

## LIMIT CYCLES FOR DISCONTINUOUS PLANAR PIECEWISE LINEAR DIFFERENTIAL SYSTEMS SEPARATED BY ONE STRAIGHT LINE AND HAVING A CENTER

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ABSTRACT. From the beginning of this century more than thirty papers have been published studying the limit cycles of the discontinuous piecewise linear differential systems with two pieces separated by a straight line, but it remains open the following question: *what is the maximum number of limit cycles that this class of differential systems can have?* Here we prove that when one of the linear differential systems has a center, real or virtual, then the discontinuous piecewise linear differential system has at most two limit cycles.

### 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

An isolated periodic orbit in the set of all periodic orbits of a differential system in the plane  $\mathbb{R}^2$  is a *limit cycle*. The study of the limit cycles of the differential systems started at the end of the 19th century with Poincaré [25]. Many phenomena of the real world are related with the existence of limit cycles, some examples are the van der Pol oscillator [28, 29], or the Belousov–Zhabotinskii reaction [3, 31], and many other examples can be found in the book [4], or in the survey [22].

The objective of this paper is to study the limit cycles of the discontinuous piecewise linear differential systems separated by a straight line.

Some of the first studies on the discontinuous piecewise linear differential systems separated by straight lines appeared in the book of Andronov, Vitt and Khaikin [1]. The interest on such differential systems persist until nowadays, mainly for the applications that they have in electrical circuits, mechanics, economy, etc, see the books [4, 27], and the surveys [22, 30].

Limit cycles in the planar discontinuous piecewise linear differential systems can be of two kinds, crossing limit cycles or sliding limit cycles. While these last limit cycles contain some segment of the lines of discontinuity which separate the different linear differential systems (see for more details [24]), the first ones do not contain such segments, only contain isolated points of the lines of discontinuity. Here we only study the crossing limit cycles of the planar discontinuous piecewise linear differential systems separated by pieces of straight lines. In what follows when we talk about limit cycles we are talking about crossing limit cycles.

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Limit cycles of discontinuous piecewise linear differential systems separated by a straight line have been studied by many authors, see for instance [2, 8, 9, 11, 13, 14, 21, 26] and the references therein. There are examples of such systems exhibiting three limit cycles, see [5, 6, 10, 12, 17, 20], and at this moment we do not know if discontinuous piecewise linear differential systems separated by a straight line can have more than three limit cycles.

In the papers [8, 19, 18] it is proved that if one of the two linear differential systems has its equilibrium on the line of discontinuity, then the discontinuous piecewise linear differential system with two pieces separated by a straight line has at most 2 limit cycles.

At this moment there are no upper bounds for the maximum number of limit cycles that a discontinuous piecewise linear differential systems in  $\mathbb{R}^2$  with two pieces separated by a straight line can have, when there is no a singularity of the two linear differential systems on the discontinuity straight line. In the next theorem we present such upper bounds when one of the two linear differential systems has a center, real or virtual.

Our main result is the following.

**Theorem 1.** *Consider a discontinuous piecewise linear differential system in  $\mathbb{R}^2$  formed by two pieces separated by a straight line. If one of the two linear differential systems has a center, real or virtual, then the discontinuous piecewise linear differential system has at most two limit cycles. More precisely, Table 1 holds, where two limit cycles can happen only in the case that the focus is real.*

	F	S	N	N'	C
C	2	1	1	1	0

TABLE 1. Upper bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems in  $\mathbb{R}^2$  with two pieces separated by a straight line when one of the linear differential systems is a center. These upper bounds are realizable. Here C, F, S, N and N' denote center, focus, saddle, node with different eigenvalues and non-diagonalizable node with equal eigenvalues, respectively.

For proving Theorem 1 we shall use the canonical forms of all the possible configurations of the discontinuous piecewise linear differential systems in  $\mathbb{R}^2$  with two pieces separated by a straight line when both systems have a singularity, real or virtual, see section 2. These canonical forms only depend on five parameters and as we mention previously are due to Freire, Ponce and Torres, see [10]. Without these canonical forms we should have twelve parameters.

Theorem 1 is proved in section 3.

## 2. CANONICAL FORMS

Without loss of generality we assume that the two pieces in the plane where is defined a piecewise linear differential system are the left and the right half-planes

$$L = \{(x, y) \in \mathbb{R}^2 : x \leq 0\}, \quad R = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}.$$

Consequently  $x = 0$  is the straight line of separation between the two linear differential systems

$$(1) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^- & a_{12}^- \\ a_{21}^- & a_{22}^- \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix} \quad \text{if } x \leq 0,$$

and

$$(2) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix} \quad \text{if } x \geq 0,$$

where  $a_{ij}^-$ ,  $a_{ij}^+$ ,  $b_s^-$ ,  $b_s^+$ ,  $i, j, s \in \{1, 2\}$ , are real constants.

We consider the piecewise linear differential systems (1)+(2). Note that these systems depend on 12 parameters. In order to reduce the number of this parameters we use the canonical forms given in [10]. Let the discontinuous piecewise linear differential systems be

$$(3) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2\ell & -1 \\ \ell^2 - \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ a \end{pmatrix},$$

defined for the points  $(x, y) \in L$ , and

$$(4) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2r & -1 \\ r^2 - \beta^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix},$$

defined for the points  $(x, y) \in R$ , where  $\alpha, \beta \in \{i, 0, 1\}$ . Of course  $i = \sqrt{-1}$ . Notice that these discontinuous piecewise linear differential systems depend on 5 parameters. If  $\alpha = i$  the equilibrium of system (3) has eigenvalues  $\ell \pm i$ , i.e. it is a focus F if  $\ell \neq 0$ , and a center C if  $\ell = 0$ . If  $\alpha = 0$  then system (3) is a non-diagonalizable node N' with eigenvalue  $\ell \neq 0$  of multiplicity 2. If  $\alpha = 1$  then system (3) is a saddle S with eigenvalues  $\ell - 1$  and  $\ell + 1$  if  $|\ell| < 1$ , and a diagonalizable node N with eigenvalues  $\ell - 1$  and  $\ell + 1$  if  $|\ell| > 1$ . Similar results for the system (4).

Let  $U$  be an open subset of  $\mathbb{R}^2$ . A homeomorphism  $h : U \rightarrow h(U)$  is a *topological equivalence* between the discontinuous piecewise linear differential system (1)+(2) and the discontinuous piecewise linear differential system (3)+(4) when  $h$  sends trajectories of system (1)+(2) contained in  $U$  into trajectories of system (3)+(4) contained in  $h(U)$ .

By Propositions 1 and 2 of [10] when we consider  $U$  as the set formed by the trajectories that do not have points in common with the sliding set of the discontinuous piecewise linear differential system (1)+(2), we obtain a topological equivalence between the phase portrait of the discontinuous piecewise linear differential systems (1)+(2) and (3)+(4). Hence, since we want to study the limit cycles of the system (1)+(2) which do not intersect its sliding set, it is sufficient to study the limit cycles of the system (3)+(4).

The equilibria of systems (3) and (4) are

$$\left(-\frac{a}{\ell^2 - \alpha^2}, -\frac{2a\ell}{\ell^2 - \alpha^2}\right) \quad \text{and} \quad \left(-\frac{c}{r^2 - \beta^2}, b - \frac{2cr}{r^2 - \beta^2}\right),$$

respectively. In what follows we shall assume that  $a \neq 0$  and  $c \neq 0$ , otherwise one of the two equilibria will be on the discontinuity straight line, and we know from [18] that the discontinuous piecewise linear differential system has at most 2 limit cycles.

### 3. PROOF OF THEOREM 1

Let  $U \subset \mathbb{R}^2$  be an open set. The non-locally constant continuous function  $H: \mathbb{R}^2 \rightarrow \mathbb{R}$  is a *first integral* of a differential system in  $U$ , if  $H(x(t), y(t)) = \text{constant}$  for all values of  $t$  for which the solution  $(x(t), y(t))$  of the differential system is defined on  $U$ , i.e. if  $dH(x(t), y(t))/dt = 0$  in  $U$  provided that  $H$  is smooth.

We separate the proof of Theorem 1 in four cases.

**Case 1:** SC and NC, i.e. the discontinuous piecewise linear differential system (3)+(4) has a center C in  $R$ , a saddle S in  $L$  if  $|\ell| < 1$ , and a node of type N in  $L$  if  $|\ell| > 1$ , which can be reals or virtuals. Then in system (3)+(4) we must take  $\alpha = 1$  and  $|\ell| \neq 1$ , and  $\beta = i$  and  $r = 0$ . So the discontinuous piecewise linear differential system (3)+(4) becomes

$$(5) \quad \begin{aligned} \dot{x} &= 2\ell x - y, & \dot{y} &= (\ell^2 - 1)x + a, & \text{in } x \leq 0, \\ \dot{x} &= -y + b, & \dot{y} &= x + c, & \text{in } x \geq 0. \end{aligned}$$

The cases CS and CN can be studied in an analogous way.

We denote by  $(x_-(t), y_-(t))$  the solution of the linear differential system in  $x \leq 0$  such that  $(x_-(0), y_-(0)) = (0, y)$ , and  $(x_+(t), y_+(t))$  the solution of the linear differential system in  $x \geq 0$  such that  $(x_+(0), y_+(0)) = (0, y)$ . The homogeneous part of the linear equation (5) in  $x \leq 0$  has a fundamental matrix solution

$$A_-(t) = e^{\ell t} \begin{pmatrix} \cosh t + \ell \sinh t & -\sinh t \\ (\ell^2 - 1) \sinh t & \cosh t - \ell \sinh t \end{pmatrix}.$$

Using this expression  $A_-(t)$  and the constant variation method we get the expression of the solution of the initial value problem (5) with  $x \leq 0$  and the initial condition  $x_-(0) = 0$  and  $y_-(0) = y$  as

$$(6) \quad \begin{pmatrix} x_-(t) \\ y_-(t) \end{pmatrix} = A_-(t) \begin{pmatrix} 0 + \frac{a}{\ell^2 - 1} \\ y + \frac{2a\ell}{\ell^2 - 1} \end{pmatrix} - \begin{pmatrix} \frac{a}{\ell^2 - 1} \\ \frac{2a\ell}{\ell^2 - 1} \end{pmatrix}.$$

Solving the linear equation in (5) with  $x \geq 0$  gives

$$(7) \quad \begin{aligned} x_+(t) &= c \cos t - c + (b - y) \sin t, \\ y_+(t) &= b + (y - b) \cos t + c \sin t. \end{aligned}$$

Let  $t_+ > 0$  be the time that the solution  $(x_+(t), y_+(t))$ , starting at the point  $(0, y)$  when  $t = 0$ , enters in forward time in the half-plane  $x \geq 0$  and reaches by first time the straight line  $x = 0$ , in case that such solution exists. Similarly,

let  $-t_- < 0$  be the time that the solution  $(x_-(t), y_-(t))$ , starting at the point  $(0, y)$  when  $t = 0$ , enters in backward time in the half-plane  $x \leq 0$  and reaches by first time the straight line  $x = 0$ , in case that such solution exists. Therefore the discontinuous piecewise linear differential system (5) has limit cycles if the system

$$(8) \quad x_+(t_+) = 0, \quad x_-(-t_-) = 0, \quad y_-(-t_-) = y_+(t_+),$$

has isolated solutions. We have three equations and three unknowns  $t_+$ ,  $t_-$  and  $y$ .

Imposing  $x_-(-t_-) = 0$  in the first equation of (6), and solving it for  $y$  gives

$$(9) \quad y = a \frac{e^{\ell t_-} - \cosh t_- - \ell \sinh t_-}{(\ell^2 - 1) \sinh t_-}.$$

Writing equations (6) in

$$(10) \quad A_-(-t) \begin{pmatrix} x_-(t) + \frac{a}{\ell^2 - 1} \\ y_-(t) + \frac{2a\ell}{\ell^2 - 1} \end{pmatrix} = \begin{pmatrix} 0 + \frac{a}{\ell^2 - 1} \\ y + \frac{2a\ell}{\ell^2 - 1} \end{pmatrix},$$

where we have use the fact that  $(A_-(t))^{-1} = A_-(-t)$ . Solving the first equation of (10) with  $t = -t_-$  and  $x_-(-t_-) = 0$ , one gets

$$(11) \quad y_-(-t_-) = a \frac{\cosh t_- - \ell \sinh t_- - e^{\ell t_-}}{(\ell^2 - 1) \sinh t_-}.$$

In order for system (5) to have a limit cycle, let  $(0, Y)$  be the intersection point of the limit cycle with the  $y$ -axis. Then by equations (7) it forces that  $t_+$  must satisfy

$$\begin{pmatrix} c & b - y \\ y - b & c \end{pmatrix} \begin{pmatrix} \cos t_+ \\ \sin t_+ \end{pmatrix} = \begin{pmatrix} c \\ Y - b \end{pmatrix}.$$

Solving these two equations gives

$$(12) \quad \cot t_+ = \frac{c^2 + (y - b)(Y - b)}{c^2 + (y - b)^2}, \quad \sin t_+ = \frac{c(Y - y)}{c^2 + (y - b)^2}.$$

These two equations together with the fundamental trigonometric identity yield

$$Y = 2b - y,$$

where we have used that fact  $Y \neq y$ . Now equation (8) together with (9) and (11) is reduced to

$$0 = 2b - y - y_-(-t_-) = 2 \frac{(b(\ell^2 - 1) + a\ell) \sinh t_- - a \sinh(\ell t_-)}{(\ell^2 - 1) \sinh t_-}.$$

Note that  $\sinh t_- \neq 0$  for  $t_- > 0$ . This last equation can be written in

$$(13) \quad c_0 f_0(t_-) - c_1 f_1(t_-) = 0,$$

with

$$\begin{aligned} c_0 &= b(\ell^2 - 1) + a\ell, & f_0(t_-) &= \sinh t_-, \\ c_1 &= a, & f_1(t_-) &= \sinh(\ell t_-). \end{aligned}$$

We can assume that  $\ell \neq 0$ , otherwise  $f_0(t_-) = 0$  and equation (13) has no solutions, and consequently system (8) has no solutions.

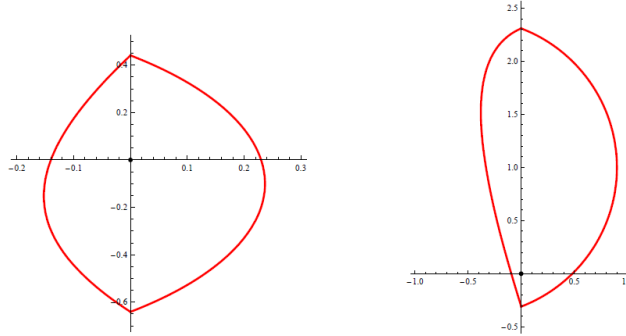
Note that equation (13) is independent of  $c$ , and it happens also in the following proof. So we could take  $c = 1$ , or  $c = -1$ ,  $c = 0$ .

The coefficients  $c_0$  and  $c_1$  are linearly independent because the parameter  $b$  appears only in one of these two coefficients. The Wronskian

$$\begin{vmatrix} f_0(t_-) & f_1(t_-) \\ f_0'(t_-) & f_1'(t_-) \end{vmatrix} = \ell \cosh(\ell t_-) \sinh t_- - \cosh t_- \sinh(\ell t_-) \neq 0,$$

for  $t_- > 0$  because at  $t_- = 0$  it takes the zero value, and the derivative of  $\ell \cosh(\ell t_-) \sinh t_- - \cosh t_- \sinh(\ell t_-)$  is  $(\ell^2 - 1) \sinh(\ell t_-) \sinh t_-$  which never vanishes when  $t_- > 0$  due to the fact that  $\sinh t_- > 0$  and  $\sinh(\ell t_-)$  has the sign of  $\text{sign}(\ell) = \pm 1$ . Therefore the functions  $f_0(t_-)$  and  $f_1(t_-)$  form an Extended Complete Chebyshev systems (see Appendix 2) and consequently equation (13) has at most one solution, and there are values of  $c_0$  and  $c_1$  for which this solution exists. Once we have a unique  $t_-$  we can obtain  $y$  from this value of  $t_-$ , and after  $t_+$ . In short we have a unique solution of system (8), and consequently a unique limit cycle for the discontinuous piecewise linear differential system (5).

In summary, if one of the linear differential systems is a center and the other is a saddle or a node of type N we have an upper bound of 1 limit cycles for such discontinuous piecewise differential systems. In Figure 1 (a) and (b) we provide a discontinuous piecewise linear differential system (14) having one limit cycle when the linear differential system different from a center is a saddle or a node of type N, respectively; see for more details on these limit cycles the Appendix 1.



(a) The limit cycle of the discontinuous piecewise linear differential system (5) for  $\ell = 1/2$ ,  $a = -1$ ,  $b = -1/10$  and  $c = 1/2$ .

(b) The limit cycle of the discontinuous piecewise linear differential system (5) for  $\ell = -2$ ,  $a = -1$ ,  $b = 1$  and  $c = 1/2$ .

FIGURE 1

**Case 2:** N'C, i.e. the discontinuous piecewise linear differential system (3)+(4) has a center C in  $R$  and a non-diagonalizable node N' in  $L$ , reals or virtuals. Then in system (3)+(4) we must take  $\alpha = 0$  and  $\ell \neq 0$ , and  $\beta = i$  and  $r = 0$ . So the discontinuous piecewise linear differential system (3)+(4) becomes

$$(14) \quad \begin{aligned} \dot{x} &= 2\ell x - y, & \dot{y} &= \ell^2 x + a, & \text{in } x \leq 0, \\ \dot{x} &= -y + b, & \dot{y} &= x + c, & \text{in } x \geq 0. \end{aligned}$$

The case CN' can be studied in a similar way.

Using the notation introduced in the proof of Case 1 we have

$$\begin{aligned} x_-(t) &= \frac{a(e^{\ell t}(1-\ell t)-1) - e^{\ell t}\ell^2 t y}{\ell^2}, \\ y_-(t) &= -\frac{a(e^{\ell t}(\ell t-2)+2) + e^{\ell t}\ell(\ell t-1)y}{\ell}, \\ x_+(t) &= c \cos t - c + (b-y) \sin t, \\ y_+(t) &= b + (y-b) \cos t + c \sin t. \end{aligned}$$

Therefore the discontinuous piecewise linear differential system (14) has limit cycles if system (8) has isolated solutions.

From equation  $x_+(t_+) = 0$  we obtain that

$$\cos t_+ = \frac{-b^2 + 2yb + c^2 - y^2}{b^2 - 2yb + c^2 + y^2},$$

and from equation  $x_-(-t_-) = 0$  we get

$$y = \frac{a(e^{\ell t_-} - \ell t_- - 1)}{\ell^2 t_-}.$$

Substituting the previous two expressions in equation  $y_+(t_+) - y(-t_-) = 0$ , and simplifying we obtain the equation

$$c_0 f_0(t_-) + c_1 f_1(t_-) = 0,$$

where

$$\begin{aligned} c_0 &= -a, & f_0(t_-) &= \sinh(\ell t_-), \\ c_1 &= \ell(a + b\ell), & f_1(t_-) &= t_-. \end{aligned}$$

Again the coefficients  $c_0$  and  $c_1$  are linearly independent because the parameter  $b$  appears only in one of these two coefficients. The Wronskian of  $f_0$  and  $f_1$  is

$$W(t_-) := \begin{vmatrix} f_0(t_-) & f_1(t_-) \\ f_0'(t_-) & f_1'(t_-) \end{vmatrix} = \sinh(\ell t_-) - \ell t_- \cosh(\ell t_-).$$

We claim that  $W(t_-) \neq 0$  for  $t_- > 0$ . Indeed, the function  $W(t_-)$  takes the value zero at  $t_- = 0$ , and its derivative  $W'(t_-) = -\ell^2 t_- \sinh(\ell t_-) \neq 0$  if  $t_- > 0$ .

Now the rest of the proof for showing that there is at most a unique solution of system (8) follows the same steps than in the proof of Case 1, and consequently the discontinuous piecewise linear differential system (14) has at most one limit cycle, and there are systems having such limit cycle.

In short, if one of the linear differential systems has a center and the other has a non-diagonalizable node  $N'$  we have an upper bound of 1 limit cycle for such discontinuous piecewise linear differential systems. In Figure 2 we provide a discontinuous piecewise linear differential system (14) having one limit cycle, see for more details on this limit cycle the Appendix 1.

**Case 3:** FC, i.e. the discontinuous piecewise linear differential system (3)+(4) has a center  $C$  in  $L$  and a focus  $F$  in  $R$ , reals or virtuals. Then in system (3)+(4) we

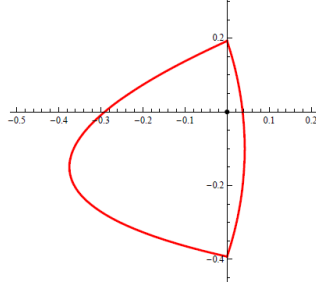


FIGURE 2. The limit cycle of the discontinuous piecewise linear differential system (14) for  $\ell = 1/5$ ,  $a = 1/10$ ,  $b = -1/10$  and  $c = 1$ .

must take  $\alpha = i$  and  $\ell \neq 0$ , and  $\beta = i$  and  $r = 0$ , and we obtain the discontinuous piecewise linear differential system

$$(15) \quad \begin{aligned} \dot{x} &= 2\ell x - y, & \dot{y} &= (\ell^2 + 1)x + a, & \text{in } x \leq 0, \\ \dot{x} &= -y + b, & \dot{y} &= x + c, & \text{in } x \geq 0. \end{aligned}$$

The case CF can be studied in a similar way.

The homogeneous part of the linear equation (15) in  $x \leq 0$  has the fundamental matrix solution

$$A_-(t) = e^{\ell t} \begin{pmatrix} \cos t + \ell \sin t & -\sin t \\ (\ell^2 + 1) \sin t & \cos t - \ell \sin t \end{pmatrix},$$

with  $(A_-(t))^{-1} = A_-(-t)$ . Then manipulating with the constant variation method one gets that the solution of the initial value problem (15) with  $x \leq 0$  and the initial condition  $x_-(0) = 0$  and  $y_-(0) = y$  satisfies

$$(16) \quad \begin{pmatrix} x_-(t) + \frac{a}{\ell^2 + 1} \\ y_-(t) + \frac{2a\ell}{\ell^2 + 1} \end{pmatrix} = A_-(t) \begin{pmatrix} 0 + \frac{a}{\ell^2 + 1} \\ y + \frac{2a\ell}{\ell^2 + 1} \end{pmatrix}.$$

Imposing  $x_-(-t_-) = 0$ , and solving the first equation of (16) in  $y$  gives

$$(17) \quad y = -a \frac{\cos t_- + \ell \sin t_- - e^{\ell t_-}}{(\ell^2 + 1) \sin t_-}.$$

Here we have used the fact that  $\sin t_- \neq 0$ . For otherwise, the first equation of (16) at  $t = -t_-$  reduces to  $a(1 + \cos t_- e^{-\ell t_-}) = 0$ , but it is impossible because  $\cos t_- = \pm 1$  and  $e^{-\ell t_-} \neq 1$  for  $\ell \neq 0$  and  $t_- > 0$ , and we have assumed that  $a \neq 0$  at the end of section 2.

For obtaining the expression of  $y_-(-t_-)$ , we write equations (16) in

$$(18) \quad A_-(-t) \begin{pmatrix} x_-(t) + \frac{a}{\ell^2 + 1} \\ y_-(t) + \frac{2a\ell}{\ell^2 + 1} \end{pmatrix} = \begin{pmatrix} 0 + \frac{a}{\ell^2 + 1} \\ y + \frac{2a\ell}{\ell^2 + 1} \end{pmatrix}.$$



Imposing  $x_-(-t_-) = 0$ , and solving the first equation of (18), which is independent of  $y$ , we obtain

$$(19) \quad y_-(-t_-) = a \frac{\cos t_- - \ell \sin t_- - e^{-\ell t_-}}{(\ell^2 + 1) \sin t_-}.$$

Since the equation in (15) with  $x \geq 0$  is the same as that in (5) with  $x \geq 0$ , it follows that

$$y_+(t_+) = 2b - y.$$

In order that the limit cycles exist, we must have

$$(20) \quad 2b - y - y_-(-t_-) = 0.$$

Inserting (17) and (19) in (20) and simplifying it, one gets

$$(21) \quad c_0 f_0(t) + c_1 f_1(t) = 0,$$

with

$$\begin{aligned} c_0 &= b(\ell^2 + 1) + a\ell, & f_0(t_-) &= \sin(t_-), \\ c_1 &= -a, & f_1(t_-) &= \sinh(\ell t_-), \end{aligned}$$

Obviously, the parameters  $c_0$  and  $c_1$  are independent, and the Wronskian of  $f_0$  and  $f_1$  is

$$W(t_-) := \begin{vmatrix} f_0(t_-) & f_1(t_-) \\ f_0'(t_-) & f_1'(t_-) \end{vmatrix} = \ell \sin t_- \cosh(\ell t_-) - \cos t_- \sinh(\ell t_-).$$

Note that the system of (15) in  $x \leq 0$  is linear and has the eigenvalues  $\ell \pm \sqrt{-1}$  at its singularity, it follows that the frequency is 1, and consequently one has  $t_- \in (0, \pi)$  for  $a < 0$  or  $t_- \in (\pi, 2\pi)$  for  $a > 0$ , because correspondingly the focus is virtual or real.

In case  $a < 0$ , because  $W(0) = 0$  and  $W'(t_-) = (\ell^2 + 1) \sin t_- \sinh(\ell t_-) \neq 0$  for  $t_- \in (0, \pi)$ , the Wronskian  $W(t_-)$  cannot vanish, and so equation (21) has at most one solution in  $t \in (0, \pi)$ . Hence system (15) has at most one limit cycle when  $a < 0$ .

In case  $a > 0$ , because  $\text{sign}(W(\pi)) = \text{sign}(\sinh(\pi\ell)) = \text{sign}(\ell)$  and  $\text{sign}(W(2\pi)) = -\text{sign}(\sinh(\pi\ell)) = -\text{sign}(\ell)$ , while  $\text{sign}(W'(t_-)) = -\text{sign}(\ell)$  for  $t_- \in (\pi, 2\pi)$ , it follows that  $W(t_-)$  has exactly a unique simple zero. Since  $f_0(t_-) = \sin(t_-) \neq 0$  for  $t_- \in (\pi, 2\pi)$ , we get from Corollary 1.4 of [23] that the functional equation (21) has at most two zeros in  $(\pi, 2\pi)$ . Consequently system (15) has at most two limit cycles when  $a > 0$ . Moreover in Figure 3 we provide a discontinuous piecewise linear differential system (15) having two limit cycles. In Appendix 1 we explicitly describe this discontinuous piecewise linear differential system with the two limit cycles.

**Case 4:** CC. This case already has been considered in Theorem 3 of [21], but i.e. the discontinuous piecewise linear differential system (3)+(4) has a center C in  $R$  and another center in  $L$ , reals or virtuals. Then in system (3)+(4) we must take  $\alpha = i$  and  $\ell = 0$ , and  $\beta = i$  and  $r = 0$ .

The first integral of the linear differential system (4) with a center is  $H_R = (x + c)^2 + (b - y)^2$ , and the first integral of the linear differential system (3) with the other center is  $H_L = (a + x)^2 + y^2$ .

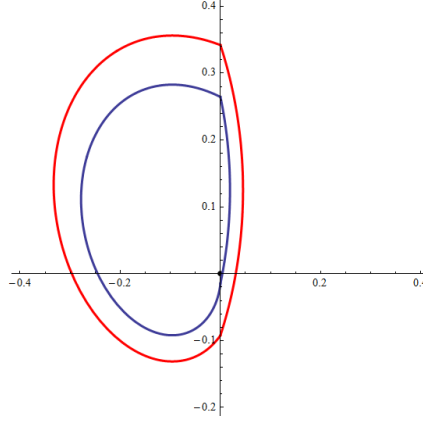


FIGURE 3. The two limit cycles of the discontinuous piecewise linear differential system (15) for  $\ell = -1/5$ ,  $a = 1/10$ ,  $b = 1/8$  and  $c = 1/2$ .

We evaluate the first integrals  $H_L$  and  $H_R$  at the points  $(0, y)$  and  $(0, Y)$  with  $y, Y \in \mathbb{R}$  and  $y \neq Y$ . If there exists a periodic solution passing through these two points, then  $y$  and  $Y$  must satisfy the system  $H_L(0, y) = H_L(0, Y)$  and  $H_R(0, y) = H_R(0, Y)$ , i.e. the system

$$(22) \quad y^2 - Y^2 = 0, \quad (b - y)^2 - (b - Y)^2 = 0.$$

For  $b \neq 0$ , system (22) has the unique solution  $y = Y$ . In this case the discontinuous piecewise linear differential systems (3)+(4) with two centers have no limit cycles. For  $b = 0$  there also exists the solution  $Y = -y$  for (22), which leads to a global center of the discontinuous piecewise linear differential system (3)+(4) and consequently there is no a limit cycle.

This completes the proof of Theorem 1.

#### APPENDIX 1: THE EXAMPLES

In Figure 1(a) is drawn the unique limit cycle of the discontinuous piecewise linear differential system

$$(23) \quad \begin{aligned} \dot{x} &= x - y, & \dot{y} &= -\frac{3}{4}x - 1, & \text{in } x \leq 0, \\ \dot{x} &= -y - \frac{1}{10}, & \dot{y} &= x + \frac{1}{2}, & \text{in } x \geq 0. \end{aligned}$$

This discontinuous piecewise linear differential system has a center in R and a saddle in L. Using the notation introduced in the proof of Case 1 then for system

(23) we have

$$\begin{aligned}x_-(t) &= \frac{e^{-t/2}}{6} (e^{2t}(2-3y) - 8e^{t/2} + 3(y+2)), \\y_-(t) &= \frac{e^{-t/2}}{12} (9(y+2) - 16e^{t/2} + e^{2t}(3y-2)), \\x_+(t) &= \frac{1}{10} (5 \cos t - (10y+1) \sin t - 5), \\y_+(t) &= \frac{1}{10} ((10y+1) + \cos t + 5 \sin t - 1).\end{aligned}$$

Then for every isolated solution  $(t_+, t_-, y)$  of system (8) the discontinuous piecewise linear differential system (23) has a limit cycle. This system has the approximated solution

$$(t_+, t_-, y) = (1.6495630077397934\dots, 1.1713632646813152\dots, -0.641020057578924\dots).$$

Using the Poincaré–Medina Theorem (see [7]), or the Newton–Kantorovich Theorem (see [15]) it is easy to prove that near this approximate solution there is really a solution, which provide the limit cycle of Figure 1 (a).

In Figure 1(b) there is a picture of the unique limit cycle of the discontinuous piecewise linear differential system

$$(24) \quad \begin{aligned}\dot{x} &= -4x - y, & \dot{y} &= 3x - 1, & \text{in } x \leq 0, \\ \dot{x} &= -y + 1, & \dot{y} &= x + \frac{1}{2}, & \text{in } x \geq 0.\end{aligned}$$

This discontinuous piecewise linear differential system has a center in  $\mathbb{R}$  and a node of type N in L. Then for system (24) we have

$$\begin{aligned}x_-(t) &= \frac{e^{-3t}}{6} (3y + 2e^{3t} - 3e^{2t}(y+1) + 1), \\y_-(t) &= \frac{e^{-3t}}{6} (-3y - 8e^{3t} + 9e^{2t}(y+1) - 1), \\x_+(t) &= \frac{1}{2} (-1 + \cos t - 2(y-1) \sin t), \\y_+(t) &= (y-1) \cos t + \frac{1}{2} (2 + \sin t).\end{aligned}$$

The discontinuous piecewise linear differential system (24) has limit cycles if system (8) has isolated solutions. This system has the solution

$$(t_+, t_-, y) = (\arccos(-41/55), \log((5 + \sqrt{21})/2), 1 - 2\sqrt{3/7}).$$

In Figure 2 is drawn the unique limit cycle of the discontinuous piecewise linear differential system

$$(25) \quad \begin{aligned}\dot{x} &= -\frac{2}{5}x - y, & \dot{y} &= \frac{1}{25}x + \frac{1}{10}, & \text{in } x \leq 0, \\ \dot{x} &= -y - \frac{1}{10}, & \dot{y} &= x + 1, & \text{in } x \geq 0.\end{aligned}$$

This discontinuous piecewise linear differential system has a center in R and a non-diagonalizable node N' in L. For this differential system we have

$$\begin{aligned} x_-(t) &= \frac{e^{t/5}(2yt + t - 5) + 5}{2}, \\ y_-(t) &= e^{t/5} \left( \frac{t}{10}(2y + 1) - y - 1 \right) + 1, \\ x_+(t) &= \cos t + \left( y - \frac{1}{10} \right) \sin t - 1, \\ y_+(t) &= \left( \frac{1}{10} - y \right) \cos t + \sin t - \frac{1}{10}. \end{aligned}$$

Again the discontinuous piecewise linear differential system (25) has limit cycles if system (8) has isolated solutions. This system has the approximated solution

$$(t_+, t_-, y) = (0.5688360788661645\dots, 5.3243427404530586\dots, -0.3923437976009292\dots).$$

In Figure 3 are drawn the two limit cycles of the discontinuous piecewise linear differential system

$$(26) \quad \begin{aligned} \dot{x} &= -\frac{2}{5}x - y, & \dot{y} &= \frac{26}{25}x + \frac{1}{10}, & \text{in } x \leq 0, \\ \dot{x} &= -y + \frac{1}{8}, & \dot{y} &= x + \frac{1}{2}, & \text{in } x \geq 0. \end{aligned}$$

This discontinuous piecewise linear differential system has a center in R and a focus in L. For this differential system we have

$$\begin{aligned} x_-(t) &= \frac{1}{52} \left( e^{-t/5}((1 - 52y_0) \sin t + 5 \cos t) - 5 \right), \\ y_-(t) &= \frac{1}{130} \left( e^{-t/5}(2(13y_0 + 6) \sin t + 5(26y_0 - 1) \cos t) + 5 \right), \\ x_+(t) &= \frac{1}{8}((1 - 8y_0) \sin t + 4 \cos t - 4), \\ y_+(t) &= \frac{1}{8}((8y_0 - 1) \cos t + 4 \sin t + 1). \end{aligned}$$

In this case system (8) has the following two approximated solutions  $(t_+, t_-, y)$ :

$$\begin{aligned} &(0.8190023138429233\dots, 4.143848527245624\dots, -0.09201912821815718\dots), \\ &(0.5439522840928744\dots, 4.749224488884674\dots, -0.01444340381074483\dots). \end{aligned}$$

Hence the discontinuous piecewise linear differential system (26) has two limit cycles.

## APPENDIX 2: EXTENDED COMPLETE CHEBYSHEV SYSTEMS

A set of functions  $\{f_0, f_1, \dots, f_n\}$  defined in an interval  $I$  is an *Extended Chebyshev* system on  $I$ , if and only if any non-zero linear combination of these functions has at most  $n$  zeros taking into account their multiplicities and this number is realizable.

If for all  $0 \leq k \leq n$  the subset of functions  $\{f_0, f_1, \dots, f_k\}$  is an Extended Chebyshev system, then we say that the set of functions  $\{f_0, f_1, \dots, f_n\}$  is an *Extended Complete Chebyshev system* on  $I$ .

The set of functions  $\{f_0, f_1, \dots, f_n\}$  is an Extended Complete Chebyshev system on  $I$  if and only if the Wronskians

$$W(f_0, \dots, f_k)(s) = \begin{vmatrix} f_0(s) & f_1(s) & \cdots & f_k(s) \\ f_0'(s) & f_1'(s) & \cdots & f_k'(s) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(s) & f_1^{(k)}(s) & \cdots & f_k^{(k)}(s) \end{vmatrix} \neq 0,$$

on  $I$  for  $k = 0, 1, \dots, n$ . For a proof of this result see the book [16].

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