \left(\frac{\rho \pm \sqrt{\Delta}}{2(\rho+1)}, 0\right) && \text{if } \Delta \ge 0,
\end{aligned}
$$

where $\Delta = \rho^2 + 4(\rho+1)(a^2-b^2)$.

If $\rho = -1$, system (7) *has the singular point $N^c = (a^2-b^2, 0)$, and the two singular points $M_{\pm}$ if $a^2-b^2 < 1$, or the unique singular point $M_{\pm} = N^c = (1, 0)$ if $a^2 - b^2 = 1$.*

(c) *For $\delta = 0$, if $\rho \notin \{-1, 0\}$ the singular points are*

$$
(15)\quad
\begin{aligned}
M^0_{\pm} &= (0, y^0_{\pm}) = \left(0, \pm\frac{a}{\sqrt{b^2-a^2}}\right) && \text{if } a^2-b^2 < 0,\\
N^0_{\pm} &= (x^0_{\pm}, 0) = \left(\pm\sqrt{\frac{a^2-b^2}{\rho+1}}, 0\right) && \text{if } \frac{a^2-b^2}{\rho+1} > 0.
\end{aligned}
$$

If $\rho = -1$ the singular points are $M^0_{\pm}$.

Proof. The proof follows easily studying the real solutions of system (7). □

Denote $\eta = a^2 - b^2$, we write the curve $\Delta = 0$ of Proposition 8 in the plane of (ρ, η) as

$$
\eta(\rho) = -\frac{\rho^2}{4(\rho+1)}, \tag{16}
$$

which is the hyperbola with the two branches $\eta_{\pm}$ corresponding to $\rho < -1$ and $\rho > -1$, respectively.

Now we define the following regions when $\delta = 1$:

$$
\begin{aligned}
R_{01} = &\ \{(\rho, a^2-b^2)\ :\ \rho < -1,\ a^2-b^2 > \eta_+(\rho)\},\\
R_{02} = &\ \{(\rho, a^2-b^2)\ :\ -1 < \rho < 0,\ a^2-b^2 < \eta_-(\rho)\},\\
R_{03} = &\ \{(\rho, a^2-b^2)\ :\ 0 < \rho,\ a^2-b^2 < \eta_-(\rho)\},\\
R_{1} = &\ \{(\rho, a^2-b^2)\ :\ \rho < -2,\ 1 < a^2-b^2 < \eta_+(\rho)\},\\
R_{2} = &\ \{(\rho, a^2-b^2)\ :\ -2 < \rho < -1,\ 1 < a^2-b^2 < \eta_+(\rho)\},\\
R_{3} = &\ \{(\rho, a^2-b^2)\ :\ -1 < \rho < 0,\ a^2-b^2 > 1\},\\
R_{4} = &\ \{(\rho, a^2-b^2)\ :\ \rho > 0,\ a^2-b^2 > 1\},\\
R_{5} = &\ \{(\rho, a^2-b^2)\ :\ \rho < -1,\ 0 < a^2-b^2 < 1\},\\
R_{6} = &\ \{(\rho, a^2-b^2)\ :\ -1 < \rho < 0,\ 0 < a^2-b^2 < 1\},\\
R_{7} = &\ \{(\rho, a^2-b^2)\ :\ 0 < \rho,\ 0 < a^2-b^2 < 1\},\\
R_{8} = &\ \{(\rho, a^2-b^2)\ :\ \rho < -1,\ a^2-b^2 < 0\},\\
R_{9} = &\ \{(\rho, a^2-b^2)\ :\ -1 < \rho < 0,\ \eta_-(\rho) < a^2-b^2 < 0\},\\
R_{10} = &\ \{(\rho, a^2-b^2)\ :\ \rho > 0,\ \eta_-(\rho) < a^2-b^2 < 0\},
\end{aligned}
$$

the curves:

$$
\begin{aligned}
L_{0} = &\ \{(\rho, a^2-b^2)\ :\rho = 0\},\\
L_{11} = &\ \{(\rho, a^2-b^2)\ :\ \rho < -2,\ a^2-b^2 = \eta_+(\rho)\},\\
L_{12} = &\ \{(\rho, a^2-b^2)\ :\ -2 < \rho < -1,\ a^2-b^2 = \eta_+(\rho)\},\\
L_{21} = &\ \{(\rho, a^2-b^2)\ :\ -1 < \rho < 0,\ a^2-b^2 = \eta_-(\rho)\},\\
L_{22} = &\ \{(\rho, a^2-b^2)\ :\ 0 < \rho,\ a^2-b^2 = \eta_-(\rho)\},\\
L_{31} = &\ \{(\rho, a^2-b^2)\ :\rho < -2,\ a^2-b^2 = 1\},\\
L_{32} = &\ \{(\rho, a^2-b^2)\ :-2 < \rho < -1,\ a^2-b^2 = 1\},\\
L_{33} = &\ \{(\rho, a^2-b^2)\ :-1 < \rho < 0,\ a^2-b^2 = 1\},\\
L_{34} = &\ \{(\rho, a^2-b^2)\ :0 < \rho,\ a^2-b^2 = 1\},\\
L_{41} = &\ \{(\rho, a^2-b^2)\ :\rho = -1,\ a^2-b^2 < 0\},\\
L_{42} = &\ \{(\rho, a^2-b^2)\ :\rho = -1,\ 0 < a^2-b^2 < 1\},\\
L_{43} = &\ \{(\rho, a^2-b^2)\ :\rho = -1,\ 1 < a^2-b^2\},\\
L_{0+} = &\ \{(\rho, a^2-b^2)\ :\rho = 0,\ 0 < a^2-b^2\},\\
L_{0-} = &\ \{(\rho, a^2-b^2)\ :\rho = 0,\ a^2-b^2 < 0\},
\end{aligned}
$$

and the points $P_1 = (-1, 1)$ and $P_2 = (-2, 1)$, see Figure 6. We also define the following regions when $\delta = 0$:

$$
\begin{aligned}
\Omega_1 = &\ \{(\rho, a^2-b^2)\ :\ \rho < -1,\ a^2-b^2 < 0\},\\
\Omega_2 = &\ \{(\rho, a^2-b^2)\ :\ -1 < \rho < 0,\ a^2-b^2 < 0\},\\
\Omega_3 = &\ \{(\rho, a^2-b^2)\ :\ 0 < \rho,\ a^2-b^2 < 0\},\\
\Omega_4 = &\ \{(\rho, a^2-b^2)\ :\ 0 < \rho,\ a^2-b^2 > 0\},\\
\Omega_5 = &\ \{(\rho, a^2-b^2)\ :\ -1 < \rho < 0,\ a^2-b^2 > 0\},\\
\Omega_6 = &\ \{(\rho, a^2-b^2)\ :\ \rho < -1,\ a^2-b^2 > 0\},
\end{aligned}
$$

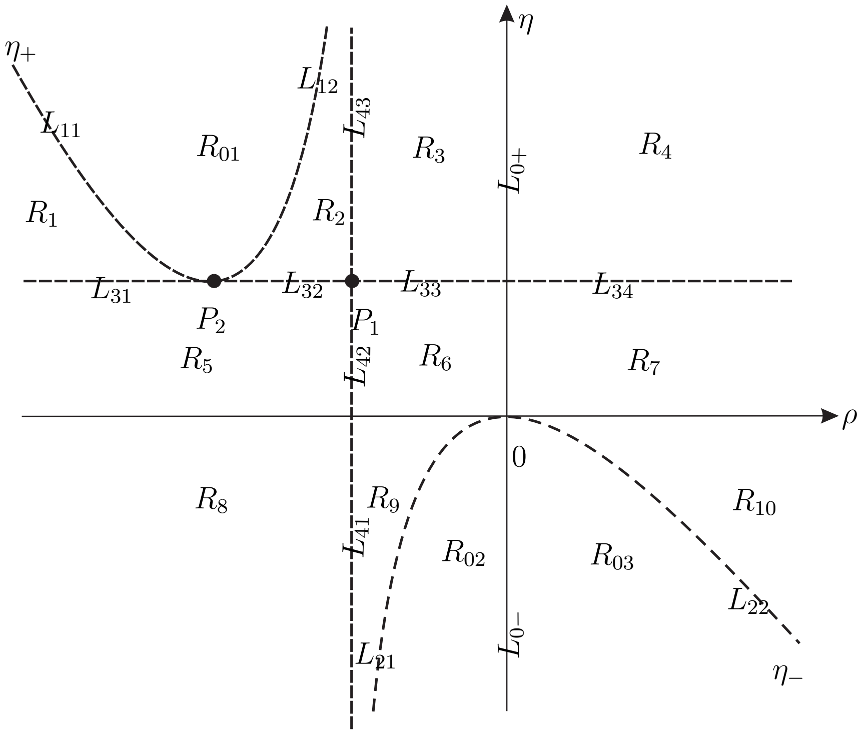

FIGURE 6. The bifurcation diagram of system (7) when $\delta = 1$.

and the straight lines L_{0+}, L_{0-} and

$$
\begin{aligned}
L_{44} = &\ \{(\rho, a^2 - b^2)\ : \rho = -1,\ a^2 - b^2 < 0\},\\
L_{45} = &\ \{(\rho, a^2 - b^2)\ : \rho = -1,\ 0 < a^2 - b^2\},
\end{aligned}
$$

shown in Figure 7.

Proposition 9. *Systems* (7) *has the following finite singular points if its parameters* $(\rho, a^2 - b^2)$ *are in*

- $(R_1 \cup R_2)$ *a saddle* N_+ *and a center* N_-*, or a center* N_+ *and a saddle* N_-*.*
- (R_5) *four singular points: a stable node* M_+*, an unstable node* M_-*, and two saddles* $N_\pm$*.*
- (R_6) *four singular points: a stable node* M_+*, an unstable node* M_-*, and a saddle* N_+ *and a center* N_-*.*
- (R_7) *four singular points: two saddles* $M_\pm$*, a center* N_+ *and a saddle* N_-*.*
- (R_9) *four singular points: a stable node* M_-*, an unstable node* M_+*, a saddle* N_+ *and a center* N_-*.*
- (R_{10}) *four singular points: three saddles* $M_\pm$ *and* N_-*, and a center* N_+*.*

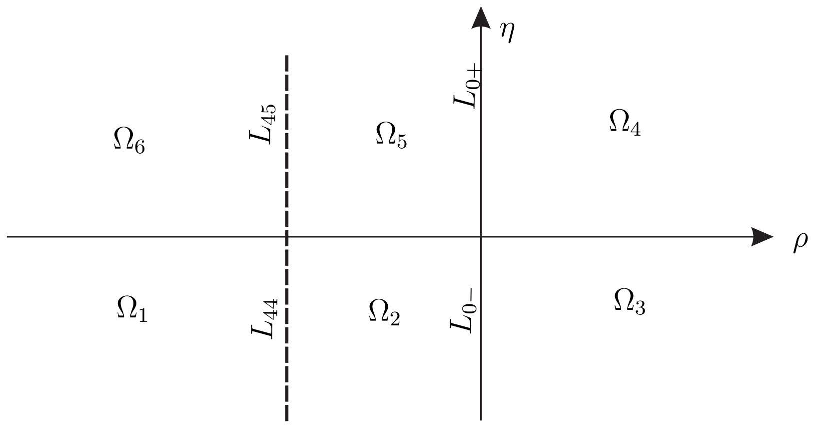

FIGURE 7. The bifurcation diagram of sysetem (7) when $\delta = 0$.

($\Omega_1 \cup R_8$) *four singular points: an unstable node M^0_+ or M_+, a stable node M^0_- or M_-, and saddles $N^0_\pm$ or $N_\pm$.*

($\Omega_2 \cup R_{02} \cup L_{44}$) *two singular points: an unstable node M^0_+ or M_+, and a stable node M^0_- or M_-.*

($\Omega_3 \cup R_{03}$) *two singular points: $M^0_\pm$ or $M_\pm$ are saddles.*

($\Omega_4 \cup R_4$) *two singular points: saddles $N^0_\pm$ or $N_\pm$.*

($\Omega_5 \cup R_3$) *two singular points: centers $N^0_\pm$ or $N_\pm$.*

($\Omega_6 \cup R_{01} \cup L_{45}$) *no singular points.*

($L_0 + \cup L_{0-}$) *all singular points on the hyperbola $\mathcal{H}$.*

($L_{11} \cup L_{12}$) *a nilpotent cusp N.*

(L_{21}) *three singular points: an unstable node M_+, a stable node M_-, and a nilpotent cusp N.*

(L_{22}) *three singular points: two saddles $M_\pm$, and a nilpotent cusp N.*

(L_{31}) *two singular points: a nilpotent singular point $M_\pm = N_- = (1, 0)$ union of one elliptic sector with one hyperbolic sector, and a hyperbolic saddle $N_+ = (-1/(\rho + 1), 0)$.*

(L_{32}) *two singular points: a nilpotent singular point $M_\pm = N_+ = (1, 0)$ union of one elliptic sector with one hyperbolic sector, and a hyperbolic saddle $N_- = (-1/(\rho + 1), 0)$.*

(L_{33}) *two singular points: a nilpotent singular point $M_\pm = N_+ = (1, 0)$ union of one elliptic sector with one hyperbolic sector, and a center $N_- = (-1/(\rho + 1), 0)$.*

(L_{34}) *two singular points: a nilpotent saddle $M_\pm = N_+ = (1, 0)$, and a hyperbolic saddle $N_- = (-1/(\rho + 1), 0)$.*

(L_{41}) *three singular points: a saddle N^c, an unstable hyperbolic node M_+, and a stable hyperbolic node M_-.*

(L_{42}) *three singular points: a saddle* N^c, *an unstable hyperbolic node* M_-, *and a stable hyperbolic node* M_+.

(L_{43}) *a center* N^c.

(P_1) $M_\pm = N_\pm = (1,0)$ *is a nilpotent singular point formed by one elliptic sector, one hyperbolic sector and two parabolic sectors.*

(P_2) $M_\pm = N_\pm = (1,0)$ *is a degenerated singular point formed by two parabolic sectors and two hyperbolic sectors.*

Proof. On L_{0+} and L_{0-} we have $\rho = 0$. Hence the straight lines $x =$ constant are invariant of system (7), and the hyperbola (4) is filled with singular points, see the phase portraits for L_{0+} and L_{0-} in Fig. 2.

In the following we always assume $\rho \neq 0$.

(**A**). Assume $\delta = 1$ and distinguish two cases in the study of the finite singular points of system (7).

Case 1: On the invariant straight line $x = 1$. There are two singular points $M_\pm$ of system (7) when $a^2 - b^2 < 1$; They coincide into a unique singular point $M(1,0)$ when $a^2 - b^2 = 1$, and no singular point when $a^2 - b^2 > 1$, see (14).

Subcase 1.1: $a^2 - b^2 < 1$. The Jacobian matrix of system (7) at $M_\pm$ are

$$\begin{pmatrix} \dfrac{\rho(a^2-b^2)y_\pm^*}{a^2} & 0 \\ 0 & -\dfrac{2(a^2-b^2)y_\pm^*}{a^2} \end{pmatrix}.$$

Therefore $M_\pm$ are saddles if $\rho > 0$, and M_+ is a stable hyperbolic node and M_- is an unstable hyperbolic node if $\rho < 0$ and $0 < a^2 - b^2 < 1$, while M_+ is an unstable hyperbolic node and M_- is a stable hyperbolic node if $\rho < 0$ and $a^2 - b^2 < 0$, see for more details Theorem 2.15 of [8] where are described the local phase portraits of the hyperbolic singular points.

Subcase 1.2: $a^2 - b^2 = 1$. The Jacobian matrix of system (7) at M is

$$J_M = \begin{pmatrix} 0 & 0 \\ \rho+2 & 0 \end{pmatrix}.$$

When $\rho \neq -2$, M is a nilpotent singular point. Using Theorem 3.5 of [8] for studying the local phase portraits of the nilpotent singular points we get that M is a nilpotent saddle if $\rho > 0$, and if $\rho < 0$ and different from -2 is union of one elliptic sector with one hyperbolic sector.

When $\rho = -2$, $N_\pm$ and M meet with each other into a degenerated singular point M. Using the polar blowing–up centered at M, i.e.

$x = r\cos\theta + 1$ and $y = r\sin\theta$, system (7) writes

$$\dot{r} = -r^2 \sin\theta(\cos^2(\theta)(a^2+1)+1), \qquad (17)$$
$$\dot{\theta} = -r\cos\theta(\cos^2(\theta)(a^2+1)-1).$$

The singular points of system (17) on $\{r = 0\}$ are located at $\theta = \pm\pi/2$, θ_k, where θ_k satisfies

$$(a^2+1)\cos^2\theta_k = 1, \quad k = 1,2,3,4,$$

and $-\pi < \theta_3 = \theta_1 - \pi < \theta_4 = \theta_2 - \pi < \theta_1 < \theta_2 = \pi - \theta_1 < \pi$. All the singularities on $S^1 \times \{0\}$ are hyperbolic. Then $(0, \pi/2)$ is a stable node $(0, -\pi/2)$ an unstable node and θ_k saddles. Doing a blowing down we obtain that M is formed by the union of two hyperbolic sectors and four parabolic sectors, see the phase portrait P_2 in Figure 2.

Case 2: Singular points on the straight line $y = 0$.

When $a^2 - b^2 < 0$ the singular points $N_\pm$ or N are located between the two branches of the hyperbola. When $a^2 - b^2 > 0$ it is easy to check that there are two singular points $N_\pm$ on $y = 0$ if and only if $(\rho, a^2 - b^2) \in R_1 \cup R_2 \cup \cdots \cup R_7 \cup L_{31} \cup \cdots \cup L_{34}$. The singular points $N_\pm$ coincide with N on $y = 0$ if and only if $(\rho, a^2 - b^2) \in L_{11} \cup L_{12} \cup L_{21} \cup L_{22}$, and with N^c if and only if $\rho = -1$. It is important for the phase portraits the location of the singular points $N_\pm$ or N and of the two branches of the hyperbola, see Figures 2, 3 and 6.

Subcase 2.1: The distribution of the singular points on $y = 0$.

If $a^2 - b^2 > 0$, it is easy to check that

$$(x_+^* \pm \sqrt{a^2-b^2})(x_-^* \pm \sqrt{a^2-b^2}) = \frac{\rho\sqrt{a^2-b^2}}{\rho+1}(\sqrt{a^2-b^2} \mp 1), \qquad (18)$$
$$(x_+^* - 1)(x_-^* - 1) = \frac{1-(a^2-b^2)}{\rho+1}.$$

which implies that in $R_1 \cup R_2$, that is when $a^2 - b^2 > 1$ and $\rho < -1$, the singular points $N_\pm$ are located at the same side of the hyperbola $\mathcal{H}$ and of the line $\mathcal{L}$ given in (5). Furthermore, we have from (14) that

$$x_-^* > \frac{\rho}{2(\rho+1)} > 1, \quad \text{for } -2 < \rho < -1,$$

and

$$x_-^* < \frac{\rho - \sqrt{\rho^2 + 4(\rho+1)}}{2(\rho+1)} = 1, \quad \text{for } \rho < -2,$$

Similarly, in R_3, that is when $a^2 - b^2 > 1$ and $1 < \rho < 0$, the singular points $N_\pm$ are located at the two sides of the hyperbola $\mathcal{H}$ and of the line $\mathcal{L}$, respectively. In R_4, that is when $a^2 - b^2 > 1$ and $\rho > 0$, the singular points $N_\pm$ are located at the same side of the hyperbola $\mathcal{H}$, while in the two sides of the line $\mathcal{L}$, respectively.

In fact we have that $-\sqrt{a^2-b^2} < x_+^* < x_-^* < 1$ in R_1, $1 < \sqrt{a^2-b^2} < x_+^* < x_-^*$ in R_2, $x_-^* < -\sqrt{a^2-b^2} < \sqrt{a^2-b^2} < x_+^*$ in R_3, and $-\sqrt{a^2-b^2} < x_-^* < 1 < x_+^* < \sqrt{a^2-b^2}$ in R_4.

In the same way we can obtain that $-\sqrt{a^2-b^2} < x_+^* < \sqrt{a^2-b^2} < 1 < x_-^*$ in R_5, $x_-^* < -\sqrt{a^2-b^2} < x_+^* < \sqrt{a^2-b^2} < 1$ in R_6, and $-\sqrt{a^2-b^2} < x_-^* < \sqrt{a^2-b^2} < x_+^* < 1$ in R_7.

For $a^2 - b^2 < 0$ it follows from (14) and (18) that $x_+^* < 0 < 1 < x_-^*$ in R_8, $x_-^* < x_+^* < 0$ in R_9, and $0 < x_-^* < x_+^* < 1$ in R_{10}.

On the curves $L_{11} \cup L_{12} \cup L_{21} \cup L_{22}$, if $\Delta = 0$, then the singular points $N_\pm$ meet into a unique singular point N on $y = 0$, i.e. $x_\pm^* = x^*$. In view of (16) it follows that $0 < x^* < 1$ in $L_{11} \cup L_{22}$, $x^* > \sqrt{a^2-b^2}$ in L_{12}, and $x^* < 0$ in L_{21}.

If $a^2 - b^2 = 1$ then $0 < x_+^* = -1/(\rho+1) < x_-^* = 1$ in L_{31}, $1 = x_+^* < x_-^* = -1/(\rho+1)$ in L_{32}, $x_-^* = -1/(\rho+1) < -1 < x_+^* = 1$ in L_{33}, and $-1 < x_-^* = -1/(\rho+1) < x_+^* = 1$ in L_{34}.

Subcase 2.2: Classification of the singular points.

If $a^2-b^2 = 1$ there are two singular points on $y = 0$, N_- and $M = N_+$ for $\rho > -2$, while N_+ and $M = N_-$ for $\rho < -2$. The singular point M is also on the invariant straight line $x = 1$, which has been studied in Subcase 1.2. The Jacobian matrix of system (7) at the other singular point N_+ or N_- is

$$J = \begin{pmatrix} 0 & -\dfrac{\rho(\rho+2)}{a^2(\rho+1)} \\ -(\rho+2) & 0 \end{pmatrix}.$$

Using the fact that system (7) is reversible with respect to the x–axis from Remark 5, we can obtain that N_+ is a saddle in L_{31}, N_- a saddle in $L_{32} \cup L_{34}$, and N_- a center in L_{33}.

If $\rho \neq -1$ and $\Delta > 0$, then system (7) has two singular points $N_\pm = (x_\pm^*, 0)$ on $y = 0$, see (14). The Jacobian matrix of system (7) at the points $N_\pm$ is

$$J = \begin{pmatrix} 0 & \dfrac{\rho(a^2-b^2)}{a^2}(x_\pm^* - 1) \\ \dfrac{\pm\sqrt{\Delta}}{a^2-b^2} & 0 \end{pmatrix}. \tag{19}$$

It is easy from Remark 5 to prove that N_+ is a saddle and N_- a center in $R_1 \cup R_6 \cup R_9$, N_+ is a center and N_- a saddle in $R_2 \cup R_7 \cup R_{10}$, $N_\pm$ are centers in R_3, and $N_\pm$ are saddles in $R_4 \cup R_5 \cup R_8$.

If $\Delta = 0$ we have from (16) that $\eta = \eta_\pm(\rho)$, and from (19) the singular point $N = (x^*, 0)$ is nilpotent. Taking $(x, y) = (X + x^*, Y)$, after $(X, Y) = (x, y)$, and rescaling the independent variable t by $\tau = \rho^3(\rho+2)t/(8a^2(\rho+1)^2)$, we obtain

$$\begin{aligned} \dot{x} &= \quad y - \frac{2(\rho+1)}{\rho+2} xy, \\ \dot{y} &= \quad -32\frac{(\rho+1)^4 a^2}{\rho^5(\rho+2)} x^2 + \frac{2(\rho+1)}{\rho(\rho+2)} y^2. \end{aligned}$$

By Theorem 3.5 of [8] the origin of the previous system is a cusp in $L_{11} \cup L_{12} \cup L_{21} \cup L_{22}$.

If $\rho = -1$ system (7) has a singular point N^c on $y = 0$. The Jacobian matrix of system (7) at the point N^c is

$$J = \begin{pmatrix} 0 & -\dfrac{(a^2-b^2)(a^2-b^2-1)}{a^2} \\ \dfrac{1}{a^2-b^2} & 0 \end{pmatrix},$$

which implies that N^c is a saddle in $L_{41} \cup L_{42}$, and a center in L_{43}. If $a^2 - b^2 = 1$ system (7) has a unique singular point $M = N^c$ at P_1, which is union of one elliptic sector with one hyperbolic sector as in the proof in Subcase 1.2.

(**B**). Assume $\delta = 0$. There are two singular points $M_\pm^0$ on $x = 0$ and two singular points $N_\pm^0$ on $y = 0$ when $a^2 - b^2 < 0$ and $\rho < -1$, two singular points $M_\pm^0$ on $x = 0$ when $a^2 - b^2 < 0$ and $-1 \leq$, two singular points $N_\pm^0$ on $y = 0$ outside of the two branches of hyperbola when $a^2 - b^2 > 0$ and $-1 < \rho < 0$, and two singular points $N_\pm^0$ on $y = 0$ between the two branches of hyperbola when $a^2 - b^2 > 0$ and $\rho > 0$. There is no singular point when $a^2 - b^2 > 0$ and $\rho \leq -1$. See (15) in Proposition 8 and Figure 3.

The Jacobian matrices of system (7) at the points $M_\pm^0$ and $N_\pm^0$ are

$$\begin{pmatrix} \dfrac{\rho(a^2-b^2)y_\pm^0}{a^2} & 0 \\ 0 & -\dfrac{2(a^2-b^2)y_\pm^0}{a^2} \end{pmatrix}, \quad \begin{pmatrix} 0 & \dfrac{\rho(a^2-b^2)}{a^2}x_\pm^0 \\ \dfrac{2(\rho+1)}{a^2-b^2}x_\pm^0 & 0 \end{pmatrix},$$

respectively. Similarly it is easy to obtain that $M_\pm$ are saddles in Ω_3, and M_+ is an unstable node, and M_- a stable node in $\Omega_1 \cup \Omega_2 \cup L_{44}$, $N_\pm$

are saddles in $\Omega_1 \cup \Omega_4$, $N_\pm$ are centers in Ω_5, and there are no singular points in Ω_6 and L_{45}. $\square$

4.2. **The infinite singular points.**

Proposition 10. *The following two statements hold.*

(a) *If $\rho \neq -1$ system (7) has two pairs one of infinite saddles and the other of nodes if $\rho < 0$, and two pairs of nodes if $\rho > 0$.*
(b) *If $\rho = -1$ the infinity of system (7) is filled of singular points.*

Proof. Notice $a^2 - b^2 \neq 0$ for system (7). Doing the Poincaré transformation (9) and the time rescaling $t = v\tau$, system (7) in the local chart (9) is

$$\begin{aligned} \dot{u} &= -v^2 + \left(\frac{(a^2-b^2)u^2}{a^2} - \frac{1}{a^2-b^2} \right) (\delta\rho v - (\rho+1)), \\ \dot{v} &= \rho \frac{a^2-b^2}{a^2} u(\delta v - 1)v. \end{aligned} \tag{20}$$

First considering the infinite singular of system (7) with $\delta = 1$. If $\rho \neq -1$ there is two singular points $P_N \left(\pm \frac{a}{a^2-b^2}, 0 \right)$ of system (20) on $v = 0$. The eigenvalues of the Jacobian matrice at P_N are $\mp \frac{2(\rho+1)}{a}$ and $\mp \frac{\rho}{a}$, which implies that system (7) has two pairs of nodes if $\rho < -1$ or $0 < \rho$, and two pairs of infinite saddles if $-1 < \rho < 0$.

Furthermore, taking the Poincaré transformation (11) and the time rescaling $t = v\tau$, system (7) in the local chart (11) is

$$\begin{aligned} \dot{u} &= uv^2 + \left(\frac{u^2}{a^2-b^2} - \frac{a^2-b^2}{a^2} \right) (\delta\rho v - (\rho+1)u), \\ \dot{v} &= v^3 + \frac{\delta\rho u v^2}{a^2-b^2} + \left(\frac{a^2-b^2}{a^2} - \frac{(\rho+1)u^2}{a^2-b^2} \right) v. \end{aligned} \tag{21}$$

We first consider system (7) with $\delta = 1$. If $\rho \neq -1$ the origin is a singular point of (21). It is easy to get that the eigenvalues of the Jacobian matrix at the origin are $\frac{a^2-b^2}{a^2}(\rho+1)$ and $\frac{a^2-b^2}{a^2}$, which implies that system (7) has a pair of infinite saddles if $\rho < -1$, and a pair of nodes if $\rho > -1$.

If $\rho = -1$ from (20) and (21) the infinity $v = 0$ of the Poincaré disc is filled with singular points. Furthermore we claim that the orbits from the infinity will go to the singular points $M_\pm$. Now we prove the claim.

In fact, when $\rho = -1$ we reduce the common factor v of the vector field of system (20), then we writes it as

$$
\begin{aligned}
\dot{u} &= -v - \left(\frac{(a^2-b^2)u^2}{a^2} - \frac{1}{a^2-b^2} \right), \\
\dot{v} &= -\frac{a^2-b^2}{a^2} u(v-1).
\end{aligned} \tag{22}
$$

We obtain the following first integral of system (22)

$$
H(u,v) := (v-1)^{-2} \left(u^2 + \frac{2a^2}{a^2-b^2} v - \frac{a^2(1+a^2-b^2)}{(a^2-b^2)^2} \right) = (v-1)^{-2} h(u,v).
$$

If $a^2 - b^2 < 1$ system (22) has a pair of singular points $(u,v) = M_\pm$, which are in fact, the singular points of system (7) in the finite plane, see (14). Noting that $M_\pm$ are located on the invariant straight line $v - 1 = 0$, and $h(u,v)|_{M_\pm} = 0$, we know that $M_\pm$ have to be located on the curve $H(u,v) = c$ for any $c \in \mathbb{R}$, especially on the curve going through $v = 0$, see (a) in Figure 8.

We complete the proof of our claim. The phase portraits of system (7) in $L_{41} \cup L_{42}$ are shown in Figure 2. In L_{43} system (7) has no singular point on $v - 1 = 0$, so any curve $H(u,v) = c$ does not intersect with the line $v - 1 = 0$.

Next we study the infinite singular points of system (7) with $\delta = 0$. Taking $\delta = 0$ in system (20) and (21), we obtain two singular points P_N of system (20), and the singular point $(0,0)$ of system (21) if $\rho \neq -1$. In a similar way as the above, we can get that their singular points have the same properties as the systems with $\delta = 1$. But if $\rho = -1$ system (7) in the local chart (9) is

$$
\begin{aligned}
\dot{u} &= -v^2, \\
\dot{v} &= \frac{a^2-b^2}{a^2} uv.
\end{aligned} \tag{23}
$$

So the infinity $v = 0$ of the Poincaré disc is filled with singular points. System (23) is equivalent to a linear differential system with a saddle if $a^2 - b^2 < 0$, or with a center if $a^2 - b^2 > 0$, and in both case with a straight line filled by singularities, see the phase portraits (b) and (c) in Figure 8.

□

Proof of the phase portraits Figures 2 and 3 of system (7) *in Theorem 3.* According to Theorem 2, Propositions 7, 9 and 10, and using the invariant straight line $x = \delta$ with $\delta = 0, 1$ and the invariant hyperbola $\mathcal{H}$

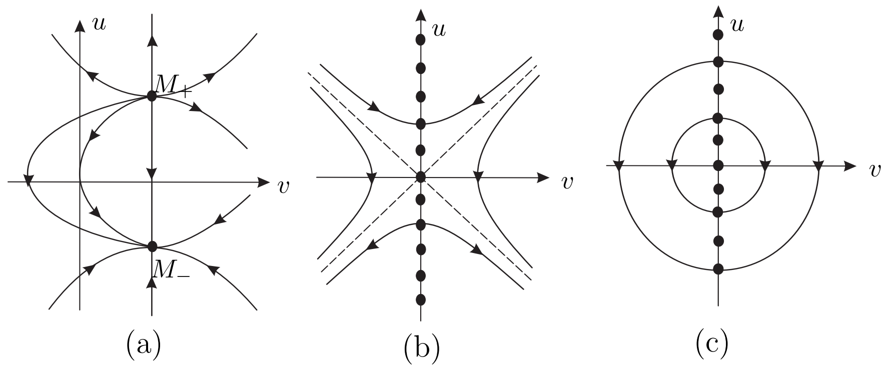

FIGURE 8. Infinite singular points of system (7) for $\rho = -1$.

in (5), we obtain the global phase portraits of system (7) in Poincaré disc described in Figure 2 and 3.

□

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