This is a preprint of: "On the multiple zeros of a real analytic function with applications to the averaging theory of differential equations", Isaac A. García, Jaume Llibre, Susanna Maza, *Nonlinearity*, vol. 31, 2666–2688, 2018.
DOI: [10.1088/1361-6544/aab592]

THE ROLE OF THE MULTIPLE ZEROS IN THE AVERAGING THEORY OF DIFFERENTIAL EQUATIONS

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Abstract. In this work we improve the classical averaging theory applied to λ-families of analytic T-periodic ordinary differential equations in standard form defined on $\mathbb{R}$. First we characterize the set of points z_0 in the phase space and the parameters λ where T-periodic solutions can be produced when we vary a small parameter ε. Second we expand the displacement map in powers of the parameter ε whose coefficients are the averaged functions. The main contribution consists in analyzing the role that have the multiple zeros $z_0 \in \mathbb{R}$ of the first non-zero averaged function. The outcome is that these multiple zeros can be of two different classes depending on whether the points (z_0, λ) belong or not to the analytic set defined by the real variety associated to the ideal generated by the averaged functions in the Noetheriang ring of all the real analytic functions at (z_0, λ). Next we are able to bound the maximum number of branches of isolated T-periodic solutions that can bifurcate from each multiple zero z_0. Sometimes these bounds depend on the cardinalities of minimal bases of the former ideal. Several examples illustrate our results.

1. Introduction and statement of the results

The method of averaging is a classical tool that allows to study the dynamics of the periodic nonlinear differential systems. It has a long history starting with the intuitive classical works of Lagrange and Laplace. Important advances of the averaging theory were made by Bogoliubov and Krylov, the reader can consult [2] for example. For a more modern exposition of the averaging theory see the book of Sanders, Verhulst and Murdock [11].

In this work we consider a family of T-periodic analytic differential equation in $\Omega \subset \mathbb{R}$ of the form

$$\dot{x} = F(t, x; \lambda, \varepsilon) = \sum_{i \geq 1} F_i(t, x; \lambda)\, \varepsilon^i, \tag{1}$$

where t is the independent variable (here called the *time*), and $x \in \Omega$ is the dependent variable with Ω a bounded open subset, $\lambda \in \mathbb{R}^p$ are the parameters of the family, for all i the function F_i is analytic in its variables and T-periodic in the t variable, and the period T is independent of the small parameter $\varepsilon \in I$ with $I \subset \mathbb{R}$ an interval containing the origin.

For each $z \in \Omega$ we denote $x(t; z, \lambda, \varepsilon)$ the solution of the Cauchy problem formed by the differential equation (1) with the initial condition $x(0; z, \lambda, \varepsilon) = z$. From the

2000 *Mathematics Subject Classification.* 37G15, 37G10, 34C07.
Key words and phrases. Averaging theory, periodic orbits, Poincaré map.

analyticity of (1) and the fact that $F(t,x;\lambda,0)=0$ one has

$$x(t;z,\lambda,\varepsilon) = z + \sum_{j\geq 1} x_j(t,z,\lambda)\,\varepsilon^j, \tag{2}$$

where $x_j(t,z,\lambda)$ are real analytic functions such that $x_j(0,z,\lambda)=0$. Assuming that $x(t;z,\lambda,\varepsilon)$ is defined in the interval $t\in[0,T]$ (this is guarantee for ε close enough to 0 since there is existence and uniqueness of solutions for the Cauchy problem on the time-scale $1/\varepsilon$), we can define the *displacement map at time* T as $d:\Omega\times\mathbb{R}^p\times I\to\Omega$ with $d(z,\lambda,\varepsilon)=x(T;z,\lambda,\varepsilon)-x(0;z,\lambda,\varepsilon)=x(T;z,\lambda,\varepsilon)-z$. Clearly, its zeros are initial conditions for the T-periodic solutions of the differential equation (1).

Integrating with respect to the time t the differential equation (1) along the solution $x(t;z,\lambda,\varepsilon)$ from 0 to t we obtain

$$x(t;z,\lambda,\varepsilon)-z=\int_0^t F(s,x(s;z,\varepsilon),\lambda,\varepsilon)\,ds,$$

from which we get

$$d(z,\lambda,\varepsilon)=\int_0^T F(t,x(t;z,\varepsilon),\lambda,\varepsilon)\,dt.$$

The displacement map is analytic at $\varepsilon=0$, so we can express it as the following series expansion

$$d(z,\lambda,\varepsilon)=\sum_{i\geq 1} f_i(z;\lambda)\,\varepsilon^i, \tag{3}$$

and from (2) we have that $f_i(z;\lambda)=x_i(T;z,\lambda)$ for all positive integer i. We call the coefficient functions $f_i(z;\lambda)$ the *averaged functions*. The way in which we can get (3) is explained with details in [6]. There we can see that the first coefficient is

$$f_1(z;\lambda)=\int_0^T F_1(t,z,\lambda)\,dt.$$

For the expression of all the other coefficients, see [6] again where the recursive expression of $x_i(t;z,\lambda)$ for $i\geq 1$ is given. We summarize these results in Theorem 21 of the Appendix.

We say that a (*complete, or positive, or negative*) *branch of* T*-periodic solutions of* (1) *bifurcates* from the point $z_0\in\Omega$ if there is an analytic function $z^*(\lambda,\varepsilon)$ (defined either for all ε in a neighborhood of zero, or just for all $\varepsilon>0$ close to zero, or $\varepsilon<0$ close to zero, respectively) such that $z^*(\lambda,0)=z_0$ and $d(z^*(\lambda,\varepsilon),\lambda,\varepsilon)\equiv 0$.

Therefore the solutions $x(t;z^*(\lambda,\varepsilon),\lambda,\varepsilon)$ of equation (1) are T-periodic and bifurcate from z_0 because $\lim_{\varepsilon\to 0^\pm} x(t;z^*(\lambda,\varepsilon),\lambda,\varepsilon)=z_0$ where the lateral limit is taken according with the complete, or positive, or negative nature of the branch.

Now we state the following easy result and, for completeness, we prove it in subsection §5.1.

Lemma 1. *Let* $z^*(\varepsilon,\lambda)$ *be a function such that* $d(z^*(\varepsilon,\lambda),\lambda,\varepsilon)\equiv 0$ *for all* $|\varepsilon|\neq 0$ *sufficiently small. Then* $f_\ell(z_0(\lambda);\lambda)=0$ *where* $z^*(0,\lambda)=z_0(\lambda)\in\Omega$ *being* ℓ *the first subindex such that* $f_\ell(z;\lambda)\not\equiv 0$.

By Lemma 1 in order to control the bifurcation of the families of T-periodic solutions of the differential system (1) for small values of $|\varepsilon|$, we need to study the zeros of the function $f_\ell(.;\lambda)$ defined in Lemma 1.

Given a particular differential equation (1) with $\lambda=\lambda^*$, let ℓ be the first positive integer such that $f_\ell\not\equiv 0$, and let $z_0\in\Omega$ be a zero of f_ℓ, i.e. $f_\ell(z_0;\lambda^*)=0$. By definition, if $\dfrac{\partial}{\partial z}f_\ell(z_0;\lambda^*)\neq 0$, then z_0 is a *simple zero*; but if $\dfrac{\partial}{\partial z}f_\ell(z_0;\lambda^*)=0$, then z_0 is a *multiple zero*.

For the simple zeros z_0 of $f_\ell(.;\lambda^*)$ one has the following well known result (see for instance [8]) consequence of the Implicit Function Theorem, for more details see subsection §5.2.

Theorem 2. *For a fixed $\lambda^*\in\mathbb{R}^p$, assume that ℓ is the first subindex such that $f_\ell(z;\lambda^*)\not\equiv 0$ and that $z_0\in\Omega$ is a simple zero of $f_\ell(.;\lambda^*)$, that is $f_\ell(z_0;\lambda^*)=0$ and $\dfrac{\partial}{\partial z}f_\ell(z_0;\lambda^*)\neq 0$. Then, for $|\varepsilon|$ sufficiently small, there exists a unique branch of T-periodic solutions $x(t;z,\lambda^*,\varepsilon)$ of equation (1) with $\lambda=\lambda^*$ bifurcating from z_0 which is complete.*

We say that a T-periodic solution $x(t;z,\lambda,\varepsilon)$ is *isolated* if there is a neighborhood $N\subset\Omega$ of z such that $x(t;\hat{z},\lambda,\varepsilon)$ is not T-periodic for all $\hat{z}\in N\backslash\{z\}$.

Recall that, as usual, a zero $z_0\in\Omega\subset\mathbb{R}$ of $f_\ell(.;\lambda^*)$ is said to be of multiplicity $\bar{k}$ if

$$f_\ell(z_0;\lambda^*)=\frac{\partial f_\ell}{\partial z}(z_0;\lambda^*)=\cdots=\frac{\partial^{\bar{k}-1}f_\ell}{\partial z^{\bar{k}-1}}(z_0;\lambda^*)=0,\quad \frac{\partial^{\bar{k}}f_\ell}{\partial z^{\bar{k}}}(z_0;\lambda^*)\neq 0.$$

For such multiple points, by using several times the Rolle theorem, in [9] it is proved the following result. The last part of the theorem is consequence of the fact that univariate real polynomials of odd degree always have an odd number (greater or equal than 1) of real roots.

Theorem 3. *For a fixed $\lambda^*\in\mathbb{R}^p$, assume that ℓ is the first subindex such that $f_\ell(z;\lambda^*)\not\equiv 0$ and that $z_0\in\Omega$ is a multiple zero of $f_\ell(.;\lambda^*)$ of multiplicity $\bar{k}$. Then the number of isolated branches of T-periodic solutions bifurcating from z_0 for equation (1) with $\lambda=\lambda^*$ and $|\varepsilon|\ll 1$ is at most $\bar{k}$. Moreover, all these branches are complete and if $\bar{k}$ is odd then the number of branches is also odd and at least one branch bifurcates from z_0.*

We call Theorems 2 and 3 the classical averaging theory. In the rest of the work when analyzing the role that multiple zeros $z_0\in\Omega$ of $f_\ell(.;\lambda^*)$ have in these bifurcations, we change the classical strategy of finding the (complete) analytic branches of T-periodic solutions of (1) bifurcating from $z_0\in\Omega$ computing the functions $z^*(\lambda,\varepsilon)$ satisfying $z^*(\lambda^*,0)=z_0$ and $d(z^*(\lambda^*,\varepsilon),\lambda^*,\varepsilon)\equiv 0$ for any ε close to zero. Instead, we opt to find the functions $\varepsilon^*(z,\lambda^*)$ with $\varepsilon^*(z_0,\lambda^*)=0$ such that $d(z,\lambda^*,\varepsilon^*(z,\lambda^*))\equiv 0$ for any z close to z_0. As far as we know, this point of view is new and, in some cases, gives improved bounds with respect to the bound $\bar{k}$ of Theorem 3 for multiple points.

Remark 4. We emphasize that, if $\dfrac{\partial \varepsilon^*}{\partial z}(z_0, \lambda^*) \neq 0$, then for each function $\varepsilon^*(z, \lambda^*)$ we count just one branch of isolated T-periodic solutions bifurcating from z_0 for any $|\varepsilon| \ll 1$, and moreover the branch is complete. On the other hand, when $\dfrac{\partial \varepsilon^*}{\partial z}(z_0, \lambda^*) = 0$, then from each function $\varepsilon^*(z, \lambda^*)$ we count one (complete) branch of isolated T-periodic solutions bifurcating from z_0 for all $|\varepsilon| \ll 1$, or two (positive or negative) branches for either $\varepsilon > 0$ or $\varepsilon < 0$ sufficiently small depending on whether the function function $\varepsilon^*(., \lambda^*)$ possesses an inflection point or a minimum or a maximum at z_0, respectively. Clearly in particular situations we can find out which of the above situations actually occurs, see for more details Proposition 11 and its proof.

In this direction our first result is the following.

Theorem 5. *For a fixed $\lambda^\dagger \in \mathbb{R}^p$, assume that ℓ is the first subindex of the displacement function* (3) *such that $f_\ell \not\equiv 0$. Let $z_0 \in \Omega$ be a multiple zero of the function $f_\ell(.; \lambda^\dagger)$. Assume also that there exists a positive integer k which is the minimum integer satisfying $f_{\ell+k}(z_0; \lambda^\dagger) \neq 0$. Then for $|\varepsilon|$ sufficiently small the number of (either positive or negative) isolated branches of T-periodic solutions that equation* (1) *with $\lambda = \lambda^\dagger$ can have bifurcating from z_0 is bounded by $2k$. Moreover, if k is odd then the number of such branches is also odd and at least one branch bifurcates from z_0.*

The proof of Theorem 5 is rather similar to that of Theorem 2 by using the Weierstrass preparation theorem instead of the Implicit Function Theorem and interchanging the role of z and ε, see the proof in subsection §5.3. Also we note that the upper bound $2k$, obtained in Theorem 5 for the maximum number of isolated branches of T-periodic solutions that equation (1) with $\lambda = \lambda^\dagger$ can have bifurcating from the multiple zero z_0, is not related with the multiplicity $\bar{k}$ of the zero z_0 as it is explained in Theorem 3. In this way we are providing a new mechanism to obtain such a maximum bound.

Remark 6. We note that the positive integer k of Theorem 5 may not exist. A typical example is a zero $(z_0, \lambda^*) \in \Omega \times \mathbb{R}^p$ of the displacement function (3) such that z_0 is an equilibrium point of the differential equation (1) with $\lambda = \lambda^*$, i.e. $F(t, z_0; \lambda^*, \varepsilon) = 0$ for all $t \in \mathbb{R}$ and $|\varepsilon| \ll 1$. For such a zero one has $f_i(z_0; \lambda^*) = 0$ for all positive integer i although $f_\ell \not\equiv 0$ for some ℓ.

Based on Remark 6 we need to develop a procedure taking also into account these kind of zeros, and to do a complementary theory for studying them.

1.1. Multiple zeros of finite-type and of infinite-type.

Assume that $f_1(z; \lambda) = \cdots = f_{\ell-1}(z; \lambda) \equiv 0$ and $f_\ell(z; \lambda) \not\equiv 0$ for some index $\ell \geq 1$, that is, the displacement map of family (1) is given by $d(z, \lambda, \varepsilon) = \sum_{i \geq \ell} f_i(z; \lambda) \varepsilon^i$.

We say that a point $(z, \lambda) = (z_0, \lambda^\dagger) \in \Omega \times \mathbb{R}^p$ is of *finite-type* if there exists an integer $k \geq 1$ such that $f_\ell(z_0; \lambda^\dagger) = \cdots = f_{\ell+k-1}(z_0; \lambda^\dagger) = 0$ but $f_{\ell+k}(z_0; \lambda^\dagger) \neq 0$. We call k the *order* of the zero $(z_0, \lambda^\dagger)$. For example, the point $(z_0, \lambda^\dagger)$ in Theorem 5 is of finite-type.

We say that a zero $(z_0, \lambda^*) \in \Omega \times \mathbb{R}^p$ of the function f_ℓ is of *infinite-type* when $f_j(z_0; \lambda^*) = 0$ for all positive integer j. Let $\mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$ be the Noetherian ring formed by all the real analytic functions at (z_0, λ^*). If $\mathbb{N}$ denotes the set of all positive integers it is clear that for (z, λ) sufficiently close to (z_0, λ^*), the sequence $\{f_j(z; \lambda)\}_{j\in\mathbb{N}} \subset \mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$, and we define the ideal $\mathcal{I} = \langle f_i(z; \lambda) : i \in \mathbb{N}\rangle$ in the ring $\mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$ as the ideal generated by all the functions $f_i(z; \lambda)$.

From the properties of the Noetherian rings there is a minimal basis of $\mathcal{I}$ of finite cardinality $m \geq 1$ formed by an initial string of averaged functions. We denote such minimal basis by

$$\{f_{j_1}(z; \lambda), \ldots , f_{j_m}(z; \lambda)\},$$

where $j_i \in \mathbb{N}$ are ordered as $\ell \leq j_1 < j_2 < \cdots < j_m$. It is clear that $\mathcal{I}$ can be minimally generated by a number of elements in $\mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$ less than m. But we abuse of notation and when we write a minimal basis B of $\mathcal{I}$ we mean a basis whose elements are averaged functions selected as follows:

(a) initially set $B = \{f_{j_1}\}$, where f_{j_1} is the first non-zero element of B;
(b) sequentially check successive elements f_r, starting with $r = j_1+1$, adjoining f_r to B if and only if $f_r \notin \langle B\rangle$, the ideal generated in $\mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$ by B.

If we denote by $\mathbf{V}_\mathbb{R}(\mathcal{I})$ the real variety of the common zeros of all the functions of the ideal $\mathcal{I}$, then clearly the infinite-type point $(z_0, \lambda^*) \in \mathbf{V}_\mathbb{R}(\mathcal{I})$. In particular, $d(z_0, \lambda^*, \varepsilon) \equiv 0$ for all $|\varepsilon| \ll 1$ which means that equation (1) with $\lambda = \lambda^*$ has a T-periodic solution starting at the fixed initial condition z_0 for all $|\varepsilon| \ll 1$.

We remark that the typical points $(z_0, \lambda^*) \in \mathbf{V}_\mathbb{R}(\mathcal{I})$ are just the equilibrium points z_0 of the differential equation (1) with $\lambda = \lambda^*$, i.e., the points $z_0 \in \Omega$ such that $F(t, z_0; \lambda^*, \varepsilon) = 0$ for all $t \in \mathbb{R}$ and $|\varepsilon| \ll 1$. See the forthcoming Hopf bifurcation section for more details.

1.2. **Main results.** Now we present our main results. The proof of the following theorem is inspired in the seminal Bautin's work [1] about Hopf bifurcations from focus and centers of polynomial planar vector fields where the role of the Poincaré-Liapunov quantities is played now by the averaged functions.

Theorem 7. *Let $d(z, \lambda, \varepsilon) = \sum_{i\geq 1} f_i(z; \lambda)\varepsilon^i$ be the displacement map associated to the family of differential equations* (1) *and let $\ell \geq 1$ be the first subindex such that the function $f_\ell(z; \lambda^*) \not\equiv 0$ for some fixed parameter value λ^*. Assume that $z_0 \in \Omega$ is a zero of the function $f_\ell(.; \lambda^*)$. Let M be an upper bound of the number of (either positive or negative) isolated branches of T-periodic solutions that the differential equation* (1) *with $\lambda = \lambda^*$ and $|\varepsilon| \ll 1$ can have bifurcating from z_0. Then the following holds:*

(i) *If (z_0, λ^*) is of finite-type with order $k \geq 1$ then $M = 2k$. When k is odd then the number of such branches is also odd and $M \geq 1$.*
(ii) *If (z_0, λ^*) is of infinite-type then $M = 2(m - 1)$. Here, m is the cardinality of the minimal basis of the ideal $\mathcal{I} = \langle f_i(z; \lambda) : i \in \mathbb{N}\rangle$ in the ring $\mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$.*

Theorem 7 is proved in subsection §5.4. Of course, statement (i) of Theorem 7 is just Theorem 5 and we include it for completeness.

When $z_0 \in \Omega$ is a zero of $f_\ell(.;\lambda^*)$, from the proof of Theorem 7 it follows that for (z,λ) sufficiently close to (z_0,λ^*) and for $|\varepsilon| \ll 1$, the displacement map $d(z;\lambda,.)$ can have at most either k or $m-1$ small isolated (either positive or negative) zeros depending on the nature of the point (z_0,λ^*). Therefore $2(m-1)$ is a bound on the number of isolated branches of T-periodic solutions of (1) with λ near λ^* and initial condition near z_0 for either $\varepsilon > 0$ or $\varepsilon < 0$ sufficiently small. This is the reason why (see subsection §1.3 and examples) we can also work with families of differential equations varying also λ and not only the perturbation small parameter ε.

Remark 8. We consider a point $(z_0,\lambda^*) \in \Omega \times \mathbb{R}^p$ of infinite-type. Since the associate ideal $\mathcal{I}$ is an ideal in the ring $\mathbb{R}\{z,\lambda\}_{(z_0,\lambda^*)}$, it is clear that $\mathcal{I}$ depends on the point (z_0,λ^*). Consequently, $\mathcal{I}$ and m also depend on (z_0,λ^*). Until now we have analyzed just one point (z_0,λ^*) of infinite-type and we have not used any notation taking care of such a dependence. In the rest of the paper we can have the situation that, for some λ^*, the function $f_\ell(.,\lambda^*)$ can have several zeros $z_r \in \Omega$ and all the points (z_r,λ^*) can be of infinite-type for all the subscripts r. In this case we will use the notation $\mathcal{I}_{(z_r,\lambda^*)}$ and $m(z_r,\lambda^*)$.

The following result is a straightforward consequence of Theorems 2 and 7 in case that, for fixed λ^*, the zero set $f_\ell^{-1}(0) = \{z_0 \in \Omega : f_\ell(z_0,\lambda^*) = 0\}$ has finite cardinality.

Corollary 9. *Let $d(z,\lambda,\varepsilon) = \sum_{i\geq 1} f_i(z;\lambda)\varepsilon^i$ be the displacement map associated to the family of differential equations* (1) *and let $\ell \geq 1$ be the first subindex such that the function $f_\ell(z;\lambda^*) \not\equiv 0$ for some fixed parameter value λ^*. Assume that the set of real zeros of the function $f_\ell(.;\lambda^*)$ in Ω is finite and given by s simple zeros, m_f multiple zeros of finite-type with orders k_j for $j = 1,\dots,m_f$, and m_c multiple zeros of infinite-type $\{z_1,\dots,z_{m_c}\} \subset \Omega$. For each $r \in \{1,\dots,m_c\}$, let $m(z_r,\lambda^*)$ be the cardinality of the minimal basis of the ideal $\mathcal{I}_{(z_r,\lambda^*)} = \langle f_i(z;\lambda) : i \in \mathbb{N}\rangle$ in the ring $\mathbb{R}\{z,\lambda\}_{(z_r,\lambda^*)}$. Then, for $|\varepsilon| \ll 1$, the number of (either positive or negative) isolated branches of T-periodic solutions that differential equation* (1) *with $\lambda = \lambda^*$ can have bifurcating from a finite point is at most $s + \sum_{i=1}^{m_f} 2k_i + \sum_{r=1}^{m_c} 2(m(z_r,\lambda^*) - 1)$.*

Joining Corollary 9 and Theorem 3 we obtain the following result.

Corollary 10. *Under the hypotheses of Corollary 9, let $\bar{k}_i$ be the multiplicity of each multiple zero of the function $f_\ell(.;\lambda^*)$ for $i = 1,\dots,m_f + m_c$. Define for $i = 1,\dots,m_f$ and for $j = 1,\dots,m_c$ the integers $\bar{m}_i^f = \min\{\bar{k}_i, 2k_i\}$ and $\bar{m}_j^c = \min\{\bar{k}_j, 2(m(z_j,\lambda^*) - 1)\}$. Then, for $|\varepsilon| \ll 1$, the number of (either positive or negative) isolated branches of T-periodic solutions that differential equation* (1) *with $\lambda = \lambda^*$ can have bifurcating from a finite point is bounded by*

$$s + \sum_{i=1}^{m_f} \bar{m}_i^f + \sum_{j=1}^{m_c} \bar{m}_j^c.$$

The bounds $2k$ or $2(m-1)$ of Theorem 7 (that also affect Corollaries 9 and 10) are based on the bounds k or $m-1$ on the number of analytic functions $\varepsilon^*(z,\lambda^*)$ such that $\varepsilon^*(z_0,\lambda^*) = 0$ and $d(z,\lambda^*,\varepsilon^*(z,\lambda^*)) \equiv 0$ for any z close to a zero z_0 of $f_\ell(.;\lambda^*)$. The origin of the duplication factor 2 in such bounds is explained in

Remark 4, but these duplications (corresponding with the case $\alpha = 2$ in the next Proposition 11) only apply when z_0 is an extremum of $\varepsilon^*(.,\lambda^*)$. Therefore, the next proposition (although not exhaustive) is useful in concrete applications to get a bound smaller than the stated in Theorem 7 and Corollaries 9 and 10. Here we only analyze some subcases of $2 \le \bar{k} > k$ which are those that interest us.

Proposition 11. *Assume there is an analytic function $\varepsilon^*(z,\lambda^*)$ with $\varepsilon^*(z_0,\lambda^*) = 0$ such that the displacement map $d(z,\lambda^*,\varepsilon^*(z,\lambda^*)) \equiv 0$ for any z close to a multiple zero z_0 of $f_\ell(.;\lambda^*)$. Let $\alpha \in \{1,2\}$ be the number of branches of isolated T-periodic solutions bifurcating from z_0 and contained in the graph of ε^*. Then the following follows:*

If z_0 is of finite-type, then

- (i) *If z_0 has order 1 then $\alpha = 2$ or $\alpha = 1$ depending on whether z_0 has even or odd multiplicity, respectively;*
- (ii) *If z_0 has order 2 and multiplicity 4 then $\alpha = 1$ or $\alpha = 2$ depending on whether z_0 is a simple or a multiple zero of $f_{\ell+1}(.;\lambda^*)$, respectively;*

If z_0 is of infinite-type, then:

- (iii) *If z_0 has multiplicity 2 then $\alpha = 1$;*
- (iv) *If z_0 has multiplicity 3 then $\alpha = 2$ or $\alpha = 1$ depending on whether z_0 is a simple or a multiple zero of $f_{\ell+1}(.;\lambda^*)$, respectively;*

Proof. We consider the reduced displacement map

$$\Delta(z,\lambda^*,\varepsilon) = \frac{d(z,\lambda^*,\varepsilon)}{\varepsilon^\ell} = f_\ell(z;\lambda^*) + \sum_{i\ge 1} f_{\ell+i}(z;\lambda^*)\varepsilon^i$$

which has the same zeroes than the displacement map for $\varepsilon \neq 0$. Then, taking derivative with respect to z in the equality $\Delta(z,\lambda^*,\varepsilon^*(z,\lambda^*)) \equiv 0$ for all $|\varepsilon| \ll 1$ and evaluating at $z = z_0$ gives

$$0 = \frac{\partial\Delta}{\partial z}(z_0,\lambda^*,0) + \frac{\partial\Delta}{\partial\varepsilon}(z_0,\lambda^*,0)\frac{\partial\varepsilon^*}{\partial z}(z_0,\lambda^*).$$

Recalling that $\dfrac{\partial f_\ell}{\partial z}(z_0;\lambda^*) = 0$, the former is written as

$$0 = f_{\ell+1}(z_0;\lambda^*)\frac{\partial\varepsilon^*}{\partial z}(z_0,\lambda^*). \tag{4}$$

In a similar way, repeating the procedure taking derivatives with respect to z again and valuating at $z = z_0$, we obtain

$$\begin{aligned} 0 \;=\; & \frac{\partial^2 f_\ell}{\partial z^2}(z_0;\lambda^*) + 2\left(\frac{\partial f_{\ell+1}}{\partial z}(z_0;\lambda^*) + f_{\ell+2}(z_0;\lambda^*)\frac{\partial\varepsilon^*}{\partial z}(z_0,\lambda^*)\right)\frac{\partial\varepsilon^*}{\partial z}(z_0,\lambda^*) + \\ & f_{\ell+1}(z_0;\lambda^*)\frac{\partial^2\varepsilon^*}{\partial z^2}(z_0,\lambda^*), \end{aligned} \tag{5}$$

and as many equations like (4) and (5) as we need. We do not display such equations since they are long but are needed to prove the result. From these equations we can check what is the first non-vanishing derivative $\dfrac{\partial^j\varepsilon^*}{\partial z^j}(z_0,\lambda^*) \neq 0$ so that we can assure that exactly two or one branches of isolated T-periodic solutions bifurcate from z_0 depending on whether j is even or odd, respectively. □

1.3. **The averaged cyclicity.** From now on we will deal with families of differential equations (1) and not with a unique member of the family as until now. So we do not fix the parameters of the family and we allow that λ varies in $\mathbb{R}^p$.

We define the *averaged cyclicity* of the full family of differential equations (1) as the maximum number of (either positive or negative) isolated branches of T-periodic solutions bifurcating from points in Ω, that is, coming from the zeros of the function $f_\ell(.;\lambda)$, defined in the statement of Lemma 1, when $|\varepsilon| \ll 1$, for any value of the parameters $\lambda \in \mathbb{R}^p$, and any initial condition $z_0 \in \Omega$. We will denote such a number as $\mathrm{Cyc}^T(F_\lambda)$, and we can compute under some finiteness assumptions an upper bound of it as follows.

First, for a fixed $j \in \mathbb{N}$ we define the open region $\Omega_j \times \Lambda_j \subset \Omega \times \mathbb{R}^p$ such that its points (z_0, λ_0) are characterized by the existence of a neighborhood $U_{(z_0,\lambda_0)} \subset \Omega_j \times \Lambda_j$ of the point (z_0, λ_0) where j is the smallest subindex such that $f_j(z;\lambda) \not\equiv 0$ for all $(z, \lambda) \in U_{(z_0,\lambda_0)}$. Observe that with this definition there can exist points $(z_c, \lambda_c) \in \Omega_j \times \Lambda_j$ which are of infinite-type because $f_i(z_c;\lambda_c) = 0$ for all $i \in \mathbb{N}$. The number of these points (z_c, λ_c) can be finite or not and also they can be isolated or not. We define $\Omega^* \times \Lambda^* \subset \Omega \times \mathbb{R}^p$ as the set of points of infinite-type, i.e.,

$$\Omega^* \times \Lambda^* = \{(z_0, \lambda_0) \in \Omega \times \mathbb{R}^p : f_j(z_0;\lambda_0) = 0 \text{ for all } j \in \mathbb{N}\}.$$

Note that $(\Omega_j \times \Lambda_j) \cap (\Omega^* \times \Lambda^*)$ can be nonempty, but always $(\Omega_j \times \Lambda_j) \cap (\Omega_i \times \Lambda_i) = \emptyset$ if $i \neq j$.

We claim that there are finitely many possible indices j of the sets $\Omega_j \times \Lambda_j$. More precisely, $1 \leq j \leq \nu < \infty$ due to the fact that the ideals $\mathcal{I}_{(z_r,\lambda^*)}$ are finitely generated by the Hilbert basis theorem. Actually, $\nu = \max_r\{m(z_r, \lambda)\}$.

On the other hand, it is clear that all the solutions of (1) with $\lambda \in \Lambda^*$ and initial condition $z \in \Omega^*$ are T-periodic for all $|\varepsilon| \ll 1$. In particular, if we have the cardinalities $\#(\Lambda^*) \geq 1$ and $\#(\Omega^*) = \infty$, then there are infinitely many T-periodic solutions of (1) with $\lambda \in \Lambda^*$ for any $|\varepsilon| \ll 1$.

Finally we shall apply Corollary 9 to each component $\Omega_j \times \Lambda_j$ starting from $j = 1$ until $j = \nu$, assuming that $\#\{(\Omega_j \times \Lambda_j) \cap (\Omega^* \times \Lambda^*)\} = m_c^{[j]} < \infty$, and that the number of points in $\Omega_j \times \Lambda_j$ which are of finite-type is $m_f^{[j]} < \infty$. In this way we obtain, for each j, a finite bound M_j on the number of T-periodic solutions of (1) with $\lambda \in \Lambda_j$ having initial condition $z_0 \in \Omega_j$ under the hypothesis that $\bigcup_{i=1}^{j-1}(\Omega_i \times \Lambda_i) = \emptyset$.

1.4. **The algorithm for computing the averaged cyclicity.**

(i) Calculate the set $\Omega_{j_1} \times \Lambda_{j_1}$ where the function $f_{j_1}(z;\lambda)$ is not identically zero and the subindex j_1 is minimum.

(ii) Compute the zero-set of the function $f_{j_1}(.;\lambda)$ on $\Omega_{j_1} \times \Lambda_{j_1}$ given by

$$f_{j_1}^{-1}(0) = \{(z_0(\lambda), \lambda) \in \Omega_{j_1} \times \Lambda_{j_1} : f_{j_1}(z_0(\lambda), \lambda) = 0\}.$$

We continue assuming the finite cardinality $\#(f_{j_1}^{-1}(0))$ of the zero-set $f_{j_1}^{-1}(0)$.

(iii) Separate, for each $\lambda \in \Lambda_{j_1}$, the simple and the multiples zeros in $f_{j_1}^{-1}(0)$. Thus we define

$$\begin{aligned} \mathcal{S}_{j_1}^{\lambda} &= \{(z_0(\lambda),\lambda) \in f_{j_1}^{-1}(0) : \frac{\partial}{\partial z} f_{j_1}(z_0(\lambda),\lambda) \neq 0\}, \\ \mathcal{M}_{j_1}^{\lambda} &= \{(z_0(\lambda),\lambda) \in f_{j_1}^{-1}(0) : \frac{\partial}{\partial z} f_{j_1}(z_0(\lambda),\lambda) = 0\}, \end{aligned}$$

and also the cardinals $\#(\mathcal{S}_{j_1}^{\lambda}) = s^{[\lambda,j_1]}$ for all $\lambda \in \Lambda_{j_1}$.

(iv) For each $\lambda \in \Lambda_{j_1}$, consider the sets of points of infinite-type $\mathcal{C}_{j_1}^{\lambda} = \mathcal{M}_{j_1}^{\lambda} \cap (\Omega^* \times \Lambda^*)$ and of finite-type $\mathcal{F}_{j_1}^{\lambda} = \mathcal{M}_{j_1}^{\lambda} \backslash (\Omega^* \times \Lambda^*)$ with finite cardinalities $\#(\mathcal{C}_{j_1}^{\lambda}) = m_c^{[\lambda,j_1]}$ and $\#(\mathcal{F}_{j_1}^{\lambda}) = m_f^{[\lambda,j_1]}$, respectively.

(v) Compute the order $k_i^{[\lambda,j_1]}$ of all the points in $\mathcal{F}_{j_1}^{\lambda}$ for $i = 1,\dots,m_f^{[\lambda,j_1]}$.

(vi) For any point $(z_i,\lambda_i) \in \mathcal{C}_{j_1}^{\lambda}$, compute a minimal basis of the ideal $\mathcal{I}_{(z_i,\lambda_i)}$ and denote its cardinality by $m(z_i,\lambda_i)$.

(vii) Then, the averaged cyclicity $\mathrm{Cyc}^T(F_\lambda)$ of family (1) in $\Omega_{j_1} \times \Lambda_{j_1}$ is finite and bounded by $\mathrm{Cyc}^T(F_\lambda) \leq M_{j_1}$ where

$$M_{j_1} = \max_{\lambda \in \Lambda_{j_1}} \left\{ s^{[\lambda,j_1]} + \sum_{i=1}^{m_f^{[\lambda,j_1]}} 2k_i^{[\lambda,j_1]} + \sum_{i=1}^{m_c^{[\lambda,j_1]}} 2\left(m(z_i,\lambda_i) - 1\right) \right\} < \infty.$$

(viii) Repeat from step (i) until step (vii) changing j_1 by the next subindex j_i for $i = 2,\dots,\nu$, assuming that the finiteness condition in step (ii) holds in all the repetitions, that is, $\#(f_{j_i}^{-1}(0)) < \infty$ for all $\lambda \in \Lambda_{j_i}$ and any i.

(ix) Finally we get an upper bound M for the averaged cyclicity $\mathrm{Cyc}^T(F_\lambda)$ of the full family of differential equations (1) in $\Omega \times \mathbb{R}^p$ given by

$$\mathrm{Cyc}^T(F_\lambda) \leq M = \max_{1 \leq i \leq \nu} \{M_{j_i}\} < \infty.$$

We note that $\mathrm{Cyc}^T(F_\lambda)$ can be unbounded when $\#(f_{j_i}^{-1}(0)) = \infty$ for some admissible i. This is the case when $\#(\Lambda^*) \geq 1$ and $\#(\Omega^*) = \infty$ producing infinitely many T-periodic solutions of (1) with $\lambda \in \Lambda^*$ for any $|\varepsilon| \ll 1$.

2. Hopf bifurcation in the plane

Consider a family of polynomial planar vector fields

$$\dot{x} = -y + P(x,y;\lambda), \quad \dot{y} = x + Q(x,y;\lambda), \tag{6}$$

with nonlinearities P and Q and parameters λ. Introducing the rescaling $(x,y) \mapsto (x/\varepsilon, y/\varepsilon)$ and next taking polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, family (6) can be written near the origin as

$$\frac{dr}{d\theta} = \mathcal{F}(\theta,r;\lambda,\varepsilon), \tag{7}$$

with $\mathcal{F}(\theta,r;\lambda,0) \equiv 0$, that is, equation (7) is written in the standard form of the averaging theory (1) with period $T = 2\pi$. Notice that the differential equation (7) is defined on the cylinder $\{(r,\theta) \in \Omega \times \mathbb{S}^1\}$ with $\Omega \subset \mathbb{R}$ an open interval containing the origin and $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Remark 12. In special cases the set $\Omega^* \times \Lambda^* \subset \Omega \times \mathbb{R}^p$ of points of infinite-type have associated ideal $\mathcal{I}$ independent on the specific point $(z_0, \lambda^*) \in \Omega^* \times \Lambda^*$ that we choose. This phenomena occurs when the sequence $\{f_j(z;\lambda)\}_{j\in\mathbb{N}} \subset \mathbb{R}[z,\lambda]$ is polynomial. Under this hypothesis $\mathcal{I}$ is a polynomial ideal in the ring $\mathbb{R}[z,\lambda]$. Therefore, $\Omega^* \times \Lambda^* = \mathbf{V}_{\mathbb{R}}(\mathcal{I})$ and we have a unique value of m independently of the point of infinite-type that we consider.

If we expand the displacement map $d(z,\lambda,1)$ of (7) with $\varepsilon = 1$ in powers of z we obtain $d(z,\lambda,1) = \sum_{i\geq 1} v_i(\lambda) z^i$ where the coefficients $v_i \in \mathbb{R}[\lambda]$ are called the *Poincaré-Liapunov constants* associated to the equilibrium point localized at origin of coordinates of the differential equation (6). The *Bautin ideal* $\mathcal{B} \subset \mathbb{R}[\lambda]$ associated to the origin of family (6) is defined as $\mathcal{B} = \langle v_i(\lambda) : i \in \mathbb{N}\rangle$. The *center variety* is defined as $\mathbf{V}_{\mathbb{R}}(\mathcal{B}) \subset \mathbb{R}^p$, and it follows that system (6) with $\lambda = \lambda^c$ has a center at the origin if and only if $\lambda^c \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$. Now we point out a relation between the Bautin ideal $\mathcal{B}$ and the ideal $\mathcal{I}$. We note that both ideals are polynomial ideals.

Proposition 13. *Let $d(z,\lambda,\varepsilon) = \sum_{j\geq 1} f_j(z;\lambda)\,\varepsilon^j$ be the displacement map associated to (7) in a neighborhood of the origin. Then $f_1(z;\lambda) \equiv 0$ and $f_j(z;\lambda) = P_j(\lambda) z^{j+1}$ where $P_j \in \mathbb{R}[\lambda]$ for all $j \in \mathbb{N}$, that is, the j-th Poincaré-Liapunov constant is $v_j(\lambda) = P_{j-1}(\lambda)$. In particular, the Bautin ideal $\mathcal{B}$ and the ideal $\mathcal{I}$ have the same cardinality in their respective minimal basis. More precisely, if $\{P_{i_1}, \ldots, P_{i_m}\}$ is a minimal basis of $\mathcal{B}$, then $\{f_{i_1}, \ldots, f_{i_m}\}$ is a minimal basis of $\mathcal{I}$.*

Proof. The structure $f_j(z;\lambda) = P_j(\lambda) z^{j+1}$ where $P_j \in \mathbb{R}[\lambda]$ for all $j \in \mathbb{N}$ is easy to check for the differential equation (7). So we will prove only the second part of the proposition.

Since $\mathcal{B} = \langle P_i(\lambda) : i \in \mathbb{N}\rangle$, let $\{P_{i_1}, \ldots, P_{i_m}\}$ be a minimal basis of $\mathcal{B}$ with cardinality m. For any $j \geq i_m$ we have $f_j(z;\lambda) = P_j(\lambda) z^{j+1}$ and, since $P_j \in \mathcal{B}$, there are polynomials $q_k(\lambda)$ such that $f_j(z;\lambda) = z^{j+1} \sum_{k=1}^m q_k(\lambda) P_{i_k}(\lambda)$. Clearly this can be rewritten as $f_j(z;\lambda) = \sum_{k=1}^m r_k(z,\lambda) f_{i_k}(z,\lambda)$ just taking $r_k(z,\lambda) = q_k(\lambda) z^{j-i_k} \in \mathbb{R}[z,\lambda]$. Thus $f_j \in \langle f_{i_1}, \ldots, f_{i_m}\rangle$ for all $j \geq i_m$. □

Note that the polynomial differential system (6) has a center at the origin when $\lambda = \lambda^c \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$ if and only if $d(z,\lambda^c,\varepsilon) \equiv 0$ for all (z,ε) in a neighborhood of $(0,0)$ or, equivalently when the functions $f_j(z;\lambda^c) \equiv 0$ for all z near the origin and all $j \in \mathbb{N}$. Clearly, for the values λ^c of the parameters we get that all the non-isolated points (z,λ^c) are of infinite-type for any $z \in \Omega$.

As usual $\mathbb{R}^+$ denotes the set of positive real numbers.

Remark 14. By construction one has $\mathcal{F}(\theta, 0; \lambda, \varepsilon) \equiv 0$, that is, $r = 0$ corresponds to the equilibrium point localized at the origin of coordinates for the full family of differential systems (6). Then it follows that $d(0,\lambda,\varepsilon) \equiv 0$ for all $\lambda \in \mathbb{R}^p$ and any $|\varepsilon| \ll 1$ or, in other words, $(z_0, \lambda) = (0, \lambda) \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$. From Proposition 13 we see that $z_0 = 0$ is always a multiple zero of $f_\ell(.;\lambda)$ for any $\lambda \in \mathbb{R}^p$. But $z_0 = 0$ has associated a finite multiplicity $\bar{k} \geq 2$ only in case that $\lambda \notin \mathbf{V}_{\mathbb{R}}(\mathcal{B})$. Hence, in this multiple zero scenario, in order to discern how many nontrivial isolated periodic solutions system (6) can have near the origin (called *small amplitude limit cycles*) when $\lambda \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$ we must first calculate m to finally obtain a bound of $2(m-1)$ for the small amplitude limit cycles of (6).

The *cyclicity* of the origin of the family of polynomial differential systems (6) is the maximum number of small amplitude limit cycles that can bifurcate from the singularity at the origin of that family. We denote such a cyclicity by $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0))$. We have a method for computing, under some assumptions, an upper bound of the cyclicity $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0))$ based on Corollary 10 applied to differential equation (7) and the algorithm developed for families in subsection §1.4.

Of course, for a fixed parameter λ^*, one has $\mathrm{Cyc}(\mathcal{X}_{\lambda^*},(0,0)) = \mathrm{Cyc}^{2\pi}(\mathcal{F}_{\lambda^*})$, and when Corollary 10 is used in the Hopf bifurcation context only two differences arise:

(a) The zero-set $f_\ell^{-1}(0)$ is the set of zeros z_0 of $f_\ell(.;\lambda^*)$ but restricted to $\Omega \cap (\mathbb{R}^+ \cup \{0\})$.
(b) For the polynomial differential system (7) we have that the functions $f_i \in \mathbb{R}[z,\lambda]$ are polynomials, hence the ideal $\mathcal{I} = \langle f_i(z;\lambda) : i \in \mathbb{N}\rangle$ is a polynomial ideal in the ring $\mathbb{R}[z,\lambda]$ instead of the ring $\mathbb{R}\{z,\lambda\}_{(z_0,\lambda^*)}$. See Remark 12.

2.1. **Quadratic systems.** Consider the quadratic polynomial differential system (simply quadratic system in what follows) in the Bautin normal form

$$
\begin{array}{rcl}
\dot{x} & = & -y + P(x,y;\lambda) = -y - A_3x^2 + (2A_2 + A_5)xy + A_6y^2, \\
\dot{y} & = & x + Q(x,y;\lambda) = x + A_2x^2 + (2A_3 + A_4)xy - A_2y^2,
\end{array} \tag{8}
$$

hence $\lambda = (A_2, A_3, A_4, A_5, A_6) \in \mathbb{R}^5$.

It is well known, see the seminal work [1] and also [15] that $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0)) = 2$ when we consider any $\lambda \in \mathbb{R}^5$. In the work [15] Żołądek makes the classification of the cyclicity $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0))$ of the origin in terms of parameters of family (8):

(I) If $(A_5 - A_6)(A_6 - A_3) \neq 0$ then $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0)) = 0$.
(II) If $(A_5 - A_6)(A_6 - A_3) = 0$ and $(5A_3 + A_4 - 5A_6)^2 + A_5^2 \neq 0$ then the cyclicity is $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0)) = 1$.
(III) If $(A_5 - A_6)(A_6 - A_3) = (5A_3 + A_4 - 5A_6)^2 + A_5^2 = 0$ then $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0)) = 2$.

Now we want to bound the cyclicity $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0))$ for the quadratic family (8) using our theory.

Using Theorem 21 of the appendix we compute the functions $f_j(z;\lambda)$ associated to the differential equation (7). Next, we let $\tilde{f}_j \equiv f_j(\mod \mathcal{I}_{j-1})$ where $\mathcal{I}_j = \langle f_1(z;\lambda),\ldots,f_j(z;\lambda)\rangle$. Thus $\tilde{f}_j$ denotes the remainder of f_j upon division by a Gröbner basis of the ideal generated by the previous f_j. Unless multiplicative constants we get

$$f_2(z;\lambda) = P_2(\lambda)z^3,\ \tilde{f}_3(z;\lambda) \equiv 0,\ \tilde{f}_4(z;\lambda) = P_4(\lambda)z^5,\ \tilde{f}_5(z;\lambda) \equiv 0,\ \tilde{f}_6(z;\lambda) = P_6(\lambda)z^7.$$

where

$$
\begin{array}{rcl}
P_2(\lambda) & = & A_5(A_3 - A_6), \\
P_4(\lambda) & = & A_2A_4(A_3 - A_6)(5A_3 + A_4 - 5A_6), \\
P_6(\lambda) & = & A_2A_4^2(A_3 - A_6)(5A_2^2 + A_4A_6 + 5A_6^2).
\end{array}
$$

We remark that the ideal $\mathcal{I}_6$ for systems (8) is not radical, i.e. $\mathcal{I}_6 \neq \sqrt{\mathcal{I}_6}$. Hence, Theorem 22 of the appendix does not work for proving that $\mathcal{I}_6$ is $\mathcal{I}$. But from Bautin's work [1] it follows that the Bautin ideal $\mathcal{B} = \langle P_2(\lambda), P_4(\lambda), P_6(\lambda)\rangle$. Thus, from Proposition 13, we conclude that $\mathcal{I} = \mathcal{I}_6$ and consequently $m = 3$.

Recalling that $f_j(0;\lambda) = 0$ for all $j \in \mathbb{N}$ and all λ, one has the point $(0,\lambda) \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$ is of infinite-type for any $\lambda \in \mathbb{R}^5$. In particular, we have the cyclicity bound $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0)) \leq 2(m-1) = 4$. Notice that this bound is not sharp and only can be improved using the multiplicity $\bar{k}$ of the zero $z_0 = 0$ if $\tilde{f}_2(z;\lambda) \not\equiv 0$.

2.2. **Cubic Sibirsky systems.** Consider a cubic differential system with cubic homogeneous nonlinearities and having a center with purely imaginary eigenvalues or a focus at the origin of coordinates. Following Sibirsky [13], see also [12] and the references therein, after a linear change of coordinates the system can be written in the following form

$$\begin{array}{rcl} \dot{x} & = & -y + P(x,y;\lambda) = -y + \beta x - (\omega + \Theta - a)x^3 - (\eta - 3\mu)x^2y \\ & & -(3\omega - 3\Theta + 2a - \xi)xy^2 - (\mu - \nu)y^3, \\ \dot{y} & = & x + Q(x,y;\lambda) = x + \beta y + (\mu + \nu)x^3 + (3\omega + 3\Theta + 2a)x^2y \\ & & f(\eta - 3\mu)xy^2 + (\omega - \Theta - a)y^3, \end{array} \tag{9}$$

where $\lambda = (\omega, \Theta, a, \eta, \mu, \xi, \nu) \in \mathbb{R}^7$ are the parameters of the family.

After Żołądek's work [16] it is known that, for system (9), $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0)) = 4$ when considering any $\lambda \in \mathbb{R}^5$. More specifically, [16] gives the following classification of the cyclicity $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0))$ in function of the parameters of family (9):

(I) If $\xi \neq 0$, then $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0)) = 0$.
(II) If $\xi = 0$ and $\nu(10a - \xi) \neq 0$, then $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0)) = 1$.
(III) If $\xi = \nu(10a - \xi) = 0$ and $(10w - \xi)^2 + 25\nu^2 \neq 0$, then $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0)) = 2$.
(IV) This case is not possible in family (9).
(V) If $\xi = w = \nu = \eta = 0$, then $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0)) = 4$.

Now we want to bound the cyclicity $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0))$ for the cubic family (9) using our theory.

We compute the first elements $f_j(z;\lambda)$, after we reduce them modulo the ideal $\mathcal{I}_{j-1}$, and thus we obtain $\tilde{f}_j(z;\lambda)$. We obtain $\tilde{f}_{2i+1}(z;\lambda) \equiv 0$ for $i = 0,1,2,3,4,5$ and, unless a multiplicative constant, the first expressions of $\tilde{f}_{2i}(z;\lambda) = P_{2i}(\lambda)z^{2i+1}$:

$$\tilde{f}_2(z;\lambda) = \xi z^3, \qquad \tilde{f}_4(z;\lambda) = a\nu z^5, \qquad \tilde{f}_6(z;\lambda) = a\Theta w z^7,$$
$$\tilde{f}_8(z;\lambda) = a^2\eta\Theta z^9, \quad \tilde{f}_{10}(z;\lambda) = a^2\Theta(a^2 - 4\Theta^2 - 4\mu^2)z^{11}.$$

Let m be the cardinality of a minimal basis of the ideal $\mathcal{I}$. Unfortunately $\mathcal{I}_{10} \neq \sqrt{\mathcal{I}_{10}}$, so we cannot use Theorem 22 to obtain that $\mathcal{I}$ is $\mathcal{I}_{10}$, and therefore that m is 5. We also note that we cannot use Theorem 23, because the primary decomposition $\mathcal{I}_{10} = \mathcal{R} \cap \mathcal{N}$ is such that the point we want to analyze $(z_0, \lambda) = (0, \lambda)$ is in the variety $\mathbf{V}_{\mathbb{R}}(\mathcal{N})$. But we can use Żołądek's results in [16] from where we know that 5 is the dimension of a minimal basis of the Bautin ideal $\mathcal{B}$. Adapting this result to our framework gives that

$$\mathcal{B} = \langle P_2(\lambda), P_4(\lambda), P_6(\lambda), P_8(\lambda), P_{10}(\lambda)\rangle = \langle \xi, a\nu, a\Theta w, a^2\eta\Theta, a^2\Theta(a^2 - 4\Theta^2 - 4\mu^2)\rangle.$$

Therefore, by using Proposition 13, we conclude that $\mathcal{I} = \mathcal{I}_{10}$, so that $m = 5$.

On the other hand, clearly $f_j(0;\lambda)=0$ for all $j\in\mathbb{N}$ and all λ, so $(0,\lambda)\in\mathbf{V}_{\mathbb{R}}(\mathcal{I})$ for any $\lambda\in\mathbb{R}^7$. This means that the cyclicity $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0))\le 2(m-1)=8$, which is not a sharp bound and only can be improved taking into account the multiplicity $\bar{k}$ of the zero $z_0=0$ of $f_\ell(.;\lambda^*)$ if $\tilde{f}_j(z;\lambda)\not\equiv 0$ for some $j\in\{2,4,6\}$.

3. Bifurcations from the period annulus

3.1. Perturbing a linear center inside the generalized Liénard systems. We shall study the maximum number of limit cycles that can bifurcate from the periodic orbits of the period annulus of a linear center perturbed inside a class of polynomial generalized Liénard differential equations of degree 7. More specifically we analyze the perturbed system

$$\dot{x}=y,\quad \dot{y}=-x-\varepsilon\left(y\hat{f}_6(x;\lambda,\varepsilon)+\hat{g}_6(x;\lambda,\varepsilon)\right), \tag{10}$$

with

$$\hat{f}_6(x;\lambda,\varepsilon)=\sum_{i=0}^{6}(A_i+B_i\varepsilon)x^i,\quad \hat{g}_6(x;\lambda,\varepsilon)=\sum_{i=1}^{6}(C_i+D_i\varepsilon)x^i.$$

Here the parameters $\lambda\in\mathbb{R}^{26}$ are the coefficients A_i,B_i,C_j,D_j for $i=0,\dots,6$ and $j=1,\dots,6$. In this example, first we see how from Theorem 3 we obtain a uniform bound on the number of bifurcating limit cycles that is either equal or sharp than the obtained using our theory.

Proposition 15. *Consider the family of quintic Liénard polynomial differential systems* (10) *(that is with $A_i=B_i=C_i=D_i=0$ for $i=5,6$) under the parameter restriction λ^* given by $A_2^2-8A_0A_4=0$ and $A_2A_4<0$. Then, for $|\varepsilon|$ sufficiently small, limit cycle bifurcations in the period annulus of the linear center can only be produced from the periodic orbit $x^2+y^2=-A_2/A_4$. Moreover, the maximum number of such bifurcating limit cycles is bounded by* 2.

Proof. Taking polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ and observing that $\dot{\theta}=-1+\mathcal{O}(\varepsilon)$, system (10) can be written as $dr/d\theta=\mathcal{F}(\theta,r;\lambda,\varepsilon)$ with $\mathcal{F}(\theta,r;\lambda,0)\equiv 0$. This differential equation is defined on the cylinder $\{(r,\theta)\in\Omega\times\mathbb{S}^1\}$ with $\Omega\subset\mathbb{R}$ and $\mathbb{S}^1=\mathbb{R}/2\pi\mathbb{Z}$. Thus we can apply to it averaging theory with period $T=2\pi$. In order to prove the proposition we need to compute the averaged cyclicity $\mathrm{Cyc}^{2\pi}(\mathcal{F}_\lambda)$ of systems (10) in $\Omega\times\Lambda$ with $\Lambda=\mathbb{R}^{18}$ and $\Omega=\mathbb{R}^+$. We recall that, perturbing a linear center by a polynomial field is other situation where the ideal $\mathcal{I}=\langle f_i(z;\lambda):i\in\mathbb{N}\rangle$ is a polynomial ideal.

Computations show that $f_1(z;\lambda)=\pi z(8A_0+2A_2z^2+A_4z^4)/8\not\equiv 0$. By assumptions $A_2A_4<0$ and the discriminant $\Delta=A_2^2-8A_0A_4=0$. Then the function $f_1(.;\lambda^*)$ only has one positive zero $z_0=\sqrt{-A_2/A_4}>0$, with multiplicity $\bar{k}=2$. Therefore, by Lemma 1, the bifurcations in the period annulus are only possible from the periodic orbit $x^2+y^2=z_0^2$ of the unperturbed linear system and, by Theorem 3, at most can bifurcate $\bar{k}=2$ limit cycles.

Further computations show that

$$\begin{aligned} f_2(z;\lambda) &= \frac{\pi}{3840} z(3840B_0 - 1920A_0C_1 + 1920A_0^2\pi - 1280A_0C_2z \\ &\quad +960B_2z^2 - 480A_2C_1z^2 - 960A_1C_2z^2 - 2640A_0C_3z^2 \\ &\quad +1920A_0A_2\pi z^2 + 320A_2C_2z^3 - 2304A_0C_4z^3 + 480B_4z^4 \\ &\quad -240A_4C_1z^4 - 800A_3C_2z^4 - 600A_2C_3z^4 - 480A_1C_4z^4 \\ &\quad +360A_2^2\pi z^4 + 1440A_0A_4\pi z^4 + 480A_4C_2z^5 - 192A_2C_4z^5 \\ &\quad -285A_4C_3z^6 - 420A_3C_4z^6 + 480A_2A_4\pi z^6 + 96A_4C_4z^7 \\ &\quad +150A_4^2\pi z^8). \end{aligned}$$

Therefore $f_2(z_0;\lambda) = \dfrac{z_0}{768A_4^3}P(\lambda)$ where

$$\begin{aligned} P(\lambda) &= 768A_4^3B_0 - 192A_2A_4^2(B_2 - A_1C_2) + 84A_2^3A_3C_4 \\ &\quad +A_2^2A_4(96B_4 - 160A_3C_2 + 3A_2C_3 - 96A_1C_4). \end{aligned}$$

So $f_2(z_0;\lambda^*) \neq 0$ if $P(\lambda^*) \neq 0$ and $f_2(z_0;\lambda^*) = 0$ otherwise. In the first case the point (z_0,λ^*) is of finite-type with order 1 while in the second case it can be either of finite-type with order $k \geq 2$ or of infinite-type. Anyway, if (z_0,λ^*) was of infinite-type, we claim that the cardinality m of a minimal basis of the ideal $\mathcal{I} = \langle f_i(z;\lambda) : i \in \mathbb{N}\rangle$ in the ring $\mathbb{R}[z,\lambda]$ is $m \geq 3$ finishing the proof after using Corollary 10.

To prove the claim first we check that, defining $\mathcal{I}_j = \langle f_i(z;\lambda) : 1 \leq i \leq j \in \mathbb{N}\rangle$, it follows $f_2 \notin \mathcal{I}_1$ and $f_3 \notin \mathcal{I}_2$. This means that $f_j(z;\lambda)$ with $j = 1,2,3$ are elements of a minimal basis of the ideal $\mathcal{I}$ and therefore $m \geq 3$. We have make these computations with polynomial ideals in the ring $\mathbb{R}[z,\lambda]$ with $\lambda = (A_0,A_1,A_2,A_3,A_4,B_0,B_1,B_2,B_3,B_4,C_1,C_2,C_3,C_4,D_1,D_2,D_3,D_4)$ and introducing the discriminant Δ as an additional generator in each $\mathcal{I}_j$.

It is worth to remark that we cannot get a bound lower than 2 in the case $P(\lambda^*) \neq 0$ for which the point (z_0,λ^*) has order 1 using statement (i) of Proposition 11 because the associated multiplicity $\bar{k}$ is even. □

In next example we show how the bounds obtained using the classical theory for multiple points stated in Theorem 3 is improved using our results.

Proposition 16. *Consider the family of septic Liénard polynomial differential systems* (10) *under the parameter restriction* λ^* *given by* $4A_4^2 - 15A_2A_6 = 0$, $8A_4^3 - 675A_0A_6^2 = 0$ *and* $A_4A_6 < 0$. *Then, for* $|\varepsilon|$ *sufficiently small, limit cycle bifurcations in the period annulus of the linear center can only be produced from the periodic orbit* $x^2 + y^2 = -\dfrac{8A_4}{15A_6}$. *Moreover, exactly one complete branch of limit cycles bifurcates when* $P(\lambda^*) \neq 0$, *where*

$$\begin{aligned} P(\lambda^*) &= 768A_4^3B_0 - 192A_2A_4^2(B_2 - A_1C_2) + 84A_2^3A_3C_4 + \\ &\quad A_2^2A_4(96B_4 - 160A_3C_2 + 3A_2C_3 - 96A_1C_4). \end{aligned}$$

Proof. As in the proof of Proposition 15, after taking polar coordinates, we write family (10) into the 2π-periodic standard form $dr/d\theta = \mathcal{F}(\theta,r;\lambda,\varepsilon)$ in $\Omega \times \Lambda$ with $\Lambda = \mathbb{R}^{26}$, $\Omega = \mathbb{R}^+$, and associated polynomial ideal $\mathcal{I} = \langle f_i(z;\lambda) : i \in \mathbb{N}\rangle$.

The first averaged function is $f_1(z;\lambda) = \pi z(64A_0+16A_2z^2+8A_4z^4+5A_6z^6)/64 \not\equiv 0$. The zeros of $f_1(.;\lambda)$ comes from a cubic equation for the unknown z^2. Using the discriminants of the cubic equations it follows that $f_1(.;\lambda)$ has a multiple zero $z_0 > 0$ of multiplicity 3 if and only if $\Delta_i = 0$ for $i = 1, 2, 3$, where $\Delta_1 = -4A_2^2A_4^2 + 32A_0A_4^3 + 20A_2^3A_6 - 180A_0A_2A_4A_6 + 675A_0^2A_6^2$, $\Delta_2 = 4A_4^2 - 15A_2A_6$ and $\Delta_3 = 16A_4^3 - 90A_2A_4A_6 + 675A_0A_6^2$. In particular, $z_0^2 = -\dfrac{8A_4}{15A_6}$ with our assumption $A_4A_6 < 0$. In order to simplify the polynomial conditions in the parameter space producing the unique zero $z_0 > 0$ of $f_1(.;\lambda)$ with multiplicity $\bar{k} = 3$ we calculate the resultants of each pair of polynomials Δ_i with respect to A_2 and, since $A_6 \neq 0$ we obtain the necessary condition $8A_4^3 - 675A_0A_6^2 = 0$ stated in the proposition. Solving for A_0 this condition and substituting into $\Delta_2 = \Delta_3 = 0$ produces the extra necessary condition $4A_4^2 - 15A_2A_6 = 0$ also stated in the proposition. Notice that we can solve for A_2 from the above equation.

In summary, when $|\varepsilon| \ll 1$, the bifurcation of limit cycles from the period annulus of the linear center is only possible from the circle $x^2 + y^2 = z_0^2$. By Theorem 3, the maximum number of complete branches of limit cycles that can bifurcate is bounded by $\bar{k} = 3$.

Now, we will use our method in order to improve the above classical bound. The next averaged function is

$$\begin{aligned} f_2(z;\lambda) \quad = \quad & \frac{\pi}{860160} z(860160B_0 - 430080A_0C_1 + 430080A_0^2\pi - 286720A_0C_2z + \\ & 215040B_2z^2 - 107520A_2C_1z^2 - 215040A1C_2z^2 - 591360A_0C_3z^2 + \\ & 430080A_0A_2\pi z^2 + 71680A_2C_2z^3 - 516096A_0C_4z^3 + 107520B_4z^4 - \\ & 53760A_4C_1z^4 - 179200A_3C_2z^4 - 134400A_2C_3z^4 - 107520A_1C_4z^4 - \\ & 663040A_0C_5z^4 + 80640A_2^2\pi z^4 + 322560A_0A_4\pi z^4 + 107520A_4C_2z^5 - \\ & 43008A_2C_4z^5 - 614400A_0C_6z^5 + 67200B_6z^6 - 33600A_6C_1z^6 - \\ & 156800A_5C_2z^6 - 63840A_4C_3z^6 - 94080A_3C_4z^6 - 150080A_2C_5z^6 - \\ & 67200A_1C_6z^6 + 107520A_2A_4\pi z^6 + 268800A_0A6\pi z^6 + 112000A6C_2z^7 + \\ & 21504A_4C_4z^7 - 92160A_2C_6z^7 - 38640A_6C_3z^8 - 84672A_5C_4z^8 - \\ & 70560A_4C_5z^8 - 60480A_3C_6z^8 + 33600A_4^2\pi z^8 + 84000A_2A_6\pi z^8 + \\ & 40320A_6C_4z^9 - 15360A_4C_6z^9 - 42280A_6C_5z^{10} - 55440A_5C_6z^{10} + \\ & 50400A_4A_6\pi z^{10} + 9600A_6C_6z^{11} + 18375A_6^2\pi z^{12}) \end{aligned}$$

Therefore one can check that $f_2(z_0;\lambda^*) \neq 0$ if and only if $P(\lambda^*) \neq 0$ where P is displayed in the statement of the proposition. Thus, we conclude that the point (z_0, λ^*) is of finite-type with order 1 only when $P(\lambda^*) \neq 0$. In this situation, from statement (i) of Theorem 7 improved by statement (i) of Proposition 11 we conclude that exactly one complete branch of limit cycles bifurcates from z_0 finishing the proof.

We emphasize that, although we will not do it, the degeneracy of the problem can be augmented imposing $P(\lambda^*) = 0$ and solving for B_0 so that we get new (and huge) parameter restrictions in the expression of $f_3(z_0;\lambda^*)$ from where we can decide if the point (z_0, λ^*) is of finite-type with order 2. $\square$

4. Polynomial ideal $\mathcal{I}$ when the parameters are fixed

Throughout this section we will pick up just one element of family (1) by fixing its parameter, say taking $\lambda = \lambda^*$. We will also work under the hypothesis that the ideal $\mathcal{I}$ is polynomial. Indeed, following the proof of Lemma 9 in [6], we know that $\mathcal{I} = \langle f_j : j \in \mathbb{N} \rangle$ is a polynomial ideal in the ring $\mathbb{R}[z]$ when equation (1) with fixed $\lambda = \lambda^*$ is a polynomial equation, i.e., when $F_i(t, x; \lambda^*)$ are polynomial in x for all $i \in \mathbb{N}$.

Under these hypothesis we will see now that Theorem 22 is strongly simplified. We recall before that, since $\mathcal{I}$ is an ideal in the ring of univariate polynomials, $\mathcal{I}$ is a principal ideal, see for instance [3]. Thus $\mathcal{I}$ is generated by one element $\mathcal{I} = \langle g \rangle$ where $g \in \mathbb{R}[z]$ is unique up to a multiplication by a nonzero constant in $\mathbb{R}$. In fact, if $\mathcal{I} \neq \{0\}$ then the generator g is a nonzero polynomial of minimum degree contained in $\mathcal{I}$. Moreover, we note that for any $p, q \in \mathbb{R}[z]$ one has $\langle p, q \rangle = \langle r \rangle$ where $r = \gcd(p, q)$ is a greatest common divisor of p and q, see again [3]. Defining $\mathcal{I}_s = \langle f_j : 1 \leq j \leq s \in \mathbb{N} \rangle$ the ideal in $\mathbb{R}[z]$ generated by the first s averaged functions, we have the following result.

Theorem 17. *Let the ideal $\mathcal{I}_s = \langle \hat{g} \rangle \subset \mathbb{R}[z]$ where all the roots of $\hat{g}$ are real and simple. Assume the equality $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) = \mathbf{V}_{\mathbb{R}}(\mathcal{I}_s)$ of real varieties holds. Then $\mathcal{I} = \mathcal{I}_s$.*

Proof. The proof consists on several steps.

(i) First, we claim that $\mathcal{I}_s$ is a radical ideal. To this end we recall that in the ring $\mathbb{R}[z]$ of univariate polynomials the nontrivial radical ideals are precisely those ideals generated by square-free polynomials, see [3]. In consequence, when $\hat{g} \notin \{0, 1\}$, $\mathcal{I}_s$ is radical if and only if $\hat{g}$ has no repeated roots over $\mathbb{C}$ or, equivalently $\hat{g}$ and its derivative $\hat{g}'$ are coprime. Since by hypothesis the roots of $\hat{g}$ are simple, then g is square-free and we prove claim (i).

(ii) Second, we claim that the equality $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) = \mathbf{V}_{\mathbb{C}}(\mathcal{I}_s)$ of complex varieties hold. Actually, since the polynomial $\hat{g}$ has no non-real roots then clearly $\mathbf{V}_{\mathbb{C}}(\mathcal{I}_s) = \mathbf{V}_{\mathbb{R}}(\mathcal{I}_s)$. Adjoining our hypothesis we obtain that $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) = \mathbf{V}_{\mathbb{C}}(\mathcal{I}_s)$. Since by definition $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) \subseteq \mathbf{V}_{\mathbb{C}}(\mathcal{I})$, the former implies that

$$\mathbf{V}_{\mathbb{C}}(\mathcal{I}_s) \subseteq \mathbf{V}_{\mathbb{C}}(\mathcal{I}). \tag{11}$$

Finally, taking into account that $\mathcal{I}_s \subseteq \mathcal{I}$, one has $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \subseteq \mathbf{V}_{\mathbb{C}}(\mathcal{I}_s)$ which combined with (11) gives the proof of claim (ii).

From the former claims (i) and (ii) and Theorem 22 applied with a number of parameters $p = 0$, we conclude that $\mathcal{I} = \mathcal{I}_s$ finishing the proof. □

Remark 18. Notice that, under the assumptions of Theorem 17, for $|\varepsilon| \ll 1$, the maximum number of isolated T-periodic solutions that can bifurcate from a point $z_0 \in \mathbb{R}$ of infinite-type is bounded by $m - 1$, where $\mathcal{I}_s$ has a minimal basis formed by averaged functions of cardinality m. In order to check whether this upper bound is sharp we can use the following classical strategy of bifurcation theory. Assume that averaged functions $\{f_{j_1}(z), \ldots, f_{j_m}(z)\}$ form a minimal basis of the ideal $\mathcal{I}$ and satisfy, for $|z - z_0| \ll 1$, the chain of inequalities $|f_{j_1}(z)| \ll |f_{j_2}(z)| \ll \cdots \ll$

$|f_{j_m}(z)| \ll 1$, with $f_{j_i}(z) f_{j_{i+1}}(z) < 0$ for $i = 1, \dots, m-1$. Then we get that $m-1$ isolated T-periodic solutions can be made to bifurcate from z_0.

4.1. **An Abel equation.** We consider the 2π-periodic Abel differential equation in standard form

$$\dot{x} = \varepsilon x \sum_{i=0}^{2} A_i(t, \varepsilon)\, x^i \tag{12}$$

defined on $\Omega = \mathbb{R}$ with coefficients

$$A_0(t,\varepsilon) = \varepsilon - 2\varepsilon^2 \cos t + \sin t, \quad A_1(t,\varepsilon) = 3\varepsilon + \cos t + \varepsilon \sin t, \quad A_2(t,\varepsilon) = 1 + \cos t + \sin t.$$

Let $z \in \mathbb{R}$ be the initial condition for the solutions of (12). Since (12) is polynomial in x, we known that the averaged functions $f_j(z)$ are polynomial. Hence, the ideal $\mathcal{I} = \langle f_j : j \in \mathbb{N} \rangle$ is an ideal in the ring $\mathbb{R}[z]$. It is easy to compute the first averaged function $f_1(z) = 2\pi z^3 \not\equiv 0$. From here we see that $z_0 = 0$ is a multiple zero of f_1. Clearly z_0 is of infinite-type since it is an equilibrium of (12) for all ε. Notice that $z_0 = 0$ has multiplicity $\bar{k} = 3$, hence at most three 2π-periodic solution can bifurcate from the equilibrium $x = 0$ for $|\varepsilon| \ll 1$, according with Theorem 3. In the next result we will improve this bound using our theory to compute the associated ideal $\mathcal{I}$ to z_0.

Proposition 19. *For $|\varepsilon| \ll 1$, there is no nonconstant 2π-periodic solution of the Abel equation* (12) *bifurcating from a finite point.*

Proof. Computing the first averaged function we obtain $f_1(z) = 2\pi z^3 \not\equiv 0$. Therefore, for $|\varepsilon| \ll 1$, the 2π-periodic solutions of the Abel equation (12) bifurcating from a finite point only can bifurcate from the equilibrium at $z_0 = 0$. Due to the fact that the multiplicity $\bar{k}$ of $z_0 = 0$ is odd, we know that at least one complete branch of 2π-periodic solutions bifurcates from $z_0 = 0$, see Theorem 3. More precisely, since $\bar{k} = 3$, the number of complete branches of 2π-periodic solutions bifurcating from $z_0 = 0$ is either one or three.

Now we compute the next averaged functions yielding $f_2(z) = \pi z(2 + 5z + 2z^2 + z^3 + 6\pi z^4)$ and $f_j \in \mathcal{I}_2$ for $j = 3, \dots, 7$, so that it is probable that $\mathcal{I}$ is just $\mathcal{I}_2$. To see that this is indeed the situation, let $g = \gcd(f_1, f_2) = z$ be the greatest common divisor of f_1 and f_2. Then $\mathcal{I}_2 = \langle \hat{g} \rangle$ with $\hat{g}(z) = z$ which only has a real simple root at $z_0 = 0$. On the other hand, since $f_j(0) = 0$ for any $j \in \mathbb{N}$ because $z = 0$ is an equilibrium, it is clear that $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) = \mathbf{V}_{\mathbb{R}}(\mathcal{I}_2) = \{z = 0\}$. In conclusion, from Theorem 17, $\mathcal{I} = \mathcal{I}_2$ and therefore $\mathcal{I}$ has a minimal basis formed by averaging functions of cardinality $m = 2$. Thus, using statement (ii) of Theorem 7, at most two positive or negative branches of 2π-periodic solutions bifurcate from the origin. We note that we cannot use Remark 18 to reach the above bound since for $|z| \ll 1$ we have $f_1(z) \sim z^3$ and $f_2(z) \sim z$, hence $f_1(z) f_2(z) > 0$.

Joining both results we conclude that there is exactly one complete branch of 2π-periodic solutions of (12) bifurcating from $z_0 = 0$. Since the origin is an equilibrium of (12) for any ε, it counts as a branch of 2π-periodic solutions and therefore the proposition follows. □

5. Proofs

First we note that the zeros of the displacement function d with $\varepsilon \neq 0$ coincide with the zeros of the reduced displacement map

$$\Delta(z, \lambda, \varepsilon) = \frac{d(z, \lambda, \varepsilon)}{\varepsilon^\ell} = f_\ell(z; \lambda) + \sum_{i \geq 1} f_{\ell+i}(z; \lambda)\varepsilon^i. \tag{13}$$

5.1. **Proof of Lemma 1.**

Proof. From $d(z^*(\varepsilon, \lambda), \lambda, \varepsilon) \equiv 0$ for all $|\varepsilon| \neq 0$ sufficiently small we see that also $\Delta(z^*(\varepsilon), \lambda, \varepsilon) \equiv 0$. Thus we have $f_\ell(z^*(\varepsilon); \lambda) + O(\varepsilon) \equiv 0$, see (13). Evaluating this condition at $\varepsilon = 0$ yields $f_\ell(z_0; \lambda) = 0$. □

5.2. **Proof of Theorem 2.**

Proof. We will analyze the reduced displacement map (13). Since z_0 is a simple zero of $f_\ell(.; \lambda^*)$, by the Implicit Function Theorem applied to $\Delta(z, \lambda^*, \varepsilon)$ in a neighborhood of $(z, \varepsilon) = (z_0, 0)$ we find a unique analytic function $z^*(\varepsilon)$ defined for $|\varepsilon| \ll 1$ and satisfying $z^*(0) = z_0$ such that $\Delta(z^*(\varepsilon), \lambda^*, \varepsilon) \equiv 0$. So Theorem 2 follows. □

5.3. **Proof of Theorem 5.**

Theorem 5 is just a consequence of Remark 4 and the following result.

Proposition 20. *Assume that $f_\ell \not\equiv 0$ and there exists $z_0 \in \Omega$ such that $f_\ell(z_0; \lambda) = 0$ and $\frac{\partial}{\partial z} f_\ell(z_0; \lambda) = 0$. Assume also that there is $k \geq 1$, the minimum integer satisfying $f_{\ell+k}(z_0; \lambda) \neq 0$. Then there are at most k functions $\varepsilon_i^*(z)$ with $i = 1, \ldots, k$ where $\varepsilon_i^*(z_0) = 0$ and satisfying $\Delta(z, \lambda, \varepsilon_i^*(z)) \equiv 0$ for all z in a sufficiently small neighborhood of z_0. Moreover, if k is odd then $\varepsilon_1^*(z)$ exists.*

Proof. Taking k derivatives of Δ with respect to ε and evaluating at $(z_0, \lambda, 0)$ gives

$$\frac{\partial^k \Delta}{\partial \varepsilon^k}(z_0, \lambda, 0) = k!\, f_{\ell+k}(z_0; \lambda) \neq 0,$$

where we have used in the last step the hypothesis $f_{\ell+k}(z_0; \lambda) \neq 0$. Then, from the Weierstrass preparation theorem (see for instance [14]), we can factorize Δ analytically around the point $(z, \lambda, \varepsilon) = (z_0, \lambda, 0)$ as

$$\Delta(z, \lambda, \varepsilon) = P_k(z, \lambda, \varepsilon) U(z, \lambda, \varepsilon), \tag{14}$$

where $U(z_0, \lambda, 0) = 1$ and P_k is a polynomial of degree k in the variable ε given by

$$P_k(z, \lambda, \varepsilon) = f_\ell(z; \lambda) + \sum_{i=1}^{k-1} a_i(z; \lambda)\varepsilon^i + \varepsilon^k,$$

where the coefficients $a_i(z; \lambda)$ are analytic functions near $z = z_0$. Due only to the degree, it is clear that there are at most k functions $\varepsilon_i^*(z, \lambda)$ with $i = 1, \ldots, k$ where $\varepsilon_i^*(z_0, \lambda) = 0$ and satisfying $P_k(z, \lambda, \varepsilon_i^*(z)) \equiv 0$, hence $\Delta(z, \lambda, \varepsilon_i^*(z)) \equiv 0$. This proves the first part of the proposition. The last part is a straightforward consequence of the continuous dependence of roots of polynomials on its coefficients if the degree of the polynomial does not change. □

5.4. **Proof of Theorem 7.**

Proof. Let (z_0, λ^*) be a point of finite-type with order $k \geq 1$. By definition, $f_{\ell+k}(z_0; \lambda^\dagger) \neq 0$. Then for (z, λ) close to $(z_0, \lambda^\dagger)$ we have $f_{\ell+k}(z; \lambda) \neq 0$, and consequently we can write

$$d(z, \lambda, \varepsilon) = \sum_{i=\ell}^{\ell+k-1} f_i(z; \lambda)\, \varepsilon^i + f_{\ell+k}(z; \lambda)\psi(z, \lambda, \varepsilon)\, \varepsilon^{\ell+k},$$

where the function ψ is analytic and satisfies $\psi(z, \lambda, 0) = 1$. Now we can make repeated application of the Rolle's Theorem as in the proof of Proposition 6.1.2 of [10] to see that the function d behaves like a polynomial in ε near $(z_0; \lambda^\dagger)$, hence the number of zeros of $d(z, \lambda, .)$ is bounded. More specifically, from there it follows that the maximum number of isolated zeros of $d(z, \lambda, .)$ coming from the zero $(z_0; \lambda^\dagger)$ of order k in the interval $(0, \hat{\varepsilon})$ with $\hat{\varepsilon} > 0$ sufficiently small is k. We note that using the same arguments we obtain that the number of isolated zeros of $d(z, \lambda, .)$ in $(-\hat{\varepsilon}, 0)$ with $\hat{\varepsilon} > 0$ sufficiently small is also bounded by k. This proves statement (i).

Now we will prove statement (ii). Let (z_0, λ^*) be a point of infinite-type. Let

$$\{f_{j_1}(z; \lambda), \ldots, f_{j_m}(z; \lambda)\},$$

a minimal basis of the ideal $\mathcal{I}$ of finite cardinality $m \geq 1$ where $\ell \leq j_1 < j_2 < \cdots < j_m$. Then, adapting Lemma 6.1.6 [10] to this context, for (z, λ) sufficiently close to (z_0, λ^*) and for ε near zero, the displacement function d can be written as

$$d(z, \lambda, \varepsilon) = \sum_{i=1}^{m} f_{j_i}(z; \lambda)\, \varepsilon^{j_i} \psi_i(z, \lambda, \varepsilon),$$

where ψ_i is an analytic functions such that $\psi_i(z, \lambda, 0) = 1$. Again following [10] (in particular Theorem 6.1.7 of [10]) we have that $d(z; \lambda, .)$ can have at most $m - 1$ small isolated (either positive or negative) zeros, that is, there are at most $m - 1$ functions either $\varepsilon_j^*(z; \lambda) \geq 0$ or $\varepsilon_j^*(z; \lambda) \leq 0$ such that $d(z, \lambda, \varepsilon_j^*(z; \lambda)) \equiv 0$ for $j = 1, \ldots, m - 1$, for all (z, λ) sufficiently close to (z_0, λ^*). This proves statement (ii). □

6. Appendix

6.1. **The expansion of the displacement map.** In order to compute the expansion in power series of ε in the displacement map $d(z, \lambda, \varepsilon) = \sum_{i\geq 1} f_i(z; \lambda)\, \varepsilon^i$, first we impose that $x(t; z, \lambda, \varepsilon)$ be a solution of (1), that is,

$$\frac{\partial x}{\partial t}(t; z, \lambda, \varepsilon) = \sum_{i\geq 1} F_i(t, x(t; z, \lambda, \varepsilon); \lambda)\, \varepsilon^i, \tag{15}$$

for all small enough $|\varepsilon|$. Taking into account (2), the left-hand side of (15) is expanded as

$$\frac{\partial x}{\partial t}(t; z, \lambda, \varepsilon) = \sum_{i\geq 1} \frac{\partial x_i}{\partial t}(t, z, \lambda)\, \varepsilon^i.$$

The power series in ε of the right-hand side of (15) is more involved. To get it, first we can perform the Taylor expansion of $F_i(t,x;\lambda)$ at $x=z$. Using (2) again we get

$$\begin{aligned} F_i(t,x(t;z,\lambda,\varepsilon);\lambda) &= F_i(t,z+\sum_{j\geq 1} x_j(t,z,\lambda)\,\varepsilon^j;\lambda) \\ &= F_i(t,z;\lambda)+\sum_{\alpha\geq 1}\frac{1}{\alpha!}\frac{\partial^\alpha F_i}{\partial x^\alpha}(t,z;\lambda)\left(\sum_{j\geq 1} x_j(t,z,\lambda)\,\varepsilon^j\right)^\alpha . \end{aligned}$$

Now we impose that equation (15) is verified equating all the coefficients of like powers of ε. From this procedure one obtains a sequence of linear differential equations for the unknown functions $x_i(t,z,\lambda)$ which can be solved with the initial conditions $x_j(0,z,\lambda)=0$. Finally we recall that $f_i(z;\lambda)=x_i(T;z,\lambda)$. In particular, the function $x_1(t,z,\lambda)$ satisfies the Cauchy problem $\dfrac{\partial x_1}{\partial t}(t,z,\lambda)=F_1(t,z,\lambda)$ with initial value $x_1(0,z,\lambda)=0$, hence $x_1(t,z,\lambda)=\int_0^t F_1(\tau,z,\lambda)\,d\tau$ and therefore

$$f_1(z;\lambda)=\int_0^T F_1(t,z,\lambda)\,dt.$$

The above algorithm can be summarized in the next result of [6].

Theorem 21. *The solution $x(t;z,\lambda,\varepsilon)$ of the T-periodic analytic equation* (1) *having initial condition $x(0;z,\lambda,\varepsilon)=z$ can be written as $x(t;z,\lambda,\varepsilon)=z+\sum_{j\geq 1} x_j(t,z,\lambda)\,\varepsilon^j$ where the $x_j(t,z,\lambda)$ can be computed recursively as follows:*

$$\begin{aligned} x_1(t,z,\lambda) &= \int_0^t F_1(\tau,z;\lambda)\,d\tau, \\ x_k(t,z,\lambda) &= \int_0^t \Bigg(F_k(\tau,z;\lambda)+\sum_{\ell=1}^{k-1}\sum_{i=1}^{\ell}\frac{1}{i!}\frac{\partial^i F_{k-\ell}}{\partial x^i}(\tau,z;\lambda) \\ &\quad \times \sum_{j_1+j_2+\cdots+j_i=\ell}\ \prod_{p=1}^{i} x_{j_p}(t,z,\lambda)\Bigg)\,d\tau, \end{aligned}$$

for all $k\geq 2$, where j_m are positive integers for all $m=1,\ldots,i$.

6.2. **Cyclicity bound theorems in averaging theory.** The results of this section are restricted to the case in which the ideal $\mathcal{I}$ is a polynomial ideal in the ring $\mathbb{R}[z,\lambda]$. The reason is because in the proofs of the forthcoming theorems we need to use Hilbert Nullstellensatz that relates complex varieties and ideals in $\mathbb{C}[z,\lambda]$.

The following result is useful to obtain a set of generators of the polynomial ideal $\mathcal{I}$ in case that $\mathcal{I}=\sqrt{\mathcal{I}}$, that is, $\mathcal{I}$ is a radical ideal. It is useful for analyzing the multiple zeros z_0 of $f_\ell(.;\lambda)$ with (z_0,λ) of infinite-type in Corollaries 9 and 10, in particular for the computation of the cardinality m of the minimal base of $\mathcal{I}$. We use the notation $\mathcal{I}_k=\langle f_i(z;\lambda): 1\leq i\leq k\rangle$. The following theorem is proved in [5], we only state it using our notation and prove it since it is a short proof.

Theorem 22 (Radical Ideal Cyclicity Bound Theorem)**.** *Let $\mathcal{I}$ be a polynomial ideal. Assume that the equality of complex varieties $\mathbf{V}_{\mathbb{C}}(\mathcal{I})=\mathbf{V}_{\mathbb{C}}(\mathcal{I}_k)$ holds in $\mathbb{C}^{p+1}$ for some $k\in\mathbb{N}$ and that $\mathcal{I}_k$ is a radical ideal. Then $\mathcal{I}=\mathcal{I}_k$.*

Proof. Suppose that $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) = \mathbf{V}_{\mathbb{C}}(\mathcal{I}_k)$. From the Strong Hilbert Nullstellensatz we know that the above equality of complex varieties is equivalent to the equality of polynomial ideals $\sqrt{\mathcal{I}} = \sqrt{\mathcal{I}_k}$. You can consult for example Proposition 3.1.16 of [10]. Then, using the assumption $\sqrt{\mathcal{I}_k} = \mathcal{I}_k$, yields

$$\mathcal{I}_k \subset \mathcal{I} \subset \sqrt{\mathcal{I}} = \sqrt{\mathcal{I}_k} = \mathcal{I}_k$$

and therefore $\mathcal{I} = \mathcal{I}_k$ finishing the proof. □

Assume that $\mathcal{I} \neq \sqrt{\mathcal{I}}$. Now Theorem 22 does not work but we still can bound the number of positive or negative isolated T-periodic solutions of equation (1) with $\lambda = \lambda^*$ that, for $|\varepsilon|$ sufficiently small, come from the zeros $z_0 \in \Omega$ of the averaged function $f_\ell(.;\lambda^*)$ when (z_0, λ^*) is a point of infinite-type and (z_0, λ^*) belongs to certain pieces of the variety $\mathbf{V}_{\mathbb{R}}(\mathcal{I})$ that we specified below. The following theorem is an adaptation to our framework of Theorem 20 in [5].

Theorem 23 (Non-Radical Ideal Cyclicity Bound Theorem)**.** *Let the ideal $\mathcal{I}$ be a polynomial ideal. Assume that the equality of complex varieties $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) = \mathbf{V}_{\mathbb{C}}(\mathcal{I}_k)$ holds in $\mathbb{C}^{p+1}$ for some $k \in \mathbb{N}$ but $\mathcal{I}_k$ is not a radical ideal. Let κ be the cardinality of a minimal basis of $\mathcal{I}_k$, hence with $\kappa \leq k$. Suppose a primary decomposition of $\mathcal{I}_k$ can be written as $\mathcal{I}_k = \mathcal{R} \cap \mathcal{N}$ where $\mathcal{R}$ is the intersection of the ideals in the decomposition that are prime and $\mathcal{N}$ is the intersection of the remaining ideals in the decomposition. Then the number of (either positive or negative) branches of isolated T-periodic solutions that can have any equation of family* (1) *corresponding to parameters $\|\lambda - \lambda^*\| \ll 1$ bifurcating, for $|\varepsilon| \ll 1$, from a point $z_0 \in \Omega$ when $(z_0, \lambda^*) \in \mathbf{V}_{\mathbb{R}}(\mathcal{I}) \setminus \mathbf{V}_{\mathbb{R}}(\mathcal{N})$ is at most $2(\kappa - 1)$.*

A key step in the proof of Theorem 23 is to use the following result from [4] based on the arguments of Proposition 1 in [7]. For a subset $S \subset \mathbb{C}^{p+1}$, we denote by $\mathbf{I}(S)$ the ideal in the ring $\mathbb{C}[z, \lambda]$ defined by $\mathbf{I}(S) = \{g \in \mathbb{C}[z, \lambda] : g(z_0, \lambda_0) = 0 \text{ for all } (z_0, \lambda_0) \in S\}$.

Proposition 24. *Suppose $I = \langle g_1, \ldots, g_\kappa \rangle$, R, and N are ideals in $\mathbb{C}[z, \lambda]$ such that R radical and $I = R \cap N$. Then, for any $g \in \mathbf{I}(\mathbf{V}_{\mathbb{C}}(I))$ and any $(z_0, \lambda_0) \in \mathbb{C}^{p+1} \setminus \mathbf{V}_{\mathbb{C}}(N)$, there exist a neighborhood U of (z_0, λ_0) in $\mathbb{C}^{p+1}$ and rational functions $h_1, \ldots, h_\kappa$ on U such that $g = h_1 g_1 + \cdots + h_\kappa g_\kappa$ on U.*

Proof of Theorem 23. The Strong Hilbert Nullstellensatz and the hypothesis $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) = \mathbf{V}_{\mathbb{C}}(\mathcal{I}_k)$ yield

$$\mathcal{I} \subset \sqrt{\mathcal{I}} = \mathbf{I}(\mathbf{V}_{\mathbb{C}}(\mathcal{I})) = \mathbf{I}(\mathbf{V}_{\mathbb{C}}(\mathcal{I}_k)).$$

From now we complexify and assume $\Omega \subset \mathbb{C}$ and parameters $\lambda \in \mathbb{C}^p$ so that the averaged functions $f_j \in \mathbb{C}[z, \lambda]$ and, in particular we have that $f_j \in \mathbf{I}(\mathbf{V}_{\mathbb{C}}(\mathcal{I}))$ for any $j \in \mathbb{N}$. Let $\{f_{i_1}(z;\lambda), \ldots, f_{i_\kappa}(z;\lambda)\}$ be a minimal basis of $\mathcal{I}_k$. Hence for any f_j and any $(z_0, \lambda^*) \in \mathbb{C}^{p+1} \setminus \mathbf{V}_{\mathbb{C}}(\mathcal{N})$, by Proposition 24 there exists a neighborhood U of (z_0, λ^*) in $\mathbb{C}^{p+1}$ and κ rational functions $h_1, \ldots, h_\kappa$ such that, as analytic functions from U to $\mathbb{C}$, $f_j = h_1 f_{i_1} + \cdots + h_\kappa f_{i_\kappa}$ is valid on U for any $j \in \mathbb{N}$. This means that working with the germs at (z_0, λ^*) of the analytic functions involved, the displacement function d can be written, for (z, λ) in a neighborhood of (z_0, λ^*)

and $|\varepsilon|$ sufficiently close to 0, as

$$d(z,\lambda,\varepsilon) = \sum_{j\geq 1} f_j(z;\lambda)\varepsilon^j = \sum_{q=1}^{\kappa} f_{i_q}(z;\lambda)[1+\psi_q(z,\lambda,\varepsilon)]\varepsilon^{i_q}$$

where ψ_q are analytic functions with $\psi_q(z,\lambda,0)=0$. Then (see for example Proposition 6.1.2 of [10]) there are at most $\kappa-1$ small (either positive or negative) zeros of $d(z,\lambda,.)$ for any (z,λ) sufficiently close to (z_0,λ^*). In other words, the number of (either positive or negative) isolated branches of T-periodic solutions that can have equation (1) with parameters $\|\lambda-\lambda^*\| \ll 1$ and $|\varepsilon| \ll 1$, bifurcating from z_0 is at most $2(\kappa-1)$. □

Acknowledgements

We thank to professor Adriana Buica, her comments which help us to improve the preliminary version of this paper.

The first and third authors are partially supported by a MINECO grant number MTM2014-53703-P and an AGAUR grant number 2014SGR 1204. The second author is partially supported by a MINECO grants MTM2016-77278-P and MTM2013-40998-P, an AGAUR grant number 2014SGR 568, and a FP7-PEOPLE-2012-IRSES number 318999.

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