

**LINEAR TYPE CENTERS OF POLYNOMIAL HAMILTONIAN  
SYSTEMS WITH NONLINEARITIES OF DEGREE 4  
SYMMETRIC WITH RESPECT TO THE Y-AXIS**

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ABSTRACT. We provide normal forms and the phase portraits in the Poincaré disk for all the linear type centers of polynomial Hamiltonian systems with nonlinearities of degree 4 symmetric with respect to the  $y$ -axis.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this work we deal with polynomial differential systems in  $\mathbb{R}^2$  of the form

$$(1) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

where the dot denotes derivative with respect to an independent real variable  $t$ , usually called the *time*. Assume the origin  $O$  is an equilibrium point of system (1).

When all the orbits of system (1) in a punctured neighborhood of the equilibrium point  $O$  are periodic, we say that the origin is a *center*. The study of the centers started with Poincaré [17] and Dulac [8], and in the present days many questions about them remain open.

If a polynomial system (1) has a center at the origin, then after a linear change of variables and a scaling of the time variable, it can be written in one of the following three forms:

$$\dot{x} = -y + P_2(x, y), \quad \dot{y} = x + Q_2(x, y),$$

called a *linear type center*,

$$\dot{x} = y + P_2(x, y), \quad \dot{y} = Q_2(x, y),$$

called a *nilpotent center*,

$$\dot{x} = P_2(x, y), \quad \dot{y} = Q_2(x, y),$$

called a *degenerate center*, where  $P_2(x, y)$  and  $Q_2(x, y)$  are polynomials without constant and linear terms.

The classification of the centers of quadratic differential systems (which all of them are linear type centers) started with the works of Dulac [8], Kapteyn [11, 12] and Bautin [3], and the characterization of their phase portraits in the Poincaré disk was due to Vulpe [20], see also Schlomiuk [19]. There are many partial results for the centers of polynomial differential systems of degree larger than 2. We must mention that Malkin [13], and Vulpe and Sibirsky [21] characterized the linear type centers of the polynomial differential systems with linear and homogeneous nonlinearities of degree 3. For polynomial differential systems of the form linear

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with homogeneous nonlinearities of degree greater than 3 the centers at the origin are not characterized, but there are partial results for degrees 4 and 5 for those linear type centers, see Chavarriga and Giné [4, 5]. On the other hand it remains a lot of work for obtaining the complete classification of the linear type centers for all polynomial differential systems of degree 3. Some interesting results on some subclasses of those systems appeared in Rousseau and Schlomiuk [18], and the ones of Żołądek [22, 23]. Recently, the linear type centers of polynomial Hamiltonian systems with nonlinearities of degree 3 were classified by Colak et al. in [7].

In this paper, we study the phase portraits in the Poincaré disk of the linear type centers of polynomial Hamiltonian systems with nonlinearities of degree 4, but since this class has 6 parameters and degree 4 his study is huge, in a first approach for studying this class we restrict our attention to the subclass of such systems which are symmetric with respect to the  $y$ -axis. More precisely, we shall classify the phase portraits in the Poincaré disk of the Hamiltonian system

$$(2) \quad \dot{x} = -y - ax^4 - 3bx^2y^2 - 5cy^4, \quad \dot{y} = x + 4ax^3y + 2bxy^3,$$

with Hamiltonian

$$(3) \quad H(x, y) = \frac{1}{2}(x^2 + y^2) + ax^4y + bx^2y^3 + cy^5,$$

and satisfying that  $a^2 + b^2 + c^2 \neq 0$ , i.e. that the polynomial system (2) has degree 4. Since  $H$  is a first integral of system (2) and the origin of this system has eigenvalues  $\pm i$ , the origin is a center. We note that system (2) is invariant under the symmetry  $(x, y, t) \rightarrow (-x, y, -t)$ , i.e. its orbits are symmetric with respect to the  $y$ -axis, and consequently the phase portrait of the system.

We note that the centers of the Hamiltonian systems with Hamiltonian

$$H(x, y) = ax^4y + bx^2y^3 + cy^5,$$

have been studied in [6], and that there are no nilpotent centers for the Hamiltonian systems with Hamiltonian

$$H(x, y) = \frac{1}{2}y^2 + ax^4y + bx^2y^3 + cy^5,$$

as it can be proved used Theorem 3.5 of [9].

In order to classify the phase portraits of systems (2) we will use the Poincaré compactification of polynomial vector fields, see subsection 2.1, which roughly speaking is to add to the plane  $\mathbb{R}^2$  its boundary, the circle  $\mathbb{S}^1$  which corresponds to the infinity, identifying this compact space with a closed disk (the Poincaré disk), and extending the flow of system (2) to this closed disk in a unique analytic way. We say that two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism of this disk which sends orbits to orbits preserving or reversing the direction of the flow, see a more precise definition in subsection 2.1. Our main result is the following one.

**Theorem 1.** *The phase portrait in the Poincaré disk of a linear type center of a polynomial Hamiltonian system with nonlinearities of degree 4 symmetric with respect to the  $y$ -axis is topologically equivalent to one of the following 30 phase portraits of Figure 1.*

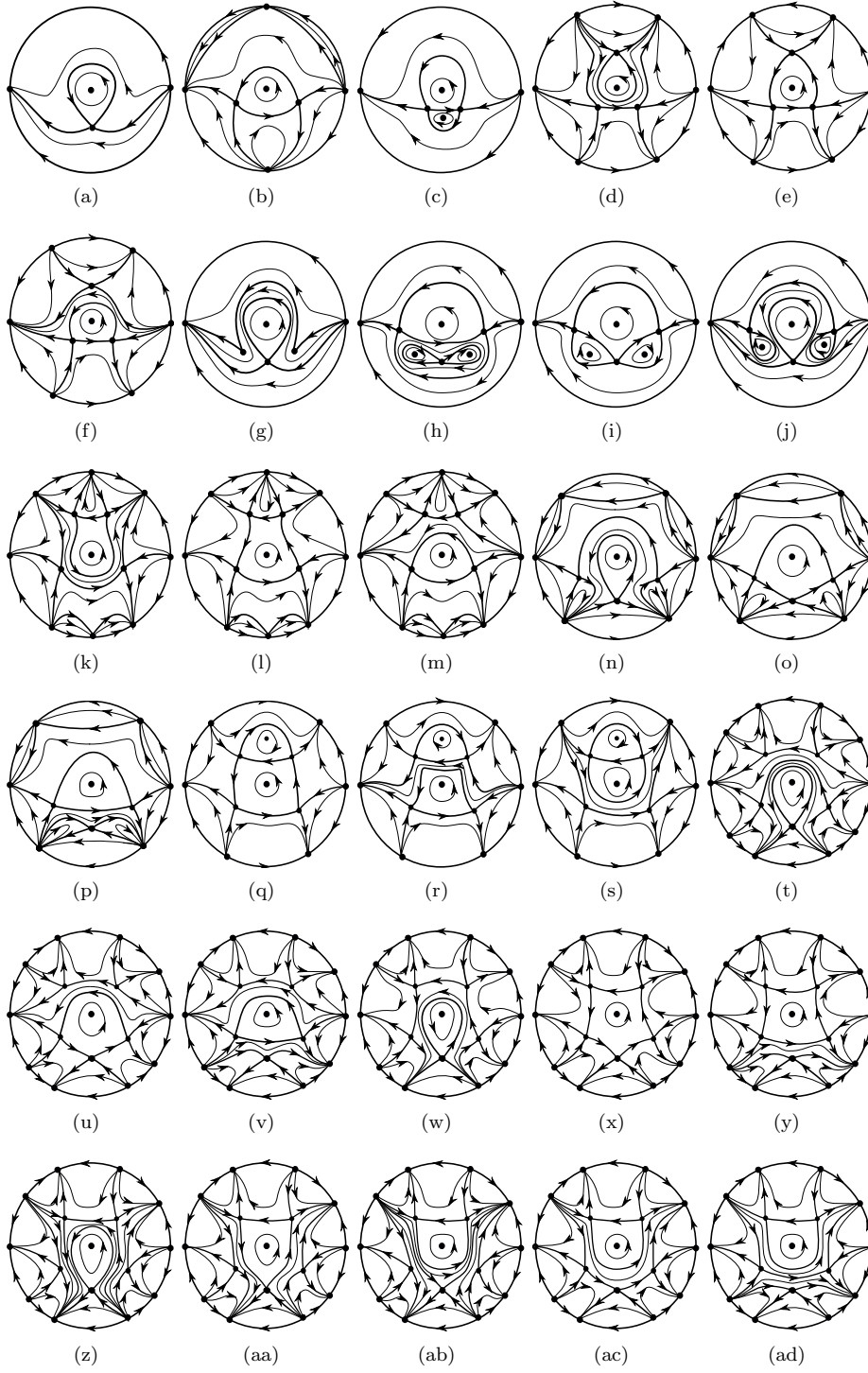


FIGURE 1. Phase portraits for the systems associated to Hamiltonian (3). The separatrices are in bold.

In section 2 we provide the notations, basic definitions and results which will allow to do the topological classification of all phase portraits for the Poincaré disk of the linear type centers for the polynomial Hamiltonian systems with nonlinearities of degree 4 symmetric with respect to the  $y$ -axis, i.e., of systems (2). A second step in this work will be to perturb the Hamiltonian centers of the phase portraits of Theorem 1 inside the class of all quartic polynomial differential systems and study how many limit cycles can bifurcate from their periodic orbits using the tools of [10]

## 2. NOTATIONS AND BASIC RESULTS

In this section we recall the basic definitions and notations that we will need for the analysis of the local phase portraits of the finite and infinite equilibria of the polynomial Hamiltonian systems (2), and for doing their phase portraits in the Poincaré disk.

We denote by  $\mathcal{P}_4(\mathbb{R}^2)$  the set of polynomial vector fields in  $\mathbb{R}^2$  of the form  $\mathcal{X}(x, y) = (P(x, y), Q(x, y))$  where  $P$  and  $Q$  are real polynomials in the variables  $x$  and  $y$  such that the maximal degree of  $P$  and  $Q$  is 4.

**2.1. Poincaré compactification.** Let  $\mathcal{X} \in \mathcal{P}_4(\mathbb{R}^2)$  be a vector field of degree 4 in  $\mathbb{R}^2$ . The *Poincaré compactified vector field*  $p(\mathcal{X})$  of  $\mathcal{X}$  is an analytic vector in  $\mathbb{S}^2$  defined in the following way, for more details see for instance Chapter 5 of [9]. Let  $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$  be the *Poincaré sphere* and  $T_y\mathbb{S}^2$  be the tangent space to  $\mathbb{S}^2$  at the point  $y$ . We identify the plane  $T_{(0,0,1)}\mathbb{S}^2$  with the  $\mathbb{R}^2$  where we have our vector field  $\mathcal{X}$ .

We define the central projection  $f : T_{(0,0,1)}\mathbb{S}^2 \rightarrow \mathbb{S}^2$  in such a way that to each point  $q$  of the plane  $T_{(0,0,1)}\mathbb{S}^2$  the map associates the two intersection points of the straight line, joining  $q$  with  $(0,0,0)$ , with the sphere  $\mathbb{S}^2$ . This map provides two copies of  $\mathcal{X}$ , one in the northern hemisphere and the other in the southern one. Denote by  $\mathcal{X}'$  the vector field  $Df \circ \mathcal{X}$  on  $\mathbb{S}^2$  except on its equator  $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ . Clearly  $\mathbb{S}^1$  is identified to the infinity of  $\mathbb{R}^2$ .

In order to extend  $\mathcal{X}'$  to a vector field on the whole  $\mathbb{S}^2$  we consider the vector field  $p(\mathcal{X}) = y_3^4 \mathcal{X}'$  in  $\mathbb{S}^2$ . In short, on  $\mathbb{S}^2 \setminus \mathbb{S}^1$  there are two symmetric copies of  $\mathcal{X}$ , and knowing the behavior of  $p(\mathcal{X})$  around  $\mathbb{S}^1$ , we know the behavior of  $\mathcal{X}$  at infinity. The infinity  $\mathbb{S}^1$  is invariant under the flow of  $p(\mathcal{X})$ .

The projection of the closed northern hemisphere of  $\mathbb{S}^2$  on  $y_3 = 0$  under  $(y_1, y_2, y_3) \mapsto (y_1, y_2)$  is called the *Poincaré disk*, and it is denoted by  $\mathbb{D}^2$ .

Here two polynomial vector fields  $\mathcal{X}$  and  $\mathcal{Y}$  on  $\mathbb{R}^2$  are *topologically equivalent* if there exists a homeomorphism on  $\mathbb{S}^2$  preserving the infinity  $\mathbb{S}^1$  carrying orbits of the flow induced by  $p(\mathcal{X})$  into orbits of the flow induced by  $p(\mathcal{Y})$ ; preserving or reversing simultaneously the sense of all orbits.

On the sphere  $\mathbb{S}^2$  we take for  $i = 1, 2, 3$  the six local charts  $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$ , and  $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ , with the diffeomorphisms  $F_i : U_i \rightarrow \mathbb{R}^2$  and  $G_i : V_i \rightarrow \mathbb{R}^2$ , which are the inverses of the central projections from the planes tangent at the points  $(1, 0, 0)$ ;  $(-1, 0, 0)$ ;  $(0, 1, 0)$ ;  $(0, -1, 0)$ ;  $(0, 0, 1)$  and  $(0, 0, -1)$ , respectively. We denote by  $z = (z_1, z_2)$  the value of  $F_i(y)$  or  $G_i(y)$  for any  $i = 1, 2, 3$ . Thus  $z$  means different coordinates in the distinct local charts. Easy computations

give for  $p(\mathcal{X})$  the next expressions:

$$(4) \quad z_2^4 \Delta(z) \left( Q \left( \frac{1}{z_2}, \frac{z_1}{z_2} \right) - z_1 P \left( \frac{1}{z_2}, \frac{z_1}{z_2} \right), -z_2 P \left( \frac{1}{z_2}, \frac{z_1}{z_2} \right) \right) \quad \text{in } U_1,$$

$$(5) \quad z_2^4 \Delta(z) \left( P \left( \frac{z_1}{z_2}, \frac{1}{z_2} \right) - z_1 Q \left( \frac{z_1}{z_2}, \frac{1}{z_2} \right), -z_2 Q \left( \frac{z_1}{z_2}, \frac{1}{z_2} \right) \right) \quad \text{in } U_2,$$

$$z^4 \Delta(z) (P(z_1, z_2), Q(z_1, z_2)) \quad \text{in } U_3,$$

where  $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-\frac{3}{2}}$ .

The expression for  $V_i$  is the same as that for  $U_i$  except for a change of sign. In the local charts with subindices  $i = 1, 2$ ,  $z_2 = 0$  always denotes the points of  $\mathbb{S}^1$ , i.e. the points of the infinity. In what follows we shall omit the factor  $\Delta(z)$  doing a convenient scaling of the vector field  $p(\mathcal{X})$ . Thus we have a polynomial vector field for  $p(\mathcal{X})$  in every local chart.

The equilibria of  $p(\mathcal{X})$  which are in the interior of the Poincaré disk are called the *finite* equilibria, which correspond with the equilibria of  $\mathcal{X}$ , and the equilibria of  $p(\mathcal{X})$  which are in  $\mathbb{S}^1$  are called the *infinite* equilibria of  $\mathcal{X}$ .

We note that studying the infinite equilibria of the local chart  $U_1$ , we obtain also the ones of the local chart  $V_1$ , and only remains to see if the origin of the local chart  $U_2$ , and consequently the origin of the local chart  $V_2$ , are infinite equilibria.

**2.2. Singular points.** Let

$$(6) \quad \mathcal{X} = (-y - ax^4 - 3bx^2y^2 - 5cy^4, x + 4ax^3y + 2bxy^3)$$

be the vector field associated to our Hamiltonian system (2) with  $a^2 + b^2 + c^2 \neq 0$ , and let  $p(\mathcal{X})$  be its Poincaré compactification. Let  $q$  be an equilibrium point of  $p(\mathcal{X})$ .

If some of the two eigenvalues  $\lambda_1$  and  $\lambda_2$  of the linear part of the vector field  $p(\mathcal{X})$  at the equilibrium point  $q$  is not zero, then this equilibrium is *elementary*.

The local phase portrait of a finite elementary equilibrium  $q$  with both eigenvalues non-zero for a Hamiltonian system is well known,  $q$  is a saddle if  $\lambda_1 \lambda_2 \neq 0$  (which in fact only can be  $\lambda_1 \lambda_2 < 0$ ), and a center if  $\text{Re} \lambda_1 = \text{Re} \lambda_2 = 0$ , for more details see for instance [2]. We recall that the flow of a Hamiltonian system in the plane preserves the area, so their finite equilibria cannot have parabolic and elliptic sectors, only hyperbolic sectors, or they must be centers. We define energy levels of the vector field (6) has the curves on which its Hamiltonian (3) is constant.

The local phase portrait of an infinite elementary equilibrium  $q$  with both eigenvalues non-zero can be studied using, for instance, Theorem 2.15 of [9].

Let  $q$  be an elementary equilibrium point with an eigenvalue zero. Then  $q$  is called a *semi-hyperbolic* equilibrium point. The local phase portrait of a finite semi-hyperbolic equilibrium point of a Hamiltonian system only can be a saddle, see again for example [9].

The local phase portrait of an infinite semi-hyperbolic equilibrium  $q$  can be described using, for instance, Theorem 2.19 of [9].

When both eigenvalues are zero but the linear part of  $p(\mathcal{X})$  at the equilibrium point  $q$  is not the zero matrix, we say that  $q$  is a *nilpotent* equilibrium point. The local phase portrait of a nilpotent equilibrium point can be studied using Theorem

3.5 of [9]. If  $q$  is nilpotent and finite, then it only can be a saddle, a cusp, or a center.

Finally, if the Jacobian matrix of  $p(\mathcal{X})$  at the equilibrium point  $q$  is identically zero, and  $q$  is isolated inside the set of all equilibrium points, then we say that  $q$  is a *linearly zero* equilibrium point. The study of the local phase portraits of such equilibria needs special changes of variables called blow-ups, see for more details Chapter 3 of [9], or [1].

**2.3. Local phase portraits of the equilibria.** We want to classify in the Poincaré disk, modulo topological equivalence, the phase portraits of  $p(\mathcal{X})$  being  $\mathcal{X}$  the polynomial Hamiltonian vector fields given in (6). For doing that we must start by classifying the local phase portraits at all finite and infinite equilibria of  $p(\mathcal{X})$  in the Poincaré disk. This will be made by using the techniques described in subsection 2.2.

**2.4. Phase portraits in the Poincaré disk.** In this subsection we shall see how to characterize the phase portraits of  $p(\mathcal{X})$  in the Poincaré disk for the polynomial Hamiltonian vector fields  $\mathcal{X}$  of (6).

A *separatrix* of  $p(\mathcal{X})$  being  $\mathcal{X}$  a Hamiltonian vector field (6) defined in the whole  $\mathbb{R}^2$  is an orbit which is either an equilibrium point, or a trajectory which lies in the boundary of a hyperbolic sector of a finite or infinite equilibrium point, or any orbit contained in the infinity  $\mathbb{S}^1$ . Neumann [15] proved that the set formed by all separatrices of  $p(\mathcal{X})$ , denoted by  $S(p(\mathcal{X}))$  is closed.

The open connected components of  $\mathbb{D}^2 \setminus S(p(\mathcal{X}))$  are called canonical regions of  $\mathcal{X}$  or of  $p(\mathcal{X})$ . A *separatrix configuration* is the union of  $S(p(\mathcal{X}))$  plus one solution chosen from each canonical region. Two separatrix configurations  $S(p(\mathcal{X}))$  and  $S(p(\mathcal{Y}))$  are *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of  $S(p(\mathcal{X}))$  into the trajectories of  $S(p(\mathcal{Y}))$ . The following result is due to Markus [14], Neumann [15] and Peixoto [16], that find it independently.

**Theorem 2.** *The phase portraits in the Poincaré disk of two compactified polynomial differential systems  $p(\mathcal{X})$  and  $p(\mathcal{Y})$  are topologically equivalent, if and only if, their separatrix configurations  $S(p(\mathcal{X}))$  and  $S(p(\mathcal{Y}))$  are topologically equivalent.*

### 3. PHASE PORTRAITS WHEN $abc = 0$

We separate the study of the phase portraits in the Poincaré disk of Hamiltonian system (2) when  $abc = 0$  in the following six cases:  $a = b = 0$  and  $c \neq 0$ ,  $a = c = 0$  and  $b \neq 0$ ,  $b = c = 0$  and  $a \neq 0$ ,  $a = 0$  and  $bc \neq 0$ ,  $b = 0$  and  $ac \neq 0$ , and  $c = 0$  and  $ab \neq 0$ .

**3.1. Case  $a = b = 0$  and  $c \neq 0$ .** Here Hamiltonian system (2) is

$$\dot{x} = -y(1 + 5cy^3), \quad \dot{y} = x.$$

Since  $c \neq 0$  doing the  $c^{-2/3}$ -symplectic change of variables  $(x, y) \mapsto (5c)^{-1/3}(x, y)$  we can assume without loss of generality that  $c = 1$  and the new Hamiltonian system is

$$(7) \quad \dot{x} = -y(1 + y^3), \quad \dot{y} = x.$$

This system only have two finite equilibria, the origin which as we know always is a center, and the saddle  $(0, -1)$ , because the two eigenvalues of the linear part at this equilibrium are  $\pm\sqrt{3}$ .

From (4) system (7) in the local chart  $U_1$  becomes

$$(8) \quad \dot{z}_1 = z_2^3 + z_1^5 + z_1^2 z_2^3, \quad \dot{z}_2 = z_1^4 z_2 + z_1 z_2^4.$$

At infinity, i.e. at  $z_2 = 0$ , this system has a unique equilibrium, the origin  $(z_1, z_2) = (0, 0)$  which is a linear zero equilibrium. For studying its local phase portrait we need to do blow-ups. Since this is the first equilibrium that we must study using the changes of variables called blow-ups we will describe them with all details in this case.

First we do the blow-up  $(z_1, z_2) \mapsto (z_1, w)$  with  $w = z_2/z_1$ , then we obtain the system

$$(9) \quad \dot{z}_1 = z_1^3(z_1^2 + w^3 + z_1^2 w^3), \quad \dot{w} = -z_1^2 w^4.$$

We note that this change of variables blow up the origin of system (8) to the whole  $w$ -axis of system (9).

Now doing a scaling of the independent variable we eliminate the common factor  $z_1^2$  between  $\dot{z}_1$  and  $\dot{w}$  of system (9), and we obtain the system

$$(10) \quad z_1' = z_1(z_1^2 + w^3 + z_1^2 w^3), \quad w' = -w^4.$$

Since the whole  $w$ -axis going back through the changes of variables goes to the origin of system (8) we must study the equilibria on such an axis. Thus, on  $z_1 = 0$  the unique equilibrium of system (10) is the origin, and again it is linearly zero. So we do a second blow-up  $(z_1, w) \rightarrow (z_1, u)$  with  $u = w/z_1$ , and we arrive to the system

$$(11) \quad z_1' = z_1^3(1 + z_1 u^3 + z_1^3 u^3), \quad u' = -z_1^2 u(1 + 2z_1 u^3 + z_1^3 u^3).$$

Doing another scaling of the independent variable we eliminate the common factor  $z_1^2$  between the two components of system (11), and we have

$$(12) \quad z_1' = z_1(1 + z_1 u^3 + z_1^3 u^3), \quad u' = -u(1 + 2z_1 u^3 + z_1^3 u^3).$$

Now, the unique equilibrium on  $z_1 = 0$  is the origin. Since the eigenvalues of the linear part of this system at the origin are  $\pm 1$ , the origin is a saddle, with the local stable separatrices on the  $u$ -axis, and the unstable ones on the  $z_1$ -axis, see Figure 2(a).

Now we must go back through the changes of variables starting at system (12) and reaching system (8). Since the scaling for passing from system (11) to system (12) is  $z_1^2$ , this change in the independent variable changes neither the orbits, nor their orientation, with the exception of the orbits on the  $u$ -axis of system (12) which become equilibria for system (11), see Figure 2(b).

Going back from system (11) to system (10), and taking into account that for system (10) we have  $z_1'|_{w=0} = z_1^3$  and  $w'|_{z_1=0} = -w^4$ , we obtain the local phase portrait at the origin of system (10), which topologically is a saddle-node, see Figure 2(c).

As in the pass from system (12) to system (11), the pass from system (10) to system (9) does not change the local phase portrait of system (10) except that the  $w$ -axis becomes filled with equilibria, see Figure 2(d).

Finally, undoing the first blow up and taking into account that for system (8) we have that  $\dot{z}_1|_{z_1=0} = z_2^3$ , we get that the local phase portrait at the origin of system (8) is the unstable node of Figure 2(e).

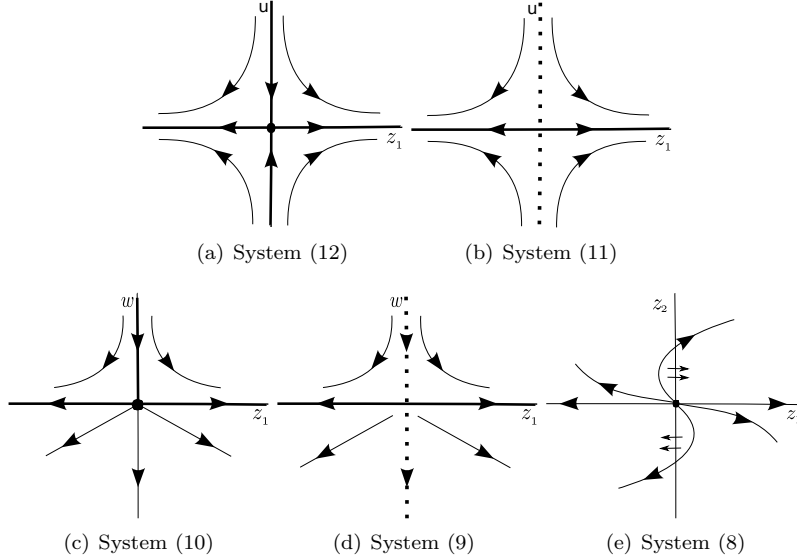


FIGURE 2. The blow-ups of the origin of the chart  $U_1$  for system (8). The dotted line represents a straight line of equilibria.

In short, in the chart  $U_1$  the unique infinite singular point is the origin which is an unstable node. Therefore, in the local chart  $V_1$  also the origin is the unique infinite singular point which is a stable node due to the fact that the vector field  $p(\mathcal{X})$  in both charts are equal except that the sign of this orbits are opposite. In order to complete the study of the infinite singular points, as we explained in subsection 2.1, only it remains to see if the origin of the local chart  $U_2$  is or not an equilibrium point.

From (5) system (7) in the local chart  $U_2$  writes

$$\dot{z}_1 = -1 - z_2^3 - z_1^2 z_2^3, \quad \dot{z}_2 = -z_1 z_2^4.$$

Hence, clearly the origin of  $U_2$  is not an equilibrium.

Thus, we can give the local dynamics on the Poincaré disk, this is shown in Figure 3.

In summary, first we know the local phase portraits of the finite singular points, a center at  $(0, 0)$  and a saddle at  $(0, -1)$  (we can compute locally an approximation of the separatrices of this elementary saddle using the eigenvectors of the saddle); second we know the local phase portraits of the infinite singular points, an unstable node at the end of the positive  $x$ -axis (the origin of  $U_1$ ) and a stable node at the end of the negative  $x$ -axis; and third that we know the behavior of system (7) on the axes, in fact,  $\dot{x}|_{x=0} = -y(1 + y^3)$ , which is negative if  $y \in (-\infty, -1) \cup (0, \infty)$  and is positive if  $y \in (-1, 0)$ ,  $\dot{y}|_{x=0} = 0$ ,  $\dot{x}|_{y=0} = 0$  and  $\dot{y}|_{y=0} = x$ .



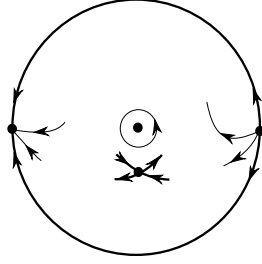


FIGURE 3. Local phase portraits for system (2) if  $a = b = 0$  and  $c \neq 0$ .

These three knowledges force that the unstable separatrix of the saddle which enters into the quadrant  $x > 0$  and  $y < 0$ , must cross the positive  $x$ -axis because it cannot intersect the stable separatrix of the saddle which also is in the quadrant  $x > 0$  and  $y < 0$ , and such unstable separatrix must intersect the positive  $y$ -axis, at this moment it connects with the stable separatrix of the saddle that initially is in the quadrant  $x < 0$  and  $y < 0$  due to the fact that the phase portrait is symmetric with respect to the  $y$ -axis. This connection forces that the phase portrait of system (7) is topologically equivalent to the one described in Figure 1(a).

Another way to see the connection of the unstable separatrix of the saddle with the stable one described in the previous paragraph is drawing the level curves of the Hamiltonian

$$H = \frac{1}{2}(x^2 + y^2) + \frac{1}{5}y^5,$$

of system (7) passing through the saddle.

**3.2. Case  $a = c = 0$  and  $b \neq 0$ .** Under these conditions, system (2) is

$$\dot{x} = -y - 3bx^2y^2, \quad \dot{y} = x + 2bxy^3,$$

doing the  $b^{-2/3}$  symplectic change of variables  $(x, y) \rightarrow b^{-1/3}(x, y)$  we get

$$(13) \quad \dot{x} = -y - 3x^2y^2, \quad \dot{y} = x + 2xy^3.$$

System (13) has three finite equilibria: the origin which we know that is a center, and other two equilibria  $\tilde{e}_{3,4} = (\pm 2^{1/6}3^{-1/2}, -2^{-1/3})$ , and these two symmetric equilibria are saddles because their eigenvalues are  $\pm\sqrt{6}$ .

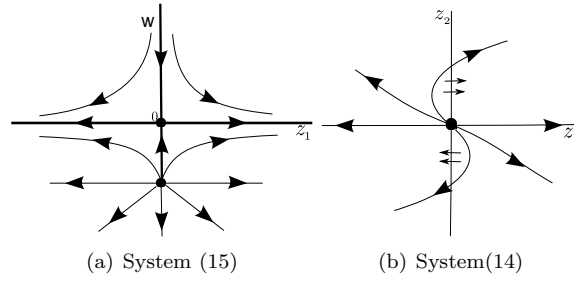
Now, we do the study at the infinity, from (4) the system in  $U_1$  is

$$(14) \quad \dot{z}_1 = 5z_1^3 + z_2^3 + z_1^2z_2^3, \quad \dot{z}_2 = 3z_1^2z_2 + z_1z_2^4.$$

At infinity, i.e.  $z_2 = 0$ , the origin is the unique equilibrium, which is degenerate, we need to apply the blow-up technique for understanding its behavior. We do the change of variables  $(z_1, z_2) \rightarrow (z_1, w)$  with  $z_2 = w \cdot z_1$  eliminating the common factor  $z_1^2$  we get

$$(15) \quad z_1' = z_1(5 + w^3 + w^3z_1^2), \quad w' = -w(2 + w^3),$$

these system have two hyperbolic equilibria, the origin and  $(0, -\sqrt[3]{2})$ . The origin has eigenvalues 5 and  $-2$ , so is a saddle, and the second equilibrium is a repeller node because its eigenvalues are 3 and 6. The local phase portrait of the behavior at the origin of system (14) is shown in Figure 4.

FIGURE 4. Blow-up of the origin of the chart  $U_1$  associated to (13)

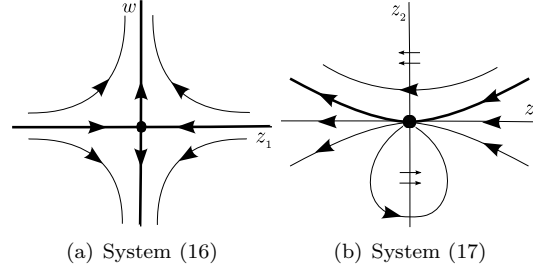
In the local chart  $U_2$  system (13) through (5) has the form

$$(16) \quad \dot{z}_1 = -z_2^3 - z_1^2(5 + z_2^3), \quad \dot{z}_2 = -z_1 z_2(2 + z_2^3),$$

and the origin is the unique equilibrium point, which is degenerate, so for understanding its local phase portrait, we need to apply the directional blow-up  $(z_1, z_2) \rightarrow (z_1, w)$  with  $z_2 = w \cdot z_1$  and eliminating the common factor  $z_1$ , and we arrive to the system

$$(17) \quad \dot{z}_1 = -z_1(5 + z_1 w^3 + z_1^3 w^3), \quad \dot{w} = w(3 + z_1 w^3).$$

System (17) has the origin as its unique equilibrium and it is a saddle, the stable separatrices in the  $z_1$ -axis and the unstable ones in the  $w$ -axis. The local phase portrait at the origin of system (16) is shown in Figure 5(b). In fact, locally it has one hyperbolic sector, one attracting parabolic sector, one repeller parabolic sector and one elliptic sector.

FIGURE 5. Blow-up of the origin of the chart  $U_2$  of system (13).

From the previous study, we can give the local dynamics at all equilibria, finite and infinite, on the Poincaré disk, this is shown in Figure 6.

Summarizing, first we observe that the local phase portrait of the finite singular points, a center at  $(0,0)$  and two symmetric (with respect to the  $y$ -axis) saddles points; second we know the local phase portrait of the four infinite equilibria, an unstable node at the end of the positive  $x$ -axis, a stable node at the end of the negative  $x$ -axis (the origin of  $U_1$  and  $V_1$ , respectively), at the end of the positive  $y$ -axis locally it has one hyperbolic sector, one attracting parabolic sector and one repeller parabolic sector, at the end of the negative  $y$ -axis the dynamics consist of

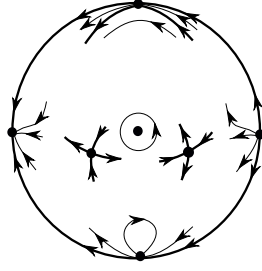


FIGURE 6. Local phase portrait for system associated to Hamiltonian function (3) when  $a = c = 0$ .

one elliptic sector, one attracting and one repeller parabolic sectors; and third the behavior of system (13) satisfies  $\dot{x}|_{y=0} = 0$ ,  $\dot{y}|_{y=0} = x$ ,  $\dot{x}|_{x=0} = -y$  and  $\dot{y}|_{x=0} = 0$ .

These three informations force that one unstable separatrix of the saddle  $e_2$  located at the region  $x > 0$  and  $y < 0$  must cross the positive  $x$ -axis because it cannot intersect the stable separatrices of itself, and such unstable separatrix must intersect the positive  $y$ -axis, at this moment it connects by symmetry with the stable separatrix of the saddle  $e_3$  with is in the region  $x < 0$  and  $y < 0$ . Analogously one stable separatrix of the saddle  $e_2$  located at the region  $x > 0$  and  $y < 0$  must cross the negative  $y$ -axis because it cannot intersect the unstable separatrices of itself, at this moment it connects with one unstable separatrix of the saddle  $e_3$  with is in the region  $x < 0$  and  $y < 0$  due to the fact that the phase portrait is symmetric with respect to the  $y$ -axis. These connections between these separatrices of the two finite saddles also can be studied using the level curves of the Hamiltonian of the system evaluated at the saddle. These connections force that the global phase portrait on the Poincaré disk of system (13) is topologically equivalent to the one of Figure 1(b).

**3.3. Case  $b = c = 0$  and  $a \neq 0$ .** The system in this case becomes

$$\dot{x} = -y - ax^4 \quad \dot{y} = x + 4ax^3y.$$

We do the  $a^{-2/3}$ -symplectic change of variables  $(x, y) \rightarrow a^{-1/3}(x, y)$  and get

$$(18) \quad \dot{x} = -y - x^4 \quad \dot{y} = x(1 + 4x^2y).$$

We study the finite equilibria of this system, in addition of the origin  $e_1$  we have two equilibria given by  $e_{2,3} = (\pm 2^{-1/3}, -2^{-4/3})$ , we know that the origin is a center, and from the linear part of system (18) have that the other two symmetric equilibria are saddles because their eigenvalues are  $\pm\sqrt{6}$ .

Now we study the infinite equilibria, from (4) the system in the local chart  $U_1$  is

$$(19) \quad \dot{z}_1 = 5z_1 + z_2^3 + z_1^2z_2^3, \quad \dot{z}_2 = z_2(1 + z_1z_2^3).$$

At infinity, i.e.  $z_2 = 0$ , system (19) has the origin as the unique equilibrium, and it is a repeller node because their eigenvalues are 5 and 1. On the other hand, from (5), the system in the local chart  $U_2$  is

$$(20) \quad \dot{z}_1 = -z_2^3 - 5z_1^4 - z_1^2z_2^3, \quad \dot{z}_2 = -z_1z_2(4z_1^2 + z_2^3),$$

and this has the origin as unique equilibrium, but it is degenerate. To analyze the behavior at the origin of  $U_2$  we need to apply the blow-up  $(z_1, z_2) \rightarrow (z_1, w)$  with  $z_2 = w \cdot z_1$ , and eliminate the common factor  $z_1^2$ , thus the system obtained is

$$(21) \quad z_1' = -z_1(5z_1 + w^3 + z_1^2 w^3), \quad w' = w(z_1 + w^3).$$

Again system (21) when  $z_1 = 0$  has the origin as the unique equilibrium which is degenerate, so we apply a second blow-up  $(z_1, w) \rightarrow (z_1, u)$  with  $w = u \cdot z_1$ , and after eliminating the common factor  $z_1$  we arrive to system

$$(22) \quad z_1' = -z_1(5 + u^3 z_1^2 + u^3 z_1^4), \quad u' = u(6 + 2u^3 z_1^2 + u^3 z_1^4).$$

At  $z_1 = 0$  system (22) has the unique equilibrium  $(0, 0)$  and it is a saddle, with the stable separatrices over the axis  $z_1$ . Figure 7 shown the local phase portrait at the origin of system (20), for obtaining this phase portrait we also have used the behavior of system (21) and (20) on the axes of coordinates.

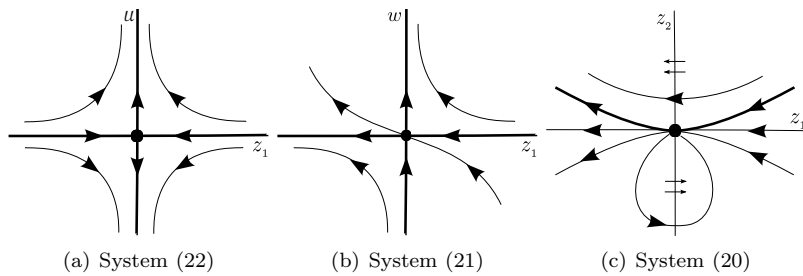


FIGURE 7. Blow-up of the origin of the chart  $U_2$  of system (18).

In summary we observe that the local phase portrait of the finite and infinite singular points are the same as in the case  $a = c = 0$  (Figure 6). The behavior of system (18) on the axes is the following,  $\dot{x}|_{y=0} = -x^4$ ,  $\dot{y}|_{y=0} = x$  and  $\dot{x}|_{x=0} = -y$ ,  $\dot{y}|_{x=0} = 0$ . Therefore, the flow must cross the  $y$ -axis orthogonally, and cross transversally the  $x$ -axis, except at the origin of coordinates. Thus, we can conclude that in this case the global phase portrait is topologically equivalent with the case  $a = c = 0$  (see Figure 1(b)).

**3.4. Case  $a = 0$  and  $bc \neq 0$ .** Here the associated system to Hamiltonian (3) is

$$\dot{x} = -y - 3bx^2y^2 - 5cy^4, \quad \dot{y} = x + 2bxy^3.$$

Since  $b \neq 0$ , without loss of generality, we can do the  $b^{-2/3}$ -symplectic change of variables  $(x, y) \mapsto b^{-1/3}(x, y)$  and we obtain the system

$$(23) \quad \dot{x} = -y - 3x^2y^2 - 5\tilde{c}y^4, \quad \dot{y} = x + 2xy^3,$$

with  $\tilde{c} = 5c/b$ , for simplicity we just use again the letter  $c$ .

The number of equilibria of system (13) depend of  $c$ , more precisely:

(i) If  $c > 2/5$ , there are only two finite equilibria,  $e_1 = (0, 0)$  and  $e_2 = (0, -(5c)^{-1/3})$ . The origin is a center as we know, and  $e_2$  is a saddle because the eigenvalues are  $\pm\sqrt{3(5c-2)/5} c^{-1/2}$ .

(ii) If  $c = 2/5$ , again there are only two finite equilibria,  $e_1 = (0, 0)$  and  $\tilde{e}_2 = (0, -2^{-1/3})$ , but now  $e_2$  is nilpotent. In order to understanding the local behavior

at  $\tilde{e}_2$  we apply Theorem 3.5 of [9] for the nilpotent equilibria. Shifting to the origin the equilibrium point  $\tilde{e}_2$  through the change of coordinates  $(x, y) \rightarrow (x, y - 2^{-1/3})$ , and rescaling the time by 1/3, system (23) takes the form

$$(24) \quad \begin{aligned} x' &= y - 2^{-2/3}x^2 - 2^{5/3}y^2 + 2^{2/3}x^2y + (2^{8/3}/3)y^3 - x^2y^2 - (2/3)y^4, \\ y' &= 2^{1/3}xy - 2^{2/3}xy^2 + (2/3)xy^3. \end{aligned}$$

Thus, by Theorem 3.5 of [9], with  $y = f(x) = 2^{-2/3}x^2 + O(x^3)$  in a neighbourhood of the point  $(0, 0)$ ,  $F(x) = 2^{-1/3}x^3 + O(x^5)$  and  $G(x) \equiv 0$ . Therefore, using the notation of Theorem 3.5 we have  $m = 3$  and  $a = 2^{-1/3}$ . So the origin of (24) is a saddle. In conclusion  $\tilde{e}_2$  is a saddle point of system (23).

(iii) If  $c < 2/5$ , there are four finite equilibria, namely the origin,  $e_2$  and  $e_{3,4} = (\pm 2^{-1/3}3^{-1/2}(2 - 5c)^{1/2}, -2^{-1/3})$ . The equilibrium  $e_2$  is a saddle if  $c < 0$  and is a center if  $0 < c < 2/5$ . The equilibrium points  $e_3$  and  $e_4$  are saddle because their eigenvalues are  $\pm\sqrt{3(2 - 5c)}$ . Thus we have concluded the local study of the finite equilibria of system (23).

Now we are going to study the infinite equilibria using the local charts given by the Poincaré compactification. From (4) we have that the system in  $U_1$  is given by

$$(25) \quad \dot{z}_1 = 5z_1^3 + 5cz_1^5 + z_2^3 + z_1^2z_2^3, \quad \dot{z}_2 = 3z_1^2z_2 + 5cz_1^4z_2 + z_1z_2^4.$$

At infinity, i.e.,  $z_2 = 0$ , when  $c > 0$  there exists only the origin as equilibrium point. If  $c < 0$  there are three equilibria, the origin and  $(\pm(-c)^{-1/2}, 0)$ . In any case, the origin is degenerate, so we need to apply a blow-up for understanding its local phase portrait. Doing the blow-up  $(z_1, z_2) \rightarrow (z_1, w)$  with  $w = z_2/z_1$  and eliminating the common factor  $z_1^2$  we get the system

$$(26) \quad \dot{z}_1 = z_1(5 + 5cz_1^2 + w^3 + w^3z_1^2), \quad w' = w(2 + w^3).$$

System (26) has two hyperbolic equilibria on  $z_1 = 0$ , the origin which is a saddle (stable separatrices on the  $w$ -axis and unstable separatrices on the  $z_1$ -axis), and the equilibrium  $(0, -2^{1/3})$  which is a repeller node (see Figure 8(a)). Now, we must go back through the change of variables starting at system (26) and reaching system (25). Undoing the blow-up and taking into account that for system (25) we have that  $\dot{z}_1|_{z_1=0} = z_2^3$  and  $\dot{z}_2|_{z_1=0} = 0$ , we obtain the local phase portrait at the origin of system (25), which topologically is an unstable node, see Figure 8(b).

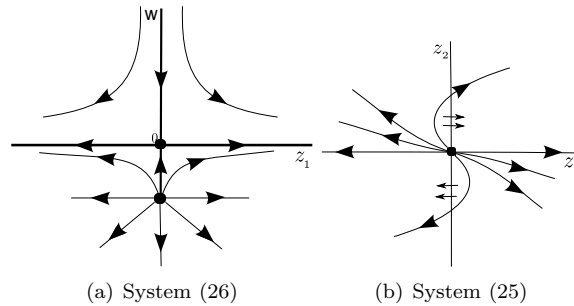


FIGURE 8. Blow-up of the origin of the chart  $U_1$  of system (23).

The equilibria  $(\pm(-c)^{-1/2}, 0)$ , have eigenvalues  $10/c$  and  $2/c$ , so they are stable nodes, since  $c < 0$ .

In the local chart  $V_1$  we have the same analysis as in the chart  $U_1$ , except that the orientations of the orbits is the converse.

From (5) system (23) in the local chart  $U_2$  is

$$\dot{z}_1 = -5c - 5z_1^2 - z_2^3 - z_1^2 z_2^3, \quad \dot{z}_2 = -z_1 z_2 (2 + z_2^3),$$

where clearly the origin of  $U_2$  is not an equilibrium.

From the previous analysis we know the local dynamics at the all equilibria on the Poincaré disk, depending on  $c$ , the local dynamics of system (23) at its equilibria is topologically equivalent to the one shown in Figure 9.

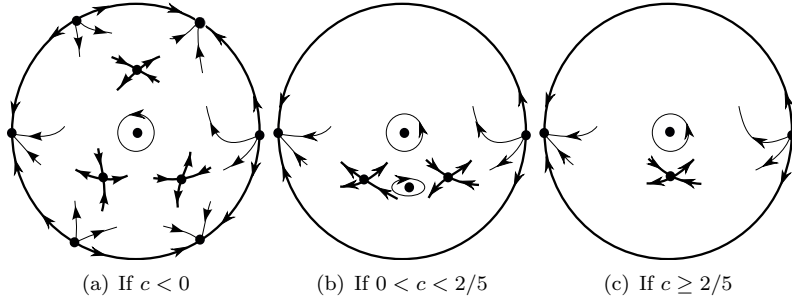


FIGURE 9. Local phase portraits for system (23) when  $a = 0$  and  $bc \neq 0$ .

Now we will determine the global phase portrait according to this local information, using the symmetry and the connections between the equilibria.

In the cases  $c > 2/5$  and  $c = 2/5$ , first we remember that in both cases there exist two finite equilibria, a center at the origin and a saddle at  $(0, -(5c)^{-1/3})$ ; second at the infinite there exist an unstable node at the end of the positive  $x$ -axis and a stable node at the end of the negative  $x$ -axis; and third the behavior of system (23) on the axes is  $\dot{x}|_{x=0} = -y(1 + 5cy^3)$ ,  $\dot{y}|_{x=0} = 0$ ,  $\dot{x}|_{y=0} = 0$  and  $\dot{y}|_{y=0} = x$ . Thus, following the same ideas as in the cases  $a = b = 0$  we arrive that the global phase portrait of system (23) with  $c \geq 2/5$  is topologically equivalent to Figure 1(a).

Next we consider the case  $0 < c < 2/5$ , first we know that there exist four finite equilibria, two centers, one at the origin and the other at  $e_2 = (0, -(5c)^{-1/3})$ , and two saddle located at  $e_{3,4} = (\pm 2^{-1/3} 3^{-1/2} \sqrt{2 - 5c}, -2^{-1/3})$ ; second at the infinite there exist an unstable node at the end of the positive  $x$ -axis and a stable node at the end of the negative  $x$ -axis; and third the behavior of system (23) on the axes is  $\dot{x}|_{x=0} = -y(1 + 5cy^3)$ ,  $\dot{y}|_{x=0} = 0$ ,  $\dot{x}|_{y=0} = 0$  and  $\dot{y}|_{y=0} = x$ .

These three informations force that one stable separatrix of the saddle  $e_3$  located at the region  $x > 0$  and  $y < 0$  must cross the negative  $y$ -axis between the center because it cannot intersect neither the unstable separatrices of itself nor the  $x$ -axis, and by symmetry such stable separatrix must connect with one unstable separatrix of the saddle  $e_4$  which is in the region  $x < 0$  and  $y < 0$ . Using the same argument we arrive that the separatrices associated to one stable separatrix and one unstable

separatrix of  $e_3$  and  $e_4$  form the boundary of the period annulus of the center  $e_2$ . Now, we take the other unstable separatrix of  $e_3$ , it must cross the positive  $x$ -axis because it cannot intersect the stable separatrices of itself, then by continuity of the flow it also must cross the positive  $y$ -axis, at this moment it connects with one stable separatrix of the saddle  $e_4$  due to the fact that the phase portrait is symmetric with respect to the  $y$ -axis. These connections force that the global phase portrait on the Poincaré disk of system (23) is topology equivalent to the one of Figure 1(c). Again the connections of the separatrices of the saddles can be studied using the level curves of the Hamiltonian.

Finally, in the case  $c < 0$ , first we remember that there exist four finite equilibria, a center at the origin and three saddles at  $e_2 = (0, -(5c)^{-1/3})$  (on the positive  $y$ -axis),  $e_{3,4} = (\pm 2^{-1/3} 3^{-1/2} \sqrt{2-5c}, -2^{-1/3})$  (on the fourth and third quadrant respectively); second at the infinite there exist six equilibria, an unstable node at the end of the positive  $x$ -axis and a stable node at the end of the negative  $x$ -axis, and other four nodes, one stable in  $U_1 \cap U_2$  and other in  $U_1 \cap V_2$ , and an unstable node in  $U_2 \cap V_1$  and  $V_2 \cap V_1$ ; an third the behavior of the system (23) on the axes is  $\dot{x}|_{x=0} = -y(1 + 5cy^3)$ ,  $\dot{y}|_{x=0} = 0$ ,  $\dot{x}|_{y=0} = 0$  and  $\dot{y}|_{y=0} = x$ .

These previous informations force that one stable separatrix of the saddle  $e_3$  must cross the negative  $y$ -axis because it neither intersects the unstable separatrices of itself nor the positive  $x$ -axis, it connects by symmetry with the unstable separatrix of the saddle  $e_4$ . The other stable separatrix of  $e_3$  must connect with an unstable orbit of the infinite equilibrium point located at the end of the positive  $x$ -axis. By symmetry the analogous happens with the other unstable separatrix of  $e_4$ .

Now, we are going to study the possible connections of the unstable separatrices of the saddle  $e_3$ . Defining the energy level associated to the equilibria  $e_2$ ,  $e_3$  (or  $e_4$ ) as  $h_2 = 3 \cdot 5^{-5/3} c^{-2/3} / 2$  and  $h_3 = (1 - c) / 2^{5/3}$ , respectively, and let  $\hat{c} = (11 - 4 \cdot 10^{1/3} - 10^{2/3}) / 15$  the value of  $c$  when  $h_2$  coincide with  $h_3$ . We note that  $h_2 < h_3$  if  $c < \hat{c}$ ,  $h_2 = h_3$  if  $c = \hat{c}$ , and  $h_2 > h_3$  if  $\hat{c} < c < 0$ . On the other hand, the equation  $H|_{y=0} = x^2/2 = h > 0$  has two solutions for any  $h > 0$ , and  $H|_{x=0} = cy^5 + y^2/2 = h$ , then applying the Descartes's rule when  $h > 0$  we obtain that  $H|_{x=0}$  has exactly one negative root and has two or zero positive roots.

In the case  $c < \hat{c}$ , for  $h = h_3$  there are no crossings of the level curves  $H = h_3$  with the positive  $y$ -axis, and for  $h = h_2$  there is exactly one point of the level curve  $H = h_2$  on the positive  $y$ -axis which is  $e_2$ . Thus, there exists a homoclinic orbit of  $e_2$  surrounding the origin. With these informations we conclude that the global phase portrait of system (23) when  $c < \hat{c}$  is topologically equivalent to the one of Figure 1(d).

In the case  $c = \hat{c}$  (bifurcation value), for  $h = h_2 = h_3$  there is one crossing point of the level curves  $H = h_2 = h_3$  on the positive  $y$ -axis and it is  $e_2$ . Thus, there exist heteroclinic orbits connecting  $e_2$ ,  $e_3$  and  $e_4$ . Using the previous analysis we are able to complete the global phase portrait of system (23) when  $c = \hat{c}$ , which is topologically equivalent to the one of Figure 1(e).

If  $\hat{c} < c < 0$ , then for  $h = h_3$  there are two crossing points of the level curves  $H = h_3$  with the positive  $y$ -axis, one of them is located between the center and the saddle  $e_2$ , and the other is located between  $e_2$  and the end of the positive  $y$ -axis, and for  $h = h_2$ , as in the case  $c < \hat{c}$ , there is exactly one point (with multiplicity 2) of the level curve  $H = h_2$  on the positive  $y$ -axis which is  $e_2$ . Thus,

there exists two heteroclinic orbits connecting  $e_3$  and  $e_4$  crossing the positive  $y$ -axis (both heteroclinic orbits connecting  $e_3$  and  $e_4$  are forming the boundary of the period annulus of the center). So the global phase portrait of system (23) in this case is topologically equivalent to the one shown in Figure 1(f).

**3.5. Case  $b = 0$  and  $ac \neq 0$ .** Under these conditions we have that the system is

$$\begin{aligned}\dot{x} &= -y - ax^4 - 5cy^4, \\ \dot{y} &= x + 4ax^3y.\end{aligned}$$

We do the  $a^{-2/3}$ -symplectic change of variables  $(x, y) \rightarrow a^{-1/3}(x, y)$  and we arrive to system

$$(27) \quad \dot{x} = -y - x^4 - 5\tilde{c}y^4, \quad \dot{y} = x + 4x^3y,$$

where  $\tilde{c} = c/a$ , by simplicity we just use again  $c$  instead of  $\tilde{c}$ .

System (27) has always two equilibria  $e_1 = (0, 0)$  (center) and  $e_2 = (0, -(5c)^{-1/3})$  (which is a saddle because its eigenvalues are  $\pm\sqrt{3}$ ). the other equilibria are given by the real roots of the polynomial  $256x^{12} - 64x^6 + 5c = 0$  with  $y = -1/(4x^2)$ . We verify that:

(i) If  $c > 4/5$  the previous polynomial do not have real solutions.

(ii) If  $c = 4/5$ , then there are two additional equilibria given by  $\tilde{e}_{3,4} = (\pm\sqrt{1/2}, -1/2)$ , both are nilpotent. We shall study their local phase portraits later.

(iii) If  $0 < c < 4/5$ , then there are four additional equilibria given by

$$e_{3,4} = \left( \pm \frac{\sqrt[6]{2 + \sqrt{4 - 5c}}}{2^{2/3}}, -\frac{1}{2^{2/3}(2 + \sqrt{4 - 5c})^{1/3}} \right),$$

and

$$e_{5,6} = \left( \pm \frac{\sqrt[6]{(2 - \sqrt{4 - 5c})}}{2^{2/3}}, -\frac{1}{2^{2/3}(2 - \sqrt{4 - 5c})^{1/3}} \right).$$

The equilibria  $e_{3,4}$  are saddles and  $e_{5,6}$  are centers.

(iv) If  $c < 0$  there exist (in addition to the origin and  $(0, -(5c)^{-1/3})$ ) two equilibria,  $e_{3,4}$  (the same that in the previous case (iii)) which are saddles because their eigenvalues are  $\pm(\sqrt{3}(-5c\sqrt{4 - 5c} - 20c + 8\sqrt{4 - 5c} + 16)^{1/2})(\sqrt{4 - 5c} + 2)^{-1}$ .

In order to complete the local study of the finite equilibria we only need to analyse the nilpotent equilibrium points  $\tilde{e}_{3,4} = (\pm\sqrt{2}, -1/2)$  for  $c = 4/5$ . By virtue of the symmetry we only study the equilibrium  $\tilde{e}_3$ , we shall apply of Theorem 3.5 [9]. Shifting the equilibrium to the origin, through the change of variables  $(x, y) \rightarrow (x + \sqrt{2}, y - 1/2)$ , and then doing the change of variables for writting the linear part of the system into its real Jordan normal  $(u, v) = (2^{-1/2}x - 2^{-1}y, x)$ , system (27) takes the form

$$\begin{aligned}\dot{u} &= v + \frac{3\sqrt{2}}{4} (2u^2 - v^2) + O(\|(u, v)\|^3), \\ \dot{v} &= 18u^2 - 3\sqrt{2}uv + O(\|(u, v)\|^3).\end{aligned}$$

Therefore using the notation of Theorem 3.5 of [9], with  $v = f(u) = -3 \cdot 2^{-1/2}u^2 + O(u^3)$  in a neighbourhood of the point  $(0, 0)$ ,  $F(u) = 18u^2 + O(u^3)$  and  $G(u) \equiv 0$ , we have  $m = 2$  and  $a = 18$ , so the origin of system (3.5) is a cusp. In conclusion  $e_3$  and  $e_4$  are cusp points of system (27).



Now we do the study of the infinite equilibria, from (4) the system in the local chart  $U_1$  is

$$(28) \quad z_1 = 5z_1 + z_2^3 + z_1^2 z_2^3 + 5cz_1^5, \quad z_2 = 5z_1(1 + cz_1^4).$$

At infinity, that is at  $z_2 = 0$ , for system (28) we have:

(i) If  $c > 0$  then the origin is the unique equilibrium, which is a repeller node, because their eigenvalues are 1 and 5.

(ii) If  $c < 0$  there are three equilibria, namely, the origin and  $(\pm(-1/c)^{1/4}, 0)$ . As previously the origin is a repeller node, and the other are attractor nodes, because their eigenvalues are  $-4$  and  $-20$ .

It is clear that in the chart  $V_1$  the origin is an attractor node and the others equilibria in the case  $c < 0$  are repeller nodes, due the fact that the vector field  $p(\mathcal{X})$  in both charts are equal except that the orientation of the orbits is the converse. The computations on the local chart  $U_2$  show that there are no more equilibria.

In summary:

(I) If  $c > 4/5$  the local phase portrait at all equilibrium points on the Poincaré disk is shown in Figure 10(d).

(II) If  $c = 4/5$  the local phase portrait on the Poincaré disk is shown in Figure 10(c).

(III) If  $0 < c < 4/5$  the local phase portrait on the Poincaré disk is shown in Figure 10(b).

(IV) If  $c < 0$  the local phase portrait on the Poincaré disk is shown in Figure 10(a).

All the possible local phase portraits for system (2) when  $b = 0$  and  $ac \neq 0$  are topologically equivalent to the one in Figure 10, and depend on  $c$ , where  $c = 0$  and  $c = 4/5$  are bifurcation values.

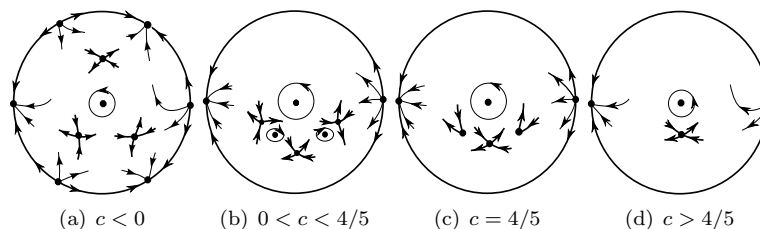


FIGURE 10. Phase portraits for system (2) when  $b = 0$  and  $ac \neq 0$ .

The analysis of the global flow in the case  $c > 4/5$  proceed in a similar way as in the cases  $a = b = 0$ , or  $a = 0$  when  $c \geq 2/5$ , thus in Figure 1(a) is represented the global flow.

For the case  $c = 4/5$ , first we note that  $\dot{x}|_{x=0} = -y(1 + 4y^3)$ ,  $\dot{y}|_{x=0} = 0$ ,  $\dot{x}|_{y=0} = -x^4$  and  $\dot{y}|_{y=0} = x$ , this information with the local dynamics forces that the unstable separatrix of  $e_2$  which is in the region  $x > 0$  and  $y < 0$  must cross the positive  $x$ -axis and then must cross the positive  $y$ -axis connecting, by the symmetry, with the stable separatrix of  $e_2$  located in the region  $x < 0$  and  $y < 0$ . Furthermore, the unstable separatrix of the cusp  $\tilde{e}_3$  located in the region  $x > 0$  and  $y < 0$  must also cross the positive  $x$ -axis and then the positive  $y$ -axis connecting with the

stable separatrix of the other cusp  $e_4$ . These connections force that the global phase portrait on the Poincaré disk is as in Figure 1(g).

For the case  $0 < c < 4/5$  we have  $\dot{x}|_{x=0} = -y(1+5cy^3)$ ,  $\dot{y}|_{x=0} = 0$ ,  $\dot{x}|_{y=0} = -x^4$  and  $\dot{y}|_{y=0} = x$ . In order to complete the global dynamics we need to know the connections of the unstable separatrix of the saddles  $e_3$  and  $e_2$ . One unstable separatrix of  $e_3$  must cross transversally the positive  $x$ -axis because it cannot intersect neither a stable separatrix of itself nor the unstable separatrix of  $e_2$ , and then by continuity must cross orthogonally the positive  $y$ -axis, at this moment it connects with one stable separatrix of  $e_4$ . Now we are going to study the possible connections of the unstable separatrix of  $e_2$  and one stable separatrix of  $e_3$ . For this we note that the energy level of  $e_2$  is  $h_2 = 3/10(5c)^{-2/3}$ , and  $e_3$  and  $e_4$  are in the same energy level  $h_3 = 3(-c + \sqrt{4-5c} + 2)(2\sqrt{4-5c} + 2)^{-5/3}$ .

The levels  $h_3$  and  $h_2$  of the saddles coincide for the bifurcation value

$$c_+ = \frac{1}{60} \left( \sqrt{48\delta + 225} + \sqrt{6 \left( -8\delta + 4215 \sqrt{\frac{3}{16\delta + 75}} + 75 \right) - 45} \right),$$

where  $\delta = (8775 + 345\sqrt{345})^{1/3} + (8775 - 345\sqrt{345})^{1/3}$ . Thus, we have three possible global dynamics for this case ( $0 < c < 4/5$ ):

(i) If  $0 < c < c_+$  one stable separatrix of  $e_3$  connects with one unstable separatrix of  $e_4$  forming the boundary of the periodic annulus of the origin. With the previous information we obtain that the global dynamics in this case is topologically equivalent to the one of Figure 1(h).

(ii) If  $c = c_+$  the separatrices of  $e_2$  located in the region  $x > 0$  and  $y < 0$  connect with the ones of  $e_3$  enclosing the center at  $e_5$  in the same region, for symmetry the same happens with  $e_6$  and  $e_4$ . In view of this information that we obtain that the global dynamics in this case is topologically equivalent to the one of Figure 1(i).

(iii) If  $c_+ < c < 4/5$  one stable separatrix of  $e_3$  is a homoclinic orbit, forming the boundary of the periodic annulus of the center  $e_5$ , connecting with one unstable separatrix of itself. Thus, we conclude that the global dynamics in this case is topologically equivalent to the one of Figure 1(j).

The study of the global flow in the case  $c < 0$  is similar to the case  $a = 0$  and  $c < 0$ , where the bifurcation value of  $c$  now is

$$c_- = \frac{1}{60} \left( \sqrt{48\delta + 225} - \sqrt{6(-8\delta + 4215 \sqrt{\frac{3}{16\delta + 75}} + 75) - 45} \right),$$

thus the global phase portrait if  $c < c_-$  is topologically equivalent to Figure 1(d), if  $c = c_-$  is topologically equivalent to Figure 1(e), and if  $c_- < c < 0$  is topologically equivalent to Figure 1(f). Thus, we have concluded the study in this case.

**3.6. Case  $c = 0$  and  $ab \neq 0$ .** In this case through the  $a^{-2/3}$ -symplectic change of variables  $(x, y) \rightarrow a^{-1/3}(x, y)$  we can assume without loss of generality that  $a = 1$  and the new Hamiltonian system is

$$(29) \quad \dot{x} = -x^4 - y - 3bx^2y^2, \quad \dot{y} = x(1 + 4x^2y + 2by^3).$$

For the equilibrium points of system (29) we have from the second the equation that  $x = 0$  or  $1 + 4x^2y + 2by^3 = 0$ . In the first case the origin is the unique equilibrium

(center) and in the second case substituting this relation in the first equation of (29) we have that the equilibrium point  $(x, y)$  must satisfy  $20b^2y^6 + 4(2b - 4)y^3 - 1 = 0$  and  $x = \pm\sqrt{-(1 + 2y^3)/(4y)}$ . Note that the discriminant of the sixth polynomial in  $y$  is always positive. After some manipulations we arrive to:

(i) If  $b > 0$  there are three finite equilibria for system (29), namely  $e_1 = (0, 0)$ ,

$$e_{2,3} = \left( \pm \frac{\sqrt{3b - \sqrt{\alpha} + 4}}{2^{5/6} \sqrt[3]{5}(b(2b + \sqrt{\alpha} - 4))^{1/6}}, \sqrt[3]{\frac{-\sqrt{\alpha} - 2b + 4}{b^2}} \right),$$

where  $\alpha = 9b^2 - 16b + 16$ . The points  $e_2$  and  $e_3$  are of the saddle type.

(ii) If  $b < 0$  there are five finite equilibria for system (29), namely  $e_1 = (0, 0)$ ,  $e_2$ ,  $e_3$  and

$$e_{4,5} = \left( \pm \frac{\sqrt{3b + \sqrt{\alpha} + 4}}{2^{5/6} \sqrt[3]{5}(-b(-2b + \sqrt{\alpha} + 4))^{1/6}}, \sqrt[3]{\frac{\sqrt{\alpha} - 2b + 4}{10b^2}} \right).$$

The points  $e_4$  and  $e_5$  are of the saddle type.

Now we are going to study the infinite equilibria. From (4) system (29) in the local chart  $U_1$  is

$$\dot{z}_1 = 5z_1 + 5bz_1^3 + z_2^3 + z_1^2z_2^3, \quad \dot{z}_2 = z_2 + 3bz_1^2z_2 + z_1z_2^4,$$

At infinite, i.e. at  $z_2 = 0$ , we have two cases: if  $b > 0$  the origin is the unique equilibrium which is a repeller node, because its eigenvalues are 1 and 5, if  $b < 0$  we have three equilibria, the origin with the same local phase portrait as before, and other two equilibria given by  $(\pm(-b)^{-1/2}, 0)$  whose eigenvalues are  $-10$  and  $-2$ , so they are attractor nodes. Therefore, in the local chart  $V_1$  also the origin is a infinite singular point which is an attractor node and there exist other two infinite singular points that are repeller nodes, due to the fact that the vector field  $p(\mathcal{X})$  in both charts are equal except that their sign are opposite.

In the chart  $U_2$ , from (5) the sistem (29) is given by

$$(30) \quad \dot{z}_1 = -5bz_1^2 - z_2^3 - 5z_1^4 - z_1^2z_2^3, \quad \dot{z}_2 = -z_1z_2(2b + 4z_1^2 + z_2^3).$$

Here, the origin if it is an equilibrium for system (30), but it is degenerate. We apply the directional blow-up  $(z_1, z_2) \rightarrow (z_1, w)$  with  $z_2 = w \cdot z_1$ , and eliminating the common factor  $z_1$ , we get

$$(31) \quad \dot{z}'_1 = -z_1(5b + 5z_1^2 + w^3z_1 + w^3z_1^3), \quad w' = w(3b + z_1^2 + w^3z_1).$$

This system has for  $z_1 = 0$  the origin as unique equilibrium, which is hyperbolic (type saddle) with eigenvalues associated equals to  $-5b$  and  $3b$ . Note that if  $b > 0$  (respectively,  $b < 0$ ) the stable separatrix is on the  $z_1$ -axis ( $w$ -axis) and the unstable separatrix is on the  $w$ -axis ( $z_1$ -axis). So it is a saddle with stable and unstable separatrices depending of the sing of  $b$ . The local phase portrait of the origin of system (30) is shown in Figure 11(b) and consists, in one hyperbolic sector, one elliptic sector, one attracting and one repeller parabolic sectors. If  $b < 0$  see the local local phase portrait of the origin of  $U_2$  in Figure 11(b), and if  $b > 0$  in Figure 12(b).

In summary, if  $b > 0$  there exists two saddles and the center at the origin as finite equilibria, and the infinite equilibria are a repeller node at the end of the positive  $x$ -axis, and an attractor node at the end of the negative  $x$ -axis.

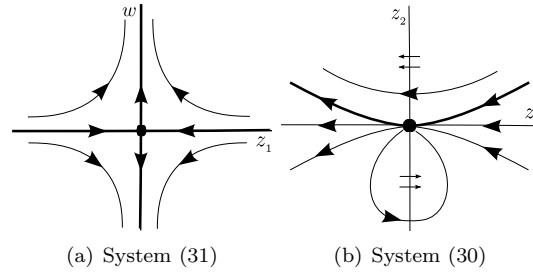


FIGURE 11. Blow-up of the origin of the chart  $U_2$  of (30) when  $c = 0$  and  $b > 0$ .

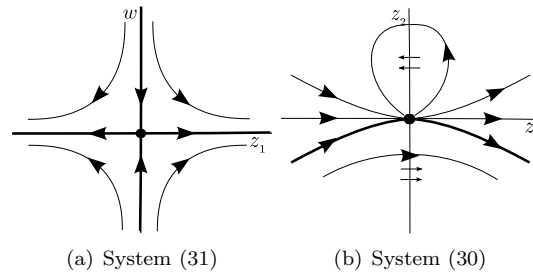


FIGURE 12. Blow-up of the origin of the chart  $U_2$  of (30) when  $c = 0$  and  $b < 0$ .

If  $b < 0$  we have that the finite equilibria are the center at the origin and four saddles, each one located in one different quadrant, and the infinite equilibria are eight, as before the ends of the  $x$ -axis have a repeller and an attractor node, furthermore there are two stable nodes in the chart  $U_1$  and two unstable nodes in the chart  $V_1$ , the local phase portraits at the end of the positive and negative  $y$ -axis is given by the origin of  $U_2$  and  $V_2$  (see Figures 11(b) and 12(b)). All the possible local phase portraits for system (3) when  $c = 0$  and  $ab \neq 0$  are topologically equivalent to the one shown in Figure 13, and depends on the sign of  $b$ .

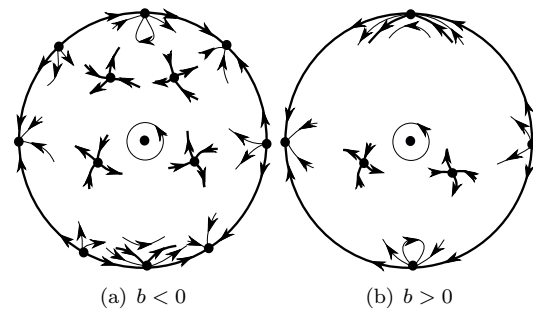


FIGURE 13. Phase portraits for system (2) when  $c = 0$  and  $ab \neq 0$ .

Since  $\dot{x}|_{x=0} = -y$ ,  $\dot{y}|_{x=0} = 0$ ,  $\dot{x}|_{y=0} = 0$  and  $\dot{y}|_{y=0} = x$ , the study of the global dynamics for the case  $b > 0$  follows in the same way that for the analysis made for the case  $a = c = 0$ , and the global phase portrait on the Poincaré disk is topologically equivalent to the one of Figure 1(b).

If  $b < 0$ , we note that the energy level of the saddle are

$$h_2 = H(e_{2,3}) = \frac{3(b(-3b + \alpha - 8) - 2\alpha + 8)}{20 \cdot 10^{2/3} b^2} \left( -\frac{2b + \alpha - 4}{b^2} \right)^{1/3},$$

and

$$h_4 = H(e_{4,5}) = \frac{3(3b^2 + (\alpha + 8)b - 2(\alpha + 4))}{20 \cdot 10^{2/3} b^2} \left( \frac{\alpha - 2b + 4}{b^2} \right)^{1/3},$$

where  $\alpha = \sqrt{9b^2 - 16b + 16}$ . Here there exists a bifurcation value  $\hat{b} = -1/(2(1 + \sqrt{5}))$  where the four saddles are in the same level of energy. We obtain that one stable separatrix of the saddle  $e_2$  located at the quadrant  $x > 0$  and  $y < 0$  must cross the negative  $y$ -axis because it cannot intersect neither the stable separatrices of itself, nor the positive  $x$ -axis, at this moment it connects with the unstable separatrix of the saddle  $e_3$ , that is in the quadrant  $x < 0$  and  $y < 0$  due to the fact that the phase portrait is the symmetric with respect to the  $y$ -axis. The same behavior happens with one unstable separatrix of  $e_4$  and one stable separatrix of  $e_5$ .

By the previous analysis we must divide the study of the dynamics in three cases:

(i) If  $b < \hat{b}$  it holds  $0 < h_4 < h_2$  so the saddles  $e_4$  and  $e_5$  are in the boundary of the period annulus of the origin and we can force that the phase portrait of system (29) in this case is topologically equivalent to the one Figure 1(k).

(ii) If  $b = \hat{b}$  the four saddles are connected through their separatrices forming the boundary of the period annulus of the center at the origin, thus the global dynamics is as the one of Figure 1(l).

(iii) If  $b > \hat{b}$  it holds  $0 < h_2 < h_4$ , which implies that the saddles  $e_2$  and  $e_3$  form the boundary of the period annulus of the center at the origin and we can complete the global dynamics as it is shown Figure 1(m).

We can conclude that in this case  $c = 0$  and  $ab \neq 0$  the global phase portrait is topologically equivalent to the ones of Figure 1(b), (k), (l) or (m).

#### 4. GENERAL CASE $abc \neq 0$

For the general case we assume that all the parameters are nonzero. Through the  $a^{-2/3}$ -symplectic change of variables  $(x, y) \rightarrow a^{-1/3}(x, y)$  we can suppose without loss of generality that  $a = 1$  and the new Hamiltonian system is

$$(32) \quad \dot{x} = -y - x^4 - 3\tilde{b}x^2y^2 - 5\tilde{c}y^4, \quad \dot{y} = x(1 + 4x^2y + 2\tilde{b}y^3),$$

where  $\tilde{b} = b/a$  and  $\tilde{c} = c/a$ , for simplicity we just use  $b$  and  $c$  instead of  $\tilde{b}$  and  $\tilde{c}$  respectively.

The equilibrium points of system (32), according to second equation (32) are characterized by  $x = 0$  or  $1 + 4x^2y + 2by^3 = 0$ . In the first case, by the first equation implies that there are two equilibria, namely,  $e_1 = (0, 0)$  that we know is always a center, and  $(0, (5c)^{-1/3})$ , which is a saddle if  $c < 0$  and  $b > 5c/2$  or  $c > 0$  and  $b < 5c/2$ , a center for  $c < 0$  and  $b < 5c/2$  or  $c > 0$  and  $b > 5c/2$ ; and it is nilpotent for  $b = 5c/2$ . In order to study the local behavior at  $e_2$  when  $b = 5c/2$  we

apply Theorem 3.5 of [9] for nilpotent equilibria. First, we move  $e_2$  to the origin by the shift  $(x, y) \rightarrow (x, y - 2^{-1/3})$ , and rescaling the time by  $1/3$ , we get that system (32) takes the form

$$\begin{aligned} x' &= y - ((5c)^{1/3}/2)x^2 - 2(5c)^{1/3}y^2 + (5c)^{1/3}x^2y + (2(5c)^{1/3}/3)y^3 - (1/3)x^4 - \\ &\quad (5c/2)x^2y^2 - (5c/3)y^4, \\ y' &= (5c)^{1/3}xy - 4/(3(5c)^{1/3})x^3 - (5c)^{1/3}xy^2 + (4/3)x^3y + (5c/3)xy^3. \end{aligned}$$

According to Theorem 3.5 of [9], using the notations of that theorem we have  $y = f(x) = ((5c)^{1/3}/2)x^2 + (x^4/3) + O(x^5)$  in a neighbourhood of the point  $(0, 0)$ ,  $F(x) = (15c - 8)/(6(5c)^{1/3})x^3 + ((5c)^{1/3}(4 - 5c)/4)x^5 + O(x^7)$  and  $G(x) \equiv 0$ . If  $c \neq 8/15$  we have that  $m = 3$  and  $a = (15c - 8)/(6(5c)^{1/3})$ , in the case  $c = 8/15$  the value of  $m$  is 5 and  $a = 4/3$ . Therefore, by Theorem 3.5 of [9] if  $c > 8/15$ , or  $c < 0$  we have that  $a > 0$  and then  $e_2$  is a saddle. If  $0 < c < 8/15$  then  $a < 0$ , thus  $e_2$  is a center or a focus, but since our system is Hamiltonian it is a center. For  $c = 8/15$  the equilibrium is a saddle.

The equilibria in the second case, i.e. when  $1 + 4x^2y + 2by^3 = 0$ , must satisfy the polynomial

$$(33) \quad 20(b^2 - 4c)u^2 + 8(b - 2)u - 1 = 0,$$

where  $u = y^3$  and

$$x = \pm \sqrt{\frac{-(1 + 2by^3)}{4y}},$$

where the radical quantity is not negative.

**(I)** First case:  $b^2 - 4c = 0$

**(I.1)** If  $b = 2$  or  $b \geq 8/5$  then there are no more equilibria.

**(I.2)** If  $b \neq 2$  and  $b < 8/5$ , then there are two more equilibria, namely,  $e_{3,4} = (\pm \sqrt{(8 - 5b)(-2 + b)^{-2/3}/8}, (8(b - 2)))^{-1/3}$ . They are saddle, because their eigenvalues are  $\pm \frac{1}{2} \sqrt{3(8 - 5b)}$ .

**(II)** Second case:  $b^2 - 4c \neq 0$ . Let  $\Delta = 16(\alpha - 20c) = 16(9b^2 - 16b - 20c + 16)$  be the discriminant of the quadratic polynomial (33).

**(II.1)** If  $\Delta < 0$  there are no more equilibria.

**(II.2)** If  $\Delta = 0$ , or  $c = c^* = \frac{1}{20}(\alpha)$ , we have two subcases.

**(II.2.1)** If  $b \geq 4/3$  there are not more equilibria.

**(II.2.2)** If  $b < 4/3$ , there exist two more equilibria, namely,

$$e_{3,4} = \left( \pm \frac{\sqrt{4 - 3b}}{27^{1/6}(b - 2)^{1/3}}, \frac{1}{4(b - 2)} \right),$$

with eigenvalues both zero, but with linear matrix not null, so it is nilpotent. After doing the computations for studying this case using Theorem 3.5 in [9] we arrive that  $e_{3,4}$  are cusp (under the notation of Theorem 3.5 we have that  $F(x) = \hat{a}x^2$  with  $\hat{a} = (9\sqrt[3]{b - 2}(3b - 4))/(2^{4/3}) < 0$  and  $G(x) = 0$ ).

**(II.3)** If  $\Delta > 0$ , i.e.  $c < c^*$ , then (33) has two real roots given by  $u_{\pm} = (\pm \sqrt{\Delta} - 2b + 4)/(10(b^2 - 4c))$ , so we have the four possible equilibria (considering  $u_{\pm} = y_{\pm}^3$ ):

$e_3 = (x_1, y_+)$ ,  $e_4 = (-x_1, y_+)$ ,  $e_5 = (x_2, y_-)$  and  $e_6 = (-x_2, y_-)$  where

$$x_1 = \pm \sqrt{\frac{(\sqrt{\Delta} - 3b + 4)}{2^{5/3}} \left( -\frac{1}{\sqrt{\Delta} - 2b + 4} \right)^{2/3}}$$

and

$$x_2 = \pm \sqrt{-\frac{(\sqrt{\Delta} + 3b - 4)}{2^{5/3}} \left( \frac{1}{\sqrt{\Delta} + 2b - 4} \right)^{2/3}}.$$

Next, we analyse when these points are well defined. More precisely  $x_1$  is real if  $4 - 3b + \sqrt{\Delta} > 0$  and  $x_2$  is real if  $4 - 3b - \sqrt{\Delta} > 0$ . Direct computations provide the following summary:

**(II.3.1)** If  $b < 4/3$  and  $2b/5 < c < c^*$  then there exist four more equilibria, namely,  $e_3, e_4, e_5$  and  $e_6$ . The equilibria  $e_{5,6}$  are always saddles, because their characteristic polynomial is in the form  $\alpha_1 \lambda^2 + \beta_1$  where  $\alpha_1 > 0$  and  $\beta_1 < 0$ . The equilibria  $e_{3,4}$  have characteristic polynomial of the same form  $\alpha_2 \lambda^2 + \beta_2$  where  $\alpha_2 > 0$ . If  $b < 0$  and  $2b/5 < c < b^2/4$ , then  $\beta < 0$  and these equilibria are saddles. If  $b < 0$  and  $b^2/4 < c$  or  $0 < b < 4/3$  and  $2b/5 < c$ , then  $\beta$  is positive and the equilibria  $e_{3,4}$  are centers.

**(II.3.2)** If  $c < 2b/5$  there exist two more equilibria,  $e_5$  and  $e_6$ , that as in the case (II.3.1) are saddles.

**(II.3.3)** If  $b > 4/3$  and  $2b/5 < c < c^*$  there are no more finite equilibria.

Explicitly, for (II.3) the new equilibria are given by

$$\begin{aligned} e_3 &= \left( \sqrt{-\frac{(\sqrt{\Delta} + 3b - 4)}{2^{5/3}} \left( \frac{1}{\sqrt{\Delta} + 2b - 4} \right)^{2/3}}, \left( \frac{\sqrt{\Delta} - 2b + 4}{10(b^2 - 4c)} \right)^{1/3} \right), \\ e_4 &= \left( -\sqrt{-\frac{(\sqrt{\Delta} + 3b - 4)}{2 \cdot 2^{2/3}} \left( \frac{1}{\sqrt{\Delta} + 2b - 4} \right)^{2/3}}, \left( \frac{\sqrt{\Delta} - 2b + 4}{10(b^2 - 4c)} \right)^{1/3} \right), \\ e_5 &= \left( \sqrt{-\frac{(\sqrt{\Delta} + 3b - 4)}{2 \cdot 2^{2/3}} \left( \frac{1}{\sqrt{\Delta} + 2b - 4} \right)^{2/3}}, \left( \frac{-\sqrt{\Delta} - 2b + 4}{10(b^2 - 4c)} \right)^{1/3} \right), \end{aligned}$$

and

$$e_6 = \left( -\sqrt{-\frac{(\sqrt{\Delta} + 3b - 4)}{2 \cdot 2^{2/3}} \left( \frac{1}{\sqrt{\Delta} + 2b - 4} \right)^{2/3}}, \left( \frac{-\sqrt{\Delta} - 2b + 4}{10(b^2 - 4c)} \right)^{1/3} \right).$$

From the analysis of the linear part of the system at these equilibria we have the following results:

If  $c < c^*$  the equilibria  $e_3$  and  $e_4$  are always saddles because their characteristic polynomials have the form  $\alpha \lambda^2 + \beta$ , where  $\alpha \beta < 0$ .

On the other hand the equilibria  $e_5$  and  $e_6$  have different behavior in relation to the parameters, are saddle if  $0 < b \leq 4/3$  and  $b^2/4 < c < 2b/5$ , or  $4/3 < b < 2$  and  $b^2/4 < c < \alpha/20$ ; are centers if  $b \leq 0$  and  $c < 2b/5$  or  $0 < b \leq 2$  and  $c < b^2/4$ , or  $b > 2$  and  $c < \alpha/20$ , and the eigenvalues are nulls if  $4/3 < b < 2$  and  $c = \alpha/20$ , or  $b > 2$  and  $c = \alpha/20$ . Thus, the local study of the finite equilibria is complete.

Next, we will study the infinite equilibria. From (4) system (32) in the local chart  $U_1$  becomes

$$(34) \quad \dot{z}_1 = 5z_1 + 5bz_1^3 + 5cz_1^5 + z_2^3 + z_1^2z_2^3, \quad \dot{z}_2 = z_2 + 3bz_1^2z_2 + 5cz_1^4z_2 + z_1z_2^4,$$

At infinity ( $z_2 = 0$ ) the equilibrium points of system (34), according to second equation (32) are characterized by  $z_1 = 0$  or  $1 + bz_1^2 + cz_1^4 = 0$ . In the first case the origin is the unique equilibrium and it is a repeller node because their eigenvalues are 1 and 5. In the second case taking  $z = z_1^2$  and let  $p(z) = 1 + bz + cz^2$ , we have the following sub-cases:

**(I)** If  $b^2 - 4c < 0$  there are no more equilibria.

**(II)** If  $b^2 - 4c = 0$ , we have two sub-cases depending of the sign of  $b$ .

**(II.1)** If  $b > 0$  there are no more equilibria.

**(II.2)** If  $b < 0$ ,  $p(z)$  has a unique root given by  $z = -2/b$ , so system (34) has two more equilibria, namely,  $(\pm\sqrt{-2/b}, 0)$  and they are degenerate, so we need to shift the equilibria to the origin and then apply a blow-up for understanding the local phase portrait at these equilibrium points. First, we take  $\tilde{e}_3 = (\sqrt{-2/b}, 0)$  and translating it to the origin through  $(z_1, z_2) \mapsto (z_1 + \sqrt{-2/b}, z_2)$  we arrive to system (35)

$$(35) \quad \begin{aligned} \dot{z}_1 = & (-10b\sqrt{-2/b})z_1^2 - 20bz_1^3 + (1 - 2/b)z_2^3 + (25b^2\sqrt{-1/8b})z_1^4 \\ & + 2\sqrt{-2/b}z_1z_2^3 - (5b^2/4)z_1^5 + z_1^2z_2^3, \\ \dot{z}_2 = & (-4b\sqrt{-2/b})z_1z_2 - 12bz_1^2z_2 + (5b^2\sqrt{-2/b})z_1^3z_2 + \sqrt{-2/b}z_2^4 + z_1z_2^4 \\ & + 5b^2z_1^4z_2/4. \end{aligned}$$

After applying the directional blow-up  $(z_1, z_2) \mapsto (z_1, w)$  with  $z_2 = w \cdot z_1$  and eliminating the common factor  $z_1$  get

$$(36) \quad \begin{aligned} \dot{z}_1 = & (-10b\sqrt{-2/b})z_1 - 20bz_1^2 + (25b^2\sqrt{-2/b})z_1^3 + (5b^2/4)z_1^4 + \\ & (1 - 2/b)z_1^2w^3 + 2\sqrt{-2/b}w^3z_1^3 + w^3z_1^4 \\ \dot{w} = & (6b\sqrt{-2/b})w + 8bz_1w - (5b^2\sqrt{-2/b}/2)z_1^2w + (2/b - 1)z_1w^4 - \\ & \sqrt{-2/b}z_1^2w^4. \end{aligned}$$

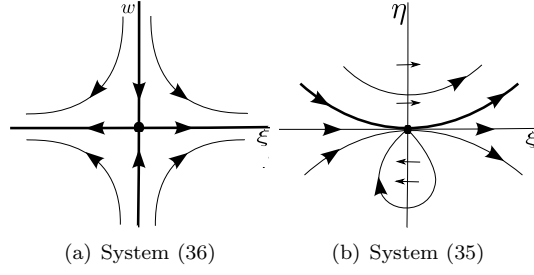
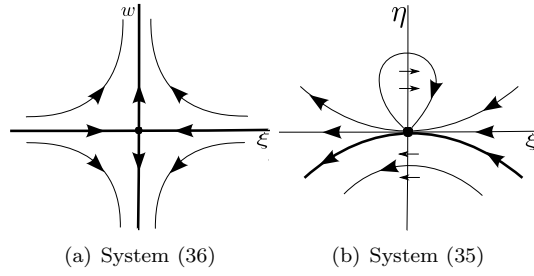
System (36) at  $z_1 = 0$  only has the origin as equilibrium and it is a saddle, with the stable separatrix on the  $w$ -axis, and the unstable separatrix on the  $z_1$ -axis (locally). Figure 14 shows the local phase portrait next to equilibrium  $\tilde{e}_3$  which consists in one hyperbolic sector, one elliptic sector, one attracting and one repeller parabolic sectors (see Figure 14(b)).

Analogously, we can find the local behavior next to the equilibria  $\tilde{e}_4 = (\sqrt{-2/b}, 0)$  as it is shown in Figure 15(b).

**(III)** If  $b^2 - 4c > 0$  the polynomial  $p(z)$  has two roots  $z_1^2 = z = (-b \pm \sqrt{b^2 - 4c})/(2c)$  giving rise to four possible equilibria. We have three sub-cases:

**(III.1)** If  $b > 0$  and  $0 < c < b^2/4$  there are no more equilibria.




 FIGURE 14. Blow-up of the equilibrium  $\tilde{e}_3$  of system (32).

 FIGURE 15. Blow-up of the equilibrium  $\tilde{e}_4$  of system (32).

**(III.2)** If  $c < 0$  there exist two more equilibria, namely,

$$p_{4,5} = \left( \pm \sqrt{(-b - \sqrt{b^2 - 4c})/(2c)}, 0 \right).$$

These equilibria at infinity are attractor nodes because their eigenvalues  $5(b^2 - 4c + b\sqrt{b^2 - 4c})/c$  and  $(b^2 - 4c + b\sqrt{b^2 - 4c})/c$  are negatives.

**(III.3)** If  $b < 0$  and  $0 < c < b^2/4$  there exist four more equilibria,  $p_{4,5}$  as in the previous case (with the same local phase portrait) and two more, namely,

$$p_{2,3} = \left( \pm \sqrt{(-b + \sqrt{b^2 - 4c})/(2c)}, 0 \right),$$

that are repeller nodes, the eigenvalues of  $p_{2,3}$  are  $5(b^2 - 4c - b\sqrt{b^2 - 4c})/c$  and  $(b^2 - 4c - b\sqrt{b^2 - 4c})/c$  and are positive.

The study in the local chart  $U_2$  does not give the origin as an equilibrium. Thus, we have completed the study of the infinite equilibria.

In order to describe the local flow on the Poincaré disk, we summarize the previous studies in Table 1 for the case  $b^2 - 4c = 0$ , in Table 2 for the case  $b^2 - 4c < 0$ , and in Table 3 for the case  $b^2 - 4c > 0$ .

In summary we have eight possible local dynamics on Poincaré disk, according to Figure 16 the cases are:

- (i) Figure 16(a): If  $b^2 - 4c = 0 \wedge b \geq 8/5$ , or  $b^2 - 4c < 0 \wedge (\Delta < 0 \vee (\Delta = 0 \wedge b \geq 4/3) \vee (\Delta > 0 \wedge b > 8/5))$ , or  $b^2 - 4c > 0 \wedge b > 8/5 \wedge 2b/5 < c$ .

TABLE 1. Equilibria of system (32) if  $b^2 - 4c = 0$ . C= Center, S=Saddle, A=Attractor Node, R=Repeller Node, D=Degenerate, Cu=Cusp.

$b$	Finite equilibria	Type (resp.)	Infinite eq. (in $U_1$ )	Type in $U_1$	Fig. 16
$b \geq 8/5$	$e_1, e_2$	C, S.	(0, 0)	R	(a)
$0 < b < 8/5$	$e_1, e_2, \tilde{e}_3, \tilde{e}_4$	C, C, S, S.	(0, 0)	R	(b)
$b < 0$	$e_1, e_2, \tilde{e}_3$ and $\tilde{e}_4$	C, S, S, S.	(0, 0), $p_4$ and $p_5$	R, D, D	(c)

TABLE 2. Equilibria of system (32) if  $b^2 - 4c < 0$ .

$\Delta$	$b$	$c$	Finite eq.	Type (resp.)	Infinite eq. (in $U_1$ )	Type in $U_1$	Fig. 16
$\Delta < 0$			$e_1$ and $e_2$	C, S	(0, 0)	R	(a)
$\Delta = 0$	$b \geq 4/3$		$e_1, e_2$	C, S	(0, 0)	R	(a)
	$b < 4/3$		$e_1, e_2, \hat{e}_3$ and $\hat{e}_4$	C, S, Cu, Cu	(0, 0)	R	(d)
$\Delta > 0$	$b > 8/5$	$2b/5 < b^2/4 < c$	$e_1, e_2$	C, S.	(0, 0)	R	(a)
	$4/3 < b < 8/5$	$b^2/4 < c < 2b/5$	$e_1, e_2, e_5$ and $e_6$	C, C, S, S.	(0, 0)	R	(b)
	$b = 4/3$	$b^2/4 < c < 2b/5$	$e_1, e_2, e_5$ and $e_6$	C, C, S, S.	(0, 0)	R	(b)
	$0 < b < 4/3$	$b^2/4 < 2b/5 < c$	$e_1, e_2, e_3, e_4, e_5$ and $e_6$	C, S, C, C, S, S	(0, 0)	R	(e)
		$b^2/4 < c < 2b/5$	$e_1, e_2, e_5$ and $e_6$	C, C, S, S	(0, 0)	R	(b)
	$b < 0$	$c > b^2/4$	$e_1, e_2, e_3, e_4, e_5$ and $e_6$	C, S, C, C, S, S	(0, 0)	R	(e)

(ii) Figure 16(b): If  $b^2 - 4c = 0 \wedge 0 < b < 8/5$ , or  $b^2 - 4c > 0 \wedge \Delta > 0 \wedge ((4/3 \leq b < 8/3 \wedge c < 2b/5) \vee (0 < b < 4/3 \wedge c < 2b/5) \vee (b^2 - 4c < 0 \wedge ((8/5 < b \wedge c < 2b/5) \vee (0 < b \leq 8/5)))$ .

(iii) Figure 16(c): If  $b^2 - 4c = 0 \wedge b < 0$ .

(iv) Figure 16(d): If  $b^2 - 4c < 0 \wedge \Delta = 0 \wedge b < 4/3$ .

(v) Figure 16(e): If  $b^2 - 4c < 0 \wedge \Delta > 0 \wedge (b < 0 \vee 0 < b < 4/3)$ .

(vi) Figure 16(f): If  $b^2 - 4c > 0 \wedge b < 0 \wedge 2b/5 < c < 0$ .

(vii) Figure 16(g): If  $b^2 - 4c > 0 \wedge b < 0 \wedge c < 2b/5$ .

(viii) Figure 16(h): If  $b^2 - 4c > 0 \wedge b < 0 \wedge c > 0$ .

Using similar arguments as in the previous phase portraits in the Poincaré disk, we obtain that the global flow associated to the parameters in the case:

TABLE 3. Equilibria of system (32) if  $b^2 - 4c > 0$  (this implies  $\Delta > 0$ ).

$b$	$c$	Finite eq.	Type (resp.)	Infinite eq. (in $U_1$ )	Type in $U_1$	Fig. 16
$8/5 < b$	$2b/5 < c < b^2/4$	$e_1, e_2$	C, S.	$(0, 0)$	R	(a)
	$c < 2b/5$	$e_1, e_2, e_5$ and $e_6$	C, C, S, S.	$(0, 0)$	R	(b)
$0 < b \leq 8/5$	$c < b^2/4$	$e_1, e_2, e_5$ and $e_6$	C, C, S, S	$(0, 0)$	R	(b)
$b < 0$	$0 < c < b^2/4$	$e_1, e_2, e_3, e_4, e_5$ and $e_6$	C, S, S, S, S, S.	$(0, 0), p_2, p_3, p_4$ and $p_5$	R, R, R, A, A	(h)
	$2b/5 < c < 0$	$e_1, e_2, e_3, e_4, e_5$ and $e_6$	C, C, S, S, S, S.	$(0, 0), p_4$ and $p_5$	R, A, A	(f)
	$c < 2b/5$	$e_1, e_2, e_5$ and $e_6$	C, S, S, S.	$(0, 0), p_4$ and $p_5$	R, A, A.	(g)

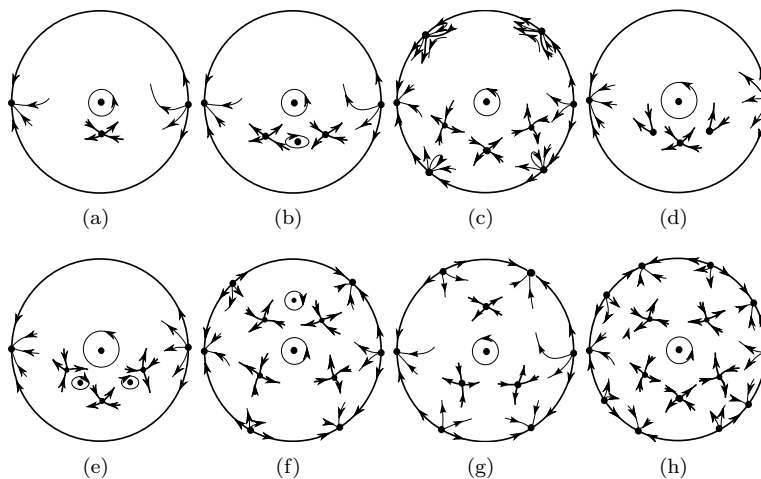

 FIGURE 16. Local phase portraits for the system associated to Hamiltonian (3) when  $abc \neq 0$ .

Figure 16(a), i.e. in the case **(i)**, the global phase portrait is topologically equivalent to the one of Figure 1(a).

Figure 16(b), i.e. in the case **(ii)**, the global phase portrait is topologically equivalent to the one of Figure 1(c).

Figure 16(d), i.e. in the case **(iv)**, the global phase portrait is topologically equivalent to the one of Figure 1(g).

Figure 16(e), i.e. in the case **(v)**, the global phase portrait is topologically equivalent to the ones of Figure 1(h), (i) or (j). In this case we have a bifurcation curve

that depends of  $b$  and  $c$  where the energy level  $h_2$  of the saddle  $e_2$  coincides with the energy level  $h_5 = h_6$  of the other four saddles  $e_5$  and  $e_6$  (resp.).

First, for case  $b^2 - 4c < 0$ ,  $\Delta > 0$ ,  $0 < b < 4/3$  and  $c > 2b/5$  we have that in the bifurcation curve  $h_2 = h_5$ , see Figure 17, the global phase portrait is topologically equivalent to Figure 1(i), in the region when  $h_2 < h_5$  the global phase portrait is topologically equivalent to the one of Figure 1(h), and in the region when  $h_2 > h_5$  the global phase portrait is topologically equivalent to the one of Figure 1(j).

Second, in the case  $b^2 - 4c < 0$ ,  $\Delta > 0$ ,  $b < 0$  and  $c > b^2/4$  we have that in the bifurcation curve, as before, the global phase portrait is topologically equivalent to the one of Figure 1(i), for values in the region  $h_2 > h_5$  the global phase portrait is topologically equivalent to the one of Figure 1(h) and for values in the region  $h_2 < h_5$  (Figure 18) the global phase portrait is topologically equivalent to the one of Figure 1(j).

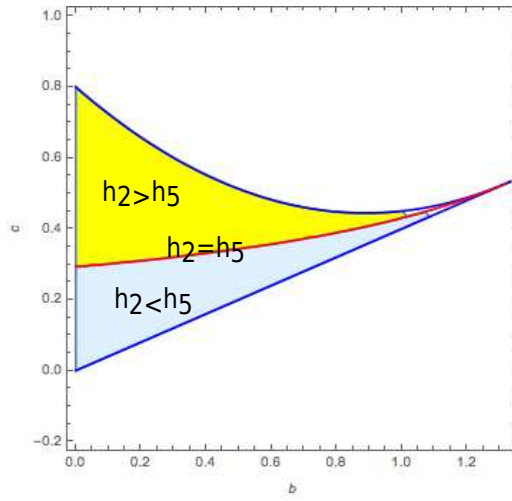


FIGURE 17. Graph of the function  $f(b, c) = h_2 - h_5$  on the  $(b, c)$ -plane (in the region of existence of the case). In case  $b^2 - 4c < 0$ ,  $\Delta > 0$ ,  $0 < b < 4/3$  and  $c > 2b/5$

Figure 16(g), i.e. when **(vii)** holds, the global phase portrait is topologically equivalent to ones of the Figure 1(d), (e) or (f). In this case we have a bifurcation curve where the energy level of the three saddles coincide, in the region that satisfy the existence conditions, see Figure 19. On this curve the global phase portrait is topologically equivalent to the one of Figure 1(e). For the parameters located in the region  $h_2 < h_5$  we have that global phase portrait is topologically equivalent to the one of Figure 1(d). And for the parameters in the region  $h_2 > h_5$  we have that global phase portrait is topologically equivalent to the one of Figure 1(f).

Next we study the global flow for those cases where the local dynamics is different than the cases already studied.

First for the parameters associated to the case of Figure 16(c), i.e. for the case **(iii)**, the behavior of the global flow have a bifurcation value given by  $b = b_1 = -2/5 + 2(-9 + 4\sqrt{6})^{1/3}(15)^{-2/3} - 2(15(-9 + 4\sqrt{6}))^{-1/3}$ , whose value corresponds to

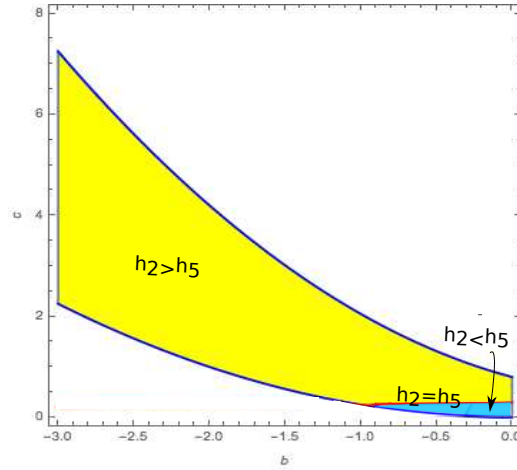


FIGURE 18. Graph of the functions  $f(b, c) = h_2 - h_5$  and its intersection with the  $(b, c)$ -plane. In the case  $b^2 - 4c < 0$ ,  $\Delta > 0$ ,  $b < 0$  and  $c > b^2/4$

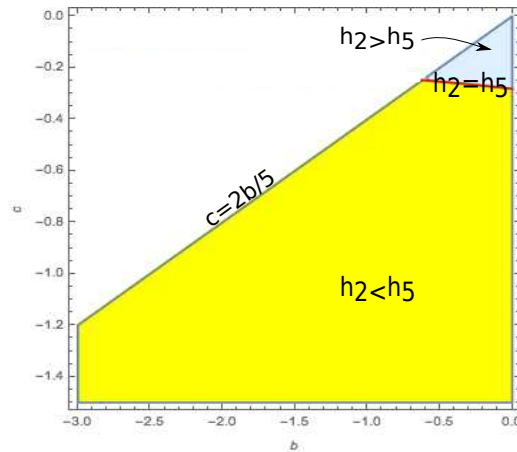


FIGURE 19. Graph of the functions  $f(b, c) = h_2 - h_5$  and its intersection with the  $(b, c)$ -plane, under the conditions of the existence of Figure 16(g), i. e. when (vii) holds.

the value of  $b$  such that the energy level  $h_{e_2} = (3 \cdot 2^{1/3}) / (5^{5/3} b^{4/3})$  of the saddle  $e_2$ , and  $h_{\tilde{e}_3} = h_{\tilde{e}_4} = (-3(b - 2)^{1/3}) / 16$  of the saddles  $\tilde{e}_3$  and  $\tilde{e}_4$  coincide. Proceeding in a similar way as for example in the case  $a = 0$  and  $bc \neq 0$ , we verify that for  $b < b_1$  there exists a homoclinic orbit of  $e_2$  forming the boundary of the period annulus of the center at the origin; there is a separatrix connecting  $\tilde{e}_3$  with  $\tilde{e}_4$  cutting the positive  $y$ -axis. These informations force that the global phase portrait in this case is topologically equivalent to the one of Figure 1(n). For  $b = b_1$  we have that the three saddles are in the boundary of the period annulus of the center,

thus the global phase portrait in this case is topologically equivalent to the one of Figure 1(o). If  $b_1 < b < 0$  the two saddles  $\tilde{e}_3$  and  $\tilde{e}_4$  are in the boundary of the period annulus of the center; the separatrices of  $e_2$  located in the region  $x > 0$  and  $y < 0$  connect with the infinite equilibrium point  $p_5$  surrounding its elliptic sector. By symmetry we obtain that the global phase portrait in this case is topologically equivalent to the one of Figure 1(p).

Second for the parameters associated to the case of Figure 16(f), when we are in the case **(vi)**, it is verified that the energy level  $h_3$  of the saddle  $e_3$  and  $e_4$  is the same, and the level of energy of the saddle  $e_5$  and  $e_6$ , called  $h_5$ , is the same. Here, a bifurcation curve appears when  $h_3 = h_5$ , i.e. the four saddle is in the same energy level, in this case two separatrices of  $e_3$  and two separatrices of  $e_4$  are forming the boundary of the period annulus of  $e_2$ ; the four saddles are in the boundary of the period annulus of the center at the origin. This forces that the global flow is topologically equivalent to the one of Figure 1(q). In the region  $h_3 > h_5$  (see Figure 20) it holds that  $0 < h_5 < h_3 < h_2$  (where  $h_2$  is the level of energy of the center  $e_2$ , and  $h = 0$  is the level of energy of the center at the origin), so the origin is enclosed by a boundary of the period annulus formed by two separatrices of each saddle  $e_5$  and  $e_6$  located in the region  $y < 0$ , on the other hand the separatrices of  $e_3$  and  $e_4$  are forming the boundary of the period annulus of the center  $e_2$ , this implies that the global phase portrait in this case is topologically equivalent to the one of Figure 1(r). Finally in the region  $h_3 < h_5$  it holds  $0 < h_3 < h_2 < h_5$ , so in this case the period annulus of the origin has the boundary formed by separatrices of  $e_3$  and  $e_4$ . Furthermore, the boundary of the period annulus of  $e_2$  is formed by separatrices of  $e_3$  and  $e_4$  too. From previous arguments it is clear that one separatrix of  $e_5$  connects with one separatrix of  $e_6$  crossing the  $y$ -axis. Thus the global phase portrait is topologically equivalent to the one of Figure 1(s).

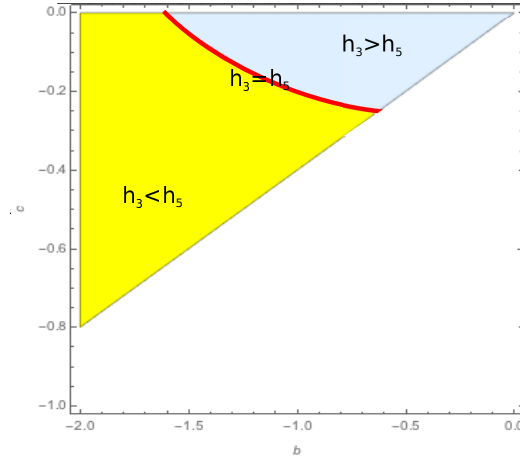


FIGURE 20. Graph of the functions  $f(b, c) = h_3 - h_5$  and its intersection with the  $(b, c)$ -plane, under the conditions of the existence of Figure 16(f), i.e. in the case **(vi)**.

Third for the parameters associated to the case of Figure 16(h), i.e. when **(viii)** holds, we note that the energy level of  $e_2$  is  $h_2 = 3/(10(5c)^{2/3})$ , for the saddle  $e_3$

and  $e_4$  is

$$h_3 = \frac{-3}{20} \frac{\sqrt{b(9b-16)} - 20c + 16 + 3b - 6}{(2\sqrt{9b^2 - 16b - 20c + 16} + 4b - 8)^{2/3}}$$

and for the saddle  $e_5$  and  $e_6$  is

$$h_5 = \frac{3}{20} \frac{\sqrt{b(9b-16)} - 20c + 16 - 3b + 6}{(-2\sqrt{9b^2 - 16b - 20c + 16} + 4b - 8)^{2/3}}.$$

For studying all the possible global dynamics we consider first the relation between  $h_3$  and  $h_5$ , in Figure 21 we divide the region of existence for this case:  $b < 0$  and  $0 < c < b^2/4$  in three subregions, when  $h_3 > h_5$ ,  $h_3 = h_5$  and  $h_3 < h_5$ . Note that the curve  $f_{35}(b, c) = h_3 - h_5$  cut the  $b$ -axis in  $b = (-1 - \sqrt{5})/2$ . Thus we separate the analysis in the three cases.

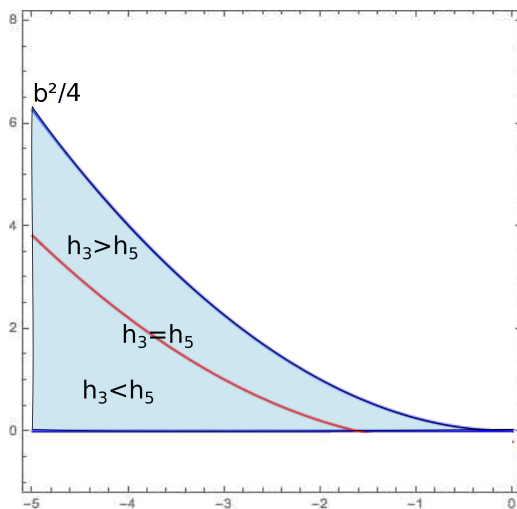


FIGURE 21. Graph of the functions  $f_{35}(b, c) = h_3 - h_5$  and its intersection with the  $(b, c)$ -plane, i.e. when (viii) holds.

- Region  $h_3 > h_5$ : In this region we note two bifurcation curves given by the functions  $f_{23}(b, c) = h_2 - h_3$  and  $f_{25}(b, c) = h_2 - h_5$  as it is shown in Figure 22.

Thus we have five regions in relation to the energy level of the saddle. In the first region it holds that  $h_2 < h_5 < h_3$ , so the boundary of the period annulus of the center at the origin is formed by separatrices of  $e_2$ , one unstable separatrix of  $e_3$  must cross the positive  $x$ -axis, and then must cross the positive  $y$ -axis connecting with a stable separatrix of  $e_4$  by the symmetry. Furthermore we know from previous analysis that one unstable separatrix of  $e_5$  connects with one stable separatrix of  $e_6$  crossing the positive  $y$ -axis. These informations show that the global phase portrait under these conditions is topologically equivalent to the one of Figure 1(t).

In the second region we have that the energy level of  $h_2$  coincide with  $h_5$ , so the three saddle on this energy level  $e_2$ ,  $e_5$  and  $e_6$  form the boundary

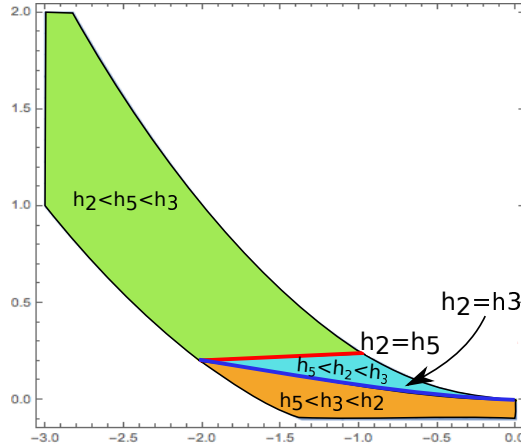


FIGURE 22. Graph of the functions  $f_{23}(b, c)$  and  $f_{25}$  in the region where  $f_{35} > 0$ .

of the period annulus of the center at the origin. As before one unstable separatrix of  $e_5$  connects with one stable separatrix of  $e_6$  crossing the positive  $y$ -axis. Thus the global phase portrait under these conditions is topologically equivalent to the one of Figure 1(u).

In the other three regions ( $h_5 < h_2 < h_3$ ,  $h_5 < h_2 = h_3$  and  $h_5 < h_3 < h_2$ ) the energy level  $h_5$  is less than the energy levels  $h_2$  and  $h_3$ . It is checked that when  $h_2 = h_3$  the level curve is formed by disconnected components and the saddle  $e_2$  is not connected with the saddle  $e_5$  or  $e_6$ . In Figure 23 it is shown the evolution of level curve  $H = h_2$  through the regions  $h_5 < h_2 < h_3$ ,  $h_5 < h_2 = h_3$  and  $h_5 < h_3 < h_2$ .

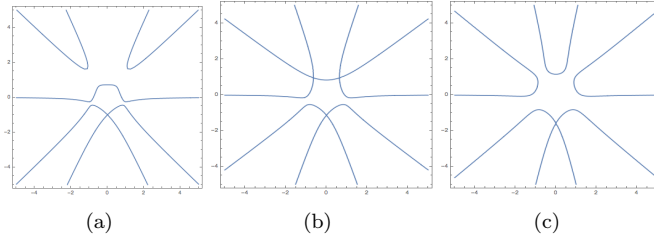


FIGURE 23. Level curve  $h_2$  passing through  $e_2$ . (a) Region  $h_5 < h_2 < h_3$ , (b) Region  $h_5 < h_2 = h_3$ , (c) Region  $h_5 < h_3 < h_2$ ,

In these last three cases some separatrices of the saddle  $e_5$  and  $e_6$  are forming the boundary of the period annulus of the center, as before  $e_5$  and  $e_6$  are connected crossing the positive  $y$ -axis and the global phase portrait is topologically equivalent to the one of Figure 1(v).

- Region  $h_3 = h_5$  Here the parameters must satisfy  $b < (-1 - \sqrt{5})/2$  and  $c = (-1 + b + b^2)/5$ , and we have three subregion that depend on the relation of the energy level  $e_2$  as it is shown in Figure 24.



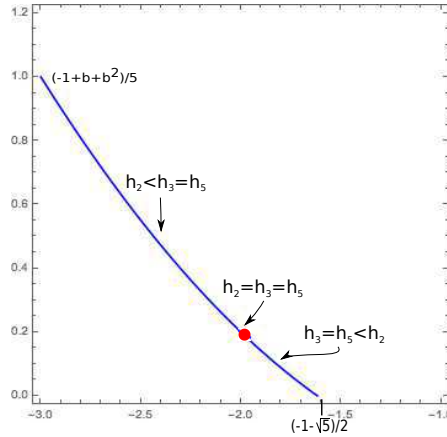


FIGURE 24. Graph of the functions  $f_{35}(b, c) = 0$ .

For the region  $h_2 < h_3 = h_5$  we verify that there exists a homoclinic orbit of  $e_2$  forming the boundary of the period annulus of the center at the origin; the saddles  $e_3, e_4, e_5$  and  $e_6$  are connected by some of their separatrices, the separatrix connecting  $e_3$  with  $e_4$  is cutting the positive  $y$ -axis. These informations force that the global phase portrait in this case is topologically equivalent to the one of Figure 1(w). For the parameters  $(b, c)$  such that the energy level coincide  $h_2 = h_3 = h_5$ , some separatrices of all the saddles  $e_2, e_3, e_4, e_5$  and  $e_6$  form the boundary of the period annulus of the center as it is shown in Figure 1(x). Thus the global phase portrait in this case is topologically equivalent to the one of Figure 1(x). In the region where the parameters  $(b, c)$  verify  $0 < h_3 = h_5 < h_2$  some separatrices of the four saddles  $e_3, e_5$  and  $e_6$  are in the boundary of the period annulus of the center; the separatrices of  $e_2$  located in the region  $x > 0$  and  $y < 0$  connect with the infinite equilibrium points  $p_3$  and  $p_5$ . By symmetry we obtain that the global phase portrait in this case is topologically equivalent to the one of Figure 1(y).

- Region  $h_3 < h_5$  In this region there are two bifurcation curves given by the functions  $f_{23}(b, c) = h_2 - h_3 = 0$  and  $f_{25}(b, c) = h_2 - h_5 = 0$ , as it is shown in Figure 25. So we have five subregion to analyse. First, if  $h_2 < h_3 < h_5$ , there exist a homoclinic orbit formed by two separatrices of  $e_2$  that form the boundary of the period annulus of the origin, next the saddle  $e_3$  and  $e_4$  are connected one of their separatrices crosses the positive  $y$ -axis, and other separatrices of each one are surrounding the period annulus of the center, passing next to  $e_2$  and continue to an infinite equilibria. Thus the separatrices of  $e_5$  and  $e_6$  cannot cross the  $y$ -axis and they connect with infinite equilibria. These information force that the global phase portrait is topologically equivalent to the one of Figure 1(z).

In the other regions always the energy levels of  $e_3$  and  $e_4$  is less or equal than the energy levels of the others saddles, so  $e_3$  and  $e_4$  are always in the boundary of the period annulus of the center. In the region where the

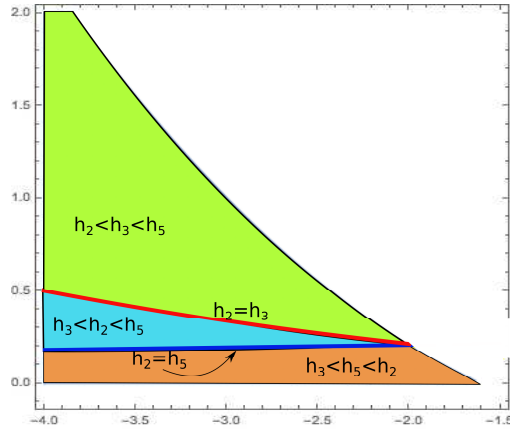


FIGURE 25. Graph of the functions  $f_{23}(b, c) = 0$  and  $f_{25}(b, c) = 0$  in the region where  $f_{35} > 0$ .

parameters  $b$  and  $c$  satisfy that  $h_2 = h_3 < h_5$  and the three saddles  $e_2$ ,  $e_3$  and  $e_4$  form the period annulus of the center at the origin, this implies that the separatrices of  $e_5$  and  $e_6$  must connect at the infinite with infinite equilibria. Thus we obtain that the global phase portrait is topologically equivalent to the one of Figure 1(aa).

In the region where  $b$  and  $c$  satisfy that  $h_3 < h_2 < h_5$  we have that the boundary of the period annulus is formed by separatrices of  $e_3$  and  $e_4$ , the level curve passing through  $e_2$  surround this annulus, two separatrices of this saddle border the annulus, approaching to the saddle  $e_3$  and  $e_4$  and then continue to an infinite equilibrium point. As before the separatrices of  $e_5$  and  $e_6$  must connect at infinite with an infinite equilibrium point, one of them first approaches to the saddle  $e_2$ . From the previous informations we conclude that the global phase portrait is topologically equivalent to the one of Figure 1(ab).

In the region where  $b$  and  $c$  satisfy that  $h_3 < h_2 = h_5$  we have, as before that some of the separatrices of the saddles  $e_3$  and  $e_4$  form the boundary of the period annulus of the center, and the saddles  $e_2$ ,  $e_5$  and  $e_6$  are connected by some of their separatrices, and their other separatrices must connect at infinite with an equilibrium point, from this the global phase portrait is topologically equivalent to Figure 1(ac).

Finally, in the region where  $h_3 < h_5 < h_2$  we have that the boundary of the period annulus is formed by separatrices of  $e_3$  and  $e_4$  surrounded for outside by two separatrices of  $e_5$  and  $e_6$ . Here one unstable separatrix of  $e_6$  cross the negative  $x$ -axis connecting with one stable separatrix of  $e_5$ . The separatrices of  $e_2$  must connect at infinite by continuity. Thus we conclude that the global phase portrait is topologically equivalent to the one of Figure 1(ad).

This complete the proof of the general case.

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