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# GLOBAL DYNAMICS OF A SD OSCILLATOR

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ABSTRACT. In this paper we derive the global bifurcation diagrams of a SD oscillator which exhibits both smooth and discontinuous dynamics depending on the value of a parameter $a$. We research all possible bifurcations of this system, including Pitchfork bifurcation, degenerate Hopf bifurcation, Homoclinic bifurcation, Double limit cycle bifurcation, Bautin bifurcation and Bogdanov-Takens bifurcation. Besides we prove that the system has at most five limit cycles. At last, we give all numerical phase portraits to illustrate our results.

## 1. INTRODUCTION AND MAIN RESULTS

In recent years SD (Smooth and Discontinuous, for short) oscillator was proposed and investigated for studying the transition from smooth to discontinuous dynamics, see for instance [2, 3, 4, 5, 16]. In those papers it is proposed the elastic beam model

$$\ddot{x} + \xi(b + x^2)\dot{x} + x\left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) = 0 \tag{1}$$

for studying this transition, where $a \geq 0$, $b$ and $\xi$ can take arbitrary real values. More precisely, the smooth dynamics appears when $a > 0$, while the discontinuous dynamic behavior occurs at $a = 0$. The global dynamics was completely studied in [5] when $a = 0$, and in [16] when $|a - 1| < \varepsilon$, $|\xi| < \varepsilon$ and $\varepsilon$ is sufficiently small.

Clearly system (1) can be rewritten as the 2-dimensional differential system

$$\begin{aligned} \dot{x} &= y - \hat{\xi}(\hat{b}x + x^3) =: y - F(x), \\ \dot{y} &= -x\left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) =: -g(x), \end{aligned} \tag{2}$$

where $\hat{\xi} = \xi/3$, $\hat{b} = 3b$, and for simplicity in what follows we still denote $\hat{\xi}$ and $\hat{b}$ by $\xi$ and $b$ respectively. Note that system (2) is invariant by the transformation $(y, t, \xi) \to (-y, -t, -\xi)$. Therefore, we only need to consider the set of parameters

$$\mathcal{G} := \{(a, b, \xi) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+\},$$

where $\mathbb{R}^+ = [0, +\infty)$.

In this paper we shall describe the dynamics of system (2). Thus, the following theorem is our main result.

**Theorem 1.** *System* (2) *has three equilibria* $E_L = (-\sqrt{1 - a^2}, -(1-a^2+b)\xi\sqrt{1 - a^2})$, $E_0 = (0, 0)$ *and* $E_R = (\sqrt{1 - a^2}, (1-a^2+b)\xi\sqrt{1 - a^2})$ *if* $a < 1$, *and only* $E_0$ *if* $a \geq 1$. *The global bifurcation diagram of system* (2) *consists of the following bifurcation surfaces:*

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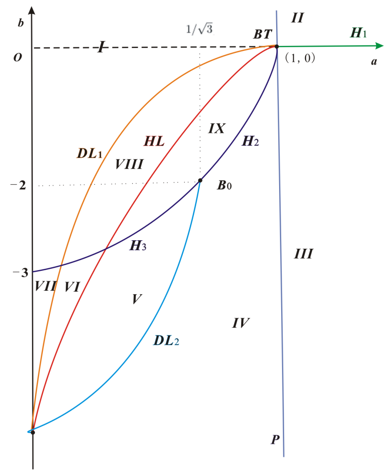



FIGURE 1. Global qualitative bifurcation diagram of system (2) on the half-plane $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$ with $\xi = \xi^* > 0$ small.

(i) *Pitchfork surface* $P := \{(a, b, \xi) \in \mathcal{G} : a = 1\}$;

(ii) *Hopf surfaces*

$$H_1 := \{(a, b, \xi) \in \mathcal{G} : a > 1, b = 0\},$$
$$H_2 := \{(a, b, \xi) \in \mathcal{G} : b = 3a^2 - 3,\ 1/\sqrt{3} < a < 1\} \text{ and}$$
$$H_3 := \{(a, b, \xi) \in \mathcal{G} : b = 3a^2 - 3,\ 0 < a < 1/\sqrt{3}\};$$

(iii) *Homoclinic surface* $HL := \{(a, b, \xi) \in \mathcal{G} : b = \varphi(a, \xi),\ 0 < a < 1\}$;

(iv) *Double limit cycle surfaces*

$$DL_1 := \{(a, b, \xi) \in \mathcal{G} : b = \phi_1(a, \xi),\ 0 < a < 1\} \text{ and}$$
$$DL_2 := \{(a, b, \xi) \in \mathcal{G} : b = \phi_2(a, \xi),\ 0 < a < 1/\sqrt{3}\};$$

(v) *Codimension* 2 *Bogdanov-Takens bifurcation with symmetry curve:*

$$BT := \{(a, b, \xi) \in \mathcal{G} : a = 1,\ b = 0\};$$

(vi) *Bautin bifurcation curve:* $B_0 := \{(a, b, \xi) \in \mathcal{G} : a = 1/\sqrt{3},\ b = -2\}$;

*where*

$$\varphi(a, \xi) < \phi_1(a, \xi) < a^2 - 1, \qquad \varphi(1, \xi) = \phi_1(1, \xi) = 0,$$
$$\phi_2(a, \xi) < \min\{\varphi(a, \xi), 3a^2 - 3\}, \qquad \phi_2(1/\sqrt{3}, \xi) = -2, \text{ and}$$
$$\varphi(0, \xi) = \phi_1(0, \xi) = \phi_2(0, \xi),$$

*as shown in Figure 1. Consequently, the complete classification of the phase portraits of system* (2) *is given in Figure 2, where*

I $\{(a, b, \xi) \in \mathcal{G} : b > \max\{3 - 3a^2,\ \phi_1(a, \xi)\},\ 0 < a < 1\}$,

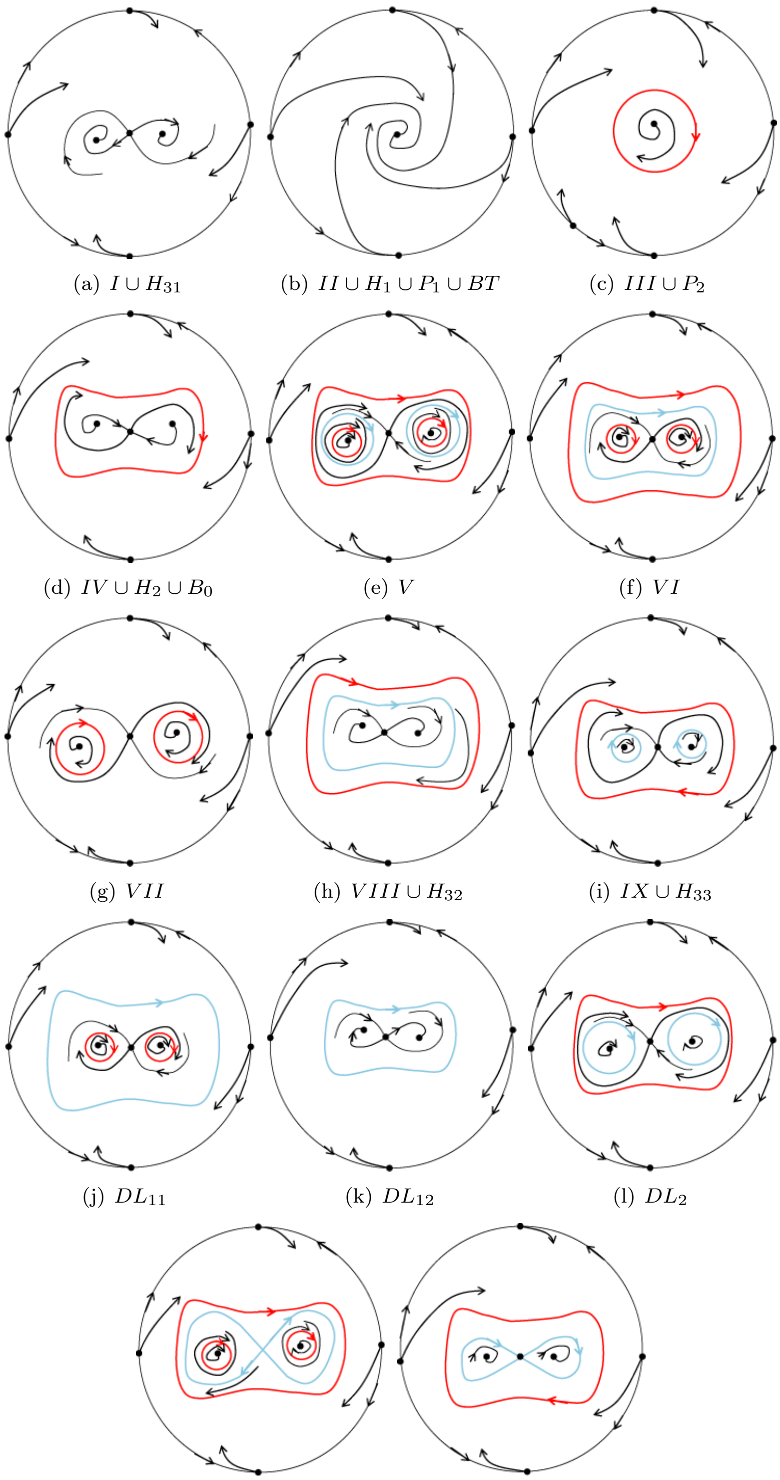
II $\{(a, b, \xi) \in \mathcal{G} : b > 0,\ a > 1\}$,

III $\{(a, b, \xi) \in \mathcal{G} : b < 0,\ a > 1\}$,

IV $\{(a, b, \xi) \in \mathcal{G} : b < \min\{3 - 3a^2,\ \phi_2(a, \xi)\},\ 0 < a < 1\}$,

V $\{(a, b, \xi) \in \mathcal{G} : \phi_2(a, \xi) < b < \min\{3 - 3a^2, \varphi(a, \xi)\},\ 0 < a < 1\}$,

VI $\{(a, b, \xi) \in \mathcal{G} : \varphi(a, \xi) < b < \min\{3 - 3a^2, \phi_1(a, \xi)\},\ 0 < a < 1\}$,



(a) $I \cup H_{31}$ (b) $II \cup H_1 \cup P_1 \cup BT$ (c) $III \cup P_2$

(d) $IV \cup H_2 \cup B_0$ (e) $V$ (f) $VI$

(g) $VII$ (h) $VIII \cup H_{32}$ (i) $IX \cup H_{33}$

(j) $DL_{11}$ (k) $DL_{12}$ (l) $DL_2$

(m) $HL_1$ (n) $HL_2$

FIGURE 2. Phase portraits of system (2) on the Poincaré disc.

VII $\{(a, b, \xi) \in \mathcal{G} : \phi_1(a, \xi) < b < 3 - 3a^2,\ 0 < a < 1\}$,
VIII $\{(a, b, \xi) \in \mathcal{G} : \max\{3 - 3a^2, \varphi(a, \xi)\} < b < \phi_1(a, \xi),\ 0 < a < 1\}$,
IX $\{(a, b, \xi) \in \mathcal{G} : 3 - 3a^2 < b < \varphi(a, \xi),\ 0 < a < 1\}$,
$HL_1$ $\{(a, b, \xi) \in \mathcal{G} : b = \varphi(a, \xi) < 3a^2 - 3,\ 0 < a < 1\}$,
$HL_2$ $\{(a, b, \xi) \in \mathcal{G} : b = \varphi(a, \xi) \geq 3a^2 - 3,\ 0 < a < 1\}$,
$DL_{11}$ $\{(a, b, \xi) \in \mathcal{G} : b = \phi_1(a, \xi) < 3a^2 - 3,\ 0 < a < 1\}$,
$DL_{12}$ $\{(a, b, \xi) \in \mathcal{G} : b = \phi_1(a, \xi) \geq 3a^2 - 3,\ 0 < a < 1\}$,
$P_1$ $\{(a, b, \xi) \in \mathcal{G} : b > 0,\ a = 1\}$,
$P_2$ $\{(a, b, \xi) \in \mathcal{G} : b < 0,\ a = 1\}$,
$H_{31}$ $\{(a, b, \xi) \in \mathcal{G} : b = 3a^2 - 3 > \phi_1(a, \xi),\ 0 < a < 1/\sqrt{3}\}$,
$H_{32}$ $\{(a, b, \xi) \in \mathcal{G} : \varphi(a, \xi) < b = 3a^2 - 3 < \phi_1(a, \xi),\ 0 < a < 1/\sqrt{3}\}$,
$H_{33}$ $\{(a, b, \xi) \in \mathcal{G} : \phi_2(a, \xi) < b = 3a^2 - 3 < \varphi(a, \xi),\ 0 < a < 1/\sqrt{3}\}$.

*Moreover, all results about limit cycles and homoclinic loops are listed in Table 1.*

| *Subsets of $\mathcal{G}$* | *large limit cycles surrounding all $E_R, E_0, E_L$* | *small limit cycles only surrounding $E_0$ $E_L$, $E_R$ respectively* | *homoclinic loops* |
|---|---|---|---|
| $I, II, H_1, H_{31}, P_1, BT$ | *0* | *0; 0; 0* | *0* |
| $III, P_2$ | *0* | *1 stable; 0; 0* | *0* |
| $IV, H_2, B_0$ | *1 stable* | *0; 0; 0* | *0* |
| $V$ | *1 stable* | *0; 2 the inner one is stable, the outer one is unstable; 2 the inner one is stable, the outer one is unstable* | *0* |
| $VI$ | *2 the inner one is unstable, the outer one is stable;* | *0; 1 stable ; 1 stable* | *0* |
| $VII$ | *0* | *0; 1 stable; 1 stable* | *0* |
| $VIII, H_{32}$ | *2 the inner one is unstable, the outer one is stable;* | *0; 0 ; 0* | *0* |
| $IX, H_{33}$ | *1 stable* | *0; 1 unstable; 1 unstable* | *0* |
| $DL_{11}$ | *1 semistable* | *0; 1 stable; 1 stable* | *0* |
| $DL_{12}$ | *1 semistable* | *0; 0; 0* | *0* |
| $DL_2$ | *1 stable* | *0; 1 semistable; 1 semistable* | *0* |
| $HL_1$ | *1 stable* | *0; 1 stable; 1 stable* | *figure-eight loop, unstable* |
| $HL_2$ | *1 stable* | *0; 0; 0* | *figure-eight loop, unstable* |

TABLE 1. Limit cycles and homoclinic loops of system (2).

In this paper we call *large limit cycles* to the ones surrounding all three equilibria, and *small limit cycles* the ones surrounding a single equilibrium. For the notions and definitions which appear in the statement of Theorem 1 see its proof.

The paper is organized as follows. In section 2 we analyze the local bifurcations, namely pitchfork bifurcation, Hopf bifurcation, Bautin bifurcation and codimension 2 Bogdanov-Takens bifurcation with symmetry. In section 3 we estimate the number of limit cycles in difference parameter regions and curves. In section 4 we study the global bifurcations, namely the different kinds of homoclinic connections and the double limit cycles. In section 5 we give the numerical phase portraits in difference parameter regions.

## 2. Local bifurcations

Computing the Jacobian matrix at equilibrium $E_0$, we have

$$J_0 := \begin{pmatrix} -b\xi & 1 \\ \frac{1}{a} - 1 & 0 \end{pmatrix}.$$

Then, at $E_0$ the determinant $det(J_0) = 1 - 1/a$ and the trace $tr(J_0) = -b\xi$, implying that $E_0$ is a saddle if $a < 1$, stable focus or node if $a > 1$ and $b > 0$, and unstable focus or node if $a > 1$ and $b < 0$. When $a = 1$, equilibrium $E_0$ is a stable node if $b > 0$, and unstable node if $b < 0$ by applying [8, Theorem 2.19]. By the symmetry of the vector field (2), equilibrium $E_L$ is of the same type as $E_R$. The Jacobian matrix at $E_R$ is

$$J_R := \begin{pmatrix} -\xi(b + 3 - 3a^2) & 1 \\ -(1 - a^2) & 0 \end{pmatrix}.$$

Calculating $det(J_R) = 1 - a^2$ and $tr(J_R) = -\xi(b + 3 - 3a^2)$. Hence $E_R$ is a stable focus or node if $a < 1$ and $b + 3 - 3a^2 > 0$, and unstable focus or node if $a < 1$ and $b + 3 - 3a^2 < 0$.

2.1. **Pitchfork bifurcation.** From the expressions of the equilibria $E_L$, $E_0$ and $E_R$ it follows immediately that a pitchfork bifurcation occurs at the origin of coordinates when $a = 1$; i.e. for $a \geq 1$ we have a unique antisaddle, while for $0 < a < 1$ from the previous antisaddle it bifurcates at $a = 1$ a saddle and two antisaddles. For more details on this kind of bifurcation see [7, 10].

2.2. **Hopf bifurcations.** There are two kinds of Hopf bifurcations, one at the equilibrium $E_0$, and the other at the equilibria $E_L$ and $E_R$, which is essentially the same bifurcation in both, because due to the invariance of system (2) with respect to the symmetry $(x, y) \mapsto (-x, -y)$, what occurs at the equilibrium point $E_L$ occurs to its symmetric $E_R$.

2.2.1 **Hopf bifurcation at** $E_0$. The next result characterizes the Hopf bifurcation at the equilibrium point $E_0$, it is proved using the averaging theory in this way we also can estimated the shape of the limit cycle bifurcating from $E_0$, and we avoid the computation of the Liapunov constant.

**Proposition 2.** *The following statements hold for the differential system* (2).

(a) *If $a > 1$, $b = 0$ and $\xi > 0$, then a Hopf bifurcation takes place at the equilibrium point located at the origin of coordinates, and the limit cycle $\gamma$ bifurcating from this equilibrium exists for $b < 0$ sufficiently small.*

(b) *For $\varepsilon > 0$ sufficiently small if $b = -\beta\varepsilon^2 < 0$, then the limit cycle $\gamma$ passes through the point*

$$\left(2\sqrt{\frac{\beta}{3}}\varepsilon + O(\varepsilon^3), 0\right).$$

*Moreover, this limit cycle is stable.*

*Proof.* Assume that $a > 1$, $\xi > 0$ and $b = -\beta\varepsilon^2 < 0$ where $\varepsilon$ is a sufficiently small parameter. Writing the differential system (2) in polar coordinates we get

$$\begin{aligned} \dot r &= \frac{r\cos\theta\sin\theta}{\sqrt{a^2+r^2\cos^2\theta}} - \xi r^3\cos^4\theta + \varepsilon^2\xi\beta\, r\cos^2\theta, \\ \dot\theta &= -1 + \frac{\cos^2\theta}{\sqrt{a^2+r^2\cos^2\theta}} + \xi r^2\sin\theta\cos^3\theta - \varepsilon^2\xi\beta\cos\theta\sin\theta. \end{aligned} \tag{3}$$

Since we want to study the Hopf bifurcation at the origin of coordinates we blow up the origin doing the scaling $r = \varepsilon R$, then differential system (3) taking as new independent variable the $\theta$ becomes

$$\begin{aligned} \frac{dR}{d\theta} = & \frac{R\sin\theta\cos\theta}{\cos^2\theta - a} + \\ & \varepsilon^2 \frac{R\left(2\xi a^2(a-1)(R^2 - 2\beta R^2\cos(2\theta)) + R^2\sin(2\theta)\right)\cos^2\theta}{4a(\cos^2\theta - a)^2} + O(\varepsilon^4), \end{aligned} \tag{4}$$

In order to apply the averaging theory described in the appendix we need that the differential equation (4) starts at least with order $\varepsilon$. So we do the change of variables $R \mapsto \rho$ defined by

$$R = \sqrt{\frac{2(1-a)}{1-2a+\cos(2\theta)}}\,\rho.$$

Then differential equation (4) in the new variable $\rho$ writes

$$\begin{aligned} \frac{d\rho}{d\theta} &= -\varepsilon^2 \frac{2(a-1)\rho\cos^2\theta}{a(1-2a+\cos(2\theta))^3}\big(2\xi a^2\left((a-1)\rho^2 - 2a\beta + \beta\right) + \\ & \qquad 2\xi a^2\left((a-1)\rho^2+\beta\right)\cos(2\theta) + \rho^2\sin(2\theta)\big) + O(\varepsilon^4) + \\ &= \varepsilon^2 F_1(\theta,\rho) + O(\varepsilon^4). \end{aligned} \tag{5}$$

Differential equation (5) is written into the normal form (37) for applying the averaging theory summarized in the appendix, using the notation of the appendix we only need to take $n = 1$, $\mathbf{x} = \rho$, $t = \theta$, $\mu = \varepsilon^2$, $F_1(t,\mathbf{x}) = F_1(\theta,\rho)$ and $T = 2\pi$, all the necessary assumptions for applying the averaging theory described in the appendix are satisfied. Then we compute

$$f_1(\rho) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,\rho)d\theta = \frac{\xi}{8}\sqrt{\frac{a}{a-1}}\rho(3\rho^2 - 4\beta).$$

Since $\beta > 0$ the averaged function $f_1(\rho)$ has a unique positive zero, $\rho = 2\sqrt{\beta/3}$ which satisfies the condition

$$D_\rho f_1\left(2\sqrt{\frac{\beta}{3}}\right) = \xi\sqrt{\frac{a}{a-1}} \neq 0. \tag{6}$$

In fact this last expression is positive because $a > 1$ and $\xi > 0$, and consequently, by the results described in the appendix the differential equation (5) has a periodic solution $\rho(\theta,\varepsilon)$ satisfying that

$$\rho(0,0) = 2\sqrt{\frac{\beta}{3}} + O(\varepsilon^2). \tag{7}$$

Moreover, this periodic solution $\rho(\theta,\varepsilon)$ is unstable because the derivative (6) is positive.

Now we will go back through the changes of variables for obtaining the periodic solution bifurcating from the equilibrium at the origin of coordinates of the differential system (2). Thus, the periodic solution $\rho(\theta,\varepsilon)$ satisfying the initial condition (7) in the variables of the differential system (4) becomes the periodic solution

$$R(\theta,\varepsilon)=\sqrt{\frac{2(1-a)}{1-2a+\cos(2\theta)}}\,\rho(\theta,\varepsilon),$$

satisfying

$$R(0,0)=2\sqrt{\frac{\beta}{3}}+O(\varepsilon^2).$$

This periodic solution in the differential system (3) becomes $(r(t,\varepsilon),\theta(t,\varepsilon))$ with

$$r(t,\varepsilon)=\sqrt{\frac{2(1-a)}{1-2a+\cos(2\theta(t,\varepsilon))}}\,\rho(\theta(t,\varepsilon),\varepsilon)\varepsilon+O(\varepsilon^3),$$

and it pass through the point

$$\left(2\sqrt{\frac{\beta}{3}}\varepsilon+O(\varepsilon^3),0\right) \tag{8}$$

in the coordinates $(r,\theta)$. Finally, this periodic solution in the coordinates of system (2) is the periodic solution $(x(t,\varepsilon),y(t,\varepsilon))$ given by

$$\varepsilon\sqrt{\frac{2(1-a)}{1-2a+\cos(2\theta(t,\varepsilon))}}\,\rho(\theta(t,\varepsilon),\varepsilon)\big(\cos(\theta(t,\varepsilon)),\sin(\theta(t,\varepsilon))\big)+O(\varepsilon^3),$$

passing through the point (8) now in coordinates $(x,y)$. Therefore, when $\varepsilon\mapsto 0$ such periodic solution tends to the origin, so it is a periodic solution of a Hopf bifurcation.

We remark that the periodic solution $\rho(\theta,\varepsilon)$ was an unstable limit cycle, but due to the fact that $\dot\theta$ is negative in a neighborhood of the origin, when we pass the unstable limit cycle $R(\theta,\varepsilon)$ to the periodic solution $(r(t,\varepsilon),\theta(t,\varepsilon))$ it changes to a stable limit cycle. □

In short, Proposition 2 shows the existence of the Hopf bifurcation surface $H_1$.

#### 2.2.2 Hopf bifurcation at $E_L$ and $E_R$.

**Proposition 3.** *The following statements hold for the differential system* (2).

(a) *If $0<a<1$, $a\neq 1/\sqrt{3}$, $b=3(a^2-1)$ and $\xi>0$, then one limit cycle bifurcates from each one of the equilibria $E_L$ and $E_R$.*

(b) *For $\varepsilon>0$ sufficiently small if $b=3a^2-3+\beta\varepsilon^2$, then those two limit cycles exist if $\beta(1-3a^2)<0$, and they pass through the points*

$$\pm\left(\sqrt{1-a^2}+2\sqrt{\frac{\beta(1-a^2)}{3(3a^2-1)}}\,\varepsilon+O(\varepsilon^2),\xi\sqrt{1-a^2}(2a^2-2+\beta\varepsilon^2)\right).$$

*Moreover, these limit cycles are stable if $\beta<0$, and unstable if $\beta>0$.*

*Proof.* Assume $0 < a < 1$, $a \neq 1/\sqrt{3}$, $\xi > 0$ and $b = 3(a^2 - 1) + \beta\varepsilon^2$, where $\varepsilon$ is a sufficiently small parameter. We shall prove the proposition studying the Hopf bifurcation at the equilibrium point $E_R$.

We translate the equilibrium point $E_R$ to the origin of coordinates doing the change

$$(x, y) = (X + \sqrt{1-a^2}, Y + \xi\sqrt{1-a^2}(2a^2 - 2 + \beta\varepsilon^2),$$

and system (2) is transformed into

$$\begin{aligned} \dot{X} &= -\varepsilon^2\xi\beta X + Y - 3\xi\sqrt{1-a^2}X^2 - \xi X^3, \\ \dot{Y} &= \left(-X - \sqrt{1-a^2}\right)\left(1 - \frac{1}{\sqrt{X^2 + 2\sqrt{1-a^2}X + 1}}\right). \end{aligned} \tag{9}$$

Writing the differential system (9) in polar coordinates we get

$$\begin{aligned} \dot{r} = {} & -r\cos\theta\left(\xi r^2\cos^3\theta + 3\xi\sqrt{1-a^2}r\cos^2\theta - \sin\theta\right) - \left(r\cos\theta + \sqrt{1-a^2}\right) \\ & \cdot\left(1 - \frac{1}{\sqrt{r^2\cos^2\theta + 2\sqrt{1-a^2}r\cos\theta + 1}}\right)\sin\theta - \varepsilon^2\xi\beta r\cos^2\theta, \\ \dot{\theta} = {} & -1 + \xi\left(r^2\cos^2\theta + 3\sqrt{1-a^2}r\cos\theta\right)\sin\theta\cos\theta + \\ & \frac{\left(r\cos\theta + \sqrt{1-a^2}\right)\cos\theta}{r\sqrt{r^2\cos^2\theta + 2\sqrt{1-a^2}r\cos\theta + 1}} - \frac{\sqrt{1-a^2}\cos\theta}{r}. \end{aligned} \tag{10}$$

Again since we want to study the Hopf bifurcation now at the origin of coordinates we blow up the origin doing the scaling $r = \varepsilon R$, then differential system (10) taking as new independent variable the $\theta$ writes

$$\begin{aligned} \frac{dR}{d\theta} = {} & \frac{a^2R\cos\theta\sin\theta}{a^2\cos^2\theta - 1} - \varepsilon\frac{3\sqrt{1-a^2}R^2\cos^2\theta\left(2\xi\left(a^2-1\right)\cos\theta - a^2\sin\theta\right)}{2\left(a^2\cos^2\theta - 1\right)^2} \\ & +\varepsilon^2\frac{R\cos^2\theta}{32\left(a^2\cos^2\theta - 1\right)^3}\Big(54\xi a^6R^2 + 2a^6R^2\sin(2\theta) + a^6R^2\sin(4\theta) \\ & -102\xi a^4R^2 - 16\xi\beta a^4 - 72\xi^2a^4R^2\sin(2\theta) - 38a^4R^2\sin(2\theta) \\ & -36\xi^2a^4R^2\sin(4\theta) + a^4R^2\sin(4\theta) + 64\xi a^2R^2 + 48\xi\beta a^2 \\ & +2\xi a^2\left(9a^4 - 29a^2 + 20\right)R^2\cos(4\theta) + 144\xi^2a^2R^2\sin(2\theta) \\ & +32a^2R^2\sin(2\theta) + 72\xi^2a^2R^2\sin(4\theta) - 16\xi R^2 - 32\xi\beta \\ & +8\xi\left(a^2-1\right)\left(9R^2a^4 - a^2\left(11R^2 + 2\beta\right) + 2R^2\right)\cos(2\theta) \\ & -72\xi^2R^2\sin(2\theta) - 36\xi^2R^2\sin(4\theta)\Big) + O(\varepsilon^3), \end{aligned} \tag{11}$$

Again for applying the averaging theory of the appendix we need that the differential equation (11) starts at least with order $\varepsilon$. Hence we do the change of variables $R \mapsto \rho$ defined by

$$R = \sqrt{\frac{2(1-a^2)}{2 - a^2 + a^2\cos(2\theta)}}\,\rho.$$

Then differential equation (11) in the new variable $\rho$ writes

$$\frac{d\rho}{d\theta} = \varepsilon F_1(\theta,\rho) + \varepsilon^2 F_2(\theta,\rho) + O(\varepsilon^3), \tag{12}$$

where

$$\begin{aligned}
F_1(\theta,\rho) = &\ \frac{3\left(a^2-1\right)\rho^2\cos^2\theta\left(2\xi\left(a^2-1\right)\cos\theta - a^2\sin\theta\right)}{\sqrt{2}\left(a^2\cos^2\theta-1\right)^2\sqrt{2-a^2-a^2\cos(2\theta)}},\\
F_2(\theta,\rho) = &\ \frac{\left(a^2-1\right)\rho\cos^2\theta}{2\left(\cos(2\theta)a^2+a^2-2\right)^4}\Big(54e\rho^2a^6+2\rho^2\sin(2\theta)a^6+\rho^2\sin(4\theta)a^6\\
&-102e\rho^2a^4-12e\beta a^4-72e^2\rho^2\sin(2\theta)a^4-38\rho^2\sin(2\theta)a^4\\
&-36e^2\rho^2\sin(4\theta)a^4+\rho^2\sin(4\theta)a^4+64e\rho^2a^2+32e\beta a^2\\
&+2e\left(9\rho^2a^4-\left(29\rho^2+2\beta\right)a^2+20\rho^2\right)\cos(4\theta)a^2\\
&+144e^2\rho^2\sin(2\theta)a^2+32\rho^2\sin(2\theta)a^2+72e^2\rho^2\sin(4\theta)a^2\\
&-16e\rho^2-32e\beta+8e\big(9\rho^2a^6-2\left(10\rho^2+\beta\right)a^4\\
&+\left(13\rho^2+4\beta\right)a^2-2\rho^2\big)\cos(2\theta)-72e^2\rho^2\sin(2\theta)\\
&-36e^2\rho^2\sin(4\theta)\Big).
\end{aligned}$$

Differential equation (12) is already into the normal form (37) for applying the averaging theory of the appendix. Again using the notation of the appendix we take $n=1$, $\mathbf{x}=\rho$, $t=\theta$, $\mu=\varepsilon$, $F_1(t,\mathbf{x}) = F_1(\theta,\rho)$ and $T=2\pi$, and all the necessary hypotheses for applying the averaging theory of the appendix hold. Then we compute

$$f_1(\rho) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,\rho)d\theta \equiv 0.$$

Since the first averaged function $f_1(\rho)$ is identically zero, we must compute the second one $f_2(\rho)$. We start calculating

$$\int_0^\theta F_1(\theta,\rho)ds = \frac{\rho^2 N(\theta)}{2\sqrt{2-2a^2}\left(2-a^2-a^2\cos(2\theta)\right)^{3/2}},$$

where

$$\begin{aligned}
N(\theta) = &\ 3a^2(1-a^2)^{3/2}\cos\theta - a^2\sqrt{2}(2-a^2-a^2\cos(2\theta))^{3/2}\\
&+\sqrt{1-a^2}\Big((a^2-a^4)\cos(3\theta)+2\xi\big(3(a^2-3)\sin\theta\\
&+(3a^2-1)\sin(3\theta)\big)\Big).
\end{aligned}$$

Then the second averaged function

$$\begin{aligned}
f_2(\rho) = &\ \frac{1}{2\pi}\int_0^{2\pi}\left[D_\rho F_1(\theta,\rho)\int_0^\theta F_1(s,\rho)ds + F_2(\theta,\rho)\right]d\theta\\
= &\ \frac{\xi\rho\left(3\left(1-3a^2\right)\rho^2+4\beta\right)}{8\sqrt{1-a^2}}
\end{aligned}$$

has a unique positive zero, $\rho = 2\sqrt{\beta/(3(3a^2-1))}$, recall that $a \neq 1/\sqrt{3}$. This zero satisfies the condition

$$D_\rho f_2\left(2\sqrt{\frac{\beta}{3(3a^2-1)}}\right) = -\frac{\xi\beta}{\sqrt{1-a^2}} \neq 0, \tag{13}$$

by assumptions. Consequently, by the results described in the appendix the differential equation (12) has a periodic solution $\rho(\theta,\varepsilon)$ satisfying that

$$\rho(0,0) = 2\sqrt{\frac{\beta}{3(3a^2-1)}} + O(\varepsilon). \tag{14}$$

Moreover, from (13) this periodic solution $\rho(\theta,\varepsilon)$ is unstable if $\beta < 0$, and stable if $\beta > 0$.

Now we will go back through the changes of variables for obtaining the periodic solution bifurcating from the equilibrium $E_R$ of the differential system (2). Thus, the periodic solution $\rho(\theta,\varepsilon)$ satisfying the initial condition (14) in the variables of the differential system (11) becomes the periodic solution

$$R(\theta,\varepsilon) = \sqrt{\frac{2(1-a^2)}{2-a^2+a^2\cos(2\theta)}}\,\rho(\theta,\varepsilon),$$

satisfying

$$R(0,0) = 2\sqrt{\frac{\beta(1-a^2)}{3(3a^2-1)}} + O(\varepsilon).$$

This periodic solution in the differential system (10) becomes $(r(t,\varepsilon),\theta(t,\varepsilon))$ with

$$r(t,\varepsilon) = \sqrt{\frac{2(1-a^2)}{2-a^2+a^2\cos(2\theta(t,\varepsilon))}}\,\rho(\theta(t,\varepsilon),\varepsilon)\varepsilon + O(\varepsilon^2),$$

and it pass through the point

$$\left(2\sqrt{\frac{\beta(1-a^2)}{3(3a^2-1)}}\,\varepsilon + O(\varepsilon^2), 0\right) \tag{15}$$

in the coordinates $(r,\theta)$. This periodic solution in the coordinates of system (9) is the periodic solution $(X(t,\varepsilon),Y(t,\varepsilon))$ given by

$$\varepsilon\sqrt{\frac{2(1-a^2)}{2-a^2+a^2\cos(2\theta(t,\varepsilon))}}\,\rho(\theta(t,\varepsilon),\varepsilon)\big(\cos(\theta(t,\varepsilon)),\sin(\theta(t,\varepsilon))\big) + O(\varepsilon^2),$$

passing through the point (15) now in coordinates $(X,Y)$. Finally, we get the periodic solution $(x(t,\varepsilon),y(t,\varepsilon))$ given by

$$(x(t,\varepsilon),y(t,\varepsilon)) = (X(t,\varepsilon)+\sqrt{1-a^2}, Y(t,\varepsilon)+\xi\sqrt{1-a^2}(2a^2-2+\beta\varepsilon^2)),$$

and passing through the point

$$\left(\sqrt{1-a^2}+2\sqrt{\frac{\beta(1-a^2)}{3(3a^2-1)}}\,\varepsilon + O(\varepsilon^2), \xi\sqrt{1-a^2}(2a^2-2+\beta\varepsilon^2)\right)$$

in coordinates $(x,y)$. So, when $\varepsilon \mapsto 0$ such periodic solution tends to the equilibrium $E_R$, so it is a periodic solution of a Hopf bifurcation.

Again we note that the limit cycle $\rho(\theta,\varepsilon)$ was unstable if $\beta<0$, and stable if $\beta>0$, but due to the fact that $\dot{\theta}$ is negative in a neighborhood of the origin, when we pass the limit cycle $R(\theta,\varepsilon)$ to the cycle $(r(t,\varepsilon),\theta(t,\varepsilon))$ it changes its type of stability. □

In particular Proposition 3 shows the existence of the Hopf bifurcation surfaces $H_2$ and $H_3$.

2.2.3 **The Bautin bifurcation curve $B_0$.** The standard or classical Hopf bifurcation in a 2-dimensional differential system, i.e. that a limit cycle bifurcates from an equilibrium point, takes place in an equilibrium point with purely imaginary eigenvalues which is not a center because the first Liapunov constant at that equilibrium is not zero. These are the Hopf bifurcations studied in subsection 2.2.2. But when the first Liapunov constant is zero, also can bifurcate a limit cycle of the equilibrium point if the second Liapunov constant is not zero, such more degenerate Hopf bifurcation is called for some authors a *Bautin bifurcation*. See for more details about these Hopf bifurcations [11, Chapter 8].

When $b=3(a^2-1)$, with the change of variables $x_1=x-\sqrt{1-a^2}$, $y_1=y/\sqrt{1-a^2}-\xi(b+1-a^2)$ and $\tau=\sqrt{1-a^2}t$ in a small neighborhood of $E_R$, system (2) can be written as

$$\begin{aligned}\frac{dx}{dt}&=y-3\xi x^2-\frac{\xi}{\sqrt{1-a^2}}x^3=:y+\hat{f}(x),\\ \frac{dy}{dt}&=-x-\frac{3a^2x^2}{2\sqrt{1-a^2}}+\frac{(4a^2-5a^4)x^3}{2-2a^2}+O(x^4)=:-x+\hat{h}(x),\end{aligned}\tag{16}$$

where, for simplicity, we still use the variables $(x,y,t)$ instead of the new ones $(x_1,y_1,\tau)$. From [10, p. 156] we compute the first Liapunov constant at the origin of system (16) and we get $\hat{g}_1=3\xi(3a^2-1)/(8\sqrt{1-a^2})$. From the expressions of $\hat{g}_1$, we can confirm that the classical Hopf bifurcation happens when $a\neq 1/\sqrt{3}$ and $b=3a^2-3$. Clearly, when $a=1/\sqrt{3}$ and $b=-2$ we have $\hat{g}_1=0$. By [11, Chapter 8], we can obtain that the second Liapunov constant $\hat{g}_2=5\sqrt{6}\ \xi/32>0$ at those values of $a$ and $b$.

In short, system (2) exhibits a Bautin bifurcation at $a=1/\sqrt{3}$ and $b=-2$ by [11, Chapter 8], i.e. at the intersection point of the surfaces $H_2$, $H_3$ and $DL_2$, in particular $\phi_2(1/\sqrt{3},\xi)=-2$.

2.3. **Codimension 2 Bogdanov-Takens bifurcation with symmetry.** From [16] a codimension 2 Bogdanov-Takens bifurcation with symmetry happens in system (2) in a neighborhood of the curve $a=1$ and $b=0$. So, in a neigborhood of the intersection point of $P$, $H_1$, $H_2$, $HL$ and $DL_1$ of the bifurcation diagram of Figure 1, i.e. $\varphi(1,\xi)=\phi_1(1,\xi)=0$.

2.4. **The dynamics near infinity.** In this subsection we will discuss the qualitative properties of the equilibria at infinity, which describe the behavior of the orbits of system (2) when $x$ and $y$ are sufficiently large.

**Proposition 4.** *As shown in Figure 3 the differential system* (2) *with* $\xi>0$ *has four equilibria at infinity* $I_A^{\pm}, I_B^{\pm}$, *where* $I_A^{\pm}$ *are the two endpoints of the* $x$*-axis, and* $I_B^{\pm}$ *are the two endpoints of the* $y$*-axis. The equilibria* $I_A^{\pm}$ *are unstable star*

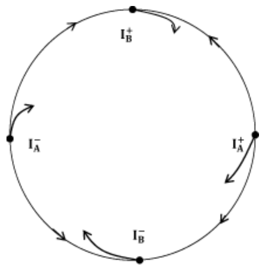



FIGURE 3. Equilibria at infinity.

*nodes, and the equilibria* $I_B^{\pm}$ *are degenerate equilibria, formed by the union of two hyperbolic sectors.*

*Proof.* Doing the Poincaré transformation $x = 1/z$, $y = u/z$, system (2) becomes

$$\frac{du}{d\tau} = \xi u + b\xi u z^2 - u^2 z^2 - z^2 + \frac{|z|^3}{\sqrt{1 + a^2 z^2}},$$
$$\frac{dz}{d\tau} = \xi z + b\xi z^3 - u z^3,$$

where $dt = z^2 d\tau$. Obviously, this system has a unique equilibrium $A : (0, 0)$ on the $u$-axis (the infinity), where $A$ is an unstable star node.

Doing the other Poincaré transformation $x = v/z, y = 1/z$, system (2) writes as

$$\begin{aligned} \frac{dv}{dt} &= z^2 - b\xi v z^2 - \xi v^3 + v^2 z^2 - \frac{v^2 |z|^3}{\sqrt{v^2 + a^2 z^2}} := \Psi_1(v, z), \\ \frac{dz}{dt} &= v z^3 - \frac{v z^3 |z|}{\sqrt{v^2 + a^2 z^2}} := \Psi_2(v, z), \end{aligned} \tag{17}$$

where $dt = z^2 d\tau$. In this local chart we only need to study the equilibrium $B : (0, 0)$ of system (17), which corresponds to two equilibria $I_{B^+}$ and $I_{B^-}$ at infinity of the system (2) at the endpoints of the positive and negative $y$-semiaxes, respectively. By Lemmas 1 and 3 of [14, Chapter 2] we only need to discuss the orbits along characteristic directions of system (17) at $B$.

Applying the polar coordinate changes $x = r\cos\theta$ and $y = r\sin\theta$, system (17) can be written

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{H(\theta) + o(1)}{G(\theta) + o(1)}, \quad \text{as } r \to 0,$$

where

$$G(\theta) = -\sin^3\theta\sqrt{\cos^2\theta + a^2\sin^2\theta}, \quad H(\theta) = \sin^2\theta\cos\theta\sqrt{\cos^2\theta + a^2\sin^2\theta}.$$

Hence a necessary condition for $\theta = \theta_0$ to be an characteristic direction is $G(\theta_0) = 0$, which has exactly two roots 0 and $\pi$. Except these two directions, there are no directions along which system (17) has orbits connecting $B$.

Notice that the vector field (17) is symmetric with respect to the $v$–axis. Thus, we only need to discuss the orbits connecting the origin $B$ of (17) in the half plane $z \geq 0$. We will construct some related open quasi–sectors to determine how many orbits of (17) connecting $B$ in the first and the second quadrants.

Observing that system (17) has four horizontal isoclines: the $v$-axis, the $z$-axis and

$$\mathcal{H}^{\pm} := \left\{(u,z) \in \mathbb{R}^2 : v = \pm\sqrt{1-a^2}z,\ 0 < r < \ell,\ a < 1\right\},$$

where $\ell > 0$ is a sufficiently small constant. Set

$$V^{\pm} := \left\{(v,z) \in \mathbb{R}^2 : z = 0,\ \pm v > 0,\ 0 < r < \ell\right\}.$$

The possible vertical isocline is

$$\mathcal{V} := \left\{(v,z) \in \mathbb{R}^2 : v = \xi^{-\frac{1}{3}} z^{\frac{2}{3}} + o(z^{\frac{2}{3}}),\ 0 < r < \ell\right\}.$$

Obviously, the isocline $\mathcal{V}$ is tangent to the $v$–axis at the origin. Set

$$\mathcal{L}^{\pm} := \left\{(v,z) \in \mathbb{R}^2 : z = \pm\, \sigma v,\ 0 < r < \ell,\ a = 1\right\},$$

where $\sigma > 0$ is a small constant. Hence, if there exist orbits of system (17) connecting $B$ along the direction of the $v$–axis in the first and the second quadrants, then near the origin must lie in the sector regions $\Delta\widehat{V^+B\mathcal{H}^+}$ or $\Delta\widehat{V^-B\mathcal{H}^-}$ if $a < 1$, and $\Delta\widehat{V^+B\mathcal{L}^+}$ or $\Delta\widehat{V^-B\mathcal{L}^-}$ if $a = 1$. The directions of vector field of (17), i.e., the directions of arrows, and the positions of the isoclines are shown in Figure 4.

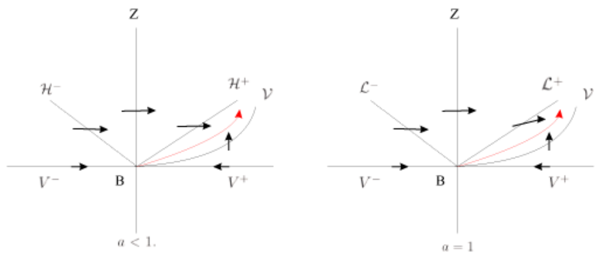



Figure 4. The vector field of system (17).

Firstly, we consider the case $a < 1$. We can check that $\dot{v} > 0$ and $\dot{z} > 0$ in $\Delta\widehat{\mathcal{V}B\mathcal{H}^+}$; $\dot{v} < 0$ and $\dot{z} > 0$ in $\Delta\widehat{V^+B\mathcal{V}}$; and $\dot{v} > 0$ and $\dot{z} < 0$ in $\Delta\widehat{V^-B\mathcal{H}^-}$.

Lemma 4 in [15] guarantees that no orbits connect $B$ in $\Delta\widehat{V^+B\mathcal{V}}$. There are also no orbits connecting $B$ in the interior of $\Delta\widehat{V^-B\mathcal{H}^-}$, because $\Psi_2(v,z)/\Psi_1(v,z)$ is not equal to the slopes of the curves tangent to the $v$–axis. On the other hand, we compute that $(\partial/\partial v)(\Psi_1(v,z)/\Psi_2(v,z)) < 0$ in the generalized normal sector $\Delta\widehat{\mathcal{V}B\mathcal{H}^+}$ of class II, i.e. $\dot{r} > 0$ in $\Delta\widehat{\mathcal{V}B\mathcal{H}^+}$ and all positive semi-orbits starting from the curves $B\mathcal{V}$ and $B\mathcal{H}^+$ go into $\Delta\widehat{\mathcal{V}B\mathcal{H}^+}$. The definition of generalized normal sectors can be seen in [15, Section 2]. Therefore, there exists a unique orbit leaving from $B$ in $\Delta\widehat{\mathcal{V}B\mathcal{H}^+}$ by Lemma 2 and Lemma 5 in [15].

Similarly, in case $a = 1$, we can also prove that exactly one orbit connects $B$ along the $v$–axis, which lies in $\Delta\widehat{\mathcal{V}B\mathcal{L}^+}$. □

## 3. Limit cycles

**Lemma 5.** *Assume that $a \geq 1$. System* (2) *has no limit cycles if $b \geq 0$, and a unique limit cycle if $b < 0$.*

*Proof.* When $b \geq 0$, since the divergence of system (2) is $f(x) = F'(x) = \xi(b + 3x^2) \geq 0$, by the Bendixson criterion (see for instance [8, Theorem 7.10]), the system (2) has no limit cycles.

When $b < 0$, the following conditions are satisfied.

(i) $g(x)$ is an odd function and $xg(x) > 0$ if $x \neq 0$;
(ii) $F(x)$ is an odd function, $F(x) < 0$ if $0 < x < \sqrt{-b}$, and $F(x) \geq 0$ if $x \geq \sqrt{-b}$;
(iii) $\int_0^\infty f(x)dx = \int_0^\infty g(x)dx = +\infty$;
(iv) $f(x)$ and $g(x)$ satisfy the Lipschitz condition in any bounded interval.

Then, by Theorem 4.1 of [18, Chapter 4], the system (2) has a unique limit cycle, which is stable. □

Note that the phase portraits (b) and (c) of Figure 2 in Theorem 1 are obtained from Lemma 5 and from the properties of equilibria.

Since the phase portrait of system (2) is symmetric with respect to the point $E_0$, the small limit cycles surrounding $E_L$ are of the same type as that surrounding $E_R$. Hence, in what follows we only consider the small limit cycles around $E_R$.

**Lemma 6.** *If $0 < a < 1$ and $b \geq a^2 - 1$, then system* (2) *has no limit cycles.*

Since the proof is similar to Lemma 4 of [5], we omit it.

Lemma 6 shows that $\phi_1(a, \xi) < a^2 - 1$ and the phase portrait (a) of Figure 2 in Theorem 1 is obtained.

Consider equation

$$\frac{dz}{dy} = y - \hat{F}(z), \ 0 \leq z \leq z_0, \tag{18}$$

where both $\hat{F}(z)$ and $\hat{F}'(z)$ are continuous in $[0, z_0)$, and $\hat{F}(0) = 0$. Let $L_J$ denote the integral curve of (18) passing through the point $P(z_J, \hat{F}(z_J))$ on the curve $y = \hat{F}(z)$. Also, let $y = \varphi_J(z)$ and $y = \tilde{\varphi}_J(z)$ represent the orbit segments of $L_J$ below and above the curve $y = \hat{F}(z)$. When $0 < z < z_J$, we clearly have $\varphi_J(z) < \hat{F}(z) < \tilde{\varphi}_J(z)$ and $\varphi'_J(z) > 0 > \tilde{\varphi}'_J(z)$. Moreover, we introduce the symbol

$$V(\hat{F}(z), \varphi_J(z), \tilde{\varphi}_J(z)) = \frac{\hat{F}'(z)}{\hat{F}(z) - \varphi_J(z)} + \frac{\hat{F}'(z)}{\tilde{\varphi}_J(z) - \hat{F}(z)}. \tag{19}$$

Then, we have

$$\int_{L_J} \hat{F}'(z)dy = \int_{a_0}^{z_J} V(\hat{F}(z), \varphi_J(z), \tilde{\varphi}_J(z))dz \tag{20}$$

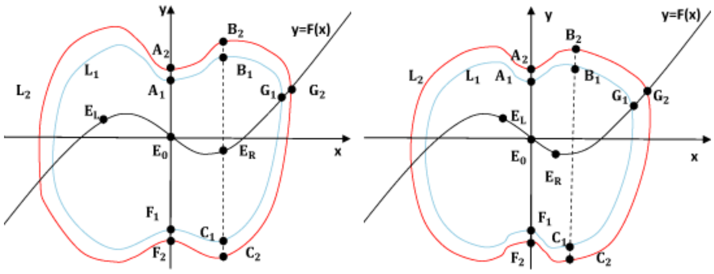
for some $a_0$.

**Lemma 7.** *[Lemma 4.5 of* [18, Chapter 4]*] For equation* (18)*, suppose there is $a_0 \geq 0$ with $\hat{F}(a_0) = 0$, and $\hat{F}(z) > 0$, $\hat{F}(z)\hat{F}'(z)$ is nondecreasing for $z > a_0$. Then*

$$\int_{a_0}^{z_Q} V(\hat{F}(z), \varphi_Q(z), \tilde{\varphi}_Q(z))dz \leq \int_{a_0}^{z_J} V(\hat{F}(z), \varphi_J(z), \tilde{\varphi}_J(z))dz, \text{ for } a_0 < z_Q < z_J.$$

Lemma 7 will be applied in the following lemma.

**Lemma 8.** *If* $0 < a < 1$ *and* $b < a^2 - 1$*, then system* (2) *has at most two large limit cycles.*




(a) $3a^2 - 3 \le b < a^2 - 1$ (b) $b < 3a^2 - 3$

Figure 5. Two large limit cycles.

*Proof.* Assume that system (2) has at least two large limit cycles surrounding the three equilibria $E_L$, $E_0$ and $E_R$, and that $L_1$ and $L_2$ are the most external limit cycles, where $L_2$ denotes the outer one.

We first consider $3a^2 - 3 \le b < a^2 - 1$. The corresponding phase portrait is shown in Figure 5(a). By the Bendixson criterium, each $L_i$ has two intersection points, denoted by $B_i$ and $C_i$ $(i = 1, 2)$ with the straight line $x = x_0$, where $x_0$ is the abscissa of the equilibrium $E_R$, as shown in Figure 5(a). By the symmetry of the phase portrait

$$2\int_{\widehat{A_iF_i}} f(x)dt = \oint_{L_i} f(x)dt = -\oint_{L_i} \mathrm{div}(y - F(x), -g(x))dt,$$

for $i = 1, 2$. On the arcs $\widehat{A_1B_1}$ and $\widehat{A_2B_2}$, let $y = y_1(x)$ and $y = y_2(x)$, respectively. In fact, for each $i = 1, 2$, we have

$$\begin{aligned}
\int_{\widehat{A_iB_i}} f(x)dt &= -\int_0^{x_0} \frac{F'(x)}{F(x) - y_i(x)}dx = -\int_0^{x_0} \frac{F'(x) - y_i'(x) + y_i'(x)}{F(x) - y_i(x)}dx \\
&= -\int_0^{x_0} \frac{1}{F(x) - y_i(x)}d(F(x) - y_i(x)) - \int_0^{x_0} \frac{y_i'(x)}{F(x) - y_i(x)}dx \\
&= -\ln\left|\frac{F(0) - y_i(x_0)}{F(0) - y_i(0)}\right| - \ln\left|\frac{F(x_0) - y_i(x_0)}{F(0) - y_i(x_0)}\right| - \int_0^{x_0} \frac{g(x)}{(F(x) - y_i(x))^2}dx \\
&= -\ln\left|\frac{F(x_0) - y_i(x_0)}{F(0) - y_i(x_0)}\right| + \int_0^{x_0} \frac{y_i'(x)}{F(0) - y_i(x)}dx - \int_0^{x_0} \frac{g(x)}{(F(x) - y_i(x))^2}dx \\
&= -\ln\left|\frac{y_i(x_0) - F(x_0)}{y_i(x_0)}\right| - \int_0^{x_0} \frac{F(x)g(x)}{y_i(x)(F(x) - y_i(x))^2}dx,
\end{aligned}$$

implying

$$\begin{aligned}
\int_{\widehat{A_2B_2}} F'(x)dt - \int_{\widehat{A_1B_1}} F'(x)dt &= \ln\left|\frac{y_1(x_0) - F(x_0)}{y_1(x_0)}\right| - \ln\left|\frac{y_2(x_0) - F(x_0)}{y_2(x_0)}\right| + \int_0^{x_0} \tilde{F}(x)dx \\
&= \ln\left|1 + \frac{(y_1(x_0) - y_2(x_0))F(x_0)}{y_1(x_0)(y_2(x_0) - F(x_0))}\right| + \int_0^{x_0} \tilde{F}(x)dx,
\end{aligned}$$

where

$$\tilde{F}(x) = \frac{g(x)F(x)}{y_1(x)(F(x)-y_1(x))^2} - \frac{g(x)F(x)}{y_2(x)(F(x)-y_2(x))^2}$$

and $y_i(x)$ is the function corresponding to the arc $\widehat{A_iB_i}$. Clearly, since $F(x) < 0 < y_1(x) < y_2(x)$ in $(0, x_0)$ we obtain

$$\frac{(y_1(x_0)-y_2(x_0))F(x_0)}{y_1(x_0)(y_2(x_0)-F(x_0))} > 0, \quad \int_0^{x_0} \tilde{F}(x)dx > 0.$$

Thus

$$\int_{\widehat{A_2B_2}} f(x)dt > \int_{\widehat{A_1B_1}} f(x)dt. \tag{21}$$

Similarly we obtain that

$$\int_{\widehat{C_2F_2}} f(x)dt - \int_{\widehat{C_1F_1}} f(x)dt > 0.$$

Setting $z = \int_0^x g(s)ds$, from (20) we get

$$\int_{\widehat{B_1G_1C_1}} F'(x)dt = \int_{\widehat{B_1G_1C_1}} \hat{F}'(z)dy = \int_{x_0}^{z_{G_1}} V(\hat{F}(z), \varphi_{G_1}(z), \tilde{\varphi}_{G_1}(z))dz.$$

By Lemma 7, in order to prove the inequality

$$\int_{z(x_0)}^{z_{G_2}} V(\hat{F}(z), \varphi_{G_2}(z), \tilde{\varphi}_{G_2}(z))dz > \int_{z(x_0)}^{z_{G_1}} V(\hat{F}(z), \varphi_{G_1}(z), \tilde{\varphi}_{G_1}(z))dz,$$

where $\hat{F}(z) = F(x) - F(x_0)$, we only need to prove that $\hat{F}(z(x_0)) = 0$, $\hat{F}(z) > 0$ and $\hat{F}(z)\hat{F}'(z)$ is nondecreasing for $z > z(x_0)$. Clearly we have $\hat{F}(z(x_0)) = 0$ and $\hat{F}(z) > 0$ for $z > z(x_0)$. Note that

$$\hat{F}(z)\hat{F}'(z) = [F(x) - F(\sqrt{1-a^2})]f(x)/g(x).$$

For $x > \sqrt{1-a^2}$ we have

$$\begin{aligned} &\frac{[F(x) - F(\sqrt{1-a^2})]f(x)}{g(x)} \\ &= \frac{\xi^2[bx + x^3 - b\sqrt{1+a^2} - (1-a^2)\sqrt{1-a^2}](b+3x^2)}{x(1-1/\sqrt{x^2+a^2})} \\ &= \frac{\xi^2[b + x^2 + (1-a^2) + \sqrt{1-a^2}x](b+3x^2)(\sqrt{x^2+a^2}+x^2+a^2)}{x(x+\sqrt{1-a^2})} \\ &= \xi^2\left(\frac{b+1-a^2}{x+\sqrt{1-a^2}} + x\right)\left(\frac{b}{x} + 3x\right)(\sqrt{x^2+a^2}+x^2+a^2), \end{aligned}$$

where the three factors of the last line are positive and increasing. Therefore, $[F(x) - F(\sqrt{1-a^2})]f(x)/g(x)$ is positive and increasing. By Lemma 7, we have that

$$\int_{\widehat{B_2G_2C_2}} f(x)dt - \int_{\widehat{B_1G_1C_1}} f(x)dt > 0.$$

Therefore

$$\oint_{L_1} \operatorname{div}(y - F(x), -g(x))dt > \oint_{L_2} \operatorname{div}(y - F(x), -g(x))dt. \tag{22}$$

Now we consider $b < 3a^2 - 3$. The corresponding phase portrait is shown in Figure 5(b). By the Bendixson criterium again, each $L_i$ has two intersection points with the straight line $x = x_0$, being $x_0$ the unique positive zero of $F'(x)$ when $x > 0$, and the intersection points are denoted by $B_i$ and $C_i$ $(i = 1, 2)$. Then the inequality (22) can be proved in a similar way to the case $3a^2 - 3 \le b < a^2 - 1$. Since $f(x) < 0$ for $0 < x < x_0$ and $y_i(x) - F(x) > 0$ $(i = 1, 2)$, we have

$$\begin{aligned} \int_{\widehat{A_2B_2}} f(x)dt - \int_{\widehat{A_1B_1}} f(x)dt &= \int_0^{x_0} \left\{ \frac{f(x)}{y_2(x) - F(x)} - \frac{f(x)}{y_1(x) - F(x)} \right\} dx \\ &= \int_0^{x_0} \frac{f(x)(y_1(x) - y_2(x))}{(y_1(x) - F(x))(y_2(x) - F(x))} dx > 0. \end{aligned} \tag{23}$$

Similarly we obtain that

$$\int_{\widehat{C_2F_2}} f(x)dt - \int_{\widehat{C_1F_1}} f(x)dt > 0.$$

Now, using again Lemma 7, we only need to prove that $\hat{F}(z(x_0)) = 0$, and $\hat{F}(z) > 0$, $\hat{F}(z)\hat{F}'(z)$ is nondecreasing for $z > z(x_0)$, where $\hat{F}(z) = F(x) - F(x_0)$. Clearly, we have $\hat{F}(z(x_0)) = 0$ and $\hat{F}(z) > 0$ for $z > z(x_0)$. Note that $\hat{F}(z)\hat{F}'(z) = [F(x) - F(\sqrt{-b/3})]f(x)/g(x)$. For $x > \sqrt{-b/3}$ we have

$$\begin{aligned} &\frac{[F(x) - F(\sqrt{-b/3})]f(x)}{g(x)} \\ &= \frac{\xi^2[bx + x^3 - b\sqrt{-b/3} - (-b/3)\sqrt{-b/3}](b + 3x^2)}{x(1 - 1/\sqrt{x^2 + a^2})} \\ &= \frac{\xi^2[2b/3 + x^2 + \sqrt{-b/3}x](b + 3x^2)(\sqrt{x^2 + a^2} + x^2 + a^2)}{x(x + \sqrt{1 - a^2})} \frac{x - \sqrt{-b/3}}{x - \sqrt{1 - a^2}} \\ &= \xi^2 \left( \frac{2b/3 - \sqrt{1 - a^2}(\sqrt{-b/3} - \sqrt{1 - a^2})}{x + \sqrt{1 - a^2}} + x + \sqrt{-b/3} - \sqrt{1 - a^2} \right) \\ &\left( \frac{b}{x} + 3x \right) (\sqrt{x^2 + a^2} + x^2 + a^2) \left( 1 + \frac{\sqrt{1 - a^2} - \sqrt{-b/3}}{x - \sqrt{1 - a^2}} \right), \end{aligned}$$

where all of the factors of the last two lines are positive and increasing. Therefore $[F(x) - F(\sqrt{-b/3})]f(x)/g(x)$ is positive and increasing. By Lemma 7, we have

$$\int_{\widehat{B_2G_2C_2}} f(x)dt - \int_{\widehat{B_1G_1C_1}} f(x)dt > 0$$

and therefore (22) follows.

However, it is impossible to have two attracting (repelling) limit cycles surrounding the same equilibrium (equilibria) adjacent one to the other. So, from the inequality (22) and the repelling of the infinity, we obtain that system (2) has at most three large limit cycles, where the outer one is stable, the middle one is semistable, the inner one is stable. Clearly, for fixed values $a$ and $\xi$, system (2) is a family of generalized rotated vector fields with respect to the parameter $b$. Assume that system (2) has exactly three large limit cycles. By Theorem 3.5 of [18, Chapter 4], the outer limit cycle and the inner one neither split, nor disappear as $b$ varies monotonically. By Theorem 3.4 of [18, Chapter 4], the middle limit cycle will bifurcate into at least one stable and one unstable cycle when $b$ varies in the suitable direction. This is a contradiction. Therefore, system (2) has at most two large limit cycles. If the two large limit cycles exist, we can obtain that the outer limit cycle is stable and the inner one is unstable. □

We consider a generic Liénard system

$$\dot{x} = y - \bar{F}(x), \qquad \dot{y} = -\bar{g}(x), \tag{24}$$

where $\bar{F}$ is $\mathcal{C}^2$, $\bar{g}$ is $\mathcal{C}^1$ and $x \in (\alpha, \beta)(\alpha, \beta$ can be $\pm\infty)$. The following proposition partly improves the result of [9, Theorem 2.1].

**Proposition 9.** *Consider system* (24)*, which satisfies* **(i)**-**(iv)***, where*

**(i):** $\bar{f}(x) = \bar{F}'(x)$ *has a unique zero* 0 *and* $\bar{f}(x) < 0$ *(resp.* $> 0$*) as* $\alpha < x < 0$ *(resp.* $0 < x < \beta$*);*

**(ii):** $\bar{F}(0) = 0$*;*

**(iii):** $x\bar{g}(x) > 0$ *for* $x \neq 0$, $x \in (\alpha, \beta)$*;*

**(iv):** *the system*

$$\bar{F}(x_1) = \bar{F}(x_2),\ \bar{\lambda}(x_1) = \bar{\lambda}(x_2) \tag{25}$$

*has at most one solution* $x_2 < 0 < x_1$*, where* $\bar{\lambda} := \bar{g}(x)/\bar{f}(x)$*.*

*Moreover, when* $\bar{F}(x_1) = \bar{F}(x_2)$, $\bar{\lambda}(x_1) > \bar{\lambda}(x_2)$ *(resp.* $\bar{F}(x_1) = \bar{F}(x_2)$, $\bar{\lambda}(x_1) < \bar{\lambda}(x_2)$*) as* $|x_1|, |x_2|$ *are small, system (24) satisfies either* **(v)** *or* **(v′)***, where*

**(v):** *the function* $\bar{F}(x)\bar{f}(x)/\bar{g}(x)$ *is decreasing (resp. increasing) for* $\alpha < x < 0$*;*

**(v′):** *the function* $\bar{F}(x)\bar{f}(x)/\bar{g}(x)$ *is increasing (resp. decreasing) for* $0 < x < \beta$ *and*

$$\lim_{x\to\alpha^+} \bar{F}(x) = \lim_{x\to\beta^-} \bar{F}(x).$$

*Then system* (24) *has at most one closed orbit in the region* $\{(x, y) \in \mathbb{R}^2 : \alpha < x < \beta\}$*. The closed orbit is simple and unstable (resp. stable) if it exists.*

*Proof.* Assume that system (24) exists a limit cycle $\gamma$, as shown in Figure 6(a). In the following we will ascertain the sign of

$$-\oint_\gamma \bar{f}(x)dt = \oint_\gamma \mathrm{dvi}(y - \bar{F}(x), -\bar{g}(x))dt.$$

Clearly $w = \bar{F}(x)$ has two inverse functions, $x_1(w)$ (respectively $x_2(w)$), on the right (resp. left) side of the origin. The functions $\bar{\lambda}(x_i(w))$ will be denoted simply by $\lambda_i(w)$.

By $w = \bar{F}(x)$, we rewrite system (24) into

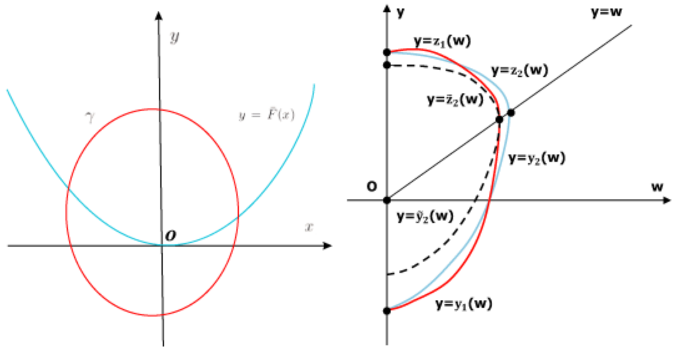
$$\dot{w} = \bar{f}(x_i(w))(y - w),\ \dot{y} = -\bar{g}(x_i(w)), \tag{26}$$

which deduces

$$\frac{dy}{dw} = \frac{\lambda_i(w)}{w - y}. \tag{27}$$

Let $y_1(w)$ and $y_2(w)$(resp. $z_1(w)$ and $z_2(w)$) be functions determined by the orbits of (26) below (resp. above) the line $y = w$, which correspond the parts of the trajectories of system (24) below (resp. above) the curve $y = \bar{F}(x)$ and depend whether they are to the left or right of the origin.

From condition **(iv)**, $\lambda_1(w) = \lambda_2(w)$ has at most one root. When the equation $\lambda_1(w) = \lambda_2(w)$ has no roots, by the comparison theorem we obtain either $z_1(w) >$




(a) Assume that system (24) exhibits a limit cycle

(b) Orbits of system (26)

FIGURE 6. Limit cycles of system (24).

$z_2(w)$ and $y_1(w) < y_2(w)$ or $z_1(w) < z_2(w)$ and $y_1(w) > y_2(w)$. By Green formula, we have

$$\oint_\gamma \bar{g}(x)dx + (y - \bar{F}(x))dy = -\iint_{\text{Int}\gamma} \bar{f}(x)dxdy = \iint_{\text{Int}\gamma, x<0} dwdy - \iint_{\text{Int}\gamma, x>0} dwdy \neq 0,$$

which contradicts $\oint_\gamma \bar{g}(x)dx + (y - \bar{F}(x))dy = 0$. When the equation $\lambda_1(w) = \lambda_2(w)$ has a unique root and the equation $z_1(w) = z_2(w)$ has no roots, we can similarly prove that (24) has no limit cycles. When the equation $\lambda_1(w) = \lambda_2(w)$ has a unique root, applying again the comparison theorem the equation $z_1(w) = z_2(w)$ has at most one root.

Now we only need to discuss the system $\lambda_1(w) = \lambda_2(w)$ and $z_1(w) = z_2(w)$ having a unique root. Here we consider that condition **(v)** holds. By $(w, y) \to (\mu w, \mu y)$ with $\mu := w_2^*/w_1^* > 1$, where $w_i^*$ is the intersection of $z_i(w)$ and the line $y = w$, system (27) deduces

$$\frac{dy}{dw} = \frac{\lambda_2(\mu w)}{\mu(w - y)}. \tag{28}$$

First, we consider that $\lambda_1(w) > \lambda_2(w)$ as $w \to 0$. Moreover, the limit cycle $\gamma$ of (24) with $x > 0$ corresponds to the broken curve in Figure 6(b) and $y_2(w)$, $z_2(w)$ are changed into two new functions, denoted $\tilde{y}_2(w)$ and $\tilde{z}_2(w)$, respectively. It is easy to check that $y_2(w) = \mu\tilde{y}_2(w/\mu)$ and $z_2(w) = \mu\tilde{z}_2(w/\mu)$. By $\mu > 1$ and the increasing of $\bar{\lambda}(w)/w$, we get $\bar{\lambda}(\mu w)/\mu > \bar{\lambda}(w)$. Furthermore, using again the comparison theorem to (27) and (28), we obtain $y_1(w) < \tilde{y}_2(w)$ and $z_1(w) > \tilde{z}_2(w)$.

Thus

$$\begin{aligned}\oint_\gamma \bar f(x)dt &= \oint_\gamma \frac{\bar f(x)}{y-\bar F(x)}dx\\ = \int_0^{w_2^*}\frac{dw}{y_2(w)-w} &- \int_0^{w_2^*}\frac{dw}{z_2(w)-w} + \int_0^{w_1^*}\frac{dw}{z_1(w)-w} - \int_0^{w_1^*}\frac{dw}{y_1(w)-w}\\ = \int_0^{w_1^*}\frac{\mu dw}{y_2(\mu w)-\mu w} &- \int_0^{w_1^*}\frac{\mu dw}{z_2(\mu w)-\mu w} + \int_0^{w_1^*}\frac{dw}{z_1(w)-w} - \int_0^{w_1^*}\frac{dw}{y_1(w)-w}\\ = \int_0^{w_1^*}\frac{(y_1-\tilde y_2)dw}{(y_1(w)-w)(\tilde y_2(w)-w)} &- \int_0^{w_1^*}\frac{(z_1-\tilde z_2)dw}{(z_1(w)-w)(\tilde z_2(w)-w)} < 0.\end{aligned}$$

So in the region $\{(x,y)\in\mathbb{R}^2 : \alpha < x < \beta\}$ $\gamma$ is unstable and simple if it exists. Moreover, it is impossible to have two attracting (repelling) limit cycles surrounding the same equilibrium adjacent one to the other. Therefore, the uniqueness has also been proved.

For the case $\lambda_1(w) < \lambda_2(w)$ as $w$ is small, we can prove that $\oint_\gamma f(x)dt > 0$ in a similar way to the case $\lambda_1(w) > \lambda_2(w)$. So, in the region $\{(x,y)\in\mathbb{R}^2 : \alpha < x < \beta\}$, the limit cycle $\gamma$ is stable and simple if it exists. Therefore, we have completed this proof. ☐

The proof of Proposition 9 gives the following corollary directly.

**Corollary 10.** *Assume that system* (24) *satisfies conditions* **(i)-(iii)** *of Proposition 9. If there is no solutions to* (25)*, then system* (24) *has no closed orbits.*

Under the preparations of above Proposition 9 and Corollary 10, we obtain the existence of small limit cycles on the parameter surfaces $H_2$ and $H_3$ as follows.

**Lemma 11.** *Assume that* $0 < a < 1$ *and* $b = 3a^2-3$*. Then system* (2) *has no small limit cycles if* $1/\sqrt{3} \le a < 1$*, and at most two small limit cycles if* $0 < a < 1/\sqrt{3}$*.*

*Proof.* By the transformation $(x,y)\to(x+\sqrt{1-a^2}, y+F(\sqrt{1-a^2}))$, system (2) can be rewritten as

$$\begin{aligned}\dot x &= y - F(x+\sqrt{1-a^2}) + F(\sqrt{1-a^2}),\\ \dot y &= -g(x+\sqrt{1-a^2}).\end{aligned}\tag{29}$$

It is easy to show that system (29) satisfies conditions **(i)-(iii)** of Proposition 9 when $x\in(-\sqrt{1-a^2},+\infty)$. Condition **(iv)** of Proposition 9 is equivalent to the fact that system

$$\begin{aligned}&F(\tilde x_1+\sqrt{1-a^2}) = F(\tilde x_2+\sqrt{1-a^2}),\\ &\frac{g(\tilde x_1+\sqrt{1-a^2})}{f(\tilde x_1+\sqrt{1-a^2})} = \frac{g(\tilde x_2+\sqrt{1-a^2})}{f(\tilde x_2+\sqrt{1-a^2})},\end{aligned}\tag{30}$$

has at most one solution, where $-\sqrt{1-a^2} < \tilde x_1 < 0 < \tilde x_2$. Clearly (30) is equivalent to the fact that the system

$$F(x_1) = F(x_2), \qquad \frac{g(x_1)}{f(x_1)} = \frac{g(x_2)}{f(x_2)}\tag{31}$$

has a unique solution when $0 < x_1 < \sqrt{1-a^2} < x_2$ and $x_j = \tilde x_j + \sqrt{1-a^2}$, $j=1,2$. Let $s := x_1+x_2$. From $F(x_1)=F(x_2)$, we have $x_1x_2 = 3(a^2-1)+s^2$.

Since $0 < x_1 < \sqrt{1-a^2} < x_2$, we obtain $\sqrt{3-3a^2} < s < 2\sqrt{1-a^2}$. Note that

$$\frac{g(x)}{f(x)} = \frac{x}{3\xi(\sqrt{x^2+a^2}+x^2+a^2)}. \tag{32}$$

From the second equality of (31) and (32), we have

$$s^2+2a^2-3-\frac{a^2 s}{x_2\sqrt{x_1^2+a^2}+x_1\sqrt{x_2^2+a^2}}=0.$$

And consequently

$$\begin{aligned}\left(x_2\sqrt{x_1^2+a^2}+x_1\sqrt{x_2^2+a^2}\right)^2 &= 2x_1^2x_2^2+a^2(x_1^2+x_2^2)+2x_1x_2\sqrt{x_1^2x_2^2+a^2(x_1^2+x_2^2)+a^4}\\ &= 2s^4+(11a^2-12)s^2+6(1-a^2)(3-2a^2)\\ &\quad +2(s^2+3a^2-3)\sqrt{s^4+(5a^2-6)s^2+(2a^2-3)^2}.\end{aligned}$$

Let $\eta := s^2$. Then, for $3(1-a^2) < \eta < 4(1-a^2)$, we have

$$\begin{aligned}&\frac{d\left[\left(x_2\sqrt{x_1^2+a^2}+x_1\sqrt{x_2^2+a^2}\right)^2/s^2\right]}{d\eta}\\ =\ & 2-\frac{6(1-a^2)(3-2a^2)}{\eta^2}+\frac{6(1-a^2)}{\eta^2}\sqrt{\eta^2+(5a^2-6)\eta+(2a^2-3)^2}+\frac{(\eta+3a^2-3)(2\eta+5a^2-6)}{\eta\sqrt{\eta^2+(5a^2-6)\eta+(2a^2-3)^2}}\\ =\ & 2+\frac{6(1-a^2)(\eta+5a^2-6)}{\eta(\sqrt{\eta^2+(5a^2-6)\eta+(2a^2-3)^2}+3-2a^2)}+\frac{(\eta+3a^2-3)(2\eta+5a^2-6)}{\eta\sqrt{\eta^2+(5a^2-6)\eta+(2a^2-3)^2}}\\ >\ & 2-\frac{6(1-a^2)}{\eta}-\frac{\eta+3a^2-3}{\eta}>0.\end{aligned}$$

Thus, if $h(\eta) := \eta+2a^2-3-a^2\sqrt{\eta}/[x_2\sqrt{x_1^2+a^2}+x_1\sqrt{x_2^2+a^2}]$, then the function $h(\eta)$ is increasing in $3(1-a^2) < \eta < 4(1-a^2)$. On the other hand, we have $h(3-3a^2) = -a-a^2 < 0$ and $h(4-4a^2) = 1-3a^2$. Therefore, when $\sqrt{3}/3 \le a < 1$, the function $h(\eta)$ has no solutions for $3(1-a^2) < \eta < 4(1-a^2)$. By Corollary 10, system (2) has no limit cycles in the region $x > 0$. When $0 < a < \sqrt{3}/3$, the function $h(\eta)$ has a unique root in $3(1-a^2) < \eta < 4(1-a^2)$. Now we only need to verify the condition **(v′)** of Proposition 9. We have

$$\begin{aligned}\frac{(F(x)-F(\sqrt{1-a^2})f(x)}{g(x)} &= 3\xi^2\frac{\sqrt{x^2+a^2}+x^2+a^2}{x}\left((3a^2-3)x+2(1-a^2)^{3/2}+x^3\right)\\ &= 3\xi^2\left(3a^2-3+\frac{2(1-a^2)^{3/2}}{x}+x^2\right)(\sqrt{x^2+a^2}+x^2+a^2).\end{aligned}$$

Then, we obtain

$$\begin{aligned}\frac{d[(F(x)-F(\sqrt{1-a^2})f(x)/(3\xi^2g(x))]}{dx} &= \left(\frac{-2(1-a^2)^{3/2}}{x^2}+2x\right)(\sqrt{x^2+a^2}+x^2+a^2)\\ &\quad+\left(3a^2-3+\frac{2(1-a^2)^{3/2}}{x}+x^2\right)\left(\frac{x}{\sqrt{x^2+a^2}}+2x\right)>0,\end{aligned}$$

for $0 < a < \sqrt{3}/3$ and $x > \sqrt{1-a^2}$, because

$$\frac{-2(1-a^2)^{3/2}}{x^2} + 2x > 0 \text{ and}$$

$$3a^2 - 3 + \frac{2(1-a^2)^{3/2}}{x} + x^2 = \frac{(x - \sqrt{1-a^2})(x^2 + \sqrt{1-a^2}x - 2(1-a^2))}{x} > 0.$$

So condition **(v′)** of Proposition 9 holds. Therefore, when $0 < a < \sqrt{3}/3$, system (2) has at most one limit cycle which lies in the region $x > 0$ by Proposition 9. □

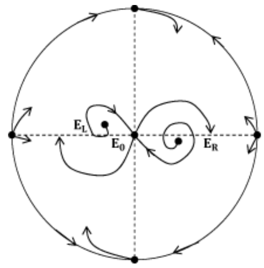



FIGURE 7. The uniqueness of limit cycles.

**Lemma 12.** *When $1/\sqrt{3} \le a < 1$ and $b = 3a^2 - 3$, system (2) has a unique limit cycle surrounding all equilibria.*

*Proof.* If $1/\sqrt{3} \le a < 1$ and $b = 3a^2 - 3$, i.e., on the parameter curve $H_2$ we obtain Figure 7 by Proposition 4 and Lemma 11, which shows the existence of a Poincaré-Bendixson annulus, i.e., any trajectory starting at a point of the boundary curves of the annulus enters (or leaves) the annulus, and inside the annulus there is no equilibrium points. So, the existence of some large limit cycles is obtained. Assume that system (2) has two large limit cycles. Let the outer limit cycle and inner one denoted by $\gamma_2$ and $\gamma_1$. Therefore $\oint_{\gamma_i} \mathrm{div}(y - F(x), -g(x))dt \le 0$ for $i = 1, 2$. By (22), $\oint_{\gamma_1} \mathrm{div}(y - F(x), -g(x))dt > \oint_{\gamma_2} \mathrm{div}(y - F(x), -g(x))dt$. However, it is impossible to have two attracting (repelling) limit cycles surrounding the same equilibrium (equilibria) adjacent one to the other. Therefore

$$\oint_{\gamma_1} \mathrm{div}(y - F(x), -g(x))dt = 0 > \oint_{\gamma_2} \mathrm{div}(y - F(x), -g(x))dt.$$

So $\gamma_1$ is a semistable limit cycle. Since the vector field of (2) is rotating with respect to $b$, by Theorem 3.4 of [18, p.211] there is a stable limit cycle $\hat{\gamma}_2$ near $\gamma_2$ for a perturbation of $b$, and two limit cycles $\hat{\gamma}_1, \tilde{\gamma}_1$ ($\hat{\gamma}_1$ is smaller than $\tilde{\gamma}_1$) near $\gamma_1$. By (22) and stabilities of equilibria, we obtain

$$\oint_{\hat{\gamma}_1} \mathrm{div}(y - F(x), -g(x))dt \le 0, \qquad \oint_{\tilde{\gamma}_1} \mathrm{div}(y - F(x), -g(x))dt \ge 0,$$

and

$$\oint_{\hat{\gamma}_2} \mathrm{div}(y - F(x), -g(x))dt < 0,$$

which contradicts $\oint_{\tilde{\gamma}_1} \operatorname{div}(y - F(x), -g(x))dt > \oint_{\tilde{\gamma}_1} \operatorname{div}(y - F(x), -g(x))dt$. Therefore, system (2) has at most one large limit cycle. Thus, the uniqueness of limit cycle is proved and the phase portrait (d) of Figure 2 in Theorem 1 is obtained. □

Moreover, we can get the phase portraits (i), (e) and (n) of Figure 2 in Theorem 1 by Proposition 3, Lemmas 8, 11, 12 and the continuity of the vector fields.

From [5] it follows that system (2) has no limit cycles for $a \to 0$ and $b = 3a^2 - 3$.

**Proposition 13.** *System* (2) *has at most two small limit cycles around the equilibria* $E_R$ *or* $E_L$ *for the values of the parameters in* $\mathcal{G}_1 := \{(a, b, \xi) \in \mathcal{G} : 0 < a < 1,\ b < 3a^2 - 3,\ 0 < \xi \ll 1\}$.

*Proof.* By the homeomorphism $(x, y) \to (x, y + F(x))$, system (2) can be rewritten as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\left(1 - \frac{1}{\sqrt{x^2 + a^2}}\right) - \xi(b + 3x^2)y. \end{aligned} \tag{33}$$

In $x > 0$ we do the change of variables $w = \sqrt{x^2 + a^2}$ and the time scaling $dt = \sqrt{1 + a^2/x^2}d\tau$ to system (33), and we get

$$\begin{aligned} \dot{w} &= y, \\ \dot{y} &= 1 - w - \frac{\xi w}{\sqrt{w^2 - a^2}}(b - 3a^2 + 3w^2)y. \end{aligned} \tag{34}$$

When $\xi = 0$ this system is the Hamiltonian system

$$\begin{aligned} \dot{w} &= y, \\ \dot{y} &= 1 - w, \end{aligned} \tag{35}$$

with the first integral

$$H(w, y) = \frac{y^2}{2} + \frac{w^2}{2} - w. \tag{36}$$

Its level curves $\Gamma_h := \{(w, y) :\ H(w, y) = h,\ -1/2 \le h < a^2/2 - a\}$ are shown in Figure 8.

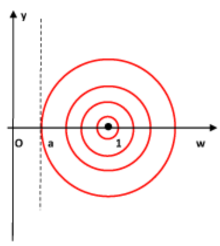



FIGURE 8. The phase portrait of the Hamiltonian system (35).

Of course $H = -1/2$ corresponds to the center $(1, 0)$, and for the values of $h$ such that $-1/2 < h < a^2/2 - a$, the curve $H(w, y) = h$ corresponds to a periodic orbit of Hamiltonian system (35) surrounding the point $(1, 0)$, which intersects the positive half $w$-axis inside the interval $(a, 1)$.

Now we consider system (34) as a perturbation of system (35) for small $\xi$. Here we only will discuss how many small limit cycles surround the equilibrium point

$(1,0)$ of system (34) when $\xi$ is sufficiently small. On the other hand, for every $h \in (-1/2, a^2/2-a)$ the orbit $\Gamma_h$ intersects the the segment $L_1 : (a,1)$ of the $w$-axis at exactly one point $Q_h(x(h),0)$. Therefore, the segment $L_1$ can be parameterized by $h \in (-1/2, a^2/2-a)$.

For every $h \in (-1/2, a^2/2-a)$, we consider the trajectory of system (33) passing through the point $Q_h(x(h),0) \in L_1$. This trajectory goes forward and backward until it intersects the positive $w$-axis at points $Q_1$ and $Q_2$, respectively, as in Figure 9. We denote the piece of trajectory from $Q_2$ to $Q_1$ by $\gamma(h,\xi,a,b)$. Then $\gamma(h,\xi,a,b)$ is a periodic orbit if and only if $Q_1 = Q_2$. From (36), we have

$$\frac{\partial H(w,y)}{\partial w} = w-1 \neq 0, \text{ if } |w| \neq 1.$$

Hence $Q_1 = Q_2$ if and only if $H(Q_1) = H(Q_2)$.

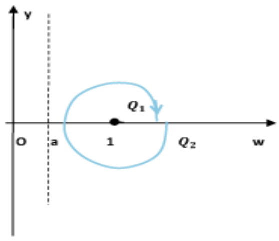



FIGURE 9. The perturbation of (35).

On the other hand, along the orbits of system (35) we have

$$\frac{dH(w,y)}{dt}dt = -\frac{\xi w}{\sqrt{w^2-a^2}}(b-3a^2+3w^2)y^2dt = -\frac{\xi w}{\sqrt{w^2-a^2}}(b-3a^2+3w^2)ydw.$$

This implies that

$$\begin{aligned} H(Q_1)-H(Q_2) &= \int_{t(Q_2)}^{t(Q_1)} \left.\frac{dH(w,y)}{dt}\right|_{(35)} dt \\ &= -\xi \int_{\gamma(h,\xi,a,b)} \frac{w}{\sqrt{w^2-a^2}}(b-3a^2+3w^2)ydw = -\xi F(h,\xi,a,b). \end{aligned}$$

Therefore $\gamma(h,\xi,a,b)$ is a periodic orbit if and only if $F(h,\xi,a,b)=0$.

We consider $F(h,\xi,a,b)$ as a perturbation of $F(h,0,a,b)$. The function $F(h,0,a,b)$ is given by

$$F(h,0,a,b) = (b-3a^2)I_1(h) + 3I_3(h),$$

where

$$I_i(h) = \int_{\Gamma_h} \frac{w^i}{\sqrt{w^2-a^2}}\, y\, dw \quad \text{for} \quad i=1,3.$$

The orientation of $\Gamma_h$ is determined by the direction of the vector field (35). By the Green's formula

$$I_1(h) = \int_{\Gamma_h} \frac{w}{\sqrt{w^2-a^2}}\, y\, dw = \iint_{D(h)} \frac{w}{\sqrt{w^2-a^2}}\, dwdy > 0, \quad h \in (-1/2, a^2/2-a),$$

where $D(h)$ is the region surrounded by $\Gamma_h$.

It is easy to show that

$$\lim_{h\to -1/2} I_1(h) = \lim_{h\to -1/2} I_3(h) = 0.$$

By the Mean Value Theorem of integrals, we have

$$\lim_{h\to -1/2}\frac{I_3(h)}{I_1(h)} = \lim_{h\to -1/2}\frac{\displaystyle\iint_{D(h)}\frac{w^3}{\sqrt{w^2-a^2}}dwdy}{\displaystyle\iint_{D(h)}\frac{w}{\sqrt{w^2-a^2}}dwdy} = \lim_{h\to -1/2}\bar{w}^2(h) = 1,$$

where $(\bar{w}(h),\bar{y}(h)) \in D(h)$ and $D(h)$ shrinks to the point $(1,0)$ as $h \to -1/2$.

Now we define the function

$$u(h) = \begin{cases} \dfrac{I_3(h)}{I_1(h)} & \text{if } h \in (-1/2, a^2/2 - a), \\ 1 & \text{if } h = -1/2. \end{cases}$$

It is continuous in the interval $[-1/2, a^2/2-a]$. Therefore, to determine the existence and the number of small limit cycles for system (34) for $\xi$ sufficiently small, we only need to study the existence and the number of zeros of the function $F(h,\xi,a,b)$ in the interval $h(-1/2, a^2/2-a)$. In addition, since

$$F(h,\xi,a,b) \approx F(h,0,a,b) = I_1(h)\left(b - 3a^2 + 3\frac{I_3(h)}{I_1(h)}\right),$$

for $\xi$ sufficiently small, we obtain that the behavior of the function $u = u(h)$, as a ratio of two Abelian integrals, is crucial in our discussion.

From the notations in [12, Corollary 2], taking

$$\Phi(w) = \frac{1}{2}w^2 - w,\ \Psi(y) = \frac{y^2}{2},\ f_1(w) = \frac{w}{\sqrt{w^2-a^2}},\ f_3(w) = \frac{w^3}{\sqrt{w^2-a^2}}$$

for system (34). Note that $H(w,y) = \Phi(w) + \Psi(y)$ and $I_i(h) = \int_{\Gamma_h} f_i(w)\ y\,dw$, $i = 1,3$. We can calculate

$$M(\tilde{w},w) + M(w,\tilde{w}) = 4(w-1)^4\left\{\frac{-w^2(2-w)^2 + a^2(4-2w+w^2)}{(w^2-a^2)[(2-w)^2-a^2]} - \frac{[w^3(2-w)^3 + 2(-4+2w+3w^2-4w^3+w^4)a^2 + (2+2w-w^2)a^4]}{(w^2-a^2)^{3/2}[(2-w)^2-a^2]^{3/2}}\right\},$$

where $\tilde{w} + w = 2$, and

$$\begin{aligned} M(\tilde{w},w) = \ & [\Phi'(\tilde{w})]^2\{\Phi'(\tilde{w})(f_3'(w)f_1(w) - f_1'(w)f_3(w)) + \Phi''(w)(f_3(w)f_1(\tilde{w}) \\ & - f_1(w)f_3(\tilde{w})) + \Phi'(w)[f_3(\tilde{w})f_1'(w) - f_1(\tilde{w})f_3'(w)]\}. \end{aligned}$$

In fact, $s_1(w,a) := -w^2(2-w)^2 + a^2(4-2w+w^2)$ is decreasing if $w \in (a,1)$. Therefore, $\min\{s_1(w,a)\} = s_1(1,a) = -1+3a^2$, and $\max\{s_1(w,a)\} = s_1(a,a) = 2a^3 > 0$ as $w \in (a,1)$. Therefore $s_1(w,a) > 0$ when $1/\sqrt{3} \le a < 1$, and $s_1(w,a) = 0$ has a unique root (denoted by $w_1$) when $0 < a < 1/\sqrt{3}$. On the other hand, $s_2(w,a) := -w^3(2-w)^3 - 2(-4+2w+3w^2-4w^3+w^4)a^2 - (2+2w-w^2)a^4$ is also decreasing if $w \in (a,1)$ because $\partial s_2(w,a)/\partial w < 0$. Hence $\min\{s_2(w,a)\} = s_2(1,a) = (3a^2-1)(1-a^2)$, and $\max\{s_2(w,a)\} = s_2(a,a) = 4(a-1)(a-2)a^2 > 0$ as $w \in (a,1)$. Similarly, $s_2(w,a) > 0$ when $1/\sqrt{3} \le a < 1$, and $s_2(w,a) = 0$ has a unique root (denoted by $w_2$) when $0 < a < 1/\sqrt{3}$. Therefore, $M(\tilde{w},w) + M(w,\tilde{w}) >$

0 for all $w \in (a, 1)$ when $1/\sqrt{3} \le a < 1$. Next, we have that $w_1 < w_2$, because $s_2(w_1, a) = 2(4-6w_1+3w_1^2)a^2-6a^4 > 2(4-6w_1+3w_1^2)a^2-2a^2 = 6(1-w_1)^2a^2 > 0$. So we only need to consider the zeros of $M(\tilde{w}, w) + M(w, \tilde{w})$ for $w \in (w_1, 1)$. Moreover,

$$\frac{d\{\sqrt{(w^2-a^2)[(2-w)^2-a^2]}s_1(w,a)\}}{dw} = \left\{ \frac{(2w(2-w)+a^2)(w^2-a^2)((2-w)^2-a^2)}{\sqrt{(w^2-a^2)[(2-w)^2-a^2]}} - \frac{[w(2-w)+a^2][-w^2(2-w)^2+a^2(4-2w+w^2)]}{\sqrt{(w^2-a^2)[(2-w)^2-a^2]}} \right\} 2(w-1) < 0,$$

for $w \in (w_1, 1)$. Therefore, $\sqrt{(w^2-a^2)[(2-w)^2-a^2]}s_1(w,a)+s_2(w,a)$ is decreasing for $w \in (w_1, 1)$, which implies that $\sqrt{(w^2-a^2)[(2-w)^2-a^2]}s_1(w,a)+s_2(w,a)$ has a unique zero. Thus $M(\tilde{w}, w) + M(w, \tilde{w}) = 0$ has a unique zero for $w \in (w_1, 1)$.

By [12, Corollary 2], the inequality $M(\tilde{w}, w) + M(w, \tilde{w}) > 0$ (resp. $< 0$) induces that $u'(h) < 0$ (resp. $> 0$). So $u'(h)$ has at most one zero. Then $F(h, 0, a, b)$ has at most two zeros because $b - 3a^2 < 0$. The proof is completed. □

## 4. Global bifurcation

The aim of this section is to show that the global bifurcation surfaces $HL$, $DL_1$, $DL_2$ of homoclinic loop and double limit cycles in Figure 1 exist and how they are located in the bifurcation diagram of system (2).

By the transformation $(x, y) \to (x, y + F(x))$, system (2) can be rewritten as system (33). Then For fixed values of $a$ and $\xi$, system (2) is a rotational family of vector fields (see [18] for definitions and properties) with respect to the parameter $b$. This implies that when $b$ increases unstable limit cycles increase and stable ones decrease in size. Furthermore, the double limit cycle that is stable in its outside part splits into a pair of limit cycles.

**Lemma 14.** *The surface $DL_2$ does not lie in $\mathcal{G}_2 := \{(a, b, \xi) \in \mathcal{G} : 0 < a < 1,\ 3a^2 - 3 < b < a^2 - 1\}$ and system (2) has at most one small limit cycle around the equilibria $E_R$ or $E_L$ in $\mathcal{G}_2$.*

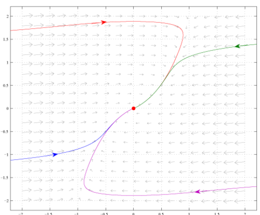
*Proof.* By Lemma 11 system (2) has no small limit cycles if $1/\sqrt{3} \le a < 1$ and $b = 3a^2-3$, and at most one small limit cycle around equilibria $E_R$ or $E_L$ if $0 < a < 1/\sqrt{3}$ and $b = 3a^2 - 3$. Therefore $DL_2$ cannot intersect with the curve $b = 3a^2 - 3$ except when $a = 1/\sqrt{3}$. Computing the trace at $E_0$ we obtain $\mathrm{tr}(J_0) = -b\xi > 0$, because $b < 0$ and $\xi > 0$. So the homoclinic loops have to be unstable if they exist by [7, Chapter 3, Theorem 3.3]. Assume that system (2) exhibits at least two small limit cycles surrounding $E_R$ for $(a, b, \xi) \in \mathcal{G}_2$. For fixed $a$ and $\xi$, we obtain that system (2) has a small semi-stable limit cycle $\Gamma_0$ when $b = b_0$ from the rotational properties of system (2) with respect to $b$. Now, given $b$, $\xi$ and a perturbation $a \to a + \varepsilon$, there exists a solution $\varphi(t, x_0, y_0)$ for $t \in (0, T)$ which lies in the small neighborhood of $\Gamma_0$ from the continuous dependence of solutions on the parameters and initial conditions, where $T$ is the periodic of $\Gamma_0$ and $\varepsilon > 0$ is small. Then using again the properties of the rotational vector fields, we can take a suitable parameter $b_0 + \varepsilon_1$ such that system (2) has a new semi-stable small limit cycle $\hat{\Gamma}_0$, because the homoclinic loops cannot be semistable, when $\varepsilon_1 > 0$ is small. By continuity system (2) has a small semi-stable limit cycle in the parameter curve either $H_2$, or

$H_3$, which leads to a contradiction. Therefore system (2) has at most one small limit cycle in $\mathcal{G}_2$. We have completed the proof. □

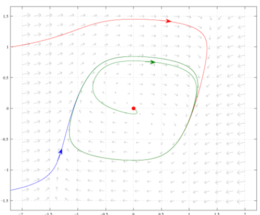
By a similar discussion as that in the proof of Lemma 14, the phase portraits (f)-(h) and (j)-(m) of system (2) in Figure 2 of Theorem 1 are obtained from the properties of rotational vector fields, continuity, Lemma 14 and the results of section 3.

**Remark 1.** *The surface $DL_2$ of double small limit cycles is the graph of a function $b = \varphi_2(a, \xi)$.*

From [5] we can obtain that a pair of grazing loop are stable for $a = 0$ if they exist. However, the pair of homoclinic loops have to be unstable if they exist when $a \neq 0$. In fact, $HL$, $DL_1$, $DL_2$ have a common intersection point for the limit value $a = 0$, i.e., $\varphi(0, \xi) = \phi_1(0, \xi) = \phi_1(0, \xi) < -3$. Since system (33) is a rotational vector field with respect to the parameter $b$, the manifolds of $E_0$ move monotonically as $a, \xi$ are fixed and $b$ increases, see [5, 6]. Therefore, it is worthwhile to note that $HL$, $DL_1$ and $DL_2$ have no intersection points except at endpoints. Summarizing the previous results, we can obtain Theorem 1, as shown in Figure 1.



(a) $(\sqrt{2}, 1, 1) \in II$



(b) $(\sqrt{2}, -1, 1) \in III$

FIGURE 10. Simulations with a single equilibrium

## 5. NUMERICAL EXAMPLES

In this section we give several numerical examples of previous results.

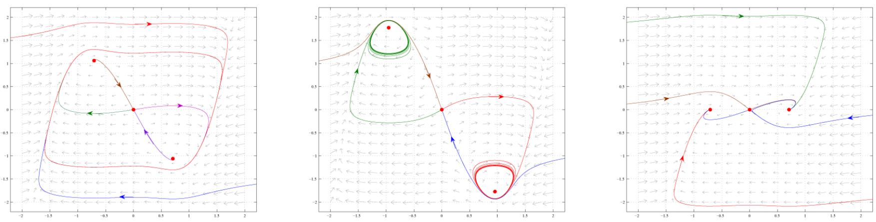
**Example 1**. Let $a = \sqrt{2}$ and $\xi = 1$. When $b = 1$ the system has a unique equilibrium $(0, 0)$, which is a sink, and no limit cycles, as shown in Figure 10(a).

However, when $b = -1$ the system has a unique equilibrium, the origin $(0, 0)$ which is a source. Furthermore, from Lemma 5, there is a unique limit cycle, which is stable, as shown in Figure 10(b).

**Example 2**. Let $\xi = 1$. When $a = \sqrt{2}/2$ and $b = -2$ the system has three equilibria and exactly one large limit cycle, as shown in Figure 11(a).

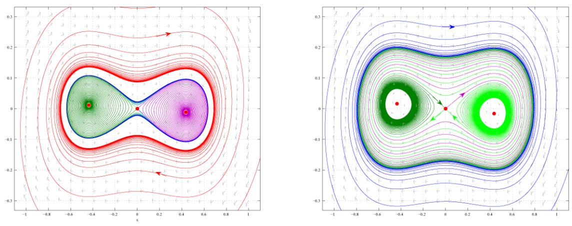
When $a = 0.3$ and $b = -2.77$ the system has three equilibria and exactly two small limit cycles, as shown in Figure 11(b).

When $a = \sqrt{2}/2$ and $b = -0.5$ the system has three equilibria and no limit cycles, as shown in Figure 11(c).

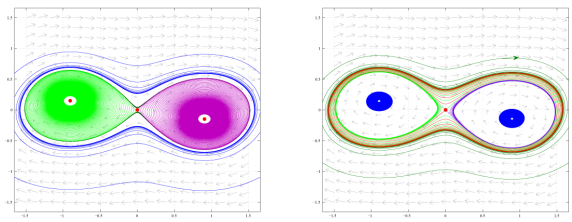


(a) $(\sqrt{2}/2, -2, 1) \in IV$ (b) $(0.3, -2.77, 1) \in VII$ (c) $(\sqrt{2}/2, -0.5, 1) \in I$

FIGURE 11. Simulations with three equilibria



(a) $(0.9, -0.46, 0.1) \in VIII$ (b) $(0.9, -0.5558, 0.1) \in IX$



(c) $(3\sqrt{2}/10, -2.461, 0.1) \in VI$ (d) $(\sqrt{5}/5, -2.41, 0.1) \in V$

FIGURE 12. Simulations with three equilibria

**Example 3**. Let $\xi = 0.1$. When $a = 0.9$ and $b = -0.46$ the system has three equilibria and exactly two large limit cycles, as shown in Figure 12(a).

When $a = 0.9$ and $b = -0.555$ the system has three equilibria, exactly two small limit cycles and one large limit cycle, as shown in Figure 12(b).

When $a = 3\sqrt{2}/10$ and $b = -2.461$ the system has three equilibria, exactly two small limit cycles and two large limit cycles, as shown in Figure 12(c).

When $a = \sqrt{5}/5$ and $b = -2.41$ the system has three equilibria, exactly four small limit cycles and one large limit cycle, as shown in Figure 12(d).

## Appendix: Averaging theory of first and second order

The averaging theory of second order for studying specifically periodic orbits can be found in [13] for $\mathbb{S}^3$ differential systems and in [1] for Lipschitz differential systems, see also Chapter 11 of [17]. Here we present a brief summary with the results that we need for studying the Hopf bifurcation of the SD oscillator systems.

Consider the differential system

$$\dot{\mathbf{x}}(t) = \mu F_1(t, \mathbf{x}) + \mu^2 F_2(t, \mathbf{x}) + \mu^3 R(t, \mathbf{x}, \mu), \tag{37}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}$, $R : \mathbb{R} \times D \times (-\mu_0, \mu_0) \to \mathbb{R}$ are continuous functions, $T$–periodic in the first variable, and $D$ is an open subset of $\mathbb{R}^n$. Assume:

(i) $F_1(t, \cdot) \in \mathbb{S}^2(D)$, $F_2(t, \cdot) \in \mathbb{S}^1(D)$ for all $t \in \mathbb{R}$, $F_1$, $F_2$, $R$, $D^2_{\mathbf{x}} F_1$,$D_{\mathbf{x}} F_2$ are locally Lipschitz with respect to $\mathbf{x}$, and $R$ is twice differentiable with respect to $\mu$.

We define the functions $F_{k0} : D \to \mathbb{R}$ for $k = 1, 2$ as follows

$$f_1(\mathbf{x}) = \frac{1}{T}\int_0^T F_1(s, \mathbf{x}) ds,$$

$$f_2(\mathbf{x}) = \frac{1}{T}\int_0^T \left[ D_{\mathbf{x}} F_1(s, \mathbf{x}) \int_0^s F_1(t, \mathbf{x}) dt + F_2(s, \mathbf{x}) \right] ds.$$

(ii) For $V \subset D$ an open and bounded set and for each $\mu \in (-\mu_0, \mu_0) \setminus \{0\}$, suppose that either $f_1(\mathbf{x}) \not\equiv 0$, there exists $a \in V$ such that $f_1(a) = 0$ and the Jacobian $\det D_{\mathbf{x}}(f_1)(a) \neq 0$; or $f_1(\mathbf{x}) \equiv 0$, there exists $a \in V$ such that $f_2(a) = 0$ and the Jacobian $\det D_{\mathbf{x}}(f_2)(a) \neq 0$.

Then for $|\mu| > 0$ sufficiently small there exists a $T$–periodic solution $\mathbf{x}(t, \mu)$ of system (37) such that $\mathbf{x}(0, \mu) \mapsto a$ when $\mu \mapsto 0$.

If for the $i$ for which $f_i(a) = 0$ the real part of all the eigenvalues of the Jacobian matrix $D_{\mathbf{x}}(f_i)(a)$ are negative, then the periodic solution $\mathbf{x}(t, \mu)$ is asymptotically stable, if some eigenvalue has a positive real part then it is unstable.

The *averaging theory of first order* takes place when $f_1(\mathbf{x}) \not\equiv 0$. If $f_1(\mathbf{x}) \equiv 0$ and $f_2(\mathbf{x}) \not\equiv 0$ we say that that we work with the *averaging theory of second order*.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## References


[1] A. Buică, J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, Bull. Sci. Math. **128** (2004), 7–22.

[2] Q. Cao, M. Wiercigroch, E. Pavlovskaia, J. Thompson, C. Grebogi, Piecewise linear approach to an archtypal oscillator for smooth and discontinouous dynamics, *Philos. Trans. R. Soc. A: Math. Phys. Eng. Sci* **366** (2008), 635-652.
[3] Q. Cao, M. Wiercigroch, E. Pavlovskaia, C. Grebogi, J. Thompson, An archetypal oscillator for smooth and discontinuous dynamics, *Phys. Rev. E* **74** (2006), 046218.
[4] Q. Cao, M. Wiercigroch, E. Pavlovskaia, C. Grebogi, J. Thompson, The limit case response of the archetypal oscillator for smooth and discontinuous dynamics, *Int. J. Non-linear Mech.* **43** (2008), 462-473.
[5] H. Chen, Global analysis on the discontinuous limit case of a smooth oscillator, *Internat. J. Bifur. Chaos*, to appear.
[6] H. Chen, X. Chen, Dynamics analyis of a cubic Liénard system with global parameters, *Nonlinearity* **28** (2015), 3535-3562.
[7] S. N. Chow, C. Li, D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge. Press, 1994.
[8] F. Dumortier, J. Llibre, J. C. Artés, *Qualitative Theory of Planar Differential Systems*, UniversiText, Springer–Verlag, New York, 2006.
[9] F. Dumortier, C. Rousseau, Cubic Liénard equations with linear damping, *Nonlinearity* **3** (1990), 1015-1039.
[10] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, 1997.
[11] Yu. A. Kuznetsov, *Elements of Applied Bifurcation Theory, 2nd Edition*, Springer, New York, 1998.
[12] C. Li, Z. Zhang, A criterion determining the monotonicity of the ratio of two Abelian integrals, *J. Diff. Equa.* **124** (1996), 407-424.
[13] J. Llibre, Averaging theory and limit cycles for quadratic systems, *Radovi Matematicki* **11** (2002), 215–228.
[14] G. Sansone, R. Conti, *Non-Linear Differential Equations*, Pergamon Press, New York, 1964.
[15] Y. Tang, W. Zhang, Generalized normal sectors and orbits in expceptional directions, *Nonlinearity* **17** (2004), 1407–1426.
[16] R. Tian, Q. Cao, S. Yang, The codimension-two bifurcation for the recent proposed SD oscillator, *Nonlinear Dyn.* **59** (2010), 19-27.
[17] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext, Springer, 1991.
[18] Z. Zhang, T. Ding, W. Huang, Z. Dong, *Qualitative Theory of Differential Equations*, Transl. Math. Monogr., Amer. Math. Soc., Providence, RI, 1992.

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