This is a preprint of: "Parrondo's dynamic paradox for the stability of non-hyperbolic fixed points", Anna Cima, Armengol Gasull, Víctor Mañosa, *Discrete Contin. Dyn. Syst.*, vol. 38(2), 889–904, 2018.
DOI: [10.3934/dcds.2018038]

Parrondo's dynamic paradox for the stability of non-hyperbolic fixed points

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Abstract

We show that for periodic non-autonomous discrete dynamical systems, even when a common fixed point for each of the autonomous associated dynamical systems is repeller, this fixed point can became a local attractor for the whole system, giving rise to a Parrondo's dynamic type paradox.

Mathematics Subject Classification 2010: 37C05, 37C25, 37C75, 39A30.
Keywords: Periodic discrete dynamical systems, non-hyperbolic points, local and global asymptotic stability, Parrondo's dynamic paradox.

1 Introduction and main results

The study of periodic discrete dynamical systems is a classical topic that has attracted the researcher's interest in the last years, among other reasons, because they are good models for describing the dynamics of biological systems under periodic fluctuations whether due to external disturbances or effects of seasonality, see [4, 15, 16, 17, 25, 26, 27] and the references therein.

These k-periodic systems can be written as

$$x_{n+1} = f_{n+1}(x_n), \tag{1}$$

with initial condition x_0, and a set of maps $\{f_m\}_{m\in\mathbb{N}}$ such that $f_m = f_\ell$ if $m \equiv \ell \,(\bmod k)$. For short, the set $\{f_1, \ldots, f_k\}$ will be called *periodic set.* We also will assume that all $f_m : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{U}$ being $\mathcal{U}$ an open set of $\mathbb{R}^n$.

It is well-known that given a periodic discrete dynamical system (1), it can be studied via the *composition map* $f_{k,k-1,\dots,1} = f_k \circ f_{k-1} \circ \cdots \circ f_1$. For instance, if all maps $f_m \in \{f_1, f_2, \dots, f_k\}$ share a common fixed point p, the nature of the steady state $x = p$ can be studied through the nature of the fixed point p of $f_{k,k-1,\dots,1}$. In the same way, the attractor of a periodic discrete dynamical system (1) is the union of attractors of some composition maps, see [17, Thms. 3 and 6].

A specially interesting case occurs when all the maps in the periodic set have a fixed point which is a *global asymptotically stable* (GAS) and the periodic system has a global asymptotically stable periodic orbit [1, 4, 6, 15]. In this setting, the simplest situation corresponds to the case when all the autonomous maps share the same fixed point which is GAS for all of them and it is also a GAS fixed point for composition map, see [31]. It is known this is not a general phenomenon, see for instance Examples 5 or 6 of next section.

We will focus on studying the stability of fixed points of k-periodic systems which are common fixed points of all the maps in the periodic set. We restrict ourselves to this setting because it is the simplest type of "periodic orbit" that a periodic dynamical system can have.

Notice that given two stable 2×2 matrices*, A_1 and A_2, it holds that $|\det(A_i)| < 1$ and hence $|\det(A_2A_1)| = |\det(A_2)\det(A_1)| < 1$. As a consequence, the fixed point of any composition map $f_{k,\dots,1}$ in $\mathbb{R}^n$ (linear or non-linear) resulting of the composition of k maps f_j with a common hyperbolic fixed point, which is asymptotically stable for all them, must be generically either asymptotically stable or a saddle, but it can never be repeller. A similar result happens with maps with a common hyperbolic repeller: generically this point is either repeller or a saddle for the composition map, but never a *local asymptotically stable* (LAS) fixed point. Hence, in this paper, to show that this third possibility may happen in both situations, we will need to deal with non-hyperbolic fixed points. We recall the definitions of LAS, GAS, repeller and semi-AS fixed point in Section 2.

The so called *Parrondo's paradox* is a paradox in game theory, that essentially says that *a combination of losing strategies becomes a winning strategy,* see [18, 23]. We will prove that in the non-hyperbolic case *the periodicity can destroy the repeller character of the common fixed points, giving rise to attracting points for the complete non-autonomous system*, showing, in consequence, the existence of a kind of *Parrondo's dynamic type paradox* for periodic discrete dynamical systems. The phenomenon that we will show is in a simpler setting than the one presented in [7], because there the authors combine periodically one-dimensional maps f_1 and f_2 to give rise to chaos or order.

We start studying the one-dimensional case. The tools for determining the stability of

*All their eigenvalues have modulus smaller than 1.

non-hyperbolic fixed points for one-dimensional analytical maps are well established, see Section 3.1. As we will see, one the key points is the computation of the so called *stability constants* for studying the stability of non-hyperbolic fixed points of one-dimensional non-orientable analytic maps.

Using these tools, in the periodic case we prove the following result which implies that, contrary to what we will show that happens in even dimensions, it is *impossible* to find *two* one-dimensional maps sharing a fixed point which is repeller, and such that the composition map has a LAS fixed point. However, it is possible to find a LAS fixed point when three or more maps sharing a repeller fixed point are composed (that is for k-periodic systems with $k \geq 3$) giving rise to the Parrondo's dynamic paradox.

Theorem A. *The following statements hold:*

(a) Consider two analytic maps $f_i : \mathcal{U} \subseteq \mathbb{R} \to \mathcal{U}$, $i = 1, 2$ having a common fixed point $p \in \mathcal{U}$ which is LAS (resp. repeller). Then, the point p is either LAS (resp. repeller) or semi-AS for the composition map $f_{2,1}$ and both possibilities may happen.

(b) There are $k \geq 3$ polynomial maps $f_i : \mathcal{U} \subseteq \mathbb{R} \to \mathcal{U}$, $i = 1, 2, \ldots, k$ sharing a common fixed point $p \in \mathcal{U}$ which is LAS (resp. repeller) for all them and such that p is a repeller (resp. a LAS) fixed point for the composition map $f_{k,k-1,\ldots,1}$.

The situations stated in item (b) of the above theorem when $k = 3$, happen for instance in Examples 1 and 2 of Section 2.

Next we consider the same problem for planar maps. Again we must pay attention to the stable/repeller character for non-hyperbolic fixed points. We restrict ourselves to maps with *elliptic* fixed points. These points are fixed points for which the eigenvalues of the associated linear part lie in the unit circle, but excluding the values ± 1. For most of them it is possible to get their Birkhoff normal form, which permits to compute the so called *Birkhoff constants* and from them the *Birkhoff stability constants.* Using these last constants it is possible to distinguish between LAS and repeller fixed points. We recall all these concepts in Section 4.1.

The problem of the local stability of *parabolic* fixed points (eigenvalues ± 1) is much more involved, see [3, 22, 28, 29] for instance, but we do not need to use them to get our results.

Now we can state our main result for the planar case, that presents again a Parrondo's type paradox in the two-periodic setting. Some simple examples of maps f_1 and f_2 illustrating it are given in Examples 7 and 8 of the next section.

Theorem B. *There exist polynomial maps f_1 and f_2 in $\mathbb{R}^2$ sharing a common fixed point p which is a LAS (resp. a repeller) fixed point for both of them, and such that p is repeller (resp. LAS) for the composition map $f_{2,1}$.*

Combining the maps that allow to prove item (b) of Theorem A and Theorem B we can prove the following result that extends these theorems to arbitrary dimensions.

Theorem C. *The following statements hold.*

(a) *For all $n \geq 1$ there exist $k \geq 3$ polynomial maps $f_i : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, for $i \in \{1, \ldots, k\}$, sharing a common fixed point p which is LAS (resp. repeller) for each map, and such that p is repeller (resp. LAS) for the composition map $f_{k,k-1,\ldots,1}$. Furthermore, for one-dimensional maps ($n = 1$), this result is optimal on k, that is, it is not possible to find only two of such maps such that the corresponding composition map $f_{2,1}$ satisfies the given properties.*

(b) *For all $n = 2m \geq 2$ there exist 2 polynomial maps $f_1, f_2 : \mathcal{U} \subseteq \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, sharing a common fixed point p which is LAS (resp. repeller) for both maps, and such that p is repeller (resp. LAS) for the composition map $f_{2,1}$.*

Although the above theorem is stated for polynomial maps, using similar techniques, it is easy to construct examples with the same properties but with less regularity, say of class $\mathcal{C}^m$ for any $m \geq 6$ (resp. $m \geq 4$) in item (a) (resp. item (b)). As we will see, this restriction comes from the use of normal forms and Taylor expansions involving terms until order five (resp. order three) in the construction of our examples.

From its statement it is natural to wonder if item (a) of the theorem could be improved for $n \geq 3$, odd, taking $k \geq 2$. We continue thinking on this question.

The paper is structured as follows. In Section 2 we collect all the explicit examples used to prove Theorems A and B and other that help to contextualize the problem that we consider. Section 3 is devoted to prove the results in the one-dimensional case. In particular, in Subsection 3.1 we give the expression of the first stability constants and we prove Theorem A. Section 4 is devoted to prove Theorems B and C.

2 Examples and some definitions

We start recalling some definitions, see [13, 14, 21].

Definition 1. *A fixed point p of a map $f : \mathcal{U} \subset \mathbb{R}^n \to \mathcal{U}$ is said to be:*

(i) Locally asymptotically stable (LAS) *if it is stable and it is locally attractive. That is, if given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $||x_0 - p|| < \delta$ *then* $||f^n(x_0) - p|| < \varepsilon$ *for all* $n \geq 1$ *(estable), and there exists* $\eta > 0$ *such that if* $||x_0 - p|| < \eta$ *then* $\lim_{n\to\infty} f^n(x_0) = p$ *(attractive). The point is* globally asymptotically stable in $\mathcal{U}$ (GAS)*, if it is LAS and* $\lim_{n\to\infty} f^n(x_0) = p$ *for all* $x_0 \in \mathcal{U}$*.*

(ii) Repeller *if there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *and for all* $x_0 \neq p$ *such that* $||x_0 - p|| < \varepsilon$*, there exists* $n = n(x_0) \in \mathbb{N}$ *such that* $||f^n(x_0) - p|| > \varepsilon$*.*

If f *is a one-dimensional map, the fixed point is called:*

(iii) Semi–asymptotically stable† *(semi-AS) from the left (resp. right) if given* $\varepsilon > 0$*, there exists* δ *such that if* $x_0 \in (p - \delta, p)$ *(resp.* $x_0 \in (p, p + \delta)$*) then* $|f^n(x_0) - p| < \varepsilon$ *for all* $n \geq 1$*, and there exists* $\eta > 0$ *such that if* $x_0 \in (p - \eta, p)$ *(resp.* $x_0 \in (p, p + \eta)$*) then* $\lim_{n\to\infty} f^n(x_0) = p$*, and there exists* $\eta > 0$ *such that if* $x_0 \in (p, p + \eta)$ *(resp.* $x_0 \in (p - \eta, p)$*) then there exists* $n \in \mathbb{N}$ *such that* $|f^n(x_0) - p| > \eta$*.*

We remark that for invertible maps, instead of definition (ii) it is simpler to say that a fixed point p is a repeller for f if p is an attractor for f^{-1}.

Next we collect several examples that illustrate the main results of this paper. We start with a one-dimensional example that gives the clue for proving item (b) of Theorem A.

Example 1. Consider the maps:

$$\begin{aligned}
f_1(x) &= -x + 3x^2 - 9x^3 + 164x^5, \\
f_2(x) &= -x + 5x^2 - 25x^3 + 1259x^5, \\
f_3(x) &= -x + 2x^2 - 4x^3 + 33x^5.
\end{aligned}$$

These maps have been chosen using the expressions of the stability constants given in Proposition 5 of next section, in such a way that they satisfy $V_3(f_i) = 0$ and $V_5(f_i) < 0$ for $i \in \{1, 2, 3\}$. Then, they have a LAS fixed point at the origin (see Theorem 4). Moreover, $f_{3,2,1}(x) = -x + 90x^4 - 48x^5 + O(6)$. Computing the stability constants for this map we obtain that $V_3(f_{3,2,1}) = 0$ and $V_5(f_{3,2,1}) = 96 > 0$. Hence, using again Theorem 4, we get that the origin is a repeller fixed point of $f_{3,2,1}$. In the proof of item (b) of Theorem A we will explain their whole process of construction.

Clearly, taking the local inverses of these maps at the origin until order five we will have an example of the other situation stated in item (b) of Theorem A, which precisely is the one that gives rise to the Parrondo's dynamic paradox. We present it in the next example.

† Also named *saddle-node* or *shunt*.

Example 2. Consider the maps

$$
\begin{aligned}
g_1(x) &= T_5(f_3^{-1}(x)) = -x + 2x^2 - 4x^3 + 31x^5,\\
g_2(x) &= T_5(f_2^{-1}(x)) = -x + 5x^2 - 25x^3 + 1241x^5,\\
g_3(x) &= T_5(f_1^{-1}(x)) = -x + 3x^2 - 9x^3 + 160x^5,
\end{aligned}
$$

where the f_j are the maps given in Example 1 and T_5 means the Taylor polynomial of degree 5 at the origin. These maps have a local repeller at the origin but the composition map $g_{3,2,1}(x) = -x + 90x^4 + 48x^5 + O(6)$ has an attractor at the origin, because its Taylor polynomial of degree 5 coincide with the one of the inverse of $f_{3,2,1}$. In fact the origin is LAS.

Remark 2. It is interesting to observe that the order in the periodic set is very important. For instance, in Example 2, we have seen that the origin of the composition map $g_{3,2,1}$ is LAS. Nevertheless, by using the stability constants, it can be seen that the origin of $g_{1,2,3}(x) = -x + 90x^4 - 72x^5 + O(6)$ is repeller.

Next example shows that even when two maps have a common GAS fixed point, the corresponding composition map does not need to have a LAS fixed point.

Example 3. Consider the maps

$$
f_1(x) = \begin{cases} -x + x^2 & x \le 1,\\ 0 & x > 1, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} -x + 2x^2 & x \le 1/2,\\ 0 & x > 1/2. \end{cases}
$$

It is easy to check that the origin is a GAS fixed point for both of them. Their corresponding composition map is

$$
f_{2,1}(x) = f_2 \circ f_1(x) = \begin{cases} 0 & x < \frac{1-\sqrt{3}}{2},\\ x + x^2 - 4x^3 + 2x^4 & x \in \left[\frac{1-\sqrt{3}}{2}, 1\right],\\ 0 & x > 1. \end{cases}
$$

It is not difficult to see that the origin is a fixed point, semi-AS from the left for the composition map (see the Theorem 3 in next section). Moreover this map also has another fixed point at $x = 1 - \sqrt{2}/2$. Hence the origin is neither a global attractor of $f_{2,1}$ nor stable. Gluing the three pieces of this example with some suitable bump functions, it is possible to obtain differentiable or $\mathcal{C}^\infty$ examples with the same features.

To end with the one-dimensional examples, and although it is out of the periodic systems framework, we consider non-periodic, non-autonomous system

$$
x_{n+1} = f_{n+1}(x_n), \tag{2}
$$

with a non-hyperbolic fixed point p. We will show that it is possible to find maps f_n sharing this common fixed point p, which is a GAS for each of them, and such that the system has an unbounded solution.

Example 4. We will construct a family of functions $\{f_n\}_{n\geq 0}$ such that the origin is GAS for each f_n but the unbounded sequence $y_n = (-1)^n(n+1)$ for $n \geq 0$ is a solution of (2).

Consider the following auxiliary map $g_a(x) = a\,(\exp(-x/a) - 1)$. Let a_0 be the solution of the equation $g_a(1) = -2$. That is,

$$a_0 = \frac{2}{2W_{-1}\big(-\exp(-1/2)/2\big) + 1} \simeq -0.7959,$$

where $W_{-1}(x)$ is the secondary branch of the Lambert W-function, [12, 24].

Now we take $f_0 := g_{a_0}$. The map f_0 has a GAS fixed point at the origin[‡] which is non-hyperbolic, since $f_0'(0) = -1$, and it satisfies $f_0(1) = -2$.

In order to construct the maps f_n for $n \geq 1$, we consider the following linear conjugations

$$h_n(x) := \frac{(-1)^n}{3}\big((2n+3)x + n\big).$$

These maps are chosen in such a way that $h_n(-2) = (-1)^{n+1}(n+2) = y_{n+1}$ and $h_n^{-1}(y_n) = 1$. Now, we define

$$f_n(x) := h_n \circ f_0 \circ h_n^{-1}(x),\ n \geq 1.$$

Obviously each map has a GAS point at the origin, because they are conjugate to f_0. Finally, observe that for all $n \in \mathbb{N}$,

$$f_n(y_n) = h_n \circ f_0 \circ h_n^{-1}(y_n) = h_n \circ f_0(1) = h_n(-2) = y_{n+1}.$$

Hence the unbounded sequence $\{y_n\}_{n\in\mathbb{N}}$ is a solution of (2) with initial condition $x_0 = 1$.

We continue this section with some simple linear two-dimensional examples.

Example 5. This first example shows two linear maps such that for each of them the origin which is GAS (in fact a super-attracting point), but the origin is a saddle point of the composition map $f_{2,1}$. Hence the origin is an unstable steady state of the periodic system, but not repeller. Set $\mathbf{x} = (x, y)$ and $f_i(\mathbf{x}) = A_i \cdot \mathbf{x}^t$, where

$$A_1 = \begin{pmatrix} 0 & 2 \\ 0 & \frac{1}{2} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 2 & 0 \end{pmatrix}.$$

[‡]Since $f_0(-\infty, 0) \subset (0, +\infty)$ and $f_0(0, +\infty) \subset (-\infty, 0)$ it is enough to check that if $x > 0$, then for $k \geq 1$ we have $f_0^{2k}([0, x]) \subset [0, a_k)$, where $\lim_{k\to+\infty} a_k = 0^+$; and if $x < 0$, then for $k \geq 1$ we have $f_0^{2k-1}([0, x]) \subset [0, b_k)$, where $\lim_{k\to+\infty} b_k = 0^+$.

Observe that $\mathrm{Spec}\,(A_1) = \mathrm{Spec}\,(A_2) = \{0, 1/2\}$, so the origin is GAS for the dynamical systems associated to f_1 and f_2. The corresponding composition map associated to the 2-periodic system is $f_{2,1}(\mathbf{x}) = A_{2,1} \cdot \mathbf{x}^t$ where

$$A_{2,1} := A_2 \cdot A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix}.$$

Since $\mathrm{Spec}\,(A_{2,1}) = \{0, 4\}$, the origin is a saddle point for $f_{2,1}$.

Example 6. In [5] the authors consider the maps $f_i(\mathbf{x}) = A_i \cdot \mathbf{x}^t$, $i = 1, 2$ where

$$A_1 = \alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

with $|\alpha| < 1$. Both maps have the origin as a GAS point because $\mathrm{Spec}\,(A_1) = \mathrm{Spec}\,(A_2) = \{\alpha\}$. Then the composition map is $f_{2,1}(\mathbf{x}) = A_{2,1} \cdot \mathbf{x}^t$ with

$$A_{2,1} = \alpha^2 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and it is such that $\mathrm{Spec}\,(A_{2,1}) = \{\left(3 \pm \sqrt{5}\right)\alpha^2/2\}$. Hence the origin is either GAS if $|\alpha| < (\sqrt{5} - 1)/2 \simeq 0.618$, or a saddle point if $(\sqrt{5} - 1)/2 < |\alpha| < 1$. Again the stability can be lost but again no repeller fixed points appear.

Another nice linear example is given in [20, p. 8]. There the author shows two linear maps, both with a stable focus at the origin, and such that the corresponding composition map has again a saddle at the origin.

We end this section with two planar examples that show that taking a periodic set with only two elements with non-hyperbolic fixed points, the stabilities of the common fixed point can be reversed for the corresponding composition maps. In fact they allow to prove Theorem B. Their constructions are detailed in Section 4.2.

Example 7. Consider the maps

$$\begin{aligned} f_1(x, y) &= \left(-y + 2x^2 + 6xy, x - 3x^2 + 2xy + 3y^2\right), \\ f_2(x, y) &= \left(\frac{x}{2} - \frac{\sqrt{3}}{2}y - x\,(x^2 + y^2), \frac{\sqrt{3}}{2}x + \frac{1}{2}y - y\,(x^2 + y^2)\right). \end{aligned}$$

As we will see in the proof of Theorem B in Section 4, the origin is a LAS fixed point for both maps f_1 and f_2, because their Birkhoff stability constants are $V_1(f_1) = V_1(f_2) = -1/2 < 0$. However the origin is a repeller fixed point for the composition map $f_{2,1}$ because $V_1(f_{2,1}) = (3\sqrt{3} - 5)/2 \simeq 0.098 > 0$.

Each map f_i is locally invertible in a neighborhood of the origin. Hence, as we did for passing from Example 1 to Example 2, by taking their local inverses f_i^{-1}, we have maps with a repeller fixed point at the origin such that their composition map has a LAS fixed point at the origin, giving an example of the remaining case considered in Theorem B. Anyway, we also include an explicit independent example.

Example 8. The origin is a repeller fixed point for both maps

$$f_1(x,y) = \left(-y + \frac{1}{3}x^2 - 8xy + \frac{5}{3}y^2, x + 4x^2 - \frac{4}{3}xy - 4y^2\right),$$
$$f_2(x,y) = \left(\frac{x}{2} - \frac{\sqrt{3}}{2}y + x(x^2+y^2), \frac{\sqrt{3}}{2}x + \frac{1}{2}y + y(x^2+y^2)\right),$$

because their corresponding Birkhoff stability constants are $V_1(f_1) = V_1(f_2) = 1/2 > 0$. Now the origin is a LAS fixed point for $f_{2,1}$, because $V_1(f_{2,1}) = 3 - 2\sqrt{3} \simeq -0.464 < 0$.

3 One-dimensional maps

3.1 Stability of fixed points

In this section we consider one-dimensional analytic maps with a fixed point that, without loss of generality, we take as the origin and we denote by $\mathcal{U}$ a neighborhood of this point. As we have already mentioned, it is clear that, from the view point of the stability problem for composition, the more interesting maps are the ones having non-hyperbolic fixed points. A summary of several results concerning this situation can also be found in [13]. Next we recall some of them and also develop some new results for the orientation reversing case.

This first result is well-known and characterizes the local dynamics at a non-hyperbolic non-oscillatory fixed points one-dimensional maps f (i.e. $f'(0) = 1$, that is, when f is *locally orientation preserving*):

Theorem 3 ([13])**.** *Let f be a $\mathcal{C}^{m+1}(\mathcal{U})$ function such that $f(0) = 0$, $f'(0) = 1$ and*

$$f(x) = x + a_m x^m + O(m+1), \text{ with } a_m \neq 0,\ m \geq 2.$$

Then:

(a) If m is even then the origin is semi-AS from the left if $a_m > 0$ and from the right if $a_m < 0$.

(b) If m is odd then the origin is repeller if $a_m > 0$ and LAS if $a_m < 0$.

The complete study of the local dynamics at a non-hyperbolic oscillatory fixed point (i.e. when $f(0) = 0$ and $f'(0) = -1$, that is when f is *locally orientation reversing*), is more involved. In [13, Thm 5.1] a result is given in terms of the derivatives of the orientation preserving map $f^2 = f \circ f$, by using Theorem 3. However in [13, Thm 5.4], to avoid using these derivatives the authors present a slightly more explicit expressions obtained using the Faà di Bruno Formula ([19]). The expressions in [13, Thm 5.1] are closely related with what we call *stability constants*, that we introduce below.

Given a $\mathcal{C}^\omega(\mathcal{U})$ function of the form

$$f(x) = -x + \sum_{j\geq 2} a_j x^j,$$

one obtains

$$f^2(x) := f \circ f(x) = x + \sum_{j\geq 3} W_j(a_2, \ldots, a_j) x^j.$$

If f is not an involution (i.e. $f^2 \neq \mathrm{Id}$), we define the *stability constant of order* $\ell \geq 3$, as V_ℓ, where

$$V_3 := W_3(a_2, a_3) \quad \text{and} \quad V_\ell := W_\ell(a_2, \ldots, a_\ell) \ \text{ if } \ W_j = 0,\, j = 3, \ldots, \ell - 1.$$

Next result shows that the first non-zero stability constant is for ℓ odd and gives the stability of the fixed point.

Theorem 4. *Let f be an analytic map in $\mathcal{U} \subseteq \mathbb{R}$ such that $f(0) = 0$, $f'(0) = -1$. If f is not an involution, then there exists $m \geq 1$ such that $V_3 = V_4 = V_5 = \cdots = V_{2m} = 0$ and $V_{2m+1} \neq 0$. Moreover, if $V_{2m+1} < 0$ (resp. $V_{2m+1} > 0$), the origin is LAS (resp. repeller).*

Proof. We start proving that the first non-zero stability constant has odd order. Suppose, to arrive to a contradiction, that $f^2(x) - x = V_{2m} x^{2m} + O(2m + 1)$ with $V_{2m} \neq 0$. Assume, for instance that $V_{2m} > 0$. Then we can consider a neighborhood of the origin $\tilde{\mathcal{U}} \subseteq \mathcal{U}$ such that for all $x \in \tilde{\mathcal{U}} \setminus \{0\}$, f is strictly monotonically decreasing and $f^2(x) - x > 0$.

Let $x_0 \in \tilde{\mathcal{U}} \setminus \{0\}$ and consider its orbit $x_n = f^n(x_0)$. We take x_0 small enough with $x_1, x_2, x_3 \in \tilde{\mathcal{U}} \setminus \{0\}$. We have that $x_2 - x_0 = f^2(x_0) - x_0 > 0$. Since f is decreasing, $f(x_2) < f(x_0)$, that is, $f^2(x_1) < x_1$, a contradiction.

Now, the theorem is a direct corollary of statement (b) of Theorem 3. ∎

Finally, we give an expression of some stability constants. It is clear that the regularity of the function can be weakened in their computation, because the only needed tools are the Taylor expansions at the origin.

Proposition 5. *Let f be a $\mathcal{C}^{12}(\mathcal{U})$ function such that $f(0) = 0$, $f'(0) = -1$. Then the first stability constants are*

$$
\begin{aligned}
V_3 =& 2\left(-{a_2}^2 - a_3\right), \\
V_5 =& 2\left(2{a_2}^4 - 3a_2a_4 - a_5\right), \\
V_7 =& 2\left(-13{a_2}^6 + 18{a_2}^3a_4 - 4a_6a_2 - 2{a_4}^2 - a_7\right), \\
V_9 =& 2\left(145{a_2}^8 - 221{a_2}^5a_4 + 35{a_2}^3a_6 + 50{a_2}^2{a_4}^2 - 5a_2a_8 - 5a_6a_4 - a_9\right), \\
V_{11} =& 2\left(-2328{a_2}^{10} + 3879{a_2}^7a_4 - 561{a_2}^5a_6 - 1263{a_2}^4{a_4}^2 + 61{a_2}^3a_8 + 171{a_2}^2a_4a_6 + \right. \\
& \left. 55a_2{a_4}^3 - 6a_2a_{10} - 6a_4a_8 - 3{a_6}^2 - a_{11}\right),
\end{aligned}
$$

Proof. The constants have been obtained by computing the first coefficient of the Taylor expansion of f^2. It is easy to check that any constant V_{2m+1} contains the monomial $-2a_{2m+1}$. Hence, once V_{2m+1} is obtained, the constant V_{2m+3} is computed by solving $V_{2m+1} = 0$ with respect the coefficient a_{2m+1} and plugging this value in the expression of W_{2m+3}. ∎

We stress that more constants can be easily obtained with very few computing time.

3.2 Proof of Theorem A

Observe that the statement (a) of Theorem A is a consequence of the following result.

Theorem 6. *Consider two analytic maps $f_i : \mathcal{U} \subseteq \mathbb{R} \to \mathcal{U}$, $i = 1, 2$ having a common non-hyperbolic fixed point $p \in \mathcal{U}$ which is LAS (resp. repeller) for both of them. Then p is either LAS (resp. repeller) or semi-AS for the composition map $f_{2,1} = f_2 \circ f_1$. More precisely,*

(a) If one of the maps f_i preserves orientation, then p is LAS (resp. repeller) for $f_{2,1}$.

(b) If both f_1 and f_2 reverse orientation, then p can be either a LAS (resp. repeller) or a semi-AS fixed point for $f_{2,1}$.

To prove it, we introduce the differentiable normal form of an analytic map f with a non-hyperbolic fixed point at 0, which is given by the next result in [8] (see also [2, 9]). A similar result for $\mathcal{C}^\infty$ maps can be found in [30, Thm 2]:

Theorem 7 (K. Chen, [8])**.** *Let f be an* analytic *diffeomorphism on $\mathbb{R}$. If f is orientation preserving (resp. reversing), and it is not an involution, then given any positive integer ℓ there exists a $\mathcal{C}^\ell$ local diffeomorphism φ on $\mathbb{R}$ such that $g = \varphi^{-1} \circ f \circ \varphi$ is in one of the normal forms:*

(a) $g(x) = \lambda x$, with $|\lambda| \neq 1$ and $\lambda > 0$ (resp. $\lambda < 0$),

(b) $g(x) = x + (\pm x)^{m+1} + cx^{2m+1}$ *(resp.* $g(x) = -x \pm x^{m+1} + cx^{2m+1}$*),*

where $c \in \mathbb{R}$*, and* m *is a positive (resp. positive even) integer.*

Next result classifies the stability of each of these normal forms. The proof follows from a straightforward application of Theorems 3 and 4.

Lemma 8. *The following statements hold.*

(a) *The map* $f(x) = x + x^{m+1} + cx^{2m+1}$*, with* $0 < m \in \mathbb{N}$ *and* $c \in \mathbb{R}$*, has a semi-AS from the left fixed point at the origin if* m *is odd, and a repeller fixed point if* m *is even.*

(b) *The map* $f(x) = x - x^{m+1} + cx^{2m+1}$*, with* $0 < m \in \mathbb{N}$ *and* $c \in \mathbb{R}$*, has a semi-AS from the right fixed point at the origin if* m *is odd, and a LAS fixed point if* m *is even.*

(c) *The map* $f(x) = -x + x^{2r+1} + cx^{4r+1}$*, with* $0 < r \in \mathbb{N}$ *and* $c \in \mathbb{R}$*, has a LAS fixed point at the origin.*

(d) *The map* $f(x) = -x - x^{2r+1} + cx^{4r+1}$*, with* $0 < r \in \mathbb{N}$ *and* $c \in \mathbb{R}$*, has a repeller fixed point at the origin.*

To prove Theorem 6, we also need the following result.

Lemma 9. *Let* f *be an analytic map in* $\mathcal{U}$ *which is not an involution, given by* $f(x) = -x + \sum_{j\geq 2} a_j x^j$*. Assume that* $a_{2j} = 0$ *and* $V_{2j+1} = 0$ *for all* $j = 1, 2, \ldots, m$*. Then* $a_{2j+1} = 0$ *for all* $j = 1, 2, \ldots, m$*.*

Proof. We prove the lemma by induction on m. If $m = 1$ then $a_2 = V_3 = 0$. By Proposition 5, since $V_3 = -2(a_2^2 + a_3) = -2\,a_3 = 0$ we get $a_3 = 0$.

Now assume that the result is true for all $j \leq m-1$ and assume that $a_{2j} = 0$ and $V_{2j+1} = 0$ for all $j = 1, 2, \ldots, m$. In particular $a_{2j} = 0$ and $V_{2j+1} = 0$ for all $j = 1, 2, \ldots, m-1$. Applying the induction hypothesis we get $a_{2j+1} = 0$ for all $j = 1, 2, \ldots, m-1$. Then, $f(x) = -x + a_{2m+1}\, x^{2m+1} + O(2m+2)$ which implies that $V_{2m+1} = -2\, a_{2m+1}$, because $f^2(x) = x - 2\, a_{2m+1}\, x^{2m+1} + O(2m+2)$. Since V_{2m+1} is zero, also a_{2m+1} must be zero. ∎

Proof of Theorem 6. We will consider only the situation where f_1 and f_2 have a LAS fixed point; the other case follows similarly.

(a) Let f_1 and f_2 be the two maps, and assume that the second one preserves orientation. Without loss of generality, we can take the first one in one of its normal forms given in Lemma 8, $f_1(x) = \pm x \mp x^{2r+1} + c\, x^{4r+1}$, and the second one, by Theorem 3, as $f_2(x) = x + a\, x^{2n+1} + O(2n+2)$ with $a < 0$. Then,

$$\begin{aligned} f_{2,1}(x) &= \pm x \mp x^{2r+1} + c\, x^{4r+1} + a\, x^{2n+1} \left(\pm 1 \mp x^{2r} + c\, x^{4r}\right)^{2n+1} + O(\min(2r+2, 2n+2)) \\ &= \pm x \mp x^{2r+1} \pm a\, x^{2n+1} + O(\min(2r+2, 2n+2)). \end{aligned}$$

When f_1 also preserves orientation, $f_{2,1}(x) = x - x^{2r+1} + a\,x^{2n+1} + O(\min(2r+2, 2n+2))$. Then, if $r \neq n$, from Theorem 3 the origin is LAS for the composition map. If $r = n$ then $f_{2,1}(x) = x + (a-1)\,x^{2n+1} + O(2n+2)$ and since $a - 1 < 0$, applying again Theorem 3 the result follows.

When f_1 reverses orientation, $f_{2,1}(x) = -x + x^{2r+1} - a\,x^{2n+1} + O(\min(2r+2, 2n+2))$. In this case

$$f_{2,1}^2(x) = x - 2x^{2r+1} + 2a\,x^{2n+1} + O(\min(2r+2, 2n+2)).$$

Applying the same tools that in the previous situation, but to $f_{2,1}^2$, the result also follows.

(b) In this case, without loss of generality, we consider f_1 written in normal form $f_1(x) = -x + x^{2r+1} + c\,x^{4r+1}$. We also consider $f_2(x) = -x + \sum_{k\geq 2} a_k x^k$ such that either $V_3 < 0$ or $V_{2j+1} = 0$ for $j = 1, 2, \ldots, m-1$ for $m \geq 2$, and $V_{2m+1} < 0$. Now we split the proof in three subcases:

(i) Assume first that $V_3 = -2(a_2^2 + a_3) < 0$. Then $f_2(x) = -x + a_2 x^2 + a_3 x^3 + O(4)$, with $a_2^2 + a_3 > 0$ and

$$\begin{aligned} f_{2,1}(x) &= x - x^{2r+1} - c\,x^{4r+1} + a_2\,x^2\left(1 - x^{2r} - c\,x^{4r}\right)^2 - a_3\,x^3\left(1 - x^{2r} - c\,x^{4r}\right)^3 + O(4) \\ &= x + a_2\,x^2 - a_3\,x^3 - x^{2r+1} + O(4). \end{aligned}$$

By Theorem 3, when $a_2 \neq 0$ the origin is semi-AS. If $a_2 = 0$, then $a_3 > 0$ and the origin is LAS for all $r \geq 1$.

(ii) In this second case we suppose that $V_3 = 0$ and that there exists $1 \leq j \leq m-1$ such that $a_{2i} = 0$ for $i = 1, 2, \ldots, j-1$ and $a_{2j} \neq 0$. Since $V_{2i+1} = 0$ for $i = 1, 2, \ldots, j-1$ applying Lemma 9 we have that $a_{2i+1} = 0$ for $i = 1, 2, \ldots, j-1$. Hence $f_2(x) = -x + a_{2j}\,x^{2j} + O(2j+1)$, and

$$\begin{aligned} f_{2,1}(x) &= x - x^{2r+1} - c\,x^{4r+1} + a_{2j}\,x^{2j}\left(1 - x^{2r} - c\,x^{4r}\right)^{2j} + O(\min(2r+2, 2j+1)) \\ &= x - x^{2r+1} + a_{2j}\,x^{2j} + O(\min(2r+2, 2j+1)). \end{aligned}$$

Hence, by Theorem 3, if $j \leq r$ then the origin is a semi-AS fixed point and if $j > r$ then it is a LAS fixed point.

(iii) Finally assume that $V_3 = 0$ and that $a_{2i} = 0$ for $i = 1, 2, \ldots, m-1$. Since $V_{2j+1} = 0$, for $j = 1, 2, \ldots, m-1$, and $V_{2m+1} < 0$, from Lemma 9 we get that $a_{2i+1} = 0$ for $i = 1, 2, \ldots, m-1$. Moreover $m \geq 2$. Consequently $f_2(x) = -x + a_{2m}\,x^{2m} + a_{2m+1}\,x^{2m+1} + O(2m+$

2). Then, $f_2^2(x) = x - 2\,a_{2m+1}\,x^{2m+1} + O(2m+2)$, and therefore $0 > V_{2m+1} = -2\,a_{2m+1}$. Moreover,

$$\begin{aligned} f_{2,1}(x) =& x - x^{2r+1} - c\,x^{4r+1} + a_{2m}\,x^{2m}\left(1 - x^{2r} - c\,x^{4r}\right)^{2m} - \\ & a_{2m+1}\,x^{2m+1}\left(1 - x^{2r} - c\,x^{4r}\right)^{2m+1} + O\big(\min(2r+2, 2m+2)\big) \\ =& x - x^{2r+1} + a_{2m}\,x^{2m} - a_{2m+1}\,x^{2m+1} + O\big(\min(2r+2, 2m+2)\big). \end{aligned}$$

Assume first that $a_{2m} \neq 0$. Then, again by Theorem 3, when $2r+1 > 2m$ then the origin is semi-AS and when $2r+1 < 2m$ the origin is LAS.

Finally, suppose that $a_{2m} = 0$. Applying once more Theorem 3 we obtain that in all cases the origin is LAS because $a_{2m+1} > 0$. ∎

Proof of Theorem A. Statement (a) is a direct consequence of Theorem 6.

(b) Observe that the maps of Examples 1 and 2 prove the statement for $k=3$. Indeed, for instance, remember that the maps of Example 1 have been chosen in such a way that the stability constats satisfy $V_3(f_i) = 0$ and $V_5(f_i) < 0$ for $i \in \{1,2,3\}$, so that they have a LAS fixed point at the origin. The stability of the origin for these maps can also be straightforwardly obtained from the first terms of the Taylor series at the origin of $f_i^2, i = 1,2$. A computation gives $f_{3,2,1}(x) = -x + 90x^4 - 48x^5 + O(6)$, so from Proposition 5 we have $V_3(f_{3,2,1}) = 0$ and $V_5(f_{3,2,1}) = 96 > 0$, and therefore the origin is a repeller fixed point of $f_{3,2,1}$. Since $f_{3,2,1}^2(x) = x + 96x^5 + O(7)$, the result also follows from Theorem 3.

Next we show how to construct the maps of Example 1. We start with some maps

$$f_i(x) = -x + a_{2,i}x^2 + a_{3,i}x^3 + a_{4,i}x^4 + a_{5,i}x^5,\ i \in \{1,2,3\}.$$

To get that $V_3(f_i) = 0$ and $V_5(f_i) < 0$, first we take $a_{3,i} = -a_{2,i}^2$, and then impose that $V_5(f_i) = 2\left(2a_{2,i}^4 - 3a_{2,i}a_{4,i} - a_{5,i}\right) = -2A_i^2$, obtaining $a_{5,i} = 2a_{2,i}^4 - 3a_{2,i}a_{4,i} + A_i^2$.

At this point we notice that

$$f_{3,2,1}(x) = -x + (a_{2,1} + a_{2,3} - a_{2,2}^2)x^2 - (a_{2,1} + a_{2,3} - a_{2,2}^2)^2x^3 + O(4),$$

hence $V_3(f_{3,2,1}) = 0$. In order to reduce parameters and simplify the expressions we take $a_{2,1} = a_{2,2}^2 - a_{2,3}$, obtaining $f_{3,2,1}(x) = -x + (3a_{2,2}^2a_{2,3} - 3a_{2,2}a_{2,3}^2 + a_{4,1} - a_{4,2} + a_{4,3})x^4 + O(5)$. Again, to reduce parameters we take $a_{2,3} = 2$ and $a_{4,1} = a_{4,2} = a_{4,3} = 0$. With this choice we get that

$$V_5(f_{3,2,1}) = -2\left(A_1^2 + A_2^2 + A_3^2 - 4a_{2,2}^3 + 24a_{2,2}^2 - 32a_{2,2}\right).$$

Taking $A_1 = \sqrt{2}$, $A_2 = 3$ and $A_3 = 1$, we obtain that

$$V_5(f_{3,2,1}) = -2\left(-4a_{2,2}^3 + 24a_{2,2}^2 - 32a_{2,2} + 12\right) = 8(a_{2,2} - 1)(a_{2,2}^2 - 5a_{2,2} + 3),$$

and therefore $V_5(f_{3,2,1}) > 0$ if and only if $a_{2,2} \in \big((5-\sqrt{13})/2, 1\big)$ or $a_{2,2} > (5+\sqrt{13})/2 \simeq 4.303$. Taking $a_{2,2} = 5$, we obtain the maps of Example 1.

Now we consider the case $k > 3$. We take the maps f_1, f_2 and f_3 given in Example 1, and for all $j \in \{4, \ldots, k\}$ we consider the maps $f_j(x) = x - x^7$, so that all them have a LAS fixed point at the origin (by Theorem 3). Then $f_{k,\ldots,4}(x) = x - (k-3)x^7 + O(13)$.

Observe that if we take any map of the form $g(x) = -x + \sum_{j=2}^{5} \alpha_j x^j + O(6)$ we obtain $f_{k,\ldots,4} \circ g(x) = -x + \sum_{j=2}^{5} \alpha_j x^j + O(6)$. Thus $f_{k,\ldots,1}(x) = f_{k,\ldots,4} \circ f_{3,2,1}(x) = -x + 90x^4 - 48x^5 + O(6)$. Therefore $V_3(f_{k,\ldots,1}) = 0$ and $V_5(f_{k,\ldots,1}) = 96 > 0$, and the origin is repeller for $f_{k,\ldots,1}$.

Similarly, consider the maps g_1, g_2 and g_3 given in Example 2 and $g_j(x) = x + x^7$ for all $j \in \{4, \ldots, k\}$. By construction, each map g_j has a repeller fixed point at the origin. Now $g_{k,\ldots,1}(x) = -x + 90x^4 + 48x^5 + O(6)$, and a computation shows that $V_3(g_{k,\ldots,1}) = 0$ and $V_5(g_{k,\ldots,1}) = -96 < 0$. Hence the origin is LAS for $g_{k,\ldots,1}$. ∎

4 Proof of Theorems B and C

We start recalling the tools that we will use to know the stability of the elliptic fixed point.

4.1 Birkhoff normal form and stability

We remark that in this article we are not interested only on the *stability* of the elliptic fixed points, that is one of the issues that people usually refers in the context of studying maps with elliptic points, via Moser twist theorem and KAM theory. Here we want to know whether these fixed points are LAS or repeller. This information is given by the so called *Birkhoff constants* that we recall next.

Given an elliptic fixed point with eigenvalues $\lambda, \bar{\lambda} = 1/\lambda$, that are not roots of unity of order ℓ for $0 < \ell \leq 2m+1$, we will say that p is a *non* $(2m+1)$*-resonant* elliptic point. Near a non $(2m+1)$-resonant elliptic fixed point, a $\mathcal{C}^{2m+2}$-map f is locally conjugated to its *Birkhoff normal form* plus some remainder terms, see [2]. This normal form is

$$f_B(z, \bar{z}) = \lambda z\Big(1 + \sum_{j=1}^{m} B_j (z\bar{z})^j\Big) + O(2m+2),$$

where $z = x + yi$ and B_j are complex numbers. The first non-vanishing number B_j is called the jth *Birkhoff constant.* Then, $V_j = \mathrm{Re}(B_j)$ will be called the jth *Birkhoff stability constant.* Both constants provide very useful dynamical information in a neighborhood of the elliptic point. In this sense, a well-known result is the following:

Lemma 10 ([11]). *For $m \in \mathbb{N}$, consider a $\mathcal{C}^{2m+2}$-map f with an elliptic fixed point $p \in \mathcal{U}$, non $(2m+1)$-resonant. Let B_m be its first non-vanishing Birkhoff constant. If $V_m = \mathrm{Re}(B_m) < 0$ (resp. $V_m = \mathrm{Re}(B_m) > 0$), then the point p is LAS (resp. repeller).*

The computation of the Birkhoff normal forms and the corresponding Birkhoff constants is a subject on which there is abundant literature, the reader is referred for instance to [2]. Next we give the expression of the first Birkhoff constant of a map with a non 3-resonant fixed point at the origin. Set

$$g(z,\bar{z}) = \lambda z + \sum_{m+j=2}^{3} a_{m,j} z^m \bar{z}^j + O(4),$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then the first Birkhoff constant is

$$B_1 = B_1(g) = \frac{P(g)}{\lambda^2 (\lambda - 1)(\lambda^2 + \lambda + 1)}, \tag{3}$$

where

$$\begin{aligned} P(g) = {}& \left(|a_{11}|^2 + a_{21}\right) \lambda^4 - a_{11}\left(2a_{20} - \bar{a}_{11}\right)\lambda^3 + \left(2|a_{02}|^2 - a_{11}a_{20} + |a_{11}|^2\right)\lambda^2 \\ & - \left(a_{11}a_{20} + a_{21}\right)\lambda + a_{11}a_{20}, \end{aligned}$$

see for instance [10, Sec. 4].

Lemma 10 allows us to utilize the Birkhoff stability constants in an analogous way as we used the stability constants for one-dimensional map. Therefore we follow a similar idea than the used to construct the maps in the proof of Theorem A, in order to prove Theorem B: we will construct two maps f_1 and f_2 such that $V_1(f_i) < 0$ (resp. positive) and $V_1(f_{2,1}) > 0$ (resp. negative).

4.2 Proof of Theorem B

Consider the maps f_1 and f_2 of Example 7. To compute their Birkhoff constants, first we write them in complex notation obtaining that

$$g_1(z,\bar{z}) = iz + (1-3i)z^2 + z\bar{z} \text{ and } g_2(z,\bar{z}) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) z - z^2\bar{z}, \tag{4}$$

are their respective equivalent complex expressions. Conversely, the real expressions of each $g_j(x,y)$ are obtained taking

$$f_j(x,y) = \left(\mathrm{Re}\left(g_j(x+yi, x-yi)\right), \mathrm{Im}\left(g_j(x+yi, x-yi)\right)\right), \quad j = 1, 2. \tag{5}$$

Next, we apply the formula (3) to the maps $g_j(z,\bar z)$, $j=1,2$. We get that their first Birkhoff constants are

$$B_1(g_1) = -\frac{1}{2} - \frac{11}{2}\,i \quad \text{ and } \quad B_1(g_2) = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i.$$

So their Birkhoff stability constants are $V_1(g_1) = V_1(g_2) = -1/2 < 0$. By Lemma 10, the origin is a LAS fixed point for both maps g_1 and g_2. Since for each $j=1,2$, $f_j(x,y)$ and $g_j(z,\bar z)$ are different expressions of the same map, the origin is LAS for both maps f_1 and f_2.

The composition map

$$g_{2,1}(z,\bar z) = g_2\circ g_1(z,\bar z) = i\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)z + \frac{1}{2}(1-3i)(1+\sqrt{3}i)z^2 + \frac{1}{2}(1+\sqrt{3}i)z\bar z - iz^2\bar z + O(4),$$

has an associated Birkhoff constant

$$B_1(g_{2,1}) = \frac{3\sqrt{3}-5}{2} + i\,\frac{3\sqrt{3}-13}{2} \simeq 0.098 - 3.902\,i,$$

and therefore $V_1(g_{2,1}) > 0$. So again by Lemma 10 the origin is a repeller fixed point for $g_{2,1}(z,\bar z)$, hence also for the composition map $f_{2,1}$, as we wanted to prove.

Next we will explain how we have found the Example 7, used in the above proof. We start with

$$g_1(z,\bar z) = \alpha z + \sum_{m+j=2}^{3} a_{m,j} z^m \bar z^j \quad \text{ and } \quad g_2(z,\bar z) = \beta z + c_{2,1} z^2 \bar z.$$

The last map has been chosen so that it only contains the cubic resonant terms that appear in the formula (3). We take $\alpha = i$ and $\beta = (1+\sqrt{3}\,i)/2$, so that the origin is non a 3-resonant elliptic fixed point for both maps. We compute the Birkhoff constant, using the formula (3), obtaining

$$\begin{aligned}
B_1(g_1) &= \frac{1-i}{2}\left(-2|a_{0,2}|^2 + 2a_{1,1}a_{2,0} + a_{2,1} + \left(a_{1,1}a_{2,0} - |a_{1,1}|^2 - a_{2,1}\right)i\right),\\
B_1(g_2) &= \frac{1-i\sqrt{3}}{2} c_{2,1}
\end{aligned}$$

and

$$\begin{aligned}
B_1(g_{2,1}) = &-\frac{1}{2}|a_{1,1}|^2 - |a_{0,2}|^2 + \frac{3}{2}a_{1,1}a_{2,0} + \frac{1}{2}c_{2,1}\\
&+\Big(\frac{\sqrt{3}}{2}a_{1,1}a_{2,0} + \frac{\sqrt{3}}{2}|a_{1,1}|^2 - \frac{\sqrt{3}}{2}c_{2,1} - |a_{0,2}|^2 - a_{1,1}a_{2,0} - |a_{1,1}|^2 - a_{2,1}\Big)i.
\end{aligned}$$

In order to reduce the parameters we set $a_{2,0} = t+si$, $a_{11} = 1$, $a_{0,2} = 0$, $a_{2,1} = 0$, $c_{2,1} = u$ where $s,t,u\in\mathbb{R}$. We get:

$$B_1(g_1) = \frac{1}{2}\left(3t+s-1+(-t+3s-1)i\right), \quad B_1(g_2) = \frac{1}{2}\left(1-\sqrt{3}\,i\right)u, \tag{6}$$

and

$$B_1(g_{2,1}) = \left(1-\frac{\sqrt{3}}{2}\right) s+\frac{3}{2}t+\frac{1}{2}u-\frac{1}{2}+\left(\frac{3}{2}s+\left(\frac{\sqrt{3}}{2}-1\right)t-\frac{\sqrt{3}}{2}u-1+\frac{\sqrt{3}}{2}\right)i \quad (7)$$

To simplify more the above expressions, we consider $s=-3t$ and $u=-1$, obtaining that $V_1(g_1)=V_1(g_2)=-1/2$, and

$$V_1(g_{2,1}) = -1+\frac{3}{2}(\sqrt{3}-1)\,t.$$

This last constant is positive for all $t>2/(3(\sqrt{3}-1))\simeq 0.911$, so taking $t=1$ we get the maps (4). Applying the formula (5) we obtain the expression of the maps of Example 7.

We have already commented before Example 8 that from the above example, simply taking the inverse maps we could construct two planar maps having a common repeller fixed point such that the corresponding composition map has a LAS fixed point. Nevertheless, next we construct a simple explicit example, namely, Example 8. In this case, to reduce parameters in the expressions (6) and (7), we take $s=2-3t$ and $u=1$, obtaining that $V_1(g_1)=V_1(g_2)=1/2$, and

$$V_1(g_{2,1}) = 2-\sqrt{3}+\frac{3}{2}(\sqrt{3}-1)\,t.$$

This constant is negative for all $t<\frac{1}{3}\left(\sqrt{3}-2\right)\left(1+\sqrt{3}\right)\simeq -0.244$. Setting $t=-2/3$, and applying the formula (5) we get the maps of this last example.

4.3 Proof of Theorem C

(a) Consider the $k\geq 3$ functions $\{f_1,f_2,\ldots,f_k\}$ used for proving item (b) of Theorem A, that is f_1, f_2 and f_3 given in Example 1, and $f_j(x)=x-x^7$ for $j=4,\ldots,k$. For any $n\in\mathbb{N}$, define the maps

$$F_j\big(x_1,x_2,\ldots,x_n\big)=\big(f_j(x_1),f_j(x_2),\ldots,f_j(x_n)\big),\quad j=1,2,\ldots,k.$$

which are from $\mathbb{R}^n$ into itself. Because the components of the above maps are uncoupled, from Theorem A we obtain that the origin is a LAS fixed point for each F_j, but a repeller fixed point for $F_{k,\ldots,1}$,

Analogously, we take the maps $g_j, j=1,\ldots,k$, given at the end of the proof of Theorem A, and define $G_j\big(x_1,x_2,\ldots,x_n\big)=\big(g_j(x_1),g_j(x_2),\ldots,g_j(x_n)\big)$, for $j=1,2,\ldots,k$. For these maps, the origin is a repeller fixed point for each G_j but it is LAS fixed point for $G_{k,\ldots,1}$.

(b) For any $m\in\mathbb{N}$ we define the two maps from R^{2m} into itself,

$$F_j\big(x_1,x_2,\ldots,x_{2m}\big)=\big(f_j(x_1,x_2),f_j(x_3,x_4),\ldots,f_j(x_{2m-1},x_{2m})\big),\quad j=1,2,$$

where the f_j are the ones appearing either in Example 7 or in Example 8. Then the result follows taking the periodic set $\{F_1, F_2\}$ and noticing that the dynamics of each consecutive pair of components of any map F_j is uncoupled.

Acknowledgements

The authors are supported by Ministry of Economy, Industry and Competitiveness of the Spanish Government through grants MINECO/FEDER MTM2016-77278-P (first and second authors) and DPI2016-77407-P (third author). The first and second authors are also supported by the grant 2014-SGR-568 from AGAUR, Generalitat de Catalunya. The third author is supported by the grant 2014-SGR-859 from AGAUR, Generalitat de Catalunya.

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