CENTERS AND UNIFORM ISOCHRONOUS CENTERS OF PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. For planar polynomial vector fields of the form

$$(-y + X(x,y))\frac{\partial}{\partial x} + (x + Y(x,y))\frac{\partial}{\partial y},$$

where X and Y start at least with terms of second order in the variables x and y, we determine necessary and sufficient conditions under which the origin is a center or a uniform isochronous centers.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

A real planar polynomial differential system is a system of the form

$$(1) \qquad \dot{x} = P(x,y), \quad \dot{y} = Q(x,y),$$

where the dot denotes derivative with respect to an independent variable here called the time t, and P and Q are real coprime polynomials in $\mathbb{R}[x,y]$. We say that the polynomial differential system (1) has *degree* $m = \max\{\deg P, \deg Q\}$.

In what follows we assume that origin $O := (0,0)$ is a singular or equilibrium point, i.e. $P(0,0) = Q(0,0) = 0$.

The equilibrium point O is a *center* if there exists an open neighborhood U of O where all the orbits contained in $U \setminus \{O\}$ are periodic.

The study of the centers of polynomial differential systems (1) has a long history. The first works are due to Dulac [4] and Poincaré [11]. Later on where developed by Lyapunov [9], Bendixson [2], Frommer [5] and many others.

Assume that the origin of the polynomial differential system (1) is a center. It is well–known that, after a linear change of variables and a constant scaling of the time variable (if necessary), system (1) can be written in one of the next three forms:

$$(2) \qquad \begin{array}{ll} \dot{x} = -y + F_1(x,y), & \dot{y} = x + F_2(x,y), \\ \dot{x} = y + F_1(x,y), & \dot{y} = F_2(x,y), \\ \dot{x} = F_1(x,y), & \dot{y} = F_2(x,y), \end{array}$$

where $F_1(x,y)$ and $F_2(x,y)$ are polynomials without constant and linear terms defined in a neighborhood of the origin. Then the origin O of the polynomial differential system (1) is called *linear type*, *nilpotent* or *degenerate* if after a linear change of variables and a scaling of the time it can be written as the first, second and third system of (2), respectively.

A center O of system (1) is a *uniform isochronous center* if the equality $x\dot{y} - y\dot{x} = a(x^2 + y^2)$ holds with $a \neq 0$; or equivalently if in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, we have that $\dot{\theta} = a$. It is known that isochronous centers and in particular isochronous uniform centers are linear type centers. This is due to the fact that blowing up the origin where the center is located for the linear type centers the origin blows up to a periodic orbit, but for nilpotent and degenerate centers the origin blows up to a graphic and consequently the periodic orbits near the origin cannot have constant period. We recall that a *graphic* is formed by finitely many singular points and orbits connecting these points in such a way that in one of the two sides of the graphic a return Poincaré map is defined. To see examples of isochronous centers which are not uniform, see for instance [8].

The following necessary and sufficient condition in order that the origin O of the first polynomial differential of system (2) be a center was obtained by Poincaré, and it was extended to analytic differential systems by Lyapunov (see for more details [6, 9]).

Theorem 1. *A planar polynomial differential system*

$$(3) \qquad \dot{x} = -y + \sum_{j=2}^{m} X_j(x, y), \quad \dot{y} = x + \sum_{j=2}^{m} Y_j(x, y),$$

of degree m has a center at the origin if and only if it has an analytic first integral of the form

$$(4) \qquad H = \sum_{j=2}^{\infty} H_j(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j(x, y),$$

where X_j, Y_j and H_j are homogenous polynomials of degree j.

One of the main objectives of the present paper is to study the centers and the uniform isochronous centers using the inverse theory (see for instance [7, 12]). More precisely, we want to determine the polynomials X_j and Y_j of system (3) in order that a function of the form (4) be a first integral of the polynomial differential system (3) in a neighborhood of the origin O.

As usual the *Poisson bracket* of the functions $f(x, y)$ and $g(x, y)$ is defined as

$$\{f, g\} := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

Our first result provides the expression of the polynomial differential system (3) in function of its first integral (4).

Theorem 2. *Given an analytic function of the form* (4) *a polynomial differential system having such function as a first integral can be written as*

$$(5) \qquad \begin{aligned} \dot{x} &= \sum_{j=2}^{m+1} g_{m+1-j}\{\Psi_j, x\} = -y + \mathcal{X}(x), \\ \dot{y} &= \sum_{j=2}^{m+1} g_{m+1-j}\{\Psi_j, y\} = x + \mathcal{X}(y), \end{aligned}$$

where $g_0 = 1$ and g_{m+1-j} is an arbitrary homogenous polynomial of degree $m+1-j \geq 0$ satisfying

$$(6) \qquad \{H_{m+l-1}, H_2\} = \sum_{j=2}^{m} \mathcal{X}_j(H_{m+l-j}), \quad l = 3, 4, \ldots,$$

where

$$\mathcal{X} = \sum_{j=2}^{m} \left(X_j \frac{\partial}{\partial x} + Y_j \frac{\partial}{\partial y} \right) = \sum_{j=2}^{m} \mathcal{X}_j$$

$$= \sum_{j=2}^{m} \left(\{H_{j+1}, \} + g_1\{H_j, \} + \ldots + g_{j-1}\{H_2, \} \right),$$

$$\Psi_j = \sum_{k=2}^{j} H_k, \quad for \quad j = 2, \ldots, m+1.$$

Theorem 2 is proved in section 2.

We remark that Theorem 2 shows that all polynomial differential systems having a first integral of the form (4) have a center at the origin, because we have proved that such systems are of the form (3) and consequently they have a focus or a center at the origin, but the existence of the first integral forces that the origin is a center. Note that this provides a new proof of the Poincaré–Liapunov Theorem (i.e. Theorem 1).

On the other hand, if a polynomial differential system has a linear type center at the origin by Theorem 1 it has an analytic first integral of the form (4), and the differential system has the structure described in Theorem 2. So this structure is necessary in order that a polynomial differential system has a center at the origin.

Now we shall put our attention on the uniform isochronous centers by using Theorem 2.

The following result is well–known, see for instance [3].

Proposition 3. *Assume that a planar polynomial differential system of degree m has a center at the origin of coordinates. Then this center is a uniform isochronous center if and only if doing a linear change of variables and a scaling of the time the system can be written as*

$$(7) \qquad \dot{x} = -y + x \sum_{n=2}^{m} \varphi_{n-1}, \quad \dot{y} = x + y \sum_{n=2}^{m} \varphi_{n-1},$$

where $\varphi_{n-1} = \varphi_{n-1}(x, y)$ is a homogenous polynomial in x and y of degree $n-1$

This result can be improved as follows. Thus, using Theorem 2 we provide a first characterization of the uniform isochronous centers.

Theorem 4. *A polynomial differential system (3) has a uniform isochronous center at the origin if and only if the system can be written as (7) with*

$$\varphi_{n-1} = \frac{1}{n+1} \sum_{j=1}^{n-1} \{H_2 \Upsilon_{n-1-j}, g_j\}$$

where $H_2 = (x^2 + y^2)/2$, g_j and Υ_j are arbitrary homogenous polynomials of degree j, and the system has a first integral of the form

$$(8) \qquad\qquad H = H_2 \left(1 + \Upsilon_1 + \Upsilon_2 + \ldots\right),$$

in a neighborhood of the origin.

The following result is well–known (see for instance [10]) and give the linearization criterion for the isochronocity of a center.

Theorem 5. *A center of an analytic system is isochronous if and only if there exits an analytic change of coordinates $X = x + o(|(x, y)|)$, $Y = y + o(|(x, y)|)$, reducing the system to the linear isochronous system $\dot{X} = -Y$, $\dot{Y} = X$.*

Corollary 6 (Linearization of uniform isochronous center)**.** *Under the assumptions of Theorem 4 we get that by the analytic change of coordinates*

$$X = x\sqrt{1 + \Upsilon_1 + \Upsilon_2 + \ldots}, \quad Y = y\sqrt{1 + \Upsilon_1 + \Upsilon_2 + \ldots},$$

the differential system (7) becomes $\dot{X} = -Y$, $\dot{Y} = X$.

Theorem 4 and Corollary 6 are proved in section 3.

In the next result we provide a second characterization of the uniform isochronous centers.

Theorem 7. *The origin of the polynomial differential system (3) is a uniform isochronous center if and only if this system can be written as (7) and*

$$x(t) = \cos t \sum_{j=1}^{\infty} \varepsilon^j s_j(t), \quad y(t) = \sin t \sum_{j=1}^{\infty} \varepsilon^j s_j(t),$$

is a periodic solution with the initial conditions $(\varepsilon, 0)$ where ε is a small parameter, and $s_j(t)$ is a convenient 2π–periodic function such that $s_1(t) = 1$, and $s_j(0) = s_j(2\pi) = 0$ for $j > 1$.

Theorem 7 is proved in section 4.

The next corollary provides a characterization of the linear type centers of polynomial differential systems when their nonlinearities are homogenous polynomials.

Corollary 8. *Under the assumptions of Theorem 2 the planar polynomial differential system of degree m*

$$(9) \qquad\qquad \dot{x} = -y + X_m, \quad \dot{y} = x + Y_m,$$

where X_m and Y_m are homogenous polynomials of degree m, has a center at the origin if and only if it can be written as

$$(10) \qquad \begin{aligned} \dot{x} &= \{H_2 + H_{m+1}, x\} + g_{m-1}\{H_2, x\} := -y + \mathcal{X}_m(x), \\ \dot{y} &= \{H_2 + H_{m+1}, y\} + g_{m-1}\{H_2, y\} := x + \mathcal{X}_m(y), \end{aligned}$$

where

$$\mathcal{X}_m = \{H_2 + H_{m+1}, \} + g_{m-1}\{H_2, \},$$

and $g_{m-1} = g_{m-1}(x, y)$ is a homogenous polynomial of degree $m - 1$ satisfying

$$(11) \qquad \mathcal{X}_m(H_k) + \{H_2, H_{m+k-1}\} = 0 \quad for \quad k = 3, 4, \ldots.$$

Corollary 8 is proved in section 5.

In [3] Conti proved the following result.

Theorem 9. *The polynomial differential system* (9) *with*

$$X_m = x\varphi_{m-1}(x,y), \quad Y_m = y\varphi_{m-1}(x,y), \quad \varphi_{m-1}(x,y) = \sum_{i+j=m-1} a_{ij}x^i y^j,$$

has a uniform isochronous center at the origin if either m is even, or m is odd and

$$\sum_{j=0}^{m-1} a_{m-1-j,j} \int_0^{2\pi} \cos^{m-1-j} t \sin^j t \, dt = 0,$$

Conti's result can be improved as follows.

Theorem 10. *The polynomial differential system* (9) *has a uniform isochronous center at the origin if and only if this system can be written as*

(12)
$$\dot{x} = -y + \frac{x}{m+1}\{H_2, g_{m-1}\},$$
$$\dot{y} = x + \frac{y}{m+1}\{H_2, g_{m-1}\},$$

where $H_2 = (x^2 + y^2)/2$ and $g_{m-1} = g_{m-1}(x,y)$ is an arbitrary homogenous polynomial of degree $m-1$ satisfying (11). *Moreover this system has the rational first integral*

$$F = \frac{H_2^{m-1}}{\left(1 + \dfrac{m-1}{m+1}g_{m-1}\right)^2}.$$

Theorem 10 is proved in section 5.

Theorem 10 characterizes the form of the polynomial uniform isochronous centers with homogeneous nonlinearities.

Note that, under the assumptions of Theorem 10 and using the notation of Theorem 4, the first integral $F^{1/(m-1)}$ has the following development as the origin

$$F^{1/(m-1)} := H = H_2 \left(1 + \Upsilon_1 + \Upsilon_2 + \ldots\right),$$

i.e. system (12) has a local first integral at the origin having H_2 as a factor, where Υ_j is a convenient homogenous polynomial of degree j.

Corollary 11 (Linearization of uniform isochronous centers with homogenous nonlinearity)**.** *Under the assumptions of Theorem 10 we get that by the analytic change of coordinates in a neighborhood of the origin*

$$X = \frac{x}{\left(1 + \dfrac{m-1}{m+1}g_{m-1}\right)^{1/(m-1)}},$$
$$Y = \frac{y}{\left(1 + \dfrac{m-1}{m+1}g_{m-1}\right)^{1/(m-1)}},$$

the differential system (12) *becomes*

$$\dot{X} = -Y, \quad \dot{Y} = X.$$

Corollary 11 is proved in section 5.

2. PROOF OF THEOREM 2

Proof of Theorem 2. Consider a general polynomial vector field of degree m that we write as

$$\mathcal{X} = \left(\sum_{j=0}^{m} X_j(x,y)\right)\frac{\partial}{\partial x} + \left(\sum_{j=0}^{m} Y_j(x,y)\right)\frac{\partial}{\partial y},$$

where X_j and Y_j for $j = 0, 1, \ldots, m$ are homogenous polynomials of degree j. Since the analytic first integral H starts with $H_2 = (x^2 + y^2)/2$, without loss of generality this implies that $X_0(x,y) = Y_0(x,y) = 0$, $X_1(x,y) = -y$ and $Y_1(x,y) = x$. Hence the following infinite number of equations follow

(13)

$$xX_2 + yY_2 = \{H_3, H_2\},$$

$$xX_3 + yY_3 + \frac{\partial H_3}{\partial x}X_2 + \frac{\partial H_3}{\partial y}Y_2 = \{H_4, H_2\},$$

$$xX_4 + yY_4 + \frac{\partial H_3}{\partial x}X_3 + \frac{\partial H_3}{\partial y}Y_3 + \frac{\partial H_4}{\partial x}X_2 + \frac{\partial H_4}{\partial y}Y_2 = \{H_5, H_2\},$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$xX_n + yY_n + \frac{\partial H_3}{\partial x}X_{n-1} + \frac{\partial H_3}{\partial y}Y_{n-1} + \ldots + \frac{\partial H_n}{\partial x}X_2 + \frac{\partial H_n}{\partial y}Y_2 = \{H_{n+1}, H_2\},$$

$$xX_{n+1} + yY_{n+1} + \frac{\partial H_3}{\partial x}X_n + \frac{\partial H_3}{\partial y}Y_n + \ldots + \frac{\partial H_{n+1}}{\partial x}X_2 + \frac{\partial H_{n+1}}{\partial y}Y_2 = \{H_{n+2}, H_2\},$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

The first equation can be rewritten as

$$x\left(X_2 + \frac{\partial H_3}{\partial y}\right) + y\left(Y_2 - \frac{\partial H_3}{\partial x}\right) = 0,$$

by solving with respect to X_2 and Y_2 we obtain the following polynomial solutions

$$X_2 = -\frac{\partial H_3}{\partial y} - yg_1 = \{H_3, x\} + g_1\{H_2, x\} := \mathcal{X}_2(x),$$

$$Y_2 = \frac{\partial H_3}{\partial x} + xg_1 = \{H_3, y\} + g_1\{H_2, y\} := \mathcal{X}_2(y),$$

where $g_1 = g_1(x, y)$ is an arbitrary homogenous polynomial of degree one. By substituting these polynomials into the second equation of (13) we get

$$x\left(X_3 + \frac{\partial H_4}{\partial y} + g_1\frac{\partial H_3}{\partial y}\right) + y\left(Y_3 - \frac{\partial H_4}{\partial x} - g_1\frac{\partial H_3}{\partial x}\right) = 0.$$

By solving this equation with respect to X_3 and Y_3 we have

$$X_3 = -\frac{\partial H_4}{\partial y} - g_1\frac{\partial H_3}{\partial y} - yg_2 = \{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2, x\} := \mathcal{X}_3(x),$$

$$Y_3 = \frac{\partial H_4}{\partial x} + g_1\frac{\partial H_3}{\partial x} + xg_2 = \{H_4, y\} + g_1\{H_3, y\} + g_2\{H_2, y\} := \mathcal{X}_3(y),$$

where $g_2 = g_2(x, y)$ is an arbitrary homogenous polynomial of degree two. By continuing this process recursively we obtain $X_4, Y_4, \ldots, X_m, Y_m$. By substituting

X_j and Y_j for $j = 1, \ldots, m$ into the rest of equations (13) we deduce the partial differential equations (6). Introducing the respectively notations we get

$$-y + X_2 + X_3 + \ldots + X_m = -y + \mathcal{X}(x) = \sum_{j=2}^{m+1} g_{m+1-j}\{\Psi_j, x\},$$

$$x + Y_2 + Y_3 + \ldots + Y_m = x + \mathcal{X}(y) = \sum_{j=2}^{m+1} g_{m+1-j}\{\Psi_j, y\},$$

with $g_0 = 1$. Thus the proof of the theorem follows. $\square$

Remark 12. *A polynomial differential system having a linear type center at the origin can written as system (5), consequently this system gives a necessary condition in order to have a linear type center, but this condition is not sufficient. Indeed it is known (see [1]) that a quadratic differential system with a center at the origin can be written as*

$$\dot{x} = -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2,$$
$$\dot{y} = x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2,$$

or equivalently (see (10) for $m = 2$)

$$\dot{x} = \{H_2 + H_3, x\} + g_1\{H_2, x\}, \quad \dot{y} = \{H_2 + H_3, y\} + g_1\{H_2, y\},$$

with

$$H_3 = \frac{1}{3}(\lambda_2 + \lambda_5)x^3 + \lambda_3 x^2 y - \frac{1}{3}(\lambda_4 + \lambda_6)y^3 - \lambda_2 xy^2,$$
$$g_1 = \lambda_4 y - \lambda_5 x.$$

It is well known that there are values of the parameters $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 for which the origin is a focus.

Proposition 13. *Let $\Psi = \Psi(x, y)$ be an arbitrary polynomial and $H_2 = (x^2 + y^2)/2$. Then the following statements hold.*
(a)

$$(14) \qquad \int_0^{2\pi} \{\Psi, H_2\}|_{x=\cos t,\, y=\sin t}\, dt = 0.$$

(b) *Under the assumptions of Theorem 2 we have that*

$$(15) \qquad \int_0^{2\pi} \mathcal{X}_j(H_{m+l-j})|_{x=\cos t,\, y=\sin t}\, dt = 0,$$

for $j = 2, \ldots, m$ and $l = 3, 4, \ldots$.

Proof. From the relations

$$\frac{d}{dt}\Psi(\cos t, \sin t) = \cos t\frac{\partial \Psi(x, y)}{\partial y} - \sin t\frac{\partial \Psi(x, y)}{\partial x}\bigg|_{x=cost, y=sint} = \{H_2, \Psi\}|_{x=\cos t, y=\sin t},$$

it follows that

$$\int_p^{2\pi} \{\Psi, H_2\}|_{x=\cos t, y=\sin t} = \Psi(\cos t, \sin t)|_0^{2\pi} = 0.$$

From (6) and (14) it follow (15). $\square$

The following result is due to Liapunov, see Theorem 1, page 276 of [9].

Theorem 14. *Let $U = U(x, y)$ be a homogenous polynomial of degree k. The linear partial differential equation*

$$y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y} := \{V, H_2\} = U,$$

has a unique homogenous polynomial solution V of degree k if k is odd; and if V is a homogenous polynomial solution when k is even then $V + c(x^2 + y^2)^{k/2}$ where $c \in \mathbb{R}$ are all the homogenous polynomial solutions.

Proposition 15. *Assuming that system (3) is given and that $\mathcal{X}_j = X_j\frac{\partial}{\partial x} + Y_j\frac{\partial}{\partial y}$, then the following statements hold.*

(a) *Condition (15) are necessary and sufficient in order that the linear first order partial differential equations (6) for $l = 3, 4, \ldots$ in the dependent variable $H_{m+l-1}(x, y)$ has solutions.*

(b) *The homogenous polynomial solution $H_{m+l-1}(x, y)$ of (6) when $m+l-1$ is odd is unique. If $H_{m+l-1}(x, y)$ is a homogenous polynomial solution of (6) when $m+l-1$ is even, then all the other homogenous polynomial solutions are of the form $H_{m+l-1}(x, y) + c(x^2 + y^2)^{(m+l-1)/2}$ with $c \in \mathbb{R}$.*

Proof of Proposition 15. Statement (a) follows from Proposition 13. Statement (b) is an easy consequence of Theorem 14. □

3. Proof of Theorem 4 and Corollary 6

First we prove the following lemma.

Lemma 16. *Under the assumptions of Theorem 2 and assuming that*

$$H_{n+1} = -\frac{1}{n+1}\left(ng_1H_n + (n-1)g_2H_{n-1} + \ldots + 2g_{n-1}H_2\right),$$

for $n > 1$. Then

$$\mathcal{X}_n(x) = \frac{x}{n+1}\sum_{j=1}^{n-1}\{H_{n+1-j}, g_j\} := x\varphi_{n-1},$$

$$\mathcal{X}_n(y) = \frac{y}{n+1}\sum_{j=1}^{n-1}\{H_{n+1-j}, g_j\} := y\varphi_{n-1}.$$

Proof. Substituting the polynomial H_{n+1} into $\mathcal{X}_n(x)$ we obtain

$$
\begin{aligned}
\mathcal{X}_n(x) = & \quad \{H_{n+1}, x\} + g_1\{H_n, x\} + \ldots + g_{n-1}\{H_2, x\} \\
= & \quad -\frac{\partial H_{n+1}}{\partial y} - g_1\frac{\partial H_n}{\partial y} - \ldots - g_{n-1}y \\
= & \quad \frac{1}{n+1}\left(n\frac{\partial H_n}{\partial y}g_1 + nH_n\frac{\partial g_1}{\partial y} - (n+1)g_1\frac{\partial H_n}{\partial y}\right. \\
& \quad \left. + \ldots + 2\frac{\partial H_2}{\partial y}g_{n-1} + 2H_2\frac{\partial g_{n-1}}{\partial y} - (n+1)g_{n-1}\frac{\partial H_2}{\partial y}\right) \\
= & \quad \frac{1}{n+1}\left(\left(x\frac{\partial H_n}{\partial x} + y\frac{\partial H_n}{\partial y}\right)\frac{\partial g_1}{\partial y} - \left(x\frac{\partial g_1}{\partial x} + y\frac{\partial g_1}{\partial y}\right)\frac{\partial H_n}{\partial y}\right) \\
& \quad + \ldots + \frac{1}{n+1}\left(\left(x\frac{\partial H_2}{\partial x} + y\frac{\partial H_2}{\partial y}\right)\frac{\partial g_{n-1}}{\partial y} - \left(x\frac{\partial g_{n-1}}{\partial x} + y\frac{\partial g_{n-1}}{\partial y}\right)\frac{\partial H_2}{\partial y}\right) \\
= & \quad \frac{x}{n+1}\sum_{j=1}^{n-1}\{H_{n+1-j}, g_j\}.
\end{aligned}
$$

Here we use the property of the homogenous polynomial $jg_j = x\dfrac{\partial g_j}{\partial y} + y\dfrac{\partial g_j}{\partial y}$. In a similar way we obtain the expression for $\mathcal{X}_n(y)$. $\qquad\square$

Proof of Theorem 4. If the origin is a linear type center of a polynomial differential system then in view of Theorem 2 we obtain that this system can be written as (5), or equivalently,

$$
\begin{aligned}
\dot{x} = & \quad -y + \mathcal{X}(x) \\
= & \quad -y - \left(\frac{\partial H_{m+1}}{\partial y} + g_1\frac{\partial H_m}{\partial y} + \ldots + g_{m-1}y\right) - \left(\frac{\partial H_m}{\partial y} + g_1\frac{\partial H_{m-1}}{\partial y} + \ldots + g_{m-2}y\right) \\
& \quad - \ldots - \left(\frac{\partial H_4}{\partial y} + g_1\frac{\partial H_3}{\partial y} + g_2\frac{\partial H_2}{\partial y}\right) - \left(\frac{\partial H_3}{\partial y} + g_1\frac{\partial H_2}{\partial y}\right), \\
\dot{y} = & \quad x + \mathcal{X}(y) \\
= & \quad x + \left(\frac{\partial H_{m+1}}{\partial x} + g_1\frac{\partial H_m}{\partial x} + \ldots + g_{m-1}x\right) + \left(\frac{\partial H_m}{\partial x} + g_1\frac{\partial H_{m-1}}{\partial x} + \ldots + g_{m-2}x\right) \\
& \quad + \ldots + \left(\frac{\partial H_4}{\partial x} + g_1\frac{\partial H_3}{\partial x} + g_2\frac{\partial H_2}{\partial x}\right) + \left(\frac{\partial H_3}{\partial x} + g_1\frac{\partial H_2}{\partial x}\right).
\end{aligned}
$$

This center is uniform isochronous center if and only if $x\dot{y} - y\dot{x} = x^2 + y^2$, consequently the following relation holds $x\mathcal{X}(y) - y\mathcal{X}(x) = 0$, i.e.

$$
\begin{aligned}
& x\left(\frac{\partial H_{m+1}}{\partial x} + g_1\frac{\partial H_m}{\partial x} + \ldots + g_{m-1}x\right) + y\left(\frac{\partial H_{m+1}}{\partial y} + g_1\frac{\partial H_m}{\partial y} + \ldots + g_{m-1}y\right) \\
& + x\left(\frac{\partial H_m}{\partial x} + g_1\frac{\partial H_{m-1}}{\partial x} + \ldots + g_{m-2}x\right) + y\left(\frac{\partial H_m}{\partial y} + g_1\frac{\partial H_{m-1}}{\partial y} + \ldots + g_{m-2}y\right) \\
& + \ldots + x\left(\frac{\partial H_3}{\partial x} + g_1\frac{\partial H_2}{\partial x}\right) + y\left(\frac{\partial H_3}{\partial y} + g_1\frac{\partial H_2}{\partial y}\right) = 0.
\end{aligned}
$$

Using the Euler theorem for the homogenous polynomials H_k of degree k for $k = 2, 3, \ldots, m + 1$ we obtain

$$(m + 1)H_{m+1} + mg_1 H_m + \ldots + 2H_2 g_{m-1}$$
$$+mH_m + (m - 1)g_1 H_{m-1} + \ldots + 2H_2 g_{m-2}$$
$$\vdots \quad \ldots \quad \vdots \quad \ldots$$
$$+5H_5 + 4H_4 g_1 + 3H_3 g_2 + 2H_2 g_3$$
$$+4H_4 + 3g_1 H_3 + 2H_2 g_2$$
$$+3H_3 + 2H_2 g_1 = 0,$$

consequently

$$
\begin{aligned}
(m + 1)H_{m+1} + mg_1 H_m + \ldots + 2H_2 g_{m-1} &= 0, \\
mH_m + (m - 1)g_1 H_{m-1} + \ldots + 2H_2 g_{m-2} &= 0, \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad &\vdots \\
4H_4 + 3g_1 H_3 + 2H_2 g_2 &= 0, \\
3H_3 + 2H_2 g_1 &= 0.
\end{aligned}
$$

(16)

By solving system (16) with respect to $H_3, H_4, \ldots, H_{m+1}$ we get

$$
\begin{aligned}
H_3 =&\ -\frac{2}{3}g_1 H_2 := \Upsilon_1 H_2, \\
H_4 =&\ -\frac{3}{4}g_1 H_3 - \frac{1}{2}g_2 H_2 = \frac{1}{4}\left(2g_1^2 - 2g_2\right)H_2 := \Upsilon_2 H_2, \\
H_5 =&\ -\frac{1}{5}\left(4g_1 H_4 + 3g_2 H_3 + 2g_3 H_2\right) \\
=&\ -\frac{1}{5}\left(2g_1^3 - 4g_1 g_2 + 2g_3\right)H_2 := \Upsilon_3 H_2, \\
H_6 =&\ -\frac{1}{6}\left(5g_1 H_5 + 4g_2 H_4 + 3g_3 H_3 + 2g_4 H_2\right) \\
=&\ \frac{1}{6}\left(2g_1^4 - 6g_1^2 g_2 + 4g_1 g_3 + 2g_2^2 - 2g_4\right)H_2 := \Upsilon_4 H_2, \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
H_{j+1} =&\ -\frac{j}{j+1}H_j - \ldots - \frac{2}{j+1}g_{j-1}H_2 \\
=&\ \frac{(-1)^j}{j+1}\left(2g_1^{j-1} - \ldots - 2g_{j-1}\right) := \Upsilon_{j-1}H_2, \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
H_{m+1} =&\ -\frac{m}{m+1}H_m - \ldots - \frac{2}{m+1}g_{m-1}H_2 \\
=&\ \frac{(-1)^m}{m+1}\left(2g_1^{m-1} - \ldots - 2g_{m-1}\right) := \Upsilon_{m-1}H_2,
\end{aligned}
$$

where Υ_j is a homogenous polynomial of degree j for $j = 1, \ldots, m - 1$.

In view of Corollary 16 $\mathcal{X}(x)$ and $\mathcal{X}(y)$ can be written as

$$\mathcal{X}(x) = \sum_{n=2}^{m} \mathcal{X}_n(x) = x \sum_{n=2}^{m} \frac{1}{n+1} \sum_{j=1}^{n-1} \{H_{n+1-j}, g_j\},$$

$$\mathcal{X}(y) = \sum_{n=2}^{m} \mathcal{X}_n(y) = y \sum_{n=2}^{m} \frac{1}{n+1} \sum_{j=1}^{n-1} \{H_{n+1-j}, g_j\}.$$

Substituting $\mathcal{X}(x)$ and $\mathcal{X}(y)$ into (5) we get

$$\dot{x} = -y + x \sum_{n=2}^{m} \frac{1}{n+1} \sum_{j=1}^{n-1} \{H_{n+1-j}, g_j\} := -y + x \sum_{j=2}^{m} \varphi_j,$$

$$\dot{y} = x + y \sum_{n=2}^{m} \frac{1}{n+1} \sum_{j=1}^{n-1} \{H_{n+1-j}, g_j\} := x + y \sum_{j=2}^{m} \varphi_j.$$

To finish the proof of the theorem we prove that conditions (6) hold. Indeed in view of the relations

$$H_j = H_2 \Upsilon_{j-2} \quad \text{for} \quad j = 3, \ldots, m+1,$$

$$\mathcal{X}_j = \varphi_{j-1}(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) \quad \text{for} \quad j = 2, \ldots, m,$$

$$\mathcal{X}_j(H_{m+l-j}) = (m+l-j)H_{m+l-j}\varphi_{j-1},$$

we get that

$$0 = \sum_{j=2}^{m} \mathcal{X}_j(H_{m+l-j}) + \{H_2, H_{l+m-1}\}$$

$$(17) \qquad = \sum_{j=2}^{m} (m+l-j)H_{m+l-j}\varphi_{j-1} + \{H_2, H_{l+m-1}\}$$

$$= H_2 \sum_{j=2}^{m} (m+l-j)\Upsilon_{m+l-j-2}\varphi_{j-1} + \{H_2, H_{m+l-1}\},$$

for $l = 3, 4, \ldots$. Hence by considering that $\{H_2, H_2\Upsilon_{m+l-3}\} = H_2\{H_2, \Upsilon_{m+l-3}\}$ we have that the solution of (17) are $H_{m+l-1} = H_2\Upsilon_{m+l-3}$, for $l = 3, 4, \ldots$ where the homogenous polynomial Υ_{m+l-3} of degree $m + l - 3$ is the solution of the first order partial differential equation

$$\{\Upsilon_{m+l-3}, H_2\} = \sum_{j=2}^{m} (m+l-j)\Upsilon_{m+l-j-2}\varphi_{j-1}.$$

From the previous results it follows that $H_j = H_2\Upsilon_{j-2}$ for $j = 3, 4, \ldots$. Consequently the first integral (4) becomes $H = H_2(1 + \Upsilon_1 + \Upsilon_2 + \ldots)$. In view of Theorem 2 the theorem is proved. $\qquad \square$

Proof of Corollary 6. By Theorem 4 the polynomial differential system (7) of degree m has the first integral $H = H_2(1 + \Upsilon_1 + \ldots) := H_2\Phi$, if and only if the following

relations hold

$$\frac{dH}{dt} = \frac{d}{dt}(H_2\Phi) = \frac{dH_2}{dt}\Phi + H_2\frac{d\Phi}{dt} = 2H_2\Phi\varphi + H_2\frac{d\Phi}{dt} = 0,$$

where $\varphi = \sum_{n=2}^{m} \varphi_{n-1}$. From $\frac{dH}{dt} = 0$, we have in a small neighborhood the origin that

$$2H_2\sqrt{\Phi}^2 + H_2 2\sqrt{\Phi}\frac{d\sqrt{\Phi}}{dt} = 0.$$

Consequently $\frac{d\sqrt{\Phi}}{dt} = -\sqrt{\Phi}\varphi$. Hence

$$\frac{dX}{dt} = (-y + x\varphi)\sqrt{\Phi} + x\frac{d\sqrt{\Phi}}{dt} = -y\sqrt{\Phi} = -Y,$$

$$\frac{dY}{dt} = (x + y\varphi)\sqrt{\Phi} + y\frac{d\sqrt{\Phi}}{dt} = x\sqrt{\Phi} = X.$$

$\square$

4. PROOF OF THEOREM 7

From Proposition 3 it follows that a differential system with a uniform isochronous center at the origin can be written as (7). Now we shall compute the solution $(x(t), y(t))$ of system (7) satisfying that $(x(0), y(0)) = (\varepsilon, 0)$. where ε is a small parameter.

We use the following complex notation

$$z(t) = x(t) + iy(t) = \sum_{j=1}^{\infty} \varepsilon^j (x_j(t) + iy_j(t)) = \sum_{j=1}^{\infty} \varepsilon^j z_j(t),$$

and we assume that the polynomial φ has the following development

$$\varphi(t) = \sum_{j=1}^{m-1} \varphi_j(x(t), y(t), \varepsilon) = \sum_{j=1}^{\infty} \varepsilon^j \tau_j(t).$$

Hence

$$\tau_1 = \varphi_1(x_1(t), y_1(t)),$$
$$\tau_2 = \varphi_1(x_2(t), y_2(t)) + \varphi_2(x_1(t), y_1(t)),$$
$$\tau_3 = \varphi_1(x_3(t), y_3(t)) + \ldots + \varphi_3(x_1(t), y_1(t)),$$
$$\vdots = \ldots \quad \ldots \vdots \quad,$$
$$\tau_j = \varphi_1(x_j(t), y_j(t)) + \ldots + \varphi_j(x_1(t), y_1(t)),$$
$$\vdots = \ldots \quad \ldots \quad,$$

By substituting $\varphi(t)$ into differential system (7) which now is of the form

$$\dot{z} = z(i + \tilde{\varphi}(z, \bar{z})),$$

we obtain an infinite number of differential systems, one for each coefficient of the different powers of ε :

(18)
$$
\begin{aligned}
\dot{z}_1 &= \quad iz_1, \\
\dot{z}_2 &= \quad iz_2 + z_1\tau_1, \\
\dot{z}_3 &= \quad iz_3 + z_1\tau_2 + z_2\tau_1, \\
&\quad \vdots \quad \vdots \quad \ldots \quad \vdots, \\
\dot{z}_k &= \quad iz_k + z_1\tau_{k-1} + z_2\tau_{k-2} + \ldots + z_{k-1}\tau_1, \\
&\quad \vdots \quad \vdots \quad \ldots \quad \vdots,
\end{aligned}
$$

where $k \geq m$, recall that m is the degree of the polynomial differential system. The solutions of (18) are

$$
\begin{aligned}
z_1 &= \quad e^{it} := e^{it}s_1, \\
z_2 &= \quad e^{it}q_1 := e^{it}s_2, \\
z_3 &= \quad e^{it}\left(\frac{1}{2!}q_1^2 + q_2\right) := e^{it}s_3, \\
z_4 &= \quad e^{it}\left(\frac{1}{3!}q_1^3 + q_1q_2 + q_3\right) := e^{it}s_4, \\
z_5 &= \quad e^{it}\left(\frac{1}{4!}q_1^4 + \frac{1}{2!}q_1^2q_2 + \frac{1}{1!}q_1q_3 + \frac{1}{2!}q_2^2 + q_4\right) := e^{it}s_5, \\
&\quad \vdots \qquad \ldots \qquad \vdots, \\
z_k &= \quad e^{it}\left(\frac{1}{(k-1)!}q_1^{k-1} + \frac{1}{(k-3)!}q_1^{k-3}q_2 + \ldots + q_{k-1}\right) := e^{it}s_k, \\
&\quad \vdots \qquad \ldots \qquad \vdots
\end{aligned}
$$

where $q_j = q_j(t) = \int_0^t \tau_j(x,y)|_{x=\cos t,\, y=\sin t}\, dt$, for $j = 1, \ldots, m-1$. Hence system (7) has a center at the origin if and only if $z_j(0) = z_j(2\pi) = 0$ for $j > 1$, i.e if and only if $s_j(0) = s_j(2\pi) = 0$, for $j > 1$ which is equivalent to $\int_0^{2\pi} \tau_j(x,y)|_{x=\cos t,\, y=\sin t}\, dt = 0$, for $j \geq 1$, On the other hand by considering the solution of system (7) is

$$
z(t) = x(t) + iy(t) = \sum_{j=1}^{\infty} \varepsilon^j z_j(t) = e^{it}\sum_{j=1}^{\infty}\varepsilon^j s_j(t),
$$

we get that

$$
x(t) = \cos t \sum_{j=1}^{\infty}\varepsilon^j s_j(t), \quad y(t) = \sin t \sum_{j=1}^{\infty}\varepsilon^j s_j(t),
$$

and it is periodic if and only if $s_j(0) = s_j(2\pi) = 0$ for $j > 1$. In short the theorem is proved.

5. Proof of Corollaries 8 and 11, and Theorem 10

Proof of Corollary 8. It follows from Theorem 2 under the conditions

$$(19) \quad g_j = \begin{cases} 0 & \text{if} \quad 1 \leq j \leq m-2, \\ g_{m-1} & \text{if} \quad j = m-1. \end{cases} \qquad H_j = \begin{cases} 0 & \text{if } 3 \leq j \leq m, \\ H_2 & \text{if } j = 2, \\ H_{m+1} & \text{if} \quad j = m+1. \end{cases}$$

$\square$

Proof of Theorem 10. Assume that the differential system (9) has a center at the origin, then in view of Corollary 8 we get that this system can be written as (10). If the center is a uniform isochronous center then, from Theorem 4 with g_j and H_j given in (19) we obtain the differential system (12). The reciprocal is obtain as follows.

From (12) it follows that

$$\frac{dH_2}{dt} = \frac{2H_2}{m+1}\{H_2, g_{m-1}\},$$

$$\frac{dg_{m-1}}{dt} = \left(1 + \frac{1}{m+1}\left(x\frac{\partial g_{m-1}}{\partial x} + y\frac{\partial g_{m-1}}{\partial y}\right)\right)\{H_2, g_{m-1}\}.$$

Since g_{m-1} is a homogenous polynomial of degree $m-1$, then

$$\frac{dH_2}{dt} = \frac{2}{m+1}H_2\{H_2, g_{m-1}\},$$

$$\frac{dg_{m-1}}{dt} = \left(1 + \frac{m-1}{m+1}g_{m-1}\right)\{H_2, g_{m-1}\}.$$

Thus the curve $H_2 = 0$ and $1 + \frac{m-1}{m+1}g_{m-1} = 0$ are invariant algebraic curves with cofactors $\frac{2}{m+1}\{H_2, g_{m-1}\}$ and $\frac{m-1}{m+1}\{H_2, g_{m-1}\}$ respectively. Therefore a first integral is

$$F = \frac{H_2^{m-1}}{\left(1 + \frac{m-1}{m+1}g_{m-1}\right)^2}.$$

Consecutively the origin is a center, and the theorem is proved . $\square$

Proof of Corollary 11. Since

$$F^{1/(m-1)} = \frac{H_2}{\left(1 + \frac{m-1}{m+1}g_{m-1}\right)^{1/(m-1)}} := H_2\Phi$$

is a first integral of system (12), we have as in the proof of Corollary 6 that $\frac{d\sqrt{\Phi}}{dt} = -\varphi\sqrt{\Phi}$. Consequently

$$\frac{dX}{dt} = (-y + x\varphi)\sqrt{\Phi} + x\frac{d\sqrt{\Phi}}{dt} = -y\sqrt{\Phi} = -Y,$$

$$\frac{dY}{dt} = (x + y\varphi)\sqrt{\Phi} + x\frac{d\sqrt{\Phi}}{dt} = x\sqrt{\Phi} = X.$$

$\square$

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