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NO PERIODIC ORBITS IN THE BIANCHI MODELS B

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Abstract. In this paper we prove that the Bianchi models B have no periodic solutions.

1. Introduction and statement of the main results

Bianchi [1, 2] was the first to classify three dimensional Lie algebras in terms of the reduction to the canonical forms for their structure constants. There are nine types of models based on the dimension n of the algebra.

a) $n = 0$ Type I;
b) $n = 1$ Type II, III;
c) $n = 2$ Type IV, V, VI, VII;
d) $n = 3$ Type $VIII$, IX.

To be of cosmological interest, this classification is called Bianchi universe models, which are spatially homogeneous cosmological models that in general are anisotropic. Taub [13] introduced this work of Bianchi into relativistic cosmology and derived the dynamic equations for the generic vacuum Bianchi geometries.

Ellis and MacCallum [8] gave this classification, introduced the type A/B of Bianchi model notation that is now in common use [Type A: $a = 0$, Type B: $a \neq 0$], see also [3]. The different types correspond to all nonequivalent sets of their structure constants. Let $\{X_1,\ X_2,\ X_3\}$ be an appropriate basis of the three dimensional Lie Algebra. Following Bogoyavlensky's approach, see [3], we can obtain the classification conditions on the property of the structure constants, and write in the form of commutation relations.

$$[X_1, X_2] = n_3 X_3, \quad [X_2, X_3] = n_1 X_1 - a X_2, \quad [X_3, X_1] = n_2 X_2 + a X_1,$$

where $a \in \mathbb{R}$, $[\ ,\ ]$ is the Lie bracket and (n_1, n_2, n_3) with $n_i \in \{+1, -1, 0\}$. In particular, for $a = 0$ we obtain models of type A and

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for $a \neq 0$ we obtain models of type B. we arrive at the following list of types A and B of homogeneous spaces. In the the table the roman numeral is labeled the type of the Bianchi classification

Type	I	II	VI_0	VII_0	$VIII$	IX
n_1	0	1	1	1	1	1
n_2	0	0	-1	1	1	1
n_3	0	0	0	0	-1	1

Table 1. The classification of Bianchi class A cosmologies.

Type	III	IV	V	VI	VII
a	1	1	1	$a \neq 1$	a
n_1	1	1	0	1	1
n_2	-1	0	0	-1	1

Table 2. The classification of Bianchi class B cosmologies.

According to Bogoyavlensky [3], for the homogeneous cosmological models of Class B Einstein's system of equations reduces to the following dynamical system in the phase space $p_i, q_i, p_\varphi, \varphi$, $i = 1, 2, 3$,

$$\begin{aligned} &\frac{dq_i}{d\tau} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q_i} - h_i, \\ &\frac{d\varphi}{d\tau} = \frac{\partial H}{\partial p_\varphi}, \qquad \frac{dp_\varphi}{d\tau} = -\frac{\partial H}{\partial \varphi} - h_\varphi, \end{aligned} \tag{1}$$

where the function H is

$$H = \frac{1}{(q_1 q_2 q_3)^{\frac{1-k}{2}}} \left(T(p_i q_i) + V_G(q_i) \right),$$

with

$$\begin{aligned} T(p_i, q_i, p_\varphi) =& 2 \sum_{1 \leq i < j \leq 3} p_i p_j q_i q_j - \sum_{i=1}^{3} p_i^2 q_i^2 - \frac{p_\varphi^2 q_1 q_2}{(n_1 q_1 - n_2 q_2)^2}, \\ V_G(q_i) =& -\frac{1}{4} \left(12 a^2 q_1 q_2 + (n_1 q_1 - n_2 q_2)^2 \right), \end{aligned}$$

and

$$h_1 = \frac{a^2 q_2}{(q_1 q_2 q_3)^{\frac{1-k}{2}}}, \quad h_2 = \frac{a^2 q_1}{(q_1 q_2 q_3)^{\frac{1-k}{2}}},$$

$$h_3 = \frac{-2a^2 q_1 q_2}{q_3 (q_1 q_2 q_3)^{\frac{1-k}{2}}}, \quad h_\varphi = \frac{a(n_1 q_1 - n_2 q_2)^2}{(q_1 q_2 q_3)^{\frac{1-k}{2}}}.$$

System (1) in an explicit form writes as

$$\begin{aligned}
\frac{dq_1}{d\tau} &= -2q_1 (q_1 q_2 q_3)^{(k-1)/2} (p_1 q_1 + p_2 q_2 - p_3 q_3),\\
\frac{dq_2}{d\tau} &= -2q_2 (q_1 q_2 q_3)^{(k-1)/2} (p_1 q_1 + p_3 q_3 - p_2 q_2),\\
\frac{dq_3}{d\tau} &= -2q_3 (q_1 q_2 q_3)^{(k-1)/2} (p_1 q_1 + p_2 q_2 - p_3 q_3),\\
\frac{dp_1}{d\tau} &= -(q_1 q_2 q_3)^{(k-1)/2} \Bigg(2p_1 (p_2 q_2 + p_3 q_3 - p_1 q_1)\\
&\quad + \frac{p_\varphi^2 (n_1 q_1 + n_2 q_2)}{(n_1 q_1 - n_2 q_2)^3} q_2 - \frac{1}{2} n_1 (n_1 q_1 - n_2 q_2) - 2a^2 q_2 \Bigg) + \frac{1-k}{2q_1} \bar{H},\\
\frac{dp_2}{d\tau} &= -(q_1 q_2 q_3)^{(k-1)/2} \Bigg(2p_2 (p_1 q_1 + p_3 q_3 - p_2 q_2)\\
&\quad - \frac{p_\varphi^2 (n_1 q_1 + n_2 q_2)}{(n_1 q_1 - n_2 q_2)^3} q_1 - \frac{1}{2} n_2 (n_1 q_1 - n_2 q_2) - 2a^2 q_1 \Bigg) + \frac{1-k}{2q_2} \bar{H},\\
\frac{dp_3}{d\tau} &= -(q_1 q_2 q_3)^{(k-1)/2} \left(2p_3 (p_1 q_1 + p_2 q_2 - p_3 q_3) - 2a^2 \frac{q_1 q_2}{q_3} \right)\\
&\quad + \frac{1-k}{2q_3} \bar{H},\\
\frac{d\varphi}{d\tau} &= -(q_1 q_2 q_3)^{(k-1)/2} \frac{2 q_1 q_2 p_\varphi}{(n_1 q_1 - n_2 q_2)^2},\\
\frac{dp_\varphi}{d\tau} &= -(q_1 q_2 q_3)^{(k-1)/2} a (n_1 q_1 - n_2 q_2)^2,
\end{aligned} \tag{2}$$

with $0 \le k \le 1$ and $\bar{H} = T + V_G$.

Using dynamical systems methods, Collins [7] first introduce phase planes with compactified boundaries, to characterize the evolution of particular Bianchi classes of universe models, and Bogoyavlensky systematically introduce its study and application in [3]. After that many dynamical properties of the Bianchi models have been studied, for example, the integrability and the existence of periodic orbits, see [4, 5, 6, 9, 10, 12] and the references quoted there.

In this paper we study the periodic solutions of the type B of Bianchi model (2). It is known that all the Bianchi class A models do not have periodic orbits. Using evolutions equations associated to these models, and showing that such equations always have some monotone function evaluated on the orbits, Wainwright and Ellis [14] prove that these models cannot exhibit periodic motion. Buzzi and Llibre [4] provide a new, direct and easier proof on the non-existence of periodic orbits for the 6 models of Bianchi class A. On the other hand for the Einstein's field equation, Llibre and Yu [11] studied the periodic orbits of the static, spherically symmetric Einstein-Yang-Mills equations. We shall mainly follow the idea of the qualitative analysis of dynamical systems shown in [4] and [11] in this paper.

Our main result is the following.

Theorem 1. *The Bianchi models B, i.e. III, IV , V, VI and VII in Table 2 have no periodic solutions.*

We shall provide the proof of Theorem 1 for the type V of the Bianchi models B in section 2, the proof of Theorem 1 for the type IV of the Bianchi models B in section 3, and the proof of Theorem 1 for the type III, VI and VII of the Bianchi models B in section 4.

2. The Bianchi V model

Consider Bianchi V model. Noticing that for Bianchi V we have $p_\varphi = 0$ and $\dot\varphi = 0$. After the change of coordinates and time

$$x_i = q_i,\ x_{i+3} = p_i q_i,\ i = 1, 2, 3,\ d\tau_0 = \frac{(q_1q_2q_3)^{\frac{k-1}{2}}}{2} d\tau,$$

we obtain a simplest model of six equations, and write it as the homogeneous polynomial differential system of degree 2.

$$\begin{aligned}
\dot x_1 &= x_1(-x_4 + x_5 + x_6),\\
\dot x_2 &= x_2(x_4 - x_5 + x_6),\\
\dot x_3 &= x_3(x_4 + x_5 - x_6),\\
\dot x_4 &= x_1x_2 + \frac{k-1}{4}(3x_1x_2 + \Lambda),\\
\dot x_5 &= x_1x_2 + \frac{k-1}{4}(3x_1x_2 + \Lambda),\\
\dot x_6 &= x_1x_2 + \frac{k-1}{4}(3x_1x_2 + \Lambda),
\end{aligned} \tag{3}$$

where $0 \le k \le 1$ and

$$\Lambda = x_4^2 + x_5^2 + x_6^2 - 2(x_4x_5 + x_4x_6 + x_5x_6).$$

Proposition 2. *The Bianchi V system* (3) *has no periodic solutions.*

Proof. Suppose that system (3) has a periodic solution $\Gamma(t) = (x_1, x_2, x_3, x_4, x_5, x_6)(t)$ with period $T > 0$. We have from (3) that $\dot x_4 = \dot x_5 = \dot x_6$, then

$$x_4 = x_6 + a \quad \text{and} \quad x_5 = x_6 + b,$$

where a and b are any constants in $\mathbb{R}$. Substituting into the first three equations of (3) we have

$$\begin{aligned}
\dot x_1 &= x_1(b - a + x_6),\\
\dot x_2 &= x_2(a - b + x_6),\\
\dot x_3 &= x_3(a + b - x_6).
\end{aligned}$$

From this differential system it follows that

$$\frac{d(x_1x_3)}{dt} = 2b(x_1x_3), \quad \frac{d(x_2x_3)}{dt} = 2a(x_2x_3).$$

Hence we get $x_1x_3 = C_1e^{2bt}$ and $x_2x_3 = C_2e^{2at}$. Since the exponential function is not periodic, so the Bianchi V system (3) has no periodic solutions. □

3. The Bianchi IV model

Consider Bianchi IV model, after the change of coordinates and time

$$x_i = q_i, \; x_{i+3} = p_iq_i, \; i = 1,2,3, \; d\tau_0 = \frac{q_1(q_1q_2q_3)^{\frac{k-1}{2}}}{2}d\tau,$$

system (2) can be written as the six-dimensional homogeneous polynomial differential system of degree 3.

$$\begin{aligned}
\dot{x}_1 &= x_1^2(-x_4+x_5+x_6),\\
\dot{x}_2 &= x_1x_2(x_4-x_5+x_6),\\
\dot{x}_3 &= x_1x_3(x_4+x_5-x_6),\\
\dot{x}_4 &= \frac{x_1^3}{4}+x_1^2x_2-2x_2(x_4-x_5+x_6)^2+\frac{k-1}{4}R,\\
\dot{x}_5 &= x_1^2x_2+2x_2(x_4-x_5+x_6)^2+\frac{k-1}{4}R,\\
\dot{x}_6 &= x_1^2x_2+\frac{k-1}{4}R,
\end{aligned} \tag{4}$$

where $0 \leq k \leq 1$ and

$$R = \frac{x_1^3}{4}+3x_1^2x_2+4x_2(x_4+x_5-2x_6)^2+x_1\Lambda.$$

Lemma 3. *System* (4) *has an invariant hyperplane* $\{x_1 = 0\}$ *and no periodic solution in this hyperplane.*

Proof. Obviously the hyperplane $\{x_1 = 0\}$ is invariant manifold for system (4).

Suppose that system (4) has a periodic solution $\Gamma(t) = (x_1, x_2, x_3, x_4, x_5, x_6)(t)$ with period $T > 0$ in $x_1 = 0$. We have that $x_1 = 0$, $x_2 = a$ and $x_3 = b$ where a, b are constants, and x_4, x_5, x_6 satisfy

$$\begin{aligned}
\dot{x}_4 &= a(k-3)(x_4-x_5+x_6)^2,\\
\dot{x}_5 &= a(k+1)(x_4-x_5+x_6)^2,\\
\dot{x}_6 &= a(k-1)(x_4-x_5+x_6)^2.
\end{aligned} \tag{5}$$

The set of singular points of (5) is the plane $x_4 - x_5 + x_6 = 0$. All the orbits of (5) shall be attracted or repelled by this plane. Moreover x_4, x_5 and x_6 are monotone. Hence if (x_4, x_5, x_6) is a periodic solution of (4), then (x_4, x_5, x_6) has to be a singular point in the phase portrait of (5), i.e. a constant vector. So $\Gamma(t)$ is a singular point instead of a periodic solution. □

Proposition 4. *Bianchi IV system* (4) *has no periodic solutions.*

Proof. In view of Lemma 3 we consider that system (4) has a periodic solution $\Gamma(t) = (x_1, x_2, x_3, x_4, x_5, x_6)(t)$ with period $T > 0$ contained in the region $x_1 \neq 0$. From (4) we have $x_4' + x_5' - 2x_6' = x_1^3/4 \neq 0$. If the function $x_4(t) + x_5(t) - 2x_6(t)$ is a periodic function, then there is some t_0 such that its derivative is zero, so $x_1^3(t_0) = 4(x_4'(t_0) + x_5'(t_0) - 2x_6'(t_0)) = 0$, which is a contradiction. So we finish the proof of Proposition 4. □

4. The Bianchi III, VI and VII model

Consider the rest of the cases, i.e. the type III, VI and VIII of Bianchi models. After the change of coordinates and time

$$x_i = q_i, \; x_{i+3} = p_i q_i, \quad i = 1, 2, 3,$$
$$\frac{d\tau_0}{d\tau} = \frac{N^3 (q_1 q_2 q_3)^{\frac{k-1}{2}}}{2},$$

where $N = x_1 - n_2 x_2$ and $n_2 = \pm 1$, system (2) can be written as the six-dimensional homogeneous polynomial differential system of degree 5.

$$\begin{aligned}
\dot{x}_1 = {} & x_1 N^3(-x_4 + x_5 + x_6), \\
\dot{x}_2 = {} & x_2 N^3(x_4 - x_5 + x_6), \\
\dot{x}_3 = {} & x_3 N^3(x_4 + x_5 - x_6), \\
\dot{x}_4 = {} & \frac{1}{4} x_1 N^4 + a^2 x_1 x_2 N^3 + \frac{k-1}{4} NS \\
& -2a^2 x_1 x_2 (x_1 + n_2 x_2)(x_4 - x_5 + x_6)^2, \\
\dot{x}_5 = {} & \frac{1}{4} N^5 - \frac{1}{4} x_1 N^4 + a^2 x_1 x_2 N^3 + \frac{k-1}{4} NS \\
& +2a^2 x_1 x_2 (x_1 + n_2 x_2)(x_4 - x_5 + x_6)^2, \\
\dot{x}_6 = {} & a^2 x_1 x_2 N^3 + \frac{k-1}{4} NS,
\end{aligned} \tag{6}$$

where $0 \leq k \leq 1$ and

$$S = N^4/4 + 4a^2 x_1 x_2 (x_1 + n_2 x_2)(x_4 + x_5 - 2x_6)^2 + N^2(3a^2 x_1 x_2 + \Lambda).$$

Lemma 5. *System* (6) *has an invariant hyperplane* $\{N = 0\}$ *and no periodic solution in this hyperplane.*

Proof. Let $\mathcal{X}$ the vector field associated to the differential system (6). Hence

$$\mathcal{X}(N) = \dot{x}_1 - n_2 \dot{x}_2 = N^3((x_1 + n_2 x_2)(-x_4 + x_5) + N x_6).$$

The hyperplane $N = 0$ is invariant by the flows of system (6).

Suppose that system (6) has a periodic solution $\Gamma(t) = (x_1, x_2, x_3, x_4, x_5, x_6)(t)$ with period $T > 0$ in $N = 0$. Then we have that $x_1 = c_1$, $x_2 = c_2$ and $x_3 = c_3$ where c_i are constants, and x_4, x_5, x_6 satisfy

$$\begin{aligned}
\dot{x}_4 = {} & -2a^2 c_1 c_2 (c_1 + n_2 c_2)(x_4 - x_5 + x_6)^2, \\
\dot{x}_5 = {} & 2a^2 c_1 c_2 (c_1 + n_2 c_2)(x_4 - x_5 + x_6)^2, \\
\dot{x}_6 = {} & 0.
\end{aligned} \tag{7}$$

Hence if (x_4, x_5, x_6) is a periodic solution of (7), using the same kind of arguments as in the proof of Lemma 3, (x_4, x_5, x_6) has to be a constant vector, i.e. a singular point in the phase portrait of (7). So $\Gamma(t)$ is a singular point instead of a periodic solution. $\square$

Proposition 6. *The type III, VI and VII of Bianchi system* (6) *has no periodic solutions.*

Proof. In view of Lemma 5, we consider that system (6) has a periodic solution $\Gamma(t) = (x_1, x_2, x_3, x_4, x_5, x_6)(t)$ with period $T > 0$ contained in the region $N \neq 0$. By the time rescaling $d\tau = N dt$, system (6) can be written as

$$
\begin{array}{rl}
x_1' = & x_1 N^2(-x_4 + x_5 + x_6),\\
x_2' = & x_2 N^2(x_4 - x_5 + x_6),\\
x_3' = & x_3 N^2(x_4 + x_5 - x_6),\\
x_4' = & \dfrac{1}{4} x_1 N^3 + a^2 x_1 x_2 N^2 + \dfrac{k-1}{4} S\\
 & -2a^2 x_1 x_2 (x_1 + n_2 x_2)(x_4 - x_5 + x_6)^2/N,\\
x_5' = & \dfrac{1}{4} N^4 - \dfrac{1}{4} x_1 N^3 + a^2 x_1 x_2 N^2 + \dfrac{k-1}{4} S\\
 & +2a^2 x_1 x_2 (x_1 + n_2 x_2)(x_4 - x_5 + x_6)^2/N,\\
x_6' = & a^2 x_1 x_2 N^2 + \dfrac{k-1}{4} S,
\end{array} \tag{8}
$$

where $' = d/d\tau$. From (8) we have $x_4'(\tau) + x_5'(\tau) - 2x_6'(\tau) = N^4/4 \neq 0$. If $\Gamma(t)$ is a periodic orbit of (6) in time variable t, then it is also a periodic orbit of (8) in time variable τ. Then the periodic function $x_4(t) + x_5(t) - 2x_6(t)$ in the variable t implies that $x_4(\tau) + x_5(\tau) - 2x_6(\tau)$ is also a periodic function in τ. Hence there is some τ_0 such that the derivative of the function $x_4(\tau) + x_5(\tau) - 2x_6(\tau)$ is zero, i.e. $N^4(\tau_0) = 4(x_4'(\tau_0) + x_5'(\tau_0) - 2x_6'(\tau_0)) = 0$, which is a contradiction. This completes the proof of Proposition 6. $\square$

Proof of Theorem 1. From Propositions 2, 4 and 6, Theorem 1 follows. $\square$

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References

[1] L. Bianchi, *Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti (On the spaces of three dimensions that admit a continuous group of movements)*, Soc. Ital. Sci. Mem. di Mat. **11** (1898), 267–352.

[2] L. Bianchi, *Lezioni sulla teoria dei gruppi continui finiti di transformazioni (Lectures on the theory of finite continuous transformation groups)*, Spoerri, Pisa (1918), 550–557.

[3] O.I. Bogoyavlensky, *Methods in the qualitative theory of dynamical systems in astrophysics and gas dynamics*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1985.

[4] C.A. Buzzi and J. Llibre, *No periodic orbits for the type A Bianchi's systems*, J. Nonlinear Math. Phys. **22**(2015), 170–179.

[5] C.A. Buzzi, J. Llibre and P.R. da Silva, *On the dynamics of the Bianchi IX System*, J. Phys. A: Math. Gen. **40** (2007), 7187–7192.

[6] J. Chavarriga and I.A. García, *Integrability and explicit solutions in some Bianchi cosmological dynamical systems*, J. Nonlinear Math. Phys. **8** (2001), 96–105.

[7] C.B. Collins, *More qualitative cosmology*, Comm. Math. Phys. **23**(1971), 137–158.

[8] G. Ellis and M. MacCallum, *A class of homogeneous cosmological models*, Comm. Math. Phys. **12**(1969), 108–141.

[9] A. Ferragut, J. Llibre and C. Pantazi, *Analytic Integrability of Bianchi A cosmological models*, J. of Geometry and Physics **62** (2012), 381–386.

[10] R.M. Gad and A.S. Alofi, *Homothetic vector field in plane symmetric Bianchi type-I cosmological model in Lyra geometry*, Modern Phys. Lett. A **29** (2014), no. 22, 1450116, 12 pp.

[11] J. Llibre and J. Yu, *On the periodic orbits of the static, spherically symmetric Einstein-Yang-Mills equations*, Comm. Math. Phys. **286**(2009), 277–281.

[12] K.E. Starkov, *Compact invariant sets of the Bianchi VIII and Bianchi IX Hamiltonian systems*, Phys. Lett. A **375** (2011), no. 36, 3184–3187.

[13] A. H. Taub, *Empty space-times admitting a three parameter group of motions*, Ann. Math. **53** (1951), 472–490.

[14] J. Wainwright and G.F.R. Ellis, *Dynamical systems in cosmology*, Cambridge University Press, Cambridge, 1997.

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