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THE SYMMETRIC PERIODIC ORBITS FOR THE CLASSICAL HELIUM ATOM

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Abstract. We analyse the existence of periodic symmetric orbits of the classical helium atom. The results obtained shows that there exists six families of periodic orbits that can be prolonged from a continuum of periodic symmetric orbits. The main technique applied in this study is the continuation method of Poincaré.

1. Introduction

The estimate of the ground-state of the energy of the helium atom was a cornerstone in the evolution of quantum mechanics. In 1980's and 1990's, (see [5] and [12]) it was presented some shortcomings of the old quantum theory, and one of the highlight points was: understand the behavior of the periodic trajectories when the classical dynamics is non-integrable or even chaotic.

In this context we study in this paper the existence of periodic trajectories of the system that governs the dynamics of the collinear helium atom and investigate when this periodic orbits can be prolonged, depending on a small parameter.

Many distinct techniques can be applied for studying the existence of periodic orbits, for example, numerical analysis, averaging theory, Melnikov functions, normal forms, variational methods, among others. However one of the first analytical studies of the existence of periodic orbits was done by Poincaré in [8].

The continuation method of Poincaré, originally presented in [8], consists in given a periodic solution for the system with a parameter equal to zero, providing conditions for extending this solution to small values of the parameter. For more details about this method see [3], for example.

In this paper we apply his method to study the symmetric periodic orbits of the collinear helium atom, that consists of two electrons of mass m_e and charge $-e$ moving on a straight line with respect to a *fixed* positively

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charged nucleus of charge $+2e$. The Hamiltonian modelizing the collinear helium system can be brought to the non-dimensionalized form

$$H_1 = \frac{1}{2}(\dot{r}_1^2 + \dot{r}_2^2) - \frac{2}{r_1} - \frac{2}{r_2} + \frac{\mu}{r_1 + r_2},$$

here $\dot{r}_1 = p_1, \dot{r}_2 = p_2$ and $\mu > 0$. For more details on the collinear charged helium atom see for instance [4], [10] and [11]. In the expression of the Hamiltonian presented in these papers appears a letter Z, that represents a charge of the nucleus of the helium atom. In the present paper we rescale the coordinates in such a way that the Hamiltonian of [4] becomes the one presented in H_1.

In the previous paper [6] we apply the same approach to study the symmetric periodic orbits for the charged collinear 3-body problem, which consists of three different masses each one with different charge, and we do not have any of the three masses fixed, as the collinear helium system has. This is a strong difference between these two systems. This difference originates completely different Hamiltonians models. In fact the Hamiltonian that models the collinear charged 3-body problem is of the form

$$H_2 = \frac{1}{2}(\mu_1 \dot{r}_1^2 + \mu_2 \dot{r}_2^2 + \mu_3 \dot{r}_1 \dot{r}_2) - \frac{e_{12}}{r_1} - \frac{e_{23}}{r_2} - \frac{e_{13}}{r_1 + r_2}$$

where the μ_i's are related with the masses of the three bodies and the e_{ij}'s with their masses and charges. Moreover we note that the terms in the Hamiltonian H_2 different from the term $\frac{1}{2}\mu_3 \dot{r}_1 \dot{r}_2$ cannot be non-dimensionalized as in the Hamiltonian H_1 because the masses m_1 and m_3, as their charges are not equal as in H_1. In this way the results about the existence and continuation of periodic orbits obtained in the paper [6] cannot be applied to the collinear helium system.

We organize this paper as follows: in Section 2 the equations that model the dynamics of the classical helium atom are described, in Section 3 we present the main results, and in Section 4 we study the symmetries of the periodic solutions of this system. Considering the parameter $\mu = 0$, the symmetric periodic orbits are studied in Section 5, and in Section 6 we apply the continuation method of Poincaré to extend the periodic solution obtained in the previous section for $\mu = 0$ to small and positive values of the parameter μ. A brief conclusion and some comments comparing the periodic orbits of the uncharged, charged symmetric three-body problem with the ones of the classical helium atom are presented in Section 7.

2. Differential equations that governs the classical motion of the helium atom

In the literature there are few rigorous results about the general three-body Coulomb problem and the main reason for this is that the equation that governs the motion are multi-dimensional, non-integrable and singular. In order to obtain a model of the equations that govern the motion an essential ingredient for the classical analysis of the three-body Coulomb problem is the regularization of its equations of motion, see [10] and [11].

We assume that the particles are in position $x_1, x_2, x_3 \in \mathbb{R}$, respectively, such that $0 < x_1 < x_2 < x_3$ and we consider the change of coordinates given by $z_1 = x_2 - x_1$ and $z_2 = x_3 - x_2$, that denotes the distance between x_2 and x_1 and x_3 and x_2. Let $L = T - U$, be the Lagrangian associated to this system, where T, U denotes the kinetic and potential energy, respectively. We introduce the variables

$$p_1 = \frac{\partial L}{\partial \dot{z}_1} \text{ and } p_2 = \frac{\partial L}{\partial \dot{z}_2}.$$

Then the Hamiltonian that governs the classical helium atom is given by

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \frac{2}{z_1} - \frac{2}{z_2} + \frac{\mu}{z_1 + z_2}. \tag{1}$$

Associated to the Hamiltonian (1) we have the system

$$\begin{aligned} \frac{dz_1}{dt} &= p_1, \\ \frac{dz_2}{dt} &= p_2, \\ \frac{dp_1}{dt} &= \frac{\mu}{(z_1+z_2)^2} - \frac{2}{z_1^2}, \\ \frac{dp_2}{dt} &= \frac{\mu}{(z_1+z_2)^2} - \frac{2}{z_2^2}. \end{aligned} \tag{2}$$

Applying the Levi-Civita transformation, see [7], given by

$$z_1 = \xi_1^2,\ z_2 = \xi_2^2,\ p_1 = \frac{\eta_1}{2\xi_1},\ p_2 = \frac{\eta_2}{\xi_2} \text{ and } dt = 4\xi_1^2\xi_2^2 ds,$$

the trajectories of the classical helium atom (2) in the new coordinates are the solution of the system

$$\begin{aligned} \frac{d\xi_1}{ds} &= \eta_1 \xi_2^2, \\ \frac{d\xi_2}{ds} &= \eta_2 \xi_1^2, \\ \frac{d\eta_1}{ds} &= \frac{\xi_2^2}{\xi_1}\left(\eta_1^2 + \frac{8\mu\xi_1^4}{\left(\xi_1^2+\xi_2^2\right)^2} - 16\right), \\ \frac{d\eta_2}{ds} &= \frac{\xi_1^2}{\xi_2}\left(\eta_2^2 + \frac{8\mu\xi_2^4}{\left(\xi_1^2+\xi_2^2\right)^2} - 16\right), \end{aligned} \tag{3}$$

on the energy level $H = h$ for some constant h. System (3) is a Hamiltonian system with a Hamiltonian G given by

$$G = 4\xi_1^2\xi_2^2\left(\frac{\eta_1^2-16}{8\xi_1^2}+\frac{\eta_2^2}{8\xi_2^2}-h+\frac{\mu}{\xi_1^2+\xi_2^2}-\frac{2}{\xi_2^2}\right),$$

with $G = 0$ if and only if $H = h$. System (3) is analytic except when $\xi_1^2 + \xi_2^2 = 0$, that corresponds to the triple collision.

We want to study the periodic orbits of the classical helium atom with binary collisions that corresponds to the parameter $\mu = 0$. Considering these periodic solutions, our objective is study the periodic solutions of system (3) for $\mu > 0$ sufficiently small, satisfying the energy relation $G = 0$. In this way in the following section we explore some symmetries involving this system.

3. Main results about the symmetric periodic orbits

Consider the involutions

$$\begin{aligned} S_1(\xi_1,\xi_2,\eta_1,\eta_2,s) &= (-\xi_1,\xi_2,\eta_1,-\eta_2,-s),\\ S_2(\xi_1,\xi_2,\eta_1,\eta_2,s) &= (\xi_1,-\xi_2,-\eta_1,\eta_2,-s),\\ S_3(\xi_1,\xi_2,\eta_1,\eta_2,s) &= (\xi_1,\xi_2,-\eta_1,-\eta_2,-s). \end{aligned}$$

We say that a solution $\varphi(s) = (\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s))$ of the differential equation is *invariant under the symmetry* S_i if $S_i(\varphi(s))$ is also a solution of differential equation for $i = 1, 2, 3$ and $\varphi(s)$ is S_i symmetric if $S_i(\varphi(s)) = \varphi(s)$, for more details, see Section 4.

As explored in [6], the periodic solutions of the differential equation (3) of the classical helium atom which are simultaneously S_1 and S_2 symmetric, are denoted by *S_{12}-symmetric periodic solutions.* In analogous way, we have S_{13}- and S_{23}-periodic solutions.

The results on the S_{12}-symmetric periodic solutions for small and positive values of μ are given in the next theorem.

Theorem 1. *Consider $h = h_1 + h_2 < 0$, p and q odd positive integers. Then the S_{12}-symmetric periodic solutions of the classical helium atom* (3) *for $\mu = 0$ and with the initial conditions*

(*a*) $\xi_1(0) = 0$, $\xi_2(0) = \sqrt{-2/h_2}$, $\eta_1(0) = 4$, $\eta_2(0) = 0$,
(*b*) *or* $\xi_1(0) = \sqrt{-2/h_1}$, $\xi_2(0) = 0$, $\eta_1(0) = 0$, $\eta_2(0) = 4$,

where $h_1 = \left(\frac{p}{q}\right)^{\frac{2}{3}} h_2$, *can be continued to a μ-parameter family of S_{12}-symmetric periodic orbits of the classical helium atom* (3) *for $\mu > 0$ small.*

The S_{13}-symmetry periodic solutions are given by the next result.

Theorem 2. *Consider $h = h_1 + h_2 < 0$, p odd and q even positive integers. Then the S_{13}-symmetric periodic solutions of the classical helium atom* (3) *for $\mu = 0$ with initial conditions*

(*a*) $\xi_1(0) = 0$, $\xi_2(0) = \sqrt{-2/h_2}$, $\eta_1(0) = 4$, $\eta_2(0) = 0$,
(*b*) *or* $\xi_1(0) = \sqrt{-2/h_1}$, $\xi_2(0) = \sqrt{-2/h_2}$, $\eta_1(0) = 0$, $\eta_2(0) = 0$,

where $h_1 = \left(\frac{p}{q}\right)^{\frac{2}{3}} h_2$, *can be continued to a μ-parameter family of S_{13}-symmetric periodic orbits of the classical helium atom* (3) *for $\mu > 0$ small.*

Finally for the S_{23}-symmetric periodic solutions, we obtain:

Theorem 3. *Consider $h = h_1 + h_2 < 0$, p even and q odd positive integers. Then the S_{23}-symmetric periodic solutions of the classical helium atom* (3) *for $\mu = 0$ with initial conditions*

(*a*) $\xi_1(0) = \sqrt{-2/h_1}$, $\xi_2(0) = 0$, $\eta_1(0) = 0$, $\eta_2(0) = 4$,
(*b*) *or* $\xi_1(0) = \sqrt{-2/h_1}$, $\xi_2(0) = \sqrt{-2/h_2}$, $\eta_1(0) = 0$, $\eta_2(0) = 0$,

where $h_1 = \left(\frac{p}{q}\right)^{\frac{2}{3}} h_2$, *can be continued to a μ-parameter family of S_{23}-symmetric periodic orbits of the classical helium atom* (3) *for $\mu > 0$ small.*

4. Symmetries of the classical helium system

The results obtained in this section are similar to the ones given in [2] and [6] for the collinear uncharged and charged three-body system, respectively. Consider the involutions

$$\begin{aligned}
Id: &\quad (\xi_1, \xi_2, \eta_1, \eta_2, s) &&\mapsto (\xi_1, \xi_2, \eta_1, \eta_2, s),\\
S_1: &\quad (\xi_1, \xi_2, \eta_1, \eta_2, s) &&\mapsto (-\xi_1, \xi_2, \eta_1, -\eta_2, -s),\\
S_2: &\quad (\xi_1, \xi_2, \eta_1, \eta_2, s) &&\mapsto (\xi_1, -\xi_2, -\eta_1, \eta_2, -s),\\
S_3: &\quad (\xi_1, \xi_2, \eta_1, \eta_2, s) &&\mapsto (\xi_1, \xi_2, -\eta_1, -\eta_2, -s),\\
S_4: &\quad (\xi_1, \xi_2, \eta_1, \eta_2, s) &&\mapsto (-\xi_1, -\xi_2, -\eta_1, -\eta_2, s),\\
S_5: &\quad (\xi_1, \xi_2, \eta_1, \eta_2, s) &&\mapsto (-\xi_1, \xi_2, -\eta_1, \eta_2, s),\\
S_6: &\quad (\xi_1, \xi_2, \eta_1, \eta_2, s) &&\mapsto (\xi_1, -\xi_2, \eta_1, -\eta_2, s),\\
S_7: &\quad (\xi_1, \xi_2, \eta_1, \eta_2, s) &&\mapsto (-\xi_1, -\xi_2, \eta_1, \eta_2, -s).
\end{aligned}$$

We say that the solution $\varphi(s) = (\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s))$ of system (3) is invariant under the symmetry S_i if $S_i(\varphi(s))$ is also a solution of system (3) with $i \in \{1, \dots, 7\}$. We say that $\varphi(s)$ is S_i-*symmetric* if $S_i(\varphi(s)) = \varphi(s)$.

Observe that the set of involutions $\{Id, S_1, \dots, S_7\}$ with the usual composition forms an abelian group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. These symmetries usually appear in Hamiltonian systems, see for example [1], [9] and [6]. Note that the symmetries S_1, S_2 and S_3 generate the other ones. So we only consider the symmetric periodic orbits with respect to the symmetries S_1, S_2 and S_3. Using similar arguments to the ones presented in [1] and [6] the following proposition holds.

Proposition 1. *Let be $\varphi(s)$ a solution of system* (3)*. The following statements hold.*

(a) *A solution $\varphi(s)$ is a S_{12}-symmetric periodic solution of period S if and only if either $\xi_1(s_0) = \eta_2(s_0) = 0$ and $\xi_2(s_0 + S/4) = \eta_1(s_0 + S/4) = 0$ and there is no $s \in (s_0, s_0 + S/4)$ such that $\xi_2(s) = \eta_1(s) = 0$, or $\xi_2(s_0) = \eta_1(s_0) = 0$ and $\xi_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0$ and there is no $s \in (s_0, s_0 + S/4)$ such that $\xi_1(s) = \eta_2(s) = 0$.*

(b) *A solution $\varphi(s)$ is a S_{13}-symmetric periodic solution of period S if and only if either $\xi_1(s_0) = \eta_2(s_0) = 0$ and $\eta_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0$ and there is no $s \in (s_0, s_0 + S/4)$ such that $\eta_1(s) = \eta_2(s) = 0$, or $\eta_1(s_0) = \eta_2(s_0) = 0$ and $\xi_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0$ and there is no $s \in (s_0, s_0 + S/4)$ such that $\xi_1(s) = \eta_2(s) = 0$.*

(c) *A solution $\varphi(s)$ is a S_{23}-symmetric periodic solution of period S if and only if either $\xi_2(s_0) = \eta_1(s_0) = 0$ and $\eta_1(s_0 + S/4) = \eta_2(s_0 + S/4) = 0$ and there is no $s \in (s_0, s_0 + S/4)$ such that $\eta_1(s) = \eta_2(s) = 0$ or, $\eta_1(s_0) = \eta_2(s_0) = 0$ and $\xi_2(s_0 + S/4) = \eta_1(s_0 + S/4) = 0$ and there is no $s \in (s_0, s_0 + S/4)$ such that $\xi_2(s) = \eta_1(s) = 0$.*

Moreover the following result, proved in [2], shows that there are no symmetric periodic solutions having more than two symmetries.

Proposition 2. *There are no periodic solutions of system* (3) *which are simultaneously S_i-symmetric for $i = 1, 2, 3$.*

5. Symmetric periodic orbits for $\mu = 0$

In the Levi-Civita coordinates $(\xi_1, \xi_2, \eta_1, \eta_2)$ system (3) with $\mu = 0$ is given by

$$
\begin{aligned}
\frac{d\xi_1}{ds} &= \eta_1\xi_2^2, \\
\frac{d\xi_2}{ds} &= \eta_2\xi_1^2, \\
\frac{d\eta_1}{ds} &= \xi_1\left(16 - \eta_2^2 + 8h\xi_2^2\right), \\
\frac{d\eta_2}{ds} &= \xi_2\left(-\eta_1^2 + 8h\xi_1^2 + 16\right),
\end{aligned} \tag{4}
$$

with the Hamiltonian

$$
H = \frac{1}{2}\left(\eta_1^2 - 16\right)\xi_2^2 + 4\xi_1^2\xi_2^2\left(\frac{\eta_2^2}{8\xi_2^2} - h - \frac{2}{\xi_2^2}\right),
$$

that can be rewritten as $H = H_1(\xi_1, \eta_1) + H_2(\xi_2, \eta_2)$ where

$$\begin{aligned} H_1(\xi_1, \eta_1) &= \frac{\eta_1^2}{8\xi_1^2} - \frac{2}{\xi_1^2}, \\ H_2(\xi_2, \eta_2) &= \frac{\eta_2^2}{8\xi_2^2} - \frac{2}{\xi_2^2}, \end{aligned}$$

if we rescale the time by $4\xi_1^2\xi_2^2$. Note that the level of energy $H = h$ satisfies that $h = h_1 + h_2$ where $H_1 = h_1$ and $H_2 = h_2$. Consider $\varphi(s) = (\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s))$ the solution of the system (4) satisfying the energy condition $H = h$, we define the new times σ and τ:

$$\begin{aligned} \frac{d\sigma}{ds} &= \xi_2^2, \quad \text{or equivalently} \quad \frac{dt}{d\sigma} = 4\xi_1^2, \\ \frac{d\tau}{ds} &= \xi_1^2, \quad \text{or equivalently} \quad \frac{dt}{d\tau} = 4\xi_2^2. \end{aligned} \tag{5}$$

Considering these new times, we obtain the system of differential equations in coordinates (ξ_1, η_1)

$$\begin{aligned} \frac{d\xi_1}{d\sigma} &= \eta_1, \\ \frac{d\eta_1}{d\sigma} &= 8h_1\xi_1, \end{aligned} \tag{6}$$

and (ξ_2, η_2) satisfies the system of differential equations

$$\begin{aligned} \frac{d\xi_2}{d\tau} &= \eta_2, \\ \frac{d\eta_2}{d\tau} &= 8h_2\xi_2. \end{aligned} \tag{7}$$

Consider the functions $G_1 = G/\xi_2^2$ and $G_2 = G/\xi_1^2$, i. e.

$$\begin{aligned} G_1 &= \frac{1}{2}\left(\eta_1^2 - 8\left(h_1\xi_1^2 + 2\right)\right), \\ G_2 &= \frac{1}{2}\left(\eta_2^2 - 8\left(h_2\xi_2^2 + 2\right)\right). \end{aligned}$$

These two functions are the Hamiltonians of system (6) and (7) respectively. Now we will study the periodic solutions of systems (6) and (7). Fixing $h_1 < 0$ we can integrate system (6) directly with the initial conditions $\xi_1(0) = \xi_{10}$ and $\eta_1(0) = \eta_{10}$, obtaining the solution $(\xi_1(\sigma), \eta_1(\sigma))$ given by

$$\begin{aligned} \xi_1(\sigma) &= \xi_{10}\cos(\omega_1\sigma) + \frac{\eta_{10}}{\omega_1}\sin(\omega_1\sigma), \\ \eta_1(\sigma) &= \eta_{10}\cos(\omega_1\sigma) - \xi_{10}\omega_1\sin(\omega_1\sigma), \end{aligned} \tag{8}$$

where $\omega_1 = \sqrt{-8h_1}$. Note that solution (8) is a periodic solution of system (6) with period $\overline{\sigma} = 2\pi/\omega_1$. Assuming that solution (8) satisfies the energy relation $G_1 = 0$ we obtain the following relationship between the initial conditions ξ_{10} and η_{10}:

$$\frac{\eta_{10}^2}{2} - 4h_1\xi_{10}^2 - 8 = 0.$$

Moreover using (5) we obtain the period of this solution in terms of the time t, i.e.

$$T_1(h_1, \alpha, \nu) = \int_0^{\overline{\sigma}} 4\xi_1^2(\sigma)d\sigma = \frac{\sqrt{8}\pi}{h_1\sqrt{-h_1}}.$$

In the same way, fixing $h_2 < 0$ we can integrate system (7) with the initial conditions $\xi_2(0) = \xi_{20}$ and $\eta_2(0) = \eta_{20}$, getting the solution $(\xi_2(\tau), \eta_2(\tau))$ given by

$$\begin{aligned} \xi_2(\tau) &= \xi_{20}\cos(\omega_2\tau) + \frac{\eta_{20}}{\omega_2}\sin(\omega_2\tau), \\ \eta_2(\tau) &= \eta_{20}\cos(\omega_2\tau) - \xi_{20}\omega_2\sin(\omega_2\tau), \end{aligned} \tag{9}$$

where $\omega_2 = \sqrt{-8h_2}$. Solution (9) is periodic with period $\overline{\tau} = 2\pi/\omega_2$. Assuming that solution (9) satisfies the energy relation $G_2 = 0$, the initial conditions ξ_{20} and η_{20} satisfy the equation

$$\frac{\eta_{20}^2}{2} - 4h_2\xi_{20}^2 - 8 = 0,$$

and by (5) the period of solution (9) in time t is given by

$$T_2(h_2, \gamma, \nu) = \int_0^{\overline{\tau}} 4\xi_2^2(\tau)d\tau = \frac{\sqrt{8}\pi}{h_2\sqrt{-h_2}}.$$

In Table 1 we summarize the relationships between the times σ, τ, t and s:

Time	σ	τ	t	s
Period	$\sigma^* = p\overline{\sigma}$	$\tau^* = q\overline{\tau}$	$T = pT_1(h_1, \alpha, \nu) = qT_2(h_2, \gamma, \nu)$	$S^* = s(T)$
Period/4	$\sigma^*/4$	$\tau^*/4$	$T/4$	$S^*/4$

TABLE 1. Period of periodic solution $\varphi(s)$ of the system (4).

The next result provide some relationship between the energy levels h_1 and h_2, the times σ, τ, t and s and the initial conditions of the periodic solutions of systems (6) and (7). The proof is similar to the proof of Proposition 3 given in [6].

Proposition 3. *Consider the periodic solutions $(\xi_1(\sigma), \eta_1(\sigma))$ and $(\xi_2(\tau), \eta_2(\tau))$ of systems (6) and (7) with periods $\overline{\sigma}$ and $\overline{\tau}$, satisfying the energy conditions $G_1 = 0$ and $G_2 = 0$, respectively. Assume that the functions $\sigma(s)$ and $\tau(s)$ given in (5) satisfy $\sigma(0) = \tau(0) = 0$ and there is no $s \in \mathbb{R}$ such that $\xi_1(\sigma(s)) = \xi_2(\tau(s)) = 0$. Then the following statements hold.*

(*a*) *The solution* $\varphi(s) = (\xi_1(\sigma(s)), \xi_2(\tau(s), \eta_1(\sigma(s), \eta_2(\tau(s))$ *of system* (4) *with initial condition* $\xi_1(0) = \xi_{10}$, $\xi_2(0) = \xi_{20}$, $\eta_1(0) = \eta_{10}$, $\eta_2(0) = \eta_{20}$ *satisfies the energy relation* $G(\xi_{10}, \xi_{20}, \eta_{10}, \eta_{20}) = 0$.

(*b*) *If* $h_1 = \left(\dfrac{p}{q}\right)^{\frac{2}{3}} h_2$ *for some* $p, q \in \mathbb{N}$ *coprime, then* $\varphi(s)$ *is a periodic solution of system* (4).

(*c*) *Let* $s(t)$ *be the inverse function of*

$$t(s) = \int_0^s 4\xi_1^2(\theta)\xi_2^2(\theta)d\theta.$$

For the h_1 *given in* (*b*) *the period and the quarter of period in times* σ, τ, t *and* s *are given in Table 1.*

Observe that by statement (*c*) of Proposition 3 we have that $dt/ds > 0$ when there are no collisions, and zero in the binary collisions. Therefore the inverse function $s = s(t)$ exists always that the system has no triple collision, and it is differentiable if there is no binary collisions. The number p in Proposition 3 represents the number of binary collisions between m_1 and m_2, and q is the number of binary collisions between the particles m_2 and m_3.

We emphasize that our interest in this paper is to study the symmetric periodic orbits of system (3) satisfying the energy relation $G = 0$. So in the following we exhibit conditions under the initial points, given in Proposition 3, to get symmetric periodic orbits.

Proposition 4. *The following statements hold.*

(*a*) *If* p *and* q *are odd then the solution* $\varphi(s)$ *given in Proposition 3 with initial conditions*

$$\text{either} \quad \xi_{10} = 0,\ \xi_{20}^* = \sqrt{\frac{2}{-h_2}},\ \eta_{10}^* = 4,\ \eta_{20} = 0,$$

$$\text{or} \quad \xi_{10}^* = \sqrt{\frac{2}{-h_1}},\ \xi_{20} = 0,\ \eta_{10} = 0,\ \eta_{20}^* = 4,$$

is a S_{12}*-symmetric periodic solution.*

(*b*) *If* p *is odd and* q *is even then the solution* $\varphi(s)$ *given in Proposition 3 with initial conditions*

$$\text{either} \quad \xi_{10}^* = 0,\ \xi_{20}^* = \sqrt{\frac{2}{-h_2}},\ \eta_{10}^* = 4,\ \eta_{20}^* = 0,$$

$$\text{or} \quad \xi_{10}^* = \sqrt{\frac{2}{-h_1}},\ \xi_{20}^* = \sqrt{\frac{2}{-h_2}},\ \eta_{10}^* = 0,\ \eta_{20}^* = 0,$$

is a S_{13}*-symmetric periodic solution.*

(c) *If p is even and q is odd then the solution $\varphi(s)$ given in Proposition 3 with initial conditions*

$$\text{either} \quad \xi_{10}^* = \sqrt{\frac{2}{-h_1}},\ \xi_{20}^* = 0,\ \eta_{10}^* = 0,\ \eta_{20}^* = 4,$$

$$\text{or} \quad \xi_{10}^* = \sqrt{\frac{2}{-h_1}},\ \xi_{20}^* = \sqrt{\frac{2}{-h_2}},\ \eta_{10}^* = 0,\ \eta_{20}^* = 0,$$

is a S_{23}-symmetric periodic solution.

Proof. Let $\varphi(s) = (\xi_1(\sigma(s)), \xi_2(\tau(s)), \eta_1(\sigma(s)), \eta_2(\tau(s)))$ be the solution of system (3). The proof follows evaluating $\varphi(s)$ at times $s = 0$ and $s = S^*/4$, where S^* is given in Table 1. □

Remark 1. *Note that the Levi-Civita transformation duplicates the number of orbits. Hence it is sufficient to consider the positive square roots of the initial conditions given in Proposition 4.*

6. Applying the continuation method of Poincaré to obtain symmetric periodic solutions for $\mu > 0$ small

In Proposition 4 we have the values for the initial conditions to get S_{12}-, S_{13}-, S_{23}-symmetric periodic orbits for system (3) when $\mu = 0$. We study the symmetric periodic orbits for small positive values of μ using the continuation method of Poincaré.

6.1. **The S_{12}-symmetric periodic solutions.**

Fixing values of $h = h_1 + h_2$, by Proposition 4 the solution $\varphi(s; \xi_{10}, \xi_{20}, \eta_{10}, \eta_{20}, \mu)$ of system (3) for $\mu = 0$ is a S_{12}-symmetric if the initial conditions satisfies

(a) either $\xi_1(0) = 0$, $\xi_2(0) = \xi_{20}^*$, $\eta_1(0) = \eta_{10}^*$, $\eta_2(0) = 0$;
(b) or $\xi_1(0) = \xi_{10}^*$, $\xi_2(0) = 0$, $\eta_1(0) = 0$, $\eta_2(0) = \eta_{20}^*$.

6.1.1. *Case* (a). By statement (a) of Proposition 1 we have that the solution φ is a S_{12}-symmetric periodic solution of the classical helium atom with period S satisfying the energy condition $G = 0$ if and only if

$$\begin{aligned} \xi_2(S/4; \xi_{20}, \eta_{10}, (\alpha, \beta, \gamma, \mu)) &= 0, \\ \eta_1(S/4; \xi_{20}, \eta_{10}, (\alpha, \beta, \gamma, \mu)) &= 0, \\ G(\xi_{20}, \eta_{10}, (\alpha, \beta, \gamma, \mu)) &= 0. \end{aligned}$$

The solution of the last equation in terms of η_{10} is given by $\eta_{10} = 4$. Thus the solution $\varphi(s; 0, \xi_{20}, \eta_{10}, 0, \mu)$ is a S_{12}-symmetric periodic solution of system (3) satisfying $G = 0$ if and only if

$$\begin{aligned} \xi_2(S/4; \xi_{20}, \mu) &= 0, \\ \eta_1(S/4; \xi_{20}, \mu) &= 0. \end{aligned} \tag{10}$$

Statement (a) of Proposition 4 provides additional information on the initial conditions and on the numbers p and q, see Table 1, for these S_{12}-symmetric periodic solutions. Thus if $p = 2m+1$, $q = 2k+1$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$, $\xi_{20} = \sqrt{-2/h_2}$ and $\eta_{10} = 4$, with m and k positive integers, then $\varphi(s; 0, \xi_{20}, \eta_{10}, 0, 0)$ is a S_{12}-symmetric solution of system (3) with $\mu = 0$ and the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left(\dfrac{p}{q}\right)^{\frac{2}{3}} h_2$ given in statement (b) of Proposition 3. Now we extend this solution to $\mu > 0$ and small. Applying the Implicit Function Theorem to system (10) in a neighbourhood of a known solution we have that if

$$\left.\begin{vmatrix} \dfrac{\partial \xi_2}{\partial s} & \dfrac{\partial \xi_2}{\partial \xi_{20}} \\ \dfrac{\partial \eta_1}{\partial s} & \dfrac{\partial \eta_1}{\partial \xi_{20}} \end{vmatrix}\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} \neq 0, \tag{11}$$

then there exist unique analytic functions $\xi_{20} = \xi_{20}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

(i) $\xi_{20}(0) = \xi_{20}^*$ and $S(0) = S^*$,

(ii) and $\varphi(s; 0, \xi_{20}, \eta_{10}, 0, \mu)$ is a S_{12}-symmetric periodic solution of (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

Note that the derivatives $\partial \xi_2/\partial s$ and $\partial \eta_1/\partial s$ are obtained evaluating the right hand of system (3) for $\mu = 0$, $s = S^*/4$ and with the initial conditions $(0, \xi_{20}^*, \eta_{10}, 0)$. So

$$\left.\frac{\partial \xi_2}{\partial s}\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = \left.\eta_2 \xi_1^2\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = (-1)^{k+1} \frac{8}{h_1} \neq 0,$$

and

$$\left.\frac{\partial \eta_1}{\partial s}\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = \left.8h_1 \xi_1 \xi_2^2\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = 0.$$

Therefore it remains to calculate $\partial \eta_1/\partial \xi_{20}$ evaluated at $s = S^*/4$, $\xi_{20} = \xi_{20}^*$ and $\mu = 0$. We obtain this value derivating the solution $\eta_1(\sigma(s); 0, \xi_{20}, \eta_{10}, 0)$ of system (4) with respect to the variable ξ_{20}, where the initial conditions $\xi_1(0) = 0$, $\xi_2(0) = \xi_{20}$, $\eta_1(0) = \eta_{10}$, $\eta_2(0) = 0$ satisfy the energy relation $G = 0$. So

$$\left.\frac{\partial \eta_1(\sigma(s); 0, \xi_{20}, \eta_{10}, 0, 0)}{\partial \xi_{20}}\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = \left.\left(\frac{\partial \eta_1}{\partial \sigma}\frac{\partial \sigma(s)}{\partial \xi_{20}} + \frac{\partial \eta_1}{\partial \xi_{20}}\right)\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}}.$$

By (5) we get the relationship between ξ_{20}, τ and σ

$$\xi_1^2(\sigma) d\sigma = \xi_2^2(\tau) d\tau.$$

Integrating this equation and assuming that $\sigma(0) = 0$ and $\tau(0) = 0$ we obtain the relation between $\sigma(s)$ and $\tau(s)$, i.e.

$$\frac{2\sqrt{2}\sin(2\omega_1\sigma(s))}{(-h_1)^{3/2}} + \frac{16\sigma(s)}{h_1} + \frac{\sqrt{2}\xi_{20}^2\sin(2\omega_2\tau(s))}{\sqrt{-h_2}} + 8\xi_{20}^2\tau(s) = 0, \tag{12}$$

where $\omega_1 = \sqrt{-8h_1}$, $\omega_2 = \sqrt{-8h_2}$ and $h = h_1 + h_2$. Derivating implicitly equation (12) with respect to the variable ξ_{20} we obtain

$$\left(\frac{\partial\sigma(s)}{\partial\xi_{20}}\right)\Bigg|_{\substack{s \,= S^*/4 \\ \xi_{20} \,= \xi_{20}^* \\ \mu \,= 0}} = (-1)^{m+1}8\sqrt{-2h_1}.$$

From (6) and (8) we get

$$\left(\frac{\partial\eta_1}{\partial\sigma}\right)\Bigg|_{\substack{s \,= S^*/4 \\ \xi_{20} \,= \xi_{20}^* \\ \mu \,= 0}} = (-1)^{m+1}8\sqrt{-2h_1},$$

and

$$\left(\frac{\partial\eta_1}{\partial\xi_{20}}\right)\Bigg|_{\substack{s \,= S^*/4 \\ \xi_{20} \,= \xi_{20}^* \\ \mu \,= 0}} = (-1)^m\frac{\sqrt{2}\pi(-h_1)^{3/2}q}{h_2}.$$

So

$$\left(\frac{\partial\eta_1(\sigma(s);0,\xi_{20},\eta_{10},0,0)}{\partial\xi_{20}}\right)\Bigg|_{\substack{s \,= S^*/4 \\ \xi_{20} \,= \xi_{20}^* \\ \mu \,= 0}} = (-1)^m\frac{2\sqrt{2}\pi(-h_1)^{3/2}q}{h_2}.$$

Finally the determinant (11) is given by

$$(-1)^{k+m}\frac{16\sqrt{2}\pi q\sqrt{-h_1}}{h_2} \neq 0.$$

6.1.2. *Case (b)*. Similarly to the previous case the solution φ is a S_{12}-symmetric periodic solution with period S satisfying the energy condition $G = 0$ if and only if

$$\begin{aligned}
\xi_1(S/4;\xi_{10},\eta_{20},\mu) &= 0,\\
\eta_2(S/4;\xi_{10},\eta_{20},\mu) &= 0,\\
G(\xi_{10},\eta_{20},\mu) &= 0.
\end{aligned}$$

The solution of $G = 0$ in terms of η_{20} is given by $\eta_{20} = 4$. So the solution $\varphi(s;\xi_{10},0,0,\eta_{20},\mu)$ is a S_{12}-symmetric periodic solution of system (3) satisfying $G = 0$ if and only if

$$\begin{aligned}
\xi_1(S/4;\xi_{10},\mu) &= 0,\\
\eta_2(S/4;\xi_{10},\mu) &= 0.
\end{aligned} \tag{13}$$

Again by statement (a) of Proposition 4 we obtain the initial conditions and the numbers p and q to have S_{12}-symmetric periodic solutions (see Table 1), i.e. $p = 2m+1$, $q = 2k+1$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$, $\xi_{10}^* = \sqrt{-2/h_1}$, $\eta_{20}^* = 4$, for $\mu = 0$ and the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left(\dfrac{p}{q}\right)^{\frac{2}{3}} h_2$ given in statement (b) of Proposition 3. As in the previous case applying the Implicit Function Theorem to system (13) in a neighborhood of a known solution we have that if

$$\left.\begin{vmatrix} \dfrac{\partial \xi_1}{\partial s} & \dfrac{\partial \xi_1}{\partial \xi_{10}} \\ \dfrac{\partial \eta_2}{\partial s} & \dfrac{\partial \eta_2}{\partial \xi_{10}} \end{vmatrix}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} \neq 0, \tag{14}$$

then there exist unique analytic functions $\xi_{10} = \xi_{10}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

(i) $\xi_{10}(0) = \xi_{10}^*$ and $S(0) = S^*$,
(ii) and $\varphi(s; \xi_{10}, 0, 0, \eta_{20}, \mu)$ is a S_{12}-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

In the same way as we work in the previous case we obtain

$$\left.\frac{\partial \xi_1}{\partial s}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} = (-1)^{m+1} \frac{8}{h_2} \neq 0,$$

and

$$\left.\frac{\partial \eta_2}{\partial s}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} = 0.$$

So it remains to calculate $\partial \eta_2 / \partial \xi_{10}$ evaluated at $s = S^*/4$, $\xi_{10} = \xi_{10}^*$ and $\mu = 0$. As in the previous case, this derivative is obtained derivating the solution $\eta_2(\tau(s); \xi_{10}, 0, 0, \eta_{20}, 0)$ of system (4) evaluated in $\xi_{10} = \xi_{10}^*$, $\xi_2(0) = 0$, $\eta_1(0) = 0$, $\eta_2(0) = \eta_{20}$, $s = S^*/4$ and satisfying the energy relation $G = 0$. Then

$$\begin{aligned} \left.\frac{\partial \eta_2(\tau(s); \xi_{10}, 0, 0, \eta_{20}, 0)}{\partial \xi_{10}}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} &= \left.\left(\frac{\partial \eta_2}{\partial \tau}\frac{\partial \tau(s)}{\partial \xi_{10}} + \frac{\partial \eta_2}{\partial \xi_{10}}\right)\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} \\ &= (-1)^{k+1} 2p\pi\sqrt{-2h_2}(h_2/h_1), \end{aligned}$$

and the determinant (14) is given by

$$(-1)^{k+m+1} \frac{16\pi p\sqrt{-2h_2}}{h_1} \neq 0.$$

6.2. **The S_{13}-symmetric periodic orbits.** By Proposition 4 the solution $\varphi(s; \xi_{10}, \xi_{20}, \eta_{10}, \eta_{20}, \mu)$ of system (3) for $\mu = 0$ and fixed values of $h = h_1 + h_2$ is a S_{13}-symmetric if the initial conditions satisfies

(a) either $\xi_1(0) = 0$, $\xi_2(0) = \xi_{20}^*$, $\eta_1(0) = \eta_{10}^*$, $\eta_2(0) = 0$,
(b) or $\xi_1(0) = \xi_{10}^*$, $\xi_2(0) = \xi_{20}^*$, $\eta_1(0) = 0$, $\eta_2(0) = 0$.

6.2.1. *Case* (a). By statement (b) of Proposition 1 we have that the solution φ is a S_{13}-symmetric periodic solution with period S satisfying the energy condition $G = 0$ if and only if

$$\begin{aligned} \eta_1(S/4; \xi_{20}, \eta_{10}, \mu) &= 0, \\ \eta_2(S/4; \xi_{20}, \eta_{10}, \mu) &= 0, \\ G(\xi_{20}, \eta_{10}, \mu) &= 0. \end{aligned}$$

Solving the last equation in terms of η_{10} we obtain $\eta_{10} = 4$. Then the solution is S_{13}-symmetric periodic satisfying $G = 0$ if and only if

$$\begin{aligned} \eta_1(S/4; \xi_{20}, \mu) &= 0, \\ \eta_2(S/4; \xi_{20}, \mu) &= 0. \end{aligned} \tag{15}$$

Statement (b) of Proposition 4 provides the initial conditions and numbers p and q, see Table 1 in order that these periodic orbits are S_{13}-symmetric, i.e. $p = 2m+1$, $q = 2k$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$, $\xi_{20} = \sqrt{-2/h_2}$ and $\eta_{10} = 4$, with m and k positive integers, then $\varphi(s; 0, \xi_{20}, \eta_{10}, 0, 0)$ is a S_{13}-symmetric solution for system (3) with $\mu = 0$. Moreover, the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left(\dfrac{p}{q}\right)^{\frac{2}{3}} h_2$ given in statement (b) of Proposition 3. Now we extend this solution for $\mu > 0$ and small. Applying the Implicit Function Theorem in system (15) in a neighborhood of a known solution we have that if

$$\left|\begin{array}{cc} \dfrac{\partial \eta_1}{\partial s} & \dfrac{\partial \eta_1}{\partial \xi_{20}} \\ \dfrac{\partial \eta_2}{\partial s} & \dfrac{\partial \eta_2}{\partial \xi_{20}} \end{array}\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} \neq 0, \tag{16}$$

then there exist unique analytic functions $\eta_{20} = \eta_{20}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

- (i) $\eta_{20}(0) = 0$ and $S(0) = S^*$,
- (ii) and $\varphi(s; 0, \xi_{20}, \eta_{10}, 0, \mu)$ is a S_{13}-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

As in the previous case we get that

$$\left.\frac{\partial \eta_1}{\partial s}\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = (-1)^m \frac{16\sqrt{-2h_1}}{h_2} \neq 0,$$

and

$$\left.\frac{\partial \eta_2}{\partial s}\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = (-1)^{k+1} \frac{16\sqrt{-2h_2}}{h_1} \neq 0.$$

Now we have to calculate the other two derivatives $\partial\eta_1/\partial\xi_{20}$ and $\partial\eta_2/\partial\xi_{20}$ evaluated at $s = S^*/4, \xi_{20} = \xi_{20}^*$ and $\mu = 0$. We obtain these values derivating the solutions $\eta_1(\sigma(s); 0, \xi_{20},\ \eta_{10}, 0)$ and $\eta_2(\tau(s); 0, \xi_{20},\ \eta_{10}, 0)$ of system (4) with respect to the variable ξ_{20}, where the initial conditions $\xi_1(0) = 0$, $\xi_2(0) = \xi_{20}$, $\eta_1(0) = \eta_{10}$, $\eta_2(0) = 0$ satisfy the energy relation $G = 0$. So

$$\left.\frac{\partial \eta_1(\sigma(s); 0, \xi_{20}, \eta_{10}, 0)}{\partial \xi_{20}}\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = \left.\left(\frac{\partial \eta_1}{\partial \sigma}\frac{\partial \sigma(s)}{\partial \xi_{20}} + \frac{\partial \eta_1}{\partial \xi_{20}}\right)\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}}$$

$$= \frac{(-1)^m\sqrt{8}(-h_1)^{3/2}}{h_2}\left(8\tfrac{\partial \tau}{\partial s}\left(\sqrt{\tfrac{2}{-h_2}}\right) + \pi q\right),$$

and

$$\left.\frac{\partial \eta_2(\tau(s); 0, \xi_{20}, \eta_{10}, 0)}{\partial \xi_{20}}\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = \left.\left(\frac{\partial \eta_2}{\partial \tau}\frac{\partial \tau(s)}{\partial \xi_{20}} + \frac{\partial \eta_2}{\partial \xi_{20}}\right)\right|_{\substack{s = S^*/4 \\ \xi_{20} = \xi_{20}^* \\ \mu = 0}} = 0.$$

Therefore the determinant (16) is

$$(-1)^{k+m+1} 64\pi q\sqrt{h_1/h_2} \neq 0.$$

6.2.2. *Case* (b). From the second part of statement (b) of Proposition 1 the solution φ is a S_{13}-symmetric periodic solution with period S satisfying the energy condition $G = 0$ if and only if

$$\begin{aligned} \xi_1(S/4; \xi_{10}, \xi_{20}, \mu) &= 0, \\ \eta_2(S/4; \xi_{10}, \xi_{20}, \mu) &= 0, \\ G(\xi_{10}, \xi_{20}, \mu) &= 0. \end{aligned}$$

The solution of the equation $G = 0$ in terms of ξ_{20} is given by $\xi_{20} = \sqrt{-2/h_2}$. So the solution satisfies $G = 0$ if and only if

$$\begin{aligned} \xi_1(S/4; \xi_{10}, \mu) &= 0, \\ \eta_2(S/4; \xi_{10}, \mu) &= 0. \end{aligned} \tag{17}$$

Again by statement (b) of Proposition 4 the solutions are S_{13}-symmetric taking $p = 2m + 1$, $q = 2k$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$, $\xi_{10}^* = \sqrt{-2/h_1}, \eta_{20}^* = 4$, for $\mu = 0$. Furthermore the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left(\frac{p}{q}\right)^{\frac{2}{3}} h_2$ given in statement (b) of Proposition 3. As in the previous case applying the Implicit Function Theorem to system (17) in a neighbourhood of a known solution we have that if

$$\left.\begin{vmatrix} \dfrac{\partial \xi_1}{\partial s} & \dfrac{\partial \xi_1}{\partial \xi_{10}} \\ \dfrac{\partial \eta_2}{\partial s} & \dfrac{\partial \eta_2}{\partial \xi_{10}} \end{vmatrix}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} \neq 0, \tag{18}$$

then there exist unique analytic functions $\xi_{10} = \xi_{10}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

(i) $\xi_{10}(0) = \xi_{10}^*$ and $S(0) = S^*$,
(ii) and $\varphi(s; \xi_{10}, 0, 0, \eta_{20}, \mu)$ is a S_{13}-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

Analogously to the previous case we obtain

$$\left.\frac{\partial \xi_1}{\partial s}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} = (-1)^m \frac{8}{h_2} \neq 0,$$

and

$$\left.\frac{\partial \eta_2}{\partial s}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} = 0.$$

So it remains to calculate $\partial \eta_2 / \partial \xi_{10}$ evaluated at $s = S^*/4, \xi_{10} = \xi_{10}^*$ and $\mu = 0$. This derivative is obtained by derivating the solution $\eta_2(\tau(s); \xi_{10}, \xi_{20}, 0, 0, 0)$ of system (4) evaluated in $\xi_{10} = \xi_{10}^*$ and $s = S^*/4$ with initial conditions $\xi_1(0) = \xi_{10},\ \xi_2(0) = \xi_{20},\ \eta_1(0) = 0$ and $\eta_2(0) = 0$, satisfying the energy relation $G = 0$. Then

$$\begin{aligned}\left.\frac{\partial \eta_2(\tau(s); \xi_{10}, \xi_{20}, 0, 0, 0)}{\partial \xi_{10}}\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^*}} &= \left.\left(\frac{\partial \eta_2}{\partial \tau}\frac{\partial \tau(s)}{\partial \xi_{10}} + \frac{\partial \eta_2}{\partial \xi_{10}}\right)\right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^*}} \\ &= (-1)^m \frac{\pi p}{h_1},\end{aligned}$$

and determinant (18) is $\dfrac{8\pi p}{h_1 h_2} \neq 0$.

6.3. **The S_{23}-symmetric periodic solutions.** By Proposition 4 the solution $\varphi(s; \xi_{10},\ \xi_{20}, \eta_{10}, \eta_{20}, \mu)$ of system (3) for $\mu = 0$ is a S_{23}-symmetric if the initial conditions satisfy

(a) either $\xi_1(0) = \xi_{10}^*, \xi_2(0) = 0, \eta_1(0) = 0, \eta_2(0) = \eta_{20}^*$;
(b) or $\xi_1(0) = \xi_{10}^*, \xi_2(0) = \xi_{20}^*, \eta_1(0) = 0, \eta_2(0) = 0$;

for fixed values of $h = h_1 + h_2$.

6.3.1. *Case* (a). By statement (c) of Proposition 1 the solution φ is a S_{23}-symmetric periodic solution with period S satisfying the energy condition $G = 0$ if and only if

$$\begin{aligned} \xi_1(S/4; \xi_{10}, \eta_{20}, \mu) &= 0, \\ \eta_2(S/4; \xi_{10}, \eta_{20}, \mu) &= 0, \\ G(\xi_{20}, \eta_{10}, \mu) &= 0. \end{aligned}$$

Solving the last equation in terms of η_{10} we obtain $\eta_{20} = 4$. So the solution $\varphi(s; \xi_{10}, 0, 0, \eta_{20}, \mu)$ is a S_{23}-symmetric periodic solution of system (3) satisfying $G = 0$ if and only if

$$
\begin{aligned}
\xi_1(S/4; \xi_{10}, \mu) &= 0, \\
\eta_2(S/4; \xi_{10}, \mu) &= 0.
\end{aligned} \tag{19}
$$

Statement (c) of Proposition 4 provides the initial conditions and the numbers p and q, see Table 1 in order that the periodic orbits be S_{23}-symmetric. More precisely, if $p = 2m$, $q = 2k+1$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$, $\xi_{10}^* = \sqrt{-2/h_1}$ and $\eta_{20}^* = 4$, with m and k positive integers, then $\varphi(s; \xi_{10}^*, 0, 0, \eta_{20}^*, 0)$ is a S_{23}-symmetric solution for system (3) with $\mu = 0$. Moreover the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left(\dfrac{p}{q}\right)^{\frac{2}{3}} h_2$ given in statement (c) of Proposition 3. We extend this solution to $\mu > 0$ and small. Applying the Implicit Function Theorem in system (19) in a neighbourhood of a known solution we have that if

$$
\left| \begin{array}{cc} \dfrac{\partial \xi_1}{\partial s} & \dfrac{\partial \xi_1}{\partial \xi_{10}} \\ \dfrac{\partial \eta_2}{\partial s} & \dfrac{\partial \eta_2}{\partial \xi_{10}} \end{array} \right| \Bigg|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} \neq 0,
$$

then there exist unique analytic functions $\eta_{10} = \eta_{10}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

(i) $\eta_{10}(0) = 0$ and $S(0) = S^*$,
(ii) and $\varphi(s; \xi_{10}, 0, 0, \eta_{20}, \mu)$ is a S_{23}-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

We have

$$
\left. \frac{\partial \xi_1}{\partial s} \right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} = 0,
$$

and

$$
\left. \frac{\partial \eta_2}{\partial s} \right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} = (-1)^k \frac{16\sqrt{-2h_2}}{h_1} \neq 0.
$$

So we have to calculate the derivative $\partial \xi_1 / \partial \xi_{10}$ evaluated at $s = S^*/4, \xi_{10} = \xi_{10}^*$ and $\mu = 0$. As in the previous cases we obtain that

$$
\begin{aligned}
\left. \frac{\partial \xi_1(\sigma(s); \xi_{10}, 0, 0, \eta_{20})}{\partial \xi_{10}} \right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} &= \left. \left(\frac{\partial \xi_1}{\partial \sigma} \frac{\partial \sigma(s)}{\partial \xi_{10}} + \frac{\partial \xi_1}{\partial \xi_{10}} \right) \right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} \\
&= (-1)^m.
\end{aligned}
$$

Therefore, the determinant (18) is $(-1)^{k+m}\dfrac{16\sqrt{-2h_2}}{h_1} \neq 0$.

6.3.2. *Case* (b). From the second part of statement (c) of Proposition 1 the solution φ is a S_{23}-symmetric periodic solution with period S satisfying the energy condition $G = 0$ if and only if

$$\begin{aligned} \xi_1(S/4; \xi_{10}, \xi_{20}, \mu) &= 0, \\ \xi_2(S/4; \xi_{10}, \xi_{20}, \mu) &= 0, \\ G(\xi_{10}, \xi_{20}, \mu) &= 0. \end{aligned}$$

Solving $G = 0$ in terms of ξ_{20} we obtain $\xi_{20} = \sqrt{-2/h_2}$. So the solution $\varphi(s; \xi_{10}, \xi_{20}, 0, 0, \mu)$ is a S_{23}-symmetric periodic solution of system (3) that satisfies $G = 0$ if and only if

$$\begin{aligned} \xi_1(S/4; \xi_{10}, \mu) &= 0, \\ \xi_2(S/4; \xi_{10}, \mu) &= 0. \end{aligned} \tag{20}$$

Again by statement (c) of Proposition 4 we have that if $p = 2m$, $q = 2k + 1$, $S = S^* = s(pT_1(h_1)) = s(qT_2(h_2))$ and $\xi_{10}^* = \sqrt{-2/h_1}, \xi_{20}^* = \sqrt{-2/h_2}$ then the solution is S_{23}-symmetric periodic, for $\mu = 0$. Furthermore the energy level $H = h = h_1 + h_2$ satisfies the condition $h_1 = \left(\dfrac{p}{q}\right)^{\frac{2}{3}} h_2$ given in statement (c) of Proposition 3. As in the previous case applying the Implicit Function Theorem to system (20) in a neighbourhood of a known solution we have that if

$$\left. \begin{vmatrix} \dfrac{\partial \xi_1}{\partial s} & \dfrac{\partial \xi_1}{\partial \xi_{10}} \\ \dfrac{\partial \xi_2}{\partial s} & \dfrac{\partial \xi_2}{\partial \xi_{10}} \end{vmatrix} \right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} \neq 0, \tag{21}$$

then there exist unique analytic functions $\xi_{10} = \xi_{10}(\mu)$ and $S = S(\mu)$ defined for $\mu \geq 0$ sufficiently small that satisfy

- (i) $\xi_{10}(0) = \xi_{10}^*$ and $S(0) = S^*$,
- (ii) and $\varphi(s; \xi_{10}, \xi_{20}, 0, 0, \mu)$ is a S_{23}-symmetric periodic solution of system (3) with period $S = S(\mu)$ satisfying the energy condition $G = 0$.

Working as in the previous cases we get

$$\left. \frac{\partial \xi_1}{\partial s} \right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} = 0,$$

and

$$\left. \frac{\partial \xi_2}{\partial s} \right|_{\substack{s = S^*/4 \\ \xi_{10} = \xi_{10}^* \\ \mu = 0}} = (-1)^k \frac{8}{h_1}.$$

So it remains to calculate $\partial\xi_1/\partial\xi_{10}$ evaluated at $s = S^*/4$, $\xi_{10} = \xi_{10}^*$ and $\mu = 0$. This derivative is obtained derivating the solution $\xi_1(\sigma(s); \xi_{10}, \xi_{20}, 0, 0, 0)$ of system (4) evaluated at $\xi_{10} = \xi_{10}^*$ and $s = S^*/4$ with initial conditions $\xi_1(0) = \xi_{10},\ \xi_2(0) = \xi_{20},\ \eta_1(0) = 0,\ \eta_2(0) = 0$ satisfying the energy relation $G = 0$. Then

$$\left.\frac{\partial\xi_1(\sigma(s); \xi_{10}, \xi_{20}, 0, 0, 0)}{\partial\xi_{10}}\right|_{\substack{s \quad = S^*/4 \\ \xi_{10} \quad = \xi_{10}^*}} = \left.\left(\frac{\partial\xi_1}{\partial\sigma}\frac{\partial\sigma(s)}{\partial\xi_{10}} + \frac{\partial\sigma_1}{\partial\xi_{10}}\right)\right|_{\substack{s \quad = S^*/4 \\ \xi_{10} \quad = \xi_{10}^*}} .$$

As the previous case, we obtain

$$\left.\frac{\partial\xi_1(\sigma(s); \xi_{10}, \xi_{20}, 0, 0, 0)}{\partial\xi_{10}}\right|_{\substack{s \quad = S^*/4 \\ \xi_{10} \quad = \xi_{10}^*}} = (-1)^m,$$

and determinant (21) is $(-1)^{k+m}\dfrac{8}{h_1}$.

7. Concluding remarks

In this paper we have studied the periodic solutions of the classical helium atom which are S_{12}-, S_{13}- and S_{23}- symmetric. Applying the continuation method of Poincaré we obtain that six families of symmetric periodic orbits can be extend from $\mu = 0$ to small positive values of μ. In [2] and [6] it was studied similar problems, but in that papers the authors studied the collinear 3-body problem in the cases where the bodies are uncharged and charged, respectively. In both cases, it was applied the continuation method of Poincaré. In the first one only three families of periodic orbits can be extended from $\mu = 0$ to μ small and positive and in the case where the bodies are charged, similar as we obtain in this paper, six families of symmetric periodic orbits can be extended.

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