

PERIODIC SOLUTIONS AND THEIR STABILITY OF SOME HIGHER-ORDER POSITIVELY HOMOGENOUS DIFFERENTIAL EQUATIONS

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ABSTRACT. In the present paper we study periodic solutions and their stability of the m -order differential equations of the form

$$x^{(m)} + f_n(x) = \mu h(t),$$

where the integers $m, n \geq 2$, $f_n(x) = \delta x^n$ or $\delta |x|^n$ with $\delta = \pm 1$, and $h(t)$ is a continuous T -periodic function of non-zero average, and μ is a positive small parameter. By using the averaging theory, we will give the existence of T -periodic solutions. Moreover, the instability and the linear stability of these periodic solutions will be obtained.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

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In this paper we are concerned with periodic solutions of some higher order homogenous or positively homogenous differential equations. A typical example is the second-order ordinary differential equation

$$\ddot{x} + x^3 = h(t). \quad (1) \quad \boxed{\text{e1}}$$

In [10] Morris proved that if $h(t)$ is a T -periodic C^1 function and its average

$$\bar{h} := \frac{1}{T} \int_0^T h(t) dt$$

is zero, then Eq. (1) has periodic solutions of period kT for all positive integer k . Later on the same result was proved by Ding and Zanolin [5] without the assumption that $\bar{h} = 0$. More recently Ortega in [11] proved that Eq. (1) has finitely many stable periodic solutions of a fixed period.

Other authors have studied more general problems related with non-autonomous differential equations, as for instance: when a periodic

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solution or an equilibrium point of an autonomous differential system persists as a periodic solution if the autonomous differential system is periodically perturbed. Thus for dimension two and for an equilibrium Buică and Ortega in [3] characterized the persistence of such periodic solutions. The Brouwer degree theory was used by these authors for obtaining their results, for these kind of problems see also the paper of Capietto, Mawhin and Zanolin [4] and the references therein. Besides the existence of periodic solutions, Ortega and Zhang in [12] have also studied the stability of the periodic solutions. For higher-order differential equations, some related works are [7, 13].

The objective of this paper is to extend the mentioned results on the periodic solutions of the second-order differential equation (1) to the m -order differential equations of the form

$$x^{(m)} + f_n(x) = \mu h(t), \quad (2) \quad \boxed{\text{x-mn}}$$

where $x^{(m)}$ denotes the m derivative of $x = x(t)$ with respect to the independent variable t . Here the integers $m \geq 2$, $n \geq 2$,

$$f_n(x) = \delta x^n \quad \text{or} \quad f_n(x) = \delta |x|^n, \quad \delta = \pm 1, \quad (3) \quad \boxed{\text{fn-x}}$$

$h(t)$ is a continuous T -periodic function with non-zero mean value

$$\bar{h} \neq 0, \quad (4) \quad \boxed{\text{h-bar}}$$

and $\mu > 0$ is a positive small parameter.

Notice that when $f_n(x) = \delta x^n$ the differential equations (2) are only *continuous* in t and smooth in x , and that when $f_n(x) = \delta |x|^n$ the differential equations (2) are only *continuous* in t and *locally-Lipschitz* in x . So in order to study the periodic solutions of these kind of differential equations and their kind of stability, we cannot use the classical version of the averaging theory because it works for C^2 differential equations, we need the improvements done in [1, 2].

As far as we know, the periodic solutions of these type of differential equations only have been studied recently when $m = 2$ in [8], and when $m = 3$ in [9]. Here we extend the study of the periodic solutions of the differential equations (2) when $m \geq 2$ is an arbitrary integer, and we also analyze the kind of stability of the periodic solutions that we find. The techniques for studying the periodic solutions of the differential equations (2) and their kind of stability are mainly based on the averaging theory, and consequently are completely different from the ones used by Morris, Capietto, Mawhin and Zanolin in the above mentioned papers.

The stability of a periodic solution of the differential equations (2) is the stability of the fixed point of the Poincaré map associated to the periodic solution. Thus, if the real part of some eigenvalue of the fixed point has modulus larger than one then the periodic solution is *asymptotically unstable*. If all the real parts of the eigenvalues of the fixed point are smaller than one then the periodic orbit is *asymptotically stable*. For more information about this kind of stability and the differences with the Liapunov stability see for instance the book [6].

For the periodic solutions of the differential equations (2) we shall study their asymptotical instability and linear stability.

1.1. Statement of the main results. Our main results are the following two theorems.

(t1) **Theorem 1.** *Consider the m -order differential equations*

$$x^{(m)} + \delta x^n = \mu h(t), \quad (5) \quad \boxed{\text{e2}}$$

where m , n , δ and $h(t)$ are as in (3) and (4). Then for $\mu > 0$ sufficiently small the following statements hold.

- (a) *If n is odd, then the differential equation (5) has one periodic solution $x(t, \mu)$ of period T such that*

$$x(0, \mu) = \mu^{1/n}(\delta \bar{h})^{1/n} + O\left(\mu^{(m+n-1)/(mn)}\right).$$

Moreover, when $m \geq 3$, or when $m = 2$ and $\delta = -1$, the periodic solution is asymptotically unstable, and when $m = 2$ and $\delta = +1$, the periodic solution is linearly stable.

- (b) *If n is even and*

$$\delta = \text{sign}(\bar{h}), \quad (6) \quad \boxed{\text{dah}}$$

then the differential equation (5) has two periodic solutions $x_{\pm}(t, \mu)$ of period T such that

$$x_{\pm}(0, \mu) = \pm \mu^{1/n}(\delta \bar{h})^{1/n} + O\left(\mu^{(m+n-1)/(mn)}\right). \quad (7) \quad \boxed{\text{sol2}}$$

Moreover, when $m \geq 3$, these periodic solutions are asymptotically unstable, and when $m = 2$, one periodic solution $x_{-\delta}(t, \mu)$ is asymptotically unstable and the other one $x_{\delta}(t, \mu)$ is linearly stable.

(t2) **Theorem 2.** *Consider the m -order differential equations*

$$x^{(m)} + \delta |x|^n = \mu h(t), \quad (8) \quad \boxed{\text{a1}}$$

where m , n , δ and $h(t)$ are as before. If (6) is satisfied, then for $\mu > 0$ sufficiently small, the differential equation (8) has two periodic

solutions $x_{\pm}(t, \mu)$ of period T which also satisfy (7). Moreover, when $m \geq 3$, these periodic solutions are asymptotically unstable, and when $m = 2$, one periodic solution $x_{-\delta}(t, \mu)$ is asymptotically unstable and the other one $x_{\delta}(t, \mu)$ is linearly stable.

These theorems show that the linearly stable periodic solutions resulted from average theory can be obtained only in case $m = 2$. Here, by the linear stability of a T -periodic solution $x = \phi(t)$ of the equation

$$\ddot{x} + f(x) = h(t),$$

it means that the linearized equation

$$\ddot{X} + f'(\phi(t))X = 0$$

is elliptic, i.e., the Floquet multipliers are different from ± 1 and have modulus 1.

Theorems 1 and 2 include as particular cases the results of [8, 9]. The proofs are given in subsections 2.2-2.5.

2. PROOFS

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We shall study the periodic solutions of the m -order periodically-driven ordinary differential equations (2). We remark that $f_n(x) = \delta x^n$ and $f_n(x) = \delta|x|^n$ are positively homogeneous

$$f_n(cx) \equiv c^n f_n(x) \quad \forall c \geq 0. \quad (9) \quad \boxed{\text{fn-x1}}$$

2.1. Equations in the normal form for applying the averaging theory. By setting

$$x_i = x^{(i-1)}, \quad i = 1, 2, \dots, m,$$

where for $i = 1$ we define $x_1 = x^{(0)} = x$, Eq. (2) is equivalent to the system

$$\begin{aligned} x'_i &= x_{i+1}, & i &= 1, 2, \dots, m-1, \\ x'_m &= \mu h(t) - f_n(x_1). \end{aligned} \quad (10) \quad \boxed{\text{x-mn1}}$$

We rescale system (10) using

$$x_i = \varepsilon^{i-1+m/(n-1)} X_i, \quad i = 1, 2, \dots, m, \quad \mu = \varepsilon^{m+m/(n-1)}, \quad (11) \quad \boxed{\text{sc}}$$

where ε is a positive parameter. Then (10) is reduced to the following system

$$X' = \varepsilon F_n(t, X), \quad (12) \quad \boxed{\text{x-mn2}}$$

where

$$\begin{aligned} X &= (X_1, X_2, \dots, X_m) \in \mathbb{R}^m, \\ F_n(t, X) &= (X_2, \dots, X_m, h(t) - f_n(X_1)) \in \mathbb{R}^m. \end{aligned}$$

Here the positive homogeneity (9) for $f_n(x)$ is used.

System (12) is written in the standard normal form for applying the averaging theory in order to study its periodic solutions and their stability, see [1, 2] for more details on the averaging theory here used.

^(s22) **2.2. Zeros of the average function.** In order to apply the averaging theory for studying the periodic solutions of the differential system (12) and consequently of the differential equation (2) we must study the zeros of the average function associated to system (12).

For any $X \in \mathbb{R}^m$ we have

$$\bar{F}_n(X) := \frac{1}{T} \int_0^T F_n(t, X) dt \equiv \begin{pmatrix} X_2 \\ \vdots \\ X_m \\ \bar{h} - f_n(X_1) \end{pmatrix}.$$

One sees that $\bar{F}_n(X)$ is C^1 in $X \in \mathbb{R}^m$. Moreover X is a zero of \bar{F}_n if and only if

$$X_* = (x_*, 0, \dots, 0),$$

where $x_* \in \mathbb{R}/\{0\}$ is determined by the scalar equation

$$f_n(x_*) = \bar{h}. \quad (13) \quad \boxed{\text{x*}}$$

Moreover the $m \times m$ Jacobian matrix of \bar{F}_n at X_* is

$$J\bar{F}_n(X_*) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -f'_n(x_*) & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (14) \quad \boxed{\text{mat}}$$

with the determinant

$$\det J\bar{F}_n(X_*) = (-1)^m f'_n(x_*). \quad (15) \quad \boxed{\text{JFx*}}$$

Case 1: n is odd and $f_n(x) = \delta x^n$. Then Eq. (13) has always the unique real solution

$$x_* = (\delta \bar{h})^{1/n}.$$

Then from (15) we have

$$\det J\bar{F}_n(X_*) = (-1)^m \delta n x_*^{n-1} = (-1)^m \delta n (\delta \bar{h})^{(n-1)/n},$$

whose sign is

$$\text{sign}(\det J\bar{F}_n(X_*)) = (-1)^m \delta. \quad (16) \quad \boxed{\text{JFx2}}$$

Case 2: n is even and $f_n(x) = \delta x^n$. In this case Eq. (13) has solutions if and only if (6) is satisfied. Under assumption (6) Eq. (13) has precisely two solutions

$$x_* = x_{\pm} = \pm(\delta\bar{h})^{1/n} = \pm|\bar{h}|^{1/n}.$$

From (15) the determinant of the associated Jacobian matrix is

$$\det J\bar{F}_n(X_{\pm}) = (-1)^m \delta n (x_{\pm})^{n-1} = \pm(-1)^m \delta n |\bar{h}|^{(n-1)/n},$$

whose signs are

$$\text{sign}(\det J\bar{F}_n(X_{\pm})) = \pm(-1)^m \delta. \quad (17) \quad \boxed{\text{JFx22}}$$

Case 3: n is odd and $f_n(x) = \delta|x|^n$. In this case Eq. (13) has solutions if and only if (6) is satisfied. Under assumption (6), Eq. (13) has precisely two solutions

$$x_* = x_{\pm} = \pm|\bar{h}|^{1/n}.$$

As

$$f'_n(x) = (\delta|x|^n)' = \delta n \text{sign}(x) |x|^{n-1},$$

we have from (15) that the determinant of the associated Jacobian matrix is

$$\det J\bar{F}_n(X_{\pm}) = (-1)^m \delta n \text{sign}(x_{\pm}) |x_{\pm}|^{n-1} = \pm(-1)^m \delta n |\bar{h}|^{(n-1)/n},$$

whose signs are

$$\text{sign}(\det J\bar{F}_n(X_{\pm})) = \pm(-1)^m \delta. \quad (18) \quad \boxed{\text{JFx24}}$$

Case 4: n is even and $f_n(x) = \delta|x|^n$. Then we have $f_n(x) = \delta|x|^n \equiv \delta x^n$, which is same as in Case 2.

2.3. Existence and asymptotic formulas for the periodic solutions. According to the results on the averaging theory, if Eq. (13) has a solution x_* , then for $0 < \varepsilon \ll 1$ the differential system (12) has a T -periodic solution $X(t, \varepsilon)$ such that

$$X(0, \varepsilon) \rightarrow (x_*, 0, \dots, 0) \text{ when } \varepsilon \rightarrow 0.$$

Due to the scaling (11) for $0 < \varepsilon \ll 1$ the differential system (2) has a T -periodic solution $x(t, \varepsilon) = \varepsilon^{m/(n-1)} X_1(t, \varepsilon)$ such that

$$X_1(0, \varepsilon) \rightarrow x_* \text{ when } \varepsilon \rightarrow 0.$$

Going back to the original parameter μ , when $0 < \mu \ll 1$, and the differential system (2) has a T -periodic solution

$$x(t, \mu) = \mu^{1/n} X_1(t, \mu)$$

such that

$$X_1(0, \mu) \rightarrow x_* \text{ when } \mu \rightarrow 0.$$

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2.4. Instability of the periodic solutions. The $m \times m$ Jacobian matrix (14) has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -c & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $c \neq 0$. One has then

$$\begin{aligned} & \det(\lambda I_{m \times m} - A) \\ = & \det \begin{pmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & -1 \\ c & 0 & 0 & \cdots & \lambda \end{pmatrix} \\ = & \lambda^m + (-1)^{m+1} c \det \begin{pmatrix} -1 & 0 & \cdots & 0 \\ \lambda & -1 & \cdots & 0 \\ & \cdots & & \\ 0 & 0 & \cdots & -1 \end{pmatrix} \\ = & \lambda^m + c. \end{aligned}$$

If $m \geq 3$, the matrix A always has at least one eigenvalue which has positive real part. On the other hand, if $m = 2$, the matrix A admits a positive eigenvalue if and only if $c < 0$.

By the instability results of [1, 2] on the periodic solutions obtained by the averaging theory, we have the following results for the periodic solution $x(t, \mu)$ obtained in the previous subsection.

The T -periodic solution $x(t, \mu)$ is always asymptotically unstable when $0 < \mu \ll 1$ for $m \geq 3$.

For $m = 2$, from (14) and (15) we conclude that $J\bar{F}_n(X_*)$ has a positive eigenvalue if and only if

$$\text{sign}(\det J\bar{F}_n(X_*)) = -1. \quad (19) \quad \boxed{\text{cm}}$$

Case 1: n is odd and $f_n(x) = \delta x^n$. By (16) and (19), we know that if $\delta = -1$, i.e. $f_n(x) = -x^n$, then the T -periodic solution $x(t, \mu)$ is asymptotically unstable when $0 < \mu \ll 1$.

Case 2: n is even and $f_n(x) = \delta x^n$. Under assumption (6), we have obtained two T -periodic solutions $x_{\pm}(t, \mu)$. Because of (17) and (19), one sees that the T -periodic solution $x_+(t, \mu)$ is asymptotically unstable if $\delta = -1$, and the T -periodic solution $x_-(t, \mu)$ is asymptotically unstable if $\delta = 1$. These are the asymptotical instability results described in Theorem 1.

Case 3: n is odd and $f_n(x) = \delta|x|^n$. Since (17) coincides with (18), we can obtain the asymptotical instability results of the T -periodic solutions $x_{-\delta}(t, \mu)$ as in Case 2. These have been stated as in Theorem 2.

Case 4: n is even and $f_n(x) = \delta|x|^n$. This can be reduced to Case 2.

(s25) **2.5. Linear stability of the periodic solutions.** To complete the proofs of Theorems 1 and 2, we need to prove the linear stability for the other periodic solution $x_{\delta}(t, \mu)$ in Cases 2 and 3 with $m = 2$. Usually speaking, the linear stability cannot be directly derived from standard results in averaging theory.

By considering Case 2 as an example, the linear stability is obtained as follows. The equation we are considering is

$$\ddot{x} + \delta x^n = \mu h(t), \quad (20) \quad \boxed{\text{e200}}$$

where n is even and $\delta \bar{h} > 0$. The t -periodic solution we are considering is $x = x_{\delta}(t, \mu)$. Then the linearization of Eq. (20) is

$$\ddot{X} + a(t, \mu)X = 0, \quad (21) \quad \boxed{\text{E200}}$$

where

$$a(t, \mu) = \delta n(x_{\delta}(t, \mu))^{n-1}.$$

By estimate (7), one can obtain

$$x_{\delta}(t, \mu) = \delta \mu^{1/n} |\bar{h}|^{1/n} (1 + o(1)) \quad \text{as } \mu \rightarrow 0.$$

By noticing that $n - 1$ is odd, we have, as $\mu \rightarrow 0$,

$$a(t, \mu) = n |\bar{h}|^{(n-1)/n} \mu^{(n-1)/n} + o(\mu^{(n-1)/n}),$$

a higher-order perturbation of the constant function

$$b_{\mu} := n |\bar{h}|^{(n-1)/n} \mu^{(n-1)/n}.$$

It is well-known that the following equation

$$\ddot{X} + b_{\mu} X = 0,$$

considered as a T -periodic Hill's equation, is elliptic. As Eq. (21) is a T -periodic Hill's equation, we know that it is also elliptic when

$0 < \mu \ll 1$. Hence the proof of Theorem 1 is complete in this case. The corresponding case in Theorem 2 can be treated similarly.

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Remark 1. *The techniques here can be easily extended to more general positively homogenous equations of the form*

$$x^{(m)} - ax_-^p + bx_+^p = \mu h(t),$$

where $m \geq 2$, $x_{\pm} = \max\{\pm x, 0\}$, a, b are constants and $p > 1$.

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