This is a preprint of: “Center problem for generalized Λ–Ω differential systems”, Jaume Llibre, Rafael Orlando Ramírez, Valentín Ramírez, *Electron. J. Differential Equations*, vol. 2018(184), 1–23, 2018.

CENTER PROBLEM FOR GENERALIZED Λ–Ω DIFFERENTIAL SYSTEMS

JAUME LLIBRE[1], RAFAEL RAMÍREZ[2] AND VALENTÍN RAMÍREZ[1]

Abstract. The Λ-Ω differential systems are the real planar polynomial differential equations of degree m of the form

$$\dot{x} = -y(1+\Lambda) + x\Omega, \qquad \dot{y} = x(1+\Lambda) + y\Omega,$$

where $\Lambda = \Lambda(x,y)$ and $\Omega = \Omega(x,y)$ are polynomials of degree at most $m-1$ such that $\Lambda(0,0) = \Omega(0,0) = 0$. We study the center problem for these Λ-Ω systems. A planar vector field with linear type center can be written as an Λ-Ω system if and only if the Poincaré-Liapunov first integral is of the form $F = \frac{1}{2}(x^2+y^2)(1+O(x,y))$.

The main objective of this paper is to study the center problem for Λ-Ω systems of degree m with $\Lambda = \mu(a_2x - a_1y)$, and $\Omega = a_1x + a_2y + \sum_{j=2}^{m-1} \Omega_j$, where μ, a_1, a_2 are constants and $\Omega_j = \Omega_j(x,y)$ is a homogenous polynomial of degree j, for $j = 2,\ldots,m-1$.

We prove the following results. Assuming that $m = 2,3,4,5$ and

$$(\mu + (m-2))(a_1^2 + a_2^2) \neq 0 \quad \text{and} \quad \sum_{j=2}^{m-2} \Omega_j \neq 0$$

then the Λ–Ω system has a weak center at the origin if and only if these systems after a linear change of variables $(x,y) \longrightarrow (X,Y)$ are invariant under the transformations $(X,Y,t) \longrightarrow (-X,Y,-t)$. If $(\mu + (m-2))(a_1^2 + a_2^2) = 0$ and $\sum_{j=1}^{m-2} \Omega_j = 0$ then the origin is a weak center. We observe that the main difficulty to prove this result for $m > 6$ is related with the huge computations.

1. Introduction

Let $\mathcal{X} = P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}$ be the real planar polynomial vector field associated to the real planar polynomial differential system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1}$$

where the dot denotes derivative with respect to an independent variables here called the time t, and P and Q are real coprime polynomials in $\mathbb{R}[x,y]$. We say that the polynomial differential system (1) has *degree* $m = \max\{\deg P, \deg Q\}$.

2010 *Mathematics Subject Classification.* 34C05, 34C07.

Key words and phrases. linear type center, Darboux first integral, Poincaré-Liapunov Theorem, Reeb integrating factor, weak center.

In what follows we assume that the origin $O := (0,0)$ is a singular or equilibrium point, i.e. $P(0,0) = Q(0,0) = 0$.

The equilibrium point O is a *center* if there exists an open neighborhood U of O where all the orbits contained in $U \setminus \{O\}$ are periodic.

We shall work with the polynomial differential systems of degree m such that

$$\dot{x} = -y + X, \quad \dot{y} = x + Y, \tag{2}$$

where $X = X(x,y)$ and $Y = Y(x,y)$ are polynomials starting at least with quadratic terms in the neighborhood of the origin, so $m = \max\{\deg X, \deg Y\} \geq 2$. The *center-focus problem* asks about conditions on the coefficients of X and Y under which the origin of system (2) is a center.

If a system (2) has a local first integral at the origin of the form

$$F = \frac{1}{2}(x^2 + y^2)\Phi(x,y),$$

where $\Phi = \Phi(x,y)$ is an analytic function such that $\Phi(0,0) = 1$, then the origin of system (2) is a center called a *weak center.* The weak center contain the uniform isochronous centers and the holomorphic isochronous centers (for a prof of these results see [12]), but they do not coincide with the all class of isochronous centers (see Remark 19 of [12]).

In this paper we shall study the particular case of differential systems (2) of the form

$$\dot{x} = -y(1 + \Lambda) + x\Omega, \qquad \dot{y} = x(1 + \Lambda) + y\Omega, \tag{3}$$

where $\Lambda = \Lambda(x,y)$ and $\Omega = \Omega(x,y)$ are polynomials such $m = \max\{\deg \Lambda, \deg \Omega\} + 1$.

By applying the inverse approach in ordinary differential equations see [10] the following theorem is proved and shows the importance of system (3) in the theory of ordinary differential equations (see Theorem 15 in [12]).

Theorem 1. *The polynomial differential system* (2) *has a weak center at the origin if and only if it can be written as* (3) *with*

$$\begin{aligned} \Lambda = & \sum_{j=2}^{m} \left(\frac{j+1}{2} \Upsilon_{j-1} + \frac{j}{2} g_1 \Upsilon_{j-2} + \ldots + \frac{3}{2} g_{j-2} \Upsilon_1 + g_{j-1} \right), \\ \Omega = & \frac{1}{2} \sum_{j=2}^{m} \Big(\{\Upsilon_{j-1}, H_2\} + g_1 \{\Upsilon_{j-2}, H_2\} + \ldots + g_{j-2} \{\Upsilon_1, H_2\} \Big), \end{aligned}$$

where g_j and Υ_j are homogenous polynomials of degree j for $j \geq 1$ and has a first integral of the form

$$H = H_2 \Phi = H_2(1 + \mu_1 \Upsilon_1 + \ldots + \mu_{m-1} \Upsilon_{m-1}),$$

where $H_2 = (x^2 + y^2)/2$, and $\mu_j = \mu_j(x,y)$ is a convenient analytic function in the neighborhood of the origin for $j = 1, \ldots, m-1$.

2. Statement of the main results

In this section we give the statements of our main results which will be proved in sections 4 and 5, also we state some conjectures.

Conjecture 2. *The polynomial differential system of degree* m

$$
\begin{aligned}
\dot{x} = & \quad -y(1+\mu(a_2x-a_1y))+x(a_1x+a_2y+\sum_{j=2}^{m-1}\Omega_j(x,y)),\\
\dot{y} = & \quad x(1+\mu(a_2x-a_1y))+y(a_1x+a_2y+\sum_{j=2}^{m-1}\Omega_j(x,y)),
\end{aligned} \tag{4}
$$

under the assumptions $(\mu+(m-2))(a_1^2+a_2^2)\neq 0$ *and* $\sum_{j=2}^{m-2}\Omega_j\neq 0$ *where* $\Omega_j = \Omega_j(x,y)$ *is a homogenous polynomial of degree* j *for* $j=2,\ldots,m-1$, *has a weak center at the origin if and only if system* (4) *after a linear change of variables* $(x,y)\longrightarrow(X,Y)$ *is invariant under the transformations* $(X,Y,t)\longrightarrow(-X,Y,-t)$. *Moreover differential system* (4) *in the variables* X,Y *becomes*

$$
\begin{aligned}
\dot{X} = & \quad -Y(1+\mu Y)+X^2\Theta(X^2,Y) = -Y(1+\mu Y)+X\{H_2,\Phi\},\\
\dot{Y} = & \quad X(1+\mu Y)+XY\Theta(X^2,Y) = X(1+\mu Y)+Y\{H_2,\Phi\},
\end{aligned}
$$

where $\Theta(X^2,Y)$ *is a polynomial of degree* $m-2$, *and* Φ *is a polynomial of degree* $m-1$ *such that* $\{H_2,\Phi\}=X\Theta(X^2,Y)$.

We observe that the case when $(\mu+(m-2))(a_1^2+a_2^2)=0$ and $\sum_{j=2}^{m-2}\Omega_j=0$ was study in [13].

Theorem 3. *Conjecture 2 holds for* $m=2,3$ *and for* $m=4$ *with* $\mu=0$.

The proof of Theorem 3 for $\mu=0$ and $m=2$ goes back to Loud [15]. The proof of Theorem 3 for $\mu=0$ and $m=3$ was done by Collins [5]. Finally the proof of Theorem 3 for $\mu=0$ and $m=4$ goes back to [1, 2, 4]. But in the proof of this last result there is some mistakes. The phase portraits of these systems are classified in [3, 8, 9]. The proof that these centers are weak centers has been done in Theorem 1.

Conjecture 4. *Assume that the polynomial differential system of degree* $m-1$

$$
\begin{aligned}
\dot{x} = & \quad -y(1+\mu(a_2x-a_1y))+x(a_1x+a_2y+\sum_{j=2}^{m-2}\Omega_j(x,y)),\\
\dot{y} = & \quad x(1+\mu(a_2x-a_1y))+y(a_1x+a_2y+\sum_{j=2}^{m-2}\Omega_j(x,y)),
\end{aligned}
$$

where $a_1a_2\neq 0$, *and* $\Omega_j=\Omega_j(x,y)$ *is a homogenous polynomial of degree* j *for* $j=2,\ldots,m-2$, *after a linear change of variables* $(x,y)\longrightarrow(X,Y)$ *it is invariant under* *the transformations* $(X, Y, t) \longrightarrow (-X, Y, -t)$. *Then the polynomial differential system of degree* m

$$\begin{aligned}\dot{x} = &\quad -y(1+\mu(a_2x - a_1y)) + x(a_1x + a_2y + \sum_{j=2}^{m-1}\Omega_j(x,y)),\\ \dot{y} = &\quad x(1+\mu(a_2x - a_1y)) + y(a_1x + a_2y + \sum_{j=2}^{m-1}\Omega_j(x,y)),\end{aligned}$$

has a weak center at the origin if and only if the system

$$\begin{aligned}\dot{x} = &\quad -y(1+\mu(a_2x - a_1y)) + x(a_1x + a_2y + \Omega_{m-1}(x,y)),\\ \dot{y} = &\quad x(1+\mu(a_2x - a_1y)) + y(a_1x + a_2y + \Omega_{m-1}(x,y)),\end{aligned} \tag{5}$$

under the assumption $(\mu + (m-2))(a_1^2 + a_2^2) \neq 0$ *and after a linear change of variables* $(x, y) \longrightarrow (X, Y)$ *it is invariant under the transformations* $(X, Y, t) \longrightarrow (-X, Y, -t)$.

We observe that the existence of the weak center of (5) was solve in [13].

Theorem 5. *Conjecture 4 holds for* $m = 3, 4, 5, 6$.

We note that when system (4) with $\mu = 0$ has a center at the origin this center is a uniform isochronous center, i.e. if we write these systems in polar coordinates (r, θ) we obtain that $\dot{\theta}$ is constant. Clearly if $\mu = 0$ then the weak centers are uniform isochronous centers.

Note that Conjecture 4 is a particular case of Conjecture 2.

3. Preliminary results

In the proofs of Theorems 3 and 5 it plays a very important role the following results and notations which we can find in [12] .

As usual the *Poisson bracket* of the functions $f(x, y)$ and $g(x, y)$ is defined as

$$\{f, g\} := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

The following result is a simple consequence of the Liapunov result given in Theorem 1, page 276 of [14] .

Corollary 6. *Let* $U = U(x, y)$ *be a homogenous polynomial of degree* m. *The linear partial differential equation* $\{H_2, V\} = U$, *has a unique homogenous polynomial solution* V *of degree* m *if* m *is odd; and if* V *is a homogenous polynomial solution when* m *is even then any other homogenous polynomial solution is of the form* $V + c(x^2 + y^2)^{m/2}$ *with* $c \in \mathbb{R}$. *Moreover, for* m *even these solutions exist if and only if* $\int_0^{2\pi} U(x, y)|_{x=\cos t,\, y=\sin t}\, dt = 0$.

Proposition 7 (see Proposition 6 of [12])**.** *The next relation holds*

$$\int_0^{2\pi} \{H_2, \Psi\}|_{x=\cos t,\, y=\sin t}\, dt = 0$$

for an arbitrary C^1 *function* $\Psi = \Psi(x, y)$ *defined in the interval* $[0, 2\pi]$.

Proposition 8 (see Proposition 24 of [12])**.** *Consider the polynomial differential system* (1) *of degree* m *which satisfies*

$$\int_0^{2\pi}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)|_{x=\cos t,\, y=\sin t}dt=0.$$

Then there exist polynomials $F=F(x,y)$ *and* $G=G(x,y)$ *of degree* $m+1$ *and* $m-1$ *respectively such that system* (1) *can be written as*

$$\dot{x}=P=\{F,x\}+(1+G)\{H_2,x\},\quad \dot{y}=Q=\{F,y\}+(1+G)\{H_2,y\},$$

with $G(0,0)=0$.

We need the following definitions and notions.

A function $V=V(x,y)$ is an *inverse integrating factor* of system (1) in an open subset $U\subset\mathbb{R}^2$ if $V\in C^1(U), V\not\equiv 0$ in U and $\dfrac{\partial}{\partial x}\left(\dfrac{P}{V}\right)+\dfrac{\partial}{\partial y}\left(\dfrac{Q}{V}\right)=0$

Theorem 9. *[Reeb 's criterion] (see for instance* [19]*). The analytic differential system* $\dot{x}=-y+\sum_{j=2}^{\infty}X_j,\quad \dot{y}=x+\sum_{j=2}^{\infty}Y_j$ *has a center at the origin if and only if there is a local nonzero analytic inverse integrating factor of the form* $V=1+h.o.t.$ *in a neighborhood of the origin.*

An analytic inverse integrating factor of the form $V=1+h.o.t.$ in a neighborhood of the origin is called a *Reeb inverse integrating factor.*

The analytic function

$$H=\sum_{j=2}^{\infty}H_j(x,y)=\frac{1}{2}(x^2+y^2)+\sum_{j=3}^{\infty}H_j(x,y),$$

where H_j is homogenous polynomials of degree $j>1$, is called the *Poincaré-Liapunov local first integral* if H is constant on the solutions of (2).

Theorem 10 (see Theorem 13 and Remark 14 of [12])**.** *Consider the polynomial vector field* $\mathcal{X}=(-y+\sum_{j=2}^{m}X_j)\dfrac{\partial}{\partial x}+(x+\sum_{j=2}^{m}Y_j)\dfrac{\partial}{\partial y}$. *Then this vector field has a Poincaré-Liapunov local first integral* H *if and only if it has a Reeb inverse integrating factor* V*. Moreover, the differential system associated to the vector field* $\mathcal{X}$ *for which* $H=(x^2+y^2)/2+h.o.t.$ *is a local first integral can be written as*

$$\begin{aligned}\dot{x}=&\ V\{H,x\}\\ =&\ \{H_{m+1},x\}+(1+g_1)\{H_m,x\}+\ldots+(1+g_1+\ldots+g_{m-1})\{H_2,x\},\\ \dot{y}=&\ V\{H,y\}\\ =&\ \{H_{m+1},y\}+(1+g_1)\{H_m,y\}+\ldots+(1+g_1+\ldots+g_{m-1})\{H_2,y\},\end{aligned}\tag{6}$$

and V and H are such that

$$V = 1+\sum_{j=1}^{\infty} g_j,$$

$$(7)\qquad H = \frac{1}{2}(x^2+y^2)+\sum_{j=2}^{\infty} H_j = \tau_1 H_{m+1}+\tau_2 H_m+\ldots+\tau_m H_2$$
$$= \int_\gamma \Big(\frac{dH_{m+1}}{V}+\frac{(1+g_1)dH_m}{V}+\ldots+\frac{(1+g_1+\ldots+g_{m-1})dH_2}{V}\Big),$$

where γ is an oriented curve (see for instance [20]*), $\tau_j=\tau_j(x,y)$ is a convenient analytic function in the neighborhood of the origin such that $\tau_j(0,0)=1$, and $g_j=g_j(x,y)$ is an arbitrary homogenous polynomial of degree j which we choose in such a way that V is the inverse Reeb integrating factor which satisfies the first order partial differential equation*

$$(8)\qquad \{H_{m+1},\frac{1}{V}\}+\{H_m,\frac{1+g_1}{V}\}+\ldots+\{H_2,\frac{1+g_1+\ldots+g_{m-1}}{V}\}=0.$$

Remark 11. *[see formula (44) and the proof of Theorem 13 of* [11]*] From* (8)*,and* (7) *the following infinite number of equations must hold*

(9)

$$\{H_{m+1},g_1\}+\{H_m,g_2\}+\ldots+\{H_3,g_{m-1}\}+\{H_2,g_m\} = 0,$$
$$\{H_{m+1},g_1^2-g_2\}+\{H_m,g_1g_2-g_3\}+\ldots+\{H_3,g_1g_{m-1}-g_m\}+\{H_2,g_1g_m+g_{m+1}\} = 0,$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Consequently

(10)

$$\int_0^{2\pi}(\{H_{m+1},g_1\}+\{H_m,g_2\}+\ldots+\{H_3,g_{m-1}\})|_{x=\cos t,y=\sin t}\,dt = 0,$$
$$\int_0^{2\pi}\big(\{H_{m+1},g_1^2-g_2\}+\{H_m,g_1g_2-g_3\}+\ldots+\{H_3,g_1g_{m-1}-g_m\}\big)|_{x=\cos t,y=\sin t}\,dt = 0,$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Conditions (9) *and* (10) *are equivalent to the following relations .*

(11)

$$\{H_{m+j+1},g_1\}+\{H_{m+j},g_2\}+\ldots+\{H_3,g_{m+j-1}\}+\{H_2,g_{m+j}\} = 0,$$
$$\int_0^{2\pi}(\{H_{m+j+1},g_1\}+\{H_{m+j},g_2\}+\ldots+\{H_3,g_{m+j-1}\})|_{x=\cos t,y=\sin t}\,dt = 0,$$

for $j\geq 0$. Theorem 10 can be applied to determine the Poincaré-Liapunov first integral, Reeb inverse integrating factor and Liapunov constants for the case when the polynomial differential system is given (see section 8 of [12]*. Indeed, given a polynomial vector field $\mathcal{X}$ of degree m with a linear type center at the origin of coordinates, using* (6) *we determine its first integral H and its Reeb inverse integrating factor. Thus, if in* (2) *$X=\sum_{j=2}^{m}X_j$ and $Y=\sum_{j=2}^{m}Y_j$ with X_j and Y_j*

homogenous polynomials of degree j, from (6) *and from the proof of Theorem 10 equating the terms of the same degree we get*

$$\begin{aligned}\{H_{j+1},x\}+g_1\{H_j,x\}+\ldots+g_{j-1}\{H_2,x\}&=X_j\\ \{H_{j+1},y\}+g_1\{H_j,y\}+\ldots+g_{j-1}\{H_2,y\}&=Y_j,\\ \{H_{k+1},x\}+g_1\{H_k,x\}+\ldots+g_{k-1}\{H_2,x\}&=0\\ \{H_{k+1},y\}+g_1\{H_k,y\}+\ldots+g_{k-1}\{H_2,y\}&=0,\end{aligned}$$

for $j=2,\ldots,m$, and $k>m$. Then the compatibility condition of these equations are

$$\begin{aligned}\{H_j,g_1\}+\ldots+\{H_2,g_{j-1}\}=&\ \frac{\partial X_j}{\partial x}+\frac{\partial Y_j}{\partial y}\quad for\quad j=2,\ldots,m,\\ \{H_k,g_1\}+\ldots+g_{k-1}\{H_2,g_{k-1}\}=&\ 0\quad for\quad k>m.\end{aligned}\tag{12}$$

for $k>1$.

If (12) *holds then by considering that H_n for $n>1$ are homogenous polynomials of degree n. Then applying Euler's Theorem for homogenous polynomials we obtain the homogenous polynomial H_n as follows*

$$\begin{aligned}H_{j+1}=&\ -\frac{1}{j+1}\left(yX_j-xX_j+jg_1H_j+\ldots+2g_{j-1}H_2\right),\\ H_{k+1}=&\ -\frac{1}{k+1}\left(kg_1H_k+\ldots+2g_{k-1}H_2\right),\end{aligned}\tag{13}$$

for $j=2,\ldots,m$, and $k>m$.

We need the following results.

Let

$$x=\kappa_1X-\kappa_2Y,\qquad y=\kappa_2X+\kappa_1Y,\tag{14}$$

be a non-degenerated linear transformation, i.e. $\kappa_1^2+\kappa_2^2\neq 0$. Then the differential system (3) becomes

$$\begin{aligned}\dot{X}=&\ -Y\left(1+\tilde{\Lambda}(X,Y)\right)+X\tilde{\Omega}(X,Y),\\ \dot{Y}=&\ X\left(1+\tilde{\Lambda}(X,Y)\right)+Y\tilde{\Omega}(X,Y),\end{aligned}\tag{15}$$

where $\tilde{\Lambda}(X,Y)=\Lambda(\kappa_1X-\kappa_2Y,\kappa_2X+\kappa_1Y)$ and $\tilde{\Omega}(X,Y)=\Omega(\kappa_1X-\kappa_2Y,\kappa_2X+\kappa_1Y)$. Here we say that system (2) is *reversible with respect to a straight line l* through the origin if it is invariant with respect to reversion about l and a reversion of time t (see for instance [6]). In particular Poincaré's Theorem is applied for the case when (2) is invariant under the transformations $(x,y,t)\longrightarrow(-x,y,-t)$, or $(x,y,t)\longrightarrow(x,-y,-t)$.

In the proof of the results which we give later on we need the Poincare's Theorem (see for instance [17], p.122.)

Theorem 12. *The origin of system* (2) *is a center if the system is reversible.*

Since a rotation with respect to the origin of coordinates is a particular transformation of type (14), when a center of system (3) is invariant with respect to a straight line it is not restrictive to assume that such straight line is the x-axis. So the center of system (3) will be invariant by the transformation $(X,Y,t) \longrightarrow (-X,Y,-t)$ or $(X,Y,t) \longrightarrow (X,-Y,-t)$. Without loss of the generality we shall study only the first case, i.e. we shall suppose that the Λ-Ω system is invariant with respect to the transformation $(X,Y,t) \longrightarrow (-X,Y,-t)$.

The following proposition is easy to prove (see for instance [18]).

Proposition 13. *Differential system* (15) *is invariant under the transformation* $(X,Y,t) \longrightarrow (-X,Y,-t)$ *if and only if it can be written as*

$$\begin{aligned} \dot X =& \quad -Y\left(1+\Theta_1(X^2,Y)\right)+X^2\Theta_2(X^2,Y),\\ \dot Y =& \quad X\left(1+\Theta_1(X^2,Y)\right)+XY\Theta_2(X^2,Y). \end{aligned} \tag{16}$$

The following result was proved Corollary 15 of [**?**].

Corollary 14. *Polynomial differential system* (16) *can be written as*

$$\begin{aligned} \dot X =& \quad -Y\left(1+\Theta_1(X^2,Y)\right)+X\{H_2,\,\Phi\},\\ \dot Y =& \quad X\left(1+\Theta_1(X^2,Y)\right)+Y\{H_2,\,\Phi\}, \end{aligned} \tag{17}$$

where $\Phi=\Phi(x,y)$ *is a polynomial of degree at most* $m-1$ *and such that* $\{H_2,\,\Phi\} = X\Theta_2(X^2,Y)$.

Corollary 15. *Any weak centers of the type*

$$\begin{aligned} \dot x =& \quad -y\,(1+\Lambda)+x\{H_2,\,\Phi\}=p,\\ \dot y =& \quad x\,(1+\Lambda)+y\{H_2,\,\Phi\}=q, \end{aligned} \tag{18}$$

satisfies that the integral of the divergence on the unit circle is zero. Moreover differential system (17) *can be written as*

$$\begin{aligned} \dot x =& \quad \{\Phi,x\}+(1+G)\{H_2,x\}:=p,\\ \dot y =& \quad \{\Phi,y\}+(1+G)\{H_2,y\}:=q, \end{aligned} \tag{19}$$

where $G=G(x,y)$ *is a polynomial of degree* $m-1$.

Proof. Indeed from the relations

$$\begin{aligned} \frac{\partial p}{\partial x}+\frac{\partial q}{\partial y} =& \quad 2\{H_2,\,\Phi\}+x\frac{\partial\{H_2,\,\Phi\}}{\partial x}+y\frac{\partial\{H_2,\,\Phi\}}{\partial y}+\{H_2,\Lambda\}\\ =& \quad \left\{H_2, 2\Phi+x\frac{\partial\Phi}{\partial x}+y\frac{\partial\Phi}{\partial y}+\Lambda\right\}, \end{aligned}$$

and Proposition 7 we obtain that

$$\int_0^{2\pi}\left(\frac{\partial p}{\partial x}+\frac{\partial q}{\partial y}\right)\Bigg|_{\substack{x=\cos t\\ x=\sin t}} dt = 0.$$

Consequently from Proposition 8 we get that (18) becomes (19). Thus the proof of corollary follows. □

4. Proof of Theorem 3

The proof of Theorem 3 for $m = 2$ and $m = 3$ follows from the proof of Theorem 7 of [13] . For $m = 4$ we prove Theorem 5 in Propositions 16.

Proposition 16. *The fourth polynomial differential system*

$$
\begin{aligned}
\dot{x} = &\ -y + x\Big(a_1x + a_2y + a_3x^2 + a_4y^2 \\
&+a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2\Big) := P, \\
\dot{y} = &\ x + y\Big(a_1x + a_2y + a_3x^2 + a_4y^2 \\
&+a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2\Big) := Q,
\end{aligned} \tag{20}
$$

where $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 \neq 0$ has a weak center at the origin if and only if after a linear change of variables $(x, y) \longrightarrow (X, Y)$ it is invariant under the transformations $(X, Y, t) \longrightarrow (-X, Y, -t)$ or $(X, Y, t) \longrightarrow (X, -Y, -t)$. Moreover,

(i) *if $a_1^2 + a_2^2 \neq 0$, then system (20) has a weak center at the origin if and only if*

$$
\begin{aligned}
a_3 + a_4 = 0, \qquad a_5a_1a_2 + (a_2^2 - a_1^2)a_4 =&\ 0, \\
a_1^3a_7 - a_1^2a_2a_9 + a_1a_2^2a_8 - a_2^3a_6 =&\ 0, \\
3a_1a_2^2a_7 - 3a_1^2a_2a_6 + (a_1^3 - 2a_1a_2^2)a_8 + (2a_1^2a_2 - a_2^3)a_9 =&\ 0.
\end{aligned} \tag{21}
$$

Consequently

(a)

$$
\begin{aligned}
a_3 + a_4 = 0, \qquad a_5 + \frac{(a_2^2 - a_1^2)}{a_1a_2}a_4 =&\ 0, \\
a_6 + \frac{1}{2a_2^3}\left(a_7(a_1^3 - 3a_2^2a_1) + a_9(a_2^3 - a_1^2a_2)\right) =&\ 0, \\
a_8 + \frac{1}{2a_2^2a_1}\left(a_7(3a_1^3 - 3a_1a_2^2) + a_9(a_2^3 - 3a_1^2a_2)\right) =&\ 0.
\end{aligned} \tag{22}
$$

when $a_1a_2 \neq 0$,

(b) $a_2 = a_3 = a_4 = a_7 = a_8 = 0$, *when $a_1 \neq 0$,*

(c) $a_1 = a_3 = a_4 = a_6 = a_9 = 0$, *when $a_2 \neq 0$.*

(ii) *If $a_1 = a_2 = 0$ and $a_4a_5 \neq 0$ then system (20) has a weak center at the origin if and only if*

$$
\begin{aligned}
a_3 + a_4 =&\ 0, \\
\lambda\, a_5 + (1 - \lambda^2)a_4 =&\ 0, \\
\lambda^3a_7 - \lambda^2a_9 + \lambda\, a_8 - a_6 =&\ 0, \\
3\lambda^2a_7 + 3\lambda a_6 + \left(\lambda^3 - 2\lambda^2\right)a_8 + \left(2\lambda^2 - 1\right)\right) a_9 =&\ 0,
\end{aligned}
$$

where $\lambda = \dfrac{a_5 + \sqrt{4a_4^2 + a_5^2}}{2a_4}$. Moreover the weak center in this case after a linear change of variables $(x, y) \longrightarrow (X, Y)$ it is invariant under the transformations $(X, Y, t) \longrightarrow (-X, Y, -t)$.

(iii) *if* $a_1 = a_2 = a_3 = a_4 = a_5 = 0$, *then the origin is a weak center.*

Proof. Sufficiency: First of all we observe that the polynomial differential system (20) after the linear change of variables (14) would be invariant under the transformation $(X, Y, t) \longrightarrow (-X, Y, -t)$ if and only if

$$(23)\qquad \begin{aligned} \kappa_2 a_1 - \kappa_1 a_2 =& \ 0, \\ \kappa_1^2 a_3 + \kappa_2^2 a_4 + \kappa_1\kappa_2 a_5 =& \ 0, \\ \kappa_2^2 a_3 + \kappa_1^2 a_4 - \kappa_1\kappa_2 a_5 =& \ 0, \\ \kappa_1^3 a_7 - \kappa_1^2\kappa_2 a_9 + \kappa_1\kappa_2^2 a_8 - \kappa_2^3 a_6 =& \ 0, \\ 3\kappa_1\kappa_2^2 a_7 - 3\kappa_1^2\kappa_2 a_6 + (\kappa_1^3 - 2\kappa_1\kappa_2^2)a_8 + (2\kappa_1^2\kappa_2 - \kappa_1\,\kappa_2^3)a_9 =& \ 0. \end{aligned}$$

We suppose that (23) holds, and consequently the origin of the new system is a center. When $a_1^2 + a_2^2 \neq 0$, after the change $x = a_1X - a_2Y, \qquad y = a_2X + a_1Y$, we get that the system has the form of system (16) with $m = 4$, here $\kappa_1 = a_1$ and $\kappa_2 = a_2$ and consequently this system is invariant under the change $(X, Y, t) \longrightarrow (-X, Y, -t)$ i.e. it is reversible. Thus in view of the Poincaré Theorem we get that the origin is a center. Hence system (20) under conditions (37) has a weak center at the origin. Thus the sufficiency under assumption (i) is proved.

When $\kappa_1\kappa_2 \neq 0$ then by solving (23) with respect to κ_1 and κ_2, and if we denote by $\kappa_1 = a_1$ and $\kappa_2 = a_2$ we obtain (22). For the case when $\kappa_2 = 0$ and $k_1 \neq 0$, then from (23) it follows that

$$(24)\qquad a_2 = a_3 = a_4 = a_7 = a_8 = 0.$$

If (24) holds then system (20) becomes

$$\begin{aligned} \dot{x} \ &= -y + x^2\Big(a_1 + a_5y + a_6x^2 + a_9y^2\Big), \\ \dot{y} \ &= x + yx\Big(a_1 + a_5y + a_6x^2 + a_9y^2\Big), \end{aligned}$$

which is invariant under the change $(x, y, t) \longrightarrow (-x, y, -t)$. If $\kappa_1 = 0$ and $k_2 \neq 0$ then from (23) it follows that

$$(25)\qquad a_1 = a_3 = a_4 = a_6 = a_9 = 0.$$

If (25) holds then (20) becomes

$$\begin{aligned} \dot{x} \ &= -y + xy\Big(a_2 + a_5x + a_7y^2 + a_8x^2\Big), \\ \dot{y} \ &= x + y^2\Big(a_2 + a_5x + a_7y^2 + a_8x^2\Big), \end{aligned}$$

which is invariant under the change $(x, y, t) \longrightarrow (x, -y, -t)$.

When $a_1 = a_2 = 0$ and $a_4a_5 \neq 0$, then by taking

$$\kappa_1 = \cos\theta := \frac{\lambda}{\sqrt{1+\lambda^2}} \quad \text{and} \quad \kappa_2 = \sin\theta := \frac{1}{\sqrt{1+\lambda^2}},$$

where λ is a solution of the equation $\lambda^2 - \dfrac{a_5}{a_4}\lambda - 1 = 0$. After the rotation $x = \cos\theta\, X - \sin\theta\, Y, \quad y = \sin\theta\, X + \cos\theta\, Y$, then in view of (23) we get that (20) becomes

$$\begin{aligned}\dot X =&\quad -Y+\frac{1+\lambda^2}{2\lambda}X^2\left(-2a_4Y+\frac{(a_9-3\lambda\, a_7)}{\sqrt{1+\lambda^2}}Y^2+\frac{\lambda^3a_7-\lambda^2\, a_9-2\lambda\, a_7}{\sqrt{1+\lambda^2}}X^2\right),\\ \dot Y =&\quad X+\frac{1+\lambda^2}{2\lambda}XY\left(-2a_4Y+\frac{(a_9-3\lambda\, a_7)}{\sqrt{1+\lambda^2}}Y^2+\frac{\lambda^3a_7-\lambda^2\, a_9-2\lambda\, a_7}{\sqrt{1+\lambda^2}}X^2\right).\end{aligned}$$

Thus this system is invariant under the change $(X, Y, t) \longrightarrow (-X, Y, -t)$, i.e. it is reversible. thus in view of the Poincaré Theorem we get that the origin is a center. Therefore the sufficiency is proved and (ii) holds.

If $a_1 = a_2 = a_3 = a_4 = a_5 = 0$, then system (20) becomes

$$\begin{aligned}\dot x &= -y+x\Big(a_6x^3+a_9xy^2+a_7y^3+a_8x^2y\Big)=-y+x\Omega_3,\\ \dot y &= x+y\Big(a_6x^3+a_9xy^2+a_7y^3+a_8x^2y\Big)=x+y\Omega_3,\end{aligned}$$

By considering that $\displaystyle\int_0^{2\pi}\Omega_3(\cos t, \sin\, t)dt = 0$, then in view of Corollary 4 of [13] we get that the origin is a weak center which in general is not reversible. Thus the sufficiency of the proposition follows.

Necessity in case (i) We shall study only the case (a). The case (b) and (c) can be studied in analogous form. Therefore we assume that $a_1a_2 \neq 0$. Now we suppose that the origin is a center of (20) and we prove that (22) holds. Indeed, from Theorem 10 it follows that differential system (20) can be written as

(26)

$$\begin{aligned}P =&\quad \{H_5,\, x\}+(1+g_1)\{H_4,x\}+(1+g_1+g_2)\{H_3,x\}+(1+g_1+g_2+g_3)\{H_2,x\}\\ =&\quad -y+x\Big(a_1x+a_2y+a_4y^2+a_3x^2+a_5xy+a_6x^3+a_7y^3+a_8x^2y+a_9xy^2\Big),\\ Q =&\quad \{H_5,\, y\}+(1+g_1)\{H_4,y\}+(1+g_1+g_2)\{H_3,y\}+(1+g_1+g_2+g_3)\{H_2,y\},\\ =&\quad x+y\Big(a_1x+a_2y+a_4y^2+a_3x^2+a_5xy+a_6x^3+a_7y^3+a_8x^2y+a_9xy^2\Big)\end{aligned}$$

In view of Corollary 6 and assisted by an algebraic computer we can obtain the solutions of (26), i.e. the homogenous polynomials H_5, H_3, g_1, g_3 of degree odd are unique and the homogenous polynomials H_4, g_2 of degree even are obtained modulo an arbitrary polynomial of the form $c(x^2 + y^2)^k$ where $k = 1, 2$. Indeed taking the homogenous part of these equations of degree two we obtain

$$\begin{aligned}\{H_3,x\}+g_1\{H_2,x\} =&\quad x(a_1x+a_2y),\\ \{H_3,y\}+g_1\{H_2,y\} =&\quad y(a_1x+a_2y).\end{aligned}$$

The solutions of these equations are

$$g_1 = 3(a_1y - a_2x), \qquad H_3 = 2H_2(a_2x - a_1y).$$

The homogenous part of (26) of degree 3 is

$$\begin{aligned}\{H_4,x\}+g_1\{H_3,x\}+g_2\{H_2,x\} =&\quad x(a_4y^2+a_3x^2+a_5xy)=x\Omega_2,\\ \{H_4,x\}+g_1\{H_3,x\}+g_2\{H_2,x\} =&\quad y(a_4y^2+a_3x^2+a_5xy)=y\Omega_2.\end{aligned}\tag{27}$$

The compatibility condition of these two last equations becomes of $\{H_3, g_1\} + \{H_2, g_2\} = 4\Omega_2$, and by considering that $\{H_3, g_1\} = \{H_2, -3(a_2x - a_1y)^2\}$ since

$$\{H_2, g_2 - 3(a_2x - a_1y)^2\} = 4\Omega_2.$$

Hence, in view of proposition 7 we get that

$$\int_0^{2\pi} \Omega_2(\cos t, \sin t)dt = 2\pi(a_3 + a_4) = 0.$$

So $a_3 + a_4 = 0$. Therefore $g_2 = 3(a_2x - a_1y)^2 - a_4xy - 2a_5x^2 + c_1H_2$, where c_1 is a constant. Then from system (27) by considering that H_4 is a homogenous polynomial of degree four we obtain the solution

$$\begin{aligned} H_4 = & -\frac{1}{4}\left(3g_1H_3 + 2g_2H_2\right) + c_1H_2^2 \\ = & \; H_2\left(3\left((a_2^2 - a^2)x^2 - a_1a_2xy\right) + a_5x^2 + 2a_4xy\right) + c_1H_2^2 \end{aligned}$$

Inserting these previous solutions g_1, H_3, g_2 and H_4 into the partial differential equations

$$\begin{aligned} & \{H_5,\, x\} + g_1\{H_4, x\} + g_2\{H_3, x\} + g_3\{H_2, x\} \\ = \; & x(a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2) = x\Omega_3 := X_4, \\ & \{H_5,\, y\} + g_1\{H_4, y\} + g_2\{H_3, y\} + g_3\{H_2, y\} \\ = \; & y(a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2) = y\Omega_3 := Y_4, \end{aligned} \tag{28}$$

we get that these differential equations have a unique solution. Indeed, in this case the compatibility condition is

$$\{H_4, g_1\} + \{H_3, g_2\} + \{H_2, g_3\} = 5\Omega_3, \tag{29}$$

because $\dfrac{\partial X_4}{\partial x} + \dfrac{\partial Y_4}{\partial x} = 5\Omega_3$, and Ω_3 is a homogenous polynomial of degree 3. Consequently there exists a unique solution g_3 of (29) such that

$$\begin{aligned} g_3 := \; & \Big(-6a_2a_1^2 - a_2^3 + \frac{11}{3}a_2a_5 - \frac{5}{3}a_1a_4 - \frac{10}{3}a_7 - \frac{5}{3}a_8\Big)x^3 \\ & + \Big((2a_1^3 - a_1a_2^2)\mu^2 + (8a_1^3 - 2a_1a_2^2 - 2a_1a_5 - a_2a_4 - 4a_1c_1)\mu \\ & + 6a_1^3 + 3a_1a_2^2 - 2a_1a_5 + 9a_2a_4 + 5a_6 - 4a_1c_1\Big)x^2y \\ & + \Big(- a_2a_1^2\mu^2 + (a_1a_4 + 4a_2c_1 + a_1a_4)\mu - 9a_2a_1^2 + 4c_1a_2 - 9a_1a_4 - 5a_7\Big)xy^2 \\ & \Big(\frac{5}{3}a_1^3\mu^2 + \frac{1}{3}(22a_1^3 - 5a_1a_5 - 5a_2a_4 - 4a_1c_1)\mu \\ & + \frac{1}{3}(21a_1^3 + 5a_1a_5 + 5a_2a_4 + 5a_9 + 10a_6 - 12a_1c_1)\Big)y^3, \end{aligned}$$

Thus the homogenous polynomial H_5 can be compute as follows

$$H_5 = -\frac{1}{5}\left(4g_1H_4 + 3g_2H_3 + 2g_3H_2\right),$$

using (28).

Hence partial differential system (28) has a solution if and only if $a_3 + a_4 = 0$. On the other hand from (9) for $m = 4$ and assuming that $a_1a_2 \neq 0$ and denoting by

$$\begin{aligned}
\lambda_1 :=& \quad a_5 - \frac{(a_1^2 - a_2^2)a_4}{a_1a_2},\\
\lambda_2 :=& \quad a_6 - \frac{1}{2a_2^3}\left(a_7(a_1^3 - 3a_2^2a_1) + a_9(a_2^3 - a_1^2a_2)\right),\\
\lambda_3 :=& \quad a_8 - \frac{1}{2a_2^2a_1}\left(a_7(3a_1^3 - 3a_1a_2^2) + a_9(a_2^3 - 3a_1^2a_2)\right).
\end{aligned}$$

From Remak 11 with $m = 4$ we get that

$$\begin{aligned}
I_1 :=& \int_0^{2\pi} (\{H_5,\, g_1\} + \{H_4,\, g_2\} + \{H_3,\, g_3\})|_{x=\cos t, y=\sin t}\, dt\\
=& \quad 3/2\pi\,(2a_1a_2\lambda_1 + 2a_2\lambda_2 - 2a_1\lambda_3) = 0.
\end{aligned}$$

Under this condition the first differential equation of (9) with $m = 4$ becomes

$$\{H_5,\, g_1\} + \{H_4,\, g_2\} + \{H_3,\, g_3\} + \{H_2, g_4\} = 0.$$

It has a solution g_4 which in view of Corollary 6 can be obtained as follows

$$g_4 = G_4(x, y) + 8c_1x(2a_4y + 2a_5x)H_2 + 4c_2H_2^2,$$

where $G_4 = G_4(x, y)$ is a convenient homogenous polynomial of degree four, c_2 is a constant. Using formula (13) with $k = 1$ $X_5 = Y_5 = 0$ we obtain the homogenous polynomial H_6 as follows

$$H_6 = -\frac{5}{6}g_1H_5 - \frac{4}{6}g_2H_4 - \frac{3}{6}g_3H_3 - \frac{2}{6}g_4H_2.$$

By considering that the integral of the homogenous polynomial of degree 5

$$\int_0^{2\pi} (\{H_6,\, g_1\} + \{H_5,\, g_2\} + \{H_4,\, g_3\} + \{H_3,\, g_4\})|_{x=\cos t, y=\sin t}\, dt \equiv 0,$$

then we obtain that there is a unique solution for the homogenous polynomial g_5 of degree 5 of the equation

$$\{H_6,\, g_1\} + \{H_5,\, g_2\} + \{H_4,\, g_3\} + \{H_3,\, g_4\} + \{H_2, g_5\} = 0,$$

which c0mes from the first equation of (11) with $m = 4$ and $j = 1$.

Using formula (13) with $k = 2$ $X_6 = Y_6 = 0$ we obtain the homogenous polynomial H_7 as follows

$$H_7 = -\frac{6}{7}g_1H_5 - \frac{5}{7}g_2H_5 - \frac{4}{7}g_3H_4 - \frac{3}{7}g_4H_3 - \frac{2}{7}g_5H_2$$

and inserting it into the next integral of the homogenous polynomials of degree 6 we get that

$$\begin{aligned}
I_2 :=& \int_0^{2\pi} (\{H_7,\, g_1\} + \{H_6,\, g_2\} + \{H_4,\, g_3\} + \{H_3,\, g_4\})|_{x=\cos t, y=\sin t}\, dt\\
=& \quad \pi\,(\nu_2\lambda_1\lambda_2 + \nu_4\lambda_1 + \nu_5\lambda_2 + \nu_6\lambda_3)\,.
\end{aligned} \tag{30}$$

where

$$
\begin{aligned}
\nu_4 = & -\frac{2\Big(4(a_1a_2)^3+16a_1a_2^5+2a_2^4a_4+(5a_1a_2^2-a_1^3)a_7+(a_1^2a_2-a_2^3)a_9\Big)}{a_2^2},\\
\nu_2 = & -4a_2, \quad \nu_5=\frac{-24a_1^3-88a_1a_2^3-8a_2^2a_4}{a_1}, \quad \nu_6=-8a_2(a_1^2+3a_2^2)
\end{aligned}
$$

By solving $I_1=0$ and $I_2=0$ and assuming that $a_1(4a_2^2+\lambda_1)+2a_2a_4\neq 0$. we get that
(31)

$$
\begin{aligned}
\lambda_2 = & \frac{a_1\lambda_1\left(-4a_1a_2^5-2a_2^4a_4+(a_1^3-5a_1a_2^2)a_7+(a_2^3-a_1^2a_2)a_9\right)}{2a_2^3(a_1(4a_2^2+\lambda_1)+2a_2a_4)},\\
\lambda_3 = & \frac{\lambda_1(-4a_1a_2^5+2a_1a_2^3\lambda_1-2_2^4a_4+(3a_1^3-15a_1a_2^2)a_7+(3a_2^3-3a_1^2a_2)a_9)}{2a_2^3(a_1(4a_2^2+\lambda_1)+2a_2a_4)}.
\end{aligned}
$$

By continuing this process we get that the following relations must hold
(32)

$$
\begin{aligned}
I_3 := & \int_0^{2\pi}(\{H_9,\,g_1\}+\{H_8,\,g_2\}+\{H_7,\,g_3\}+\ldots+\{H_3,\,g_7\})|_{x=\cos t,y=\sin t}\,dt\\
= & \; p(\lambda_1,\lambda_2,\lambda_3)=0,
\end{aligned}
$$

where p is a convenient polynomial of degree five in the variables $\lambda_1,\lambda_2,\lambda_3$. Inserting into I_3 the values of λ_2 and λ_3 from (31) we get that the following relations must hold

$$
\tilde{p}=p(\lambda_1,\lambda_2,\lambda_3)|=\lambda_1\left(e_4\lambda_1^4+e_3\lambda_1^3+e_2\lambda_2^2+e_1\lambda_1+e_0\right)=0, \tag{33}
$$

where

$$
\begin{aligned}
e_4 = & \; 6550\pi\, a_2^4a_1^4,\\
e_3 = & \; 41280\pi\, a_2^4a_1^4c_1+r_0^{(3)},\\
e_2 = & \left(99840\pi\, a_2^4a_1^4\pi\right)c_1^2+r_1^{(2)},\\
e_1 = & \left(10a_2a_1(79872a_1^3a_2^5+3993a_1^2a_2^4a_4)\right)c_1^2+r_1^{(1)},\\
e_0 = & \; \pi\,(20a_1a_2+10a_4)\left(79872a_1^3a_2^7+39936a_1^2a_2^6a_4\right)c_1^2+r_1^{(0)},
\end{aligned} \tag{34}
$$

where $r_j^{(k)}$ is a convenient polynomial of degree j in the variable c_1 for $k=0,1,2,3,$. Now we show that the polynomial $\tilde{p}$ has only one real root. Indeed from the results given in [16] we get that a quartic polynomial with real coefficients $e_4x^4+e_3x^3+e_2x^2+e_1x+e_0$ with $e_4\neq 0$ has four complex roots if
(35)

$$
\begin{aligned}
D_2 = & \; 3e_3^2-8e_2e_4\leq 0,\\
D_4 = & \; 256e_4^3e_0^3-27e_4^2e_1^4-192e_4^2e_1e_0^2e_3-27e_3^4e_0^2-6e_4e_3^2e_0e_3^2+e_2^2e_1^2e_3^2\\
& -4e_4e_2^3e_1^2+18e_2e_3^3e_1e_0+144e_4e_2e_0^2e_3^2-80e_4e_2^2e_0e_3e_1+18e_4e_2e_1^3e_3\\
& -4e_2^3e_0e_3^2-4e_3^3e_1^3+16e_4e_2^4e_0-128e_4^2e_2^2e_0^2+144e_4^2e_2e_0e_1^2>0.
\end{aligned}
$$

After some computations we can prove that for the e_j's given in (34) for $j = 0, 1, 2, 3, 4$ we get that

$$\begin{aligned} D_2 = &\ \Big(-119500800\pi^2 a_1^8 a_2^8 \Big) c_1^2 + q_1^{(2)}, \\ D_4 = &\ \Big(35842867256894590493928960000000\pi^6 a_1^{21} a_2^{27} (2a_1a_2 + a_4)^3 \Big) c_1^9 + q_8^{(4)}, \end{aligned}$$

where $q_j^{(k)}$ is a convenient polynomial of degree j in the variable c_1, for $k = 2, 4$. Taking the arbitrary constant c_1 big enough and such that $a_1a_2(2a_1a_2 + a_4)c_1 > 0$ we obtain that the polynomial $\tilde{p}$ has the unique real root $\lambda_1 = 0$, and consequently $\lambda_2 = \lambda_3 = 0$.

Finally we study the case when $2a_1a_2 + a_4$. By repeating the process of the previous case we finally obtain that from the equations $I_j = 0$ for $j = 1, 2, 3$ we get that

$$\begin{aligned} \lambda_3 = &\ \frac{3a_2}{a_1}\lambda_2, \\ 0 = &\ \lambda_2 \left(174a_2^3\lambda_2 + a_2(87a_1^2 - 29a_2^2)a_9 + a_2(261a_2^2 - 87a_1^2)a_7 \right. \\ &\ \left. + a_2^3 a_1(605a_2^2 - 995a_1^2) + 704a_1a_2^3c_1\right). \end{aligned}$$

By choosing the arbitrary constant properly, we can obtain that the unique solution of $I_j = 0$ for $j = 1, 2, 3$ is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus the origin is a weak center in this particular case. Thus the necessity of the proposition is proved.

We observe that Proposition 16 provides the necessary and sufficient conditions for the existence of quartic uniform isochronous centers. We observe that this problem was study in [4, 1, 2], but in these papers there are some mistakes. For more details see the appendix.

Proposition 16 can be generalized as follows and the proof is similar

Proposition 17. *The fourth polynomial differential system*

$$\begin{aligned} \dot{x} = &\ -y(1 + \mu(a_2x - a_1y)) + x\Big(a_1x + a_3x^2 + a_2y + a_4y^2 \\ &\ + a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2\Big), \\ \dot{y} = &\ x(1 + \mu(a_2x - a_1y)) + y\Big(a_1x + a_2y + a_3x^2 + a_4y^2 \\ &\ + a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2\Big), \end{aligned} \tag{36}$$

where $(\mu + m - 2)(a_1^2 + a_2^2) + a_3^2 + a_4^2 + a_5^2 \neq 0$ has a weak center at the origin if and only if after a linear change of variables $(x, y) \longrightarrow (X, Y)$ it is invariant under the transformations $(X, Y, t) \longrightarrow (-X, Y, -t)$ or $(X, Y, t) \longrightarrow (X, -Y, -t)$. Moreover,

(i) *if $a_1^2 + a_2^2 \neq 0$, then system (36) has a weak center at the origin if and only if*

$$\begin{aligned} a_3 + a_4 = 0, \qquad a_5a_1a_2 + (a_2^2 - a_1^2)a_4 = &\ 0, \\ a_1^3a_7 - a_1^2a_2a_9 + a_1a_2^2a_8 - a_2^3a_6 = &\ 0, \\ 3a_1a_2^2a_7 - 3a_1^2a_2a_6 + (a_1^3 - 2a_1a_2^2)a_8 + (2a_1^2a_2 - a_2^3)a_9 = &\ 0. \end{aligned} \tag{37}$$

Consequently

(a)

$$\begin{aligned} a_3+a_4=0, \qquad a_5+\frac{(a_2^2-a_1^2)}{a_1a_2}a_4 =&\ \ 0,\\ a_6+\frac{1}{2a_2^3}\left(a_7(a_1^3-3a_2^2a_1)+a_9(a_2^3-a_1^2a_2)\right) =&\ \ 0,\\ a_8+\frac{1}{2a_2^2a_1}\left(a_7(3a_1^3-3a_1a_2^2)+a_9(a_2^3-3a_1^2a_2)\right) =&\ \ 0. \end{aligned}$$

when $a_1a_2\neq 0$,

(b) $a_2=a_3=a_4=a_7=a_8=0$, *when* $a_1\neq 0$,

(c) $a_1=a_3=a_4=a_6=a_9=0$, *when* $a_2\neq 0$.

(ii) *If* $a_1=a_2=0$ *and* $a_4a_5\neq 0$ *then system* (36) *has a weak center at the origin if and only if*

$$\begin{aligned} a_3+a_4 =&\ \ 0,\\ \lambda\, a_5+(1-\lambda^2)a_4 =&\ \ 0,\\ \lambda^3a_7-\lambda^2a_9+\lambda\, a_8-a_6 =&\ \ 0,\\ 3\lambda^2a_7+3\lambda a_6+\left(\lambda^3-2\lambda^2\right)a_8+\left(2\lambda^2-1)\right)a_9 =&\ \ 0, \end{aligned}$$

where $\lambda=\dfrac{a_5+\sqrt{4a_4^2+a_5^2}}{2a_4}$. *Moreover the weak center in this case after a linear change of variables* $(x,y)\longrightarrow(X,Y)$ *it is invariant under the transformations* $(X,Y,t)\longrightarrow(-X,Y,-t)$.

(iii) *if* $a_1=a_2=a_3=a_4=a_5=0$, *then the origin is a weak center.*

(iv) $\mu+2=a_3=a_4=a_5=0$, *then the origin is a weak center.*

5. Proof of Theorem 5

The proof of Theorem 5 follows from the next propositions.

Proposition 18. *A cubic polynomial differential system*

$$(38)\qquad \begin{aligned} \dot{x} =&\ \ -y(1+\mu(a_2x-a_1y))+x(a_1x+a_2y+a_3x^2+a_4y^2+a_5xy),\\ \dot{y} =&\ \ x(1+\mu(a_2x-a_1y))+y(a_1x+a_2y+a_3x^2+a_4y^2+a_5xy), \end{aligned}$$

has a weak center at the origin if and only if

$$(39)\qquad a_3+a_4=0, \qquad a_1a_2a_5+(a_2^2-a_1^2)a_4=0,$$

Moreover system (38) *under condition* (39) *and* $(\mu+1)(a_1^2+a_2^2)\neq 0$, *after a linear change of variables* $(x,y)\longrightarrow(X,Y)$ *it is invariant under the transformations* $(X,Y,t)\longrightarrow(-X,Y,-t)$.

Proposition 38 is proved in Proposition 23 of [13].

We give the proof of Proposition 19. The proofs of Propositions 20 and 21 are analogous to the proofs of Proposition 19.

Proposition 19. *A polynomial differential system of degree four*

(40)
$$\begin{aligned}\dot{x} = &\ -y(1+\mu(a_2x-a_1y)) + x\left(a_1x+a_2y+a_4\left(y^2-x^2-\frac{(a_2^2-a_1^2)}{a_1a_2}xy\right)\right.\\ &\left. +a_6x^3+a_7y^3+a_8x^2y+a_9xy^2\right),\\ \dot{y} = &\ x(1+\mu(a_2x-a_1y)) + y\left(a_1x+a_2y+a_4\left(y^2-x^2-\frac{(a_2^2-a_1^2)}{a_1a_2}xy\right)\right.\\ &\left. +a_6x^3+a_7y^3+a_8x^2y+a_9xy^2\right),\end{aligned}$$

where $a_1a_2 \neq 0$ *has a weak center at the origin if and only if the following conditions hold.*

(41)
$$\begin{aligned}\lambda_1 := &\quad a_9 + \frac{1}{2a_2a_1^2}\left((3a_1a_2^2-a_1^3)a_8 + (3a_2^3-3a_2a_1^2)a_6\right),\\ \lambda_2 := &\quad a_7 + \frac{1}{2a_1^3}\left((a_2^3-3a_2a_1^2)a_8 + (a_2^3-3a_2a_1^2)a_6\right)\end{aligned}$$

Moreover system (40) *under conditions* (41) *and after a linear change of variables* $(x,y) \longrightarrow (X,Y)$ *it is invariant under the transformations* $(X,Y,t) \longrightarrow (-X,Y,-t)$.

Proof. Sufficiency: First we observe that the differential system (40) under the linear transformation (14) can be written as (15) with $m = 4$, and

$$\begin{aligned}\Lambda = &\quad \mu(a_2x-a_1y) = 0,\\ \Omega = &\quad a_1x+a_2y+a_4(y^2-x^2-\frac{a_2^2-a_1^2)}{a_1a_2}xy)+a_6x^3+a_7y^3+a_8xy+a_9xy^2 = 0.\end{aligned}$$

This differential system is invariant under the transformation $(X,Y,t) \longrightarrow (-X,Y,-t)$ if and only if

(42)
$$\begin{aligned}\kappa_1a_2-\kappa_2a_1 = &\quad 0,\\ \kappa_1(\kappa_1^2a_7+\kappa_2^2a_8)-\kappa_2(\kappa_2^2a_6+\kappa_1^2a_9) = &\quad 0,\\ 3\kappa_1\kappa_2(a_7\kappa_2-a_6\kappa_1)+\kappa_1(\kappa_1^2-2\kappa_2^2)a_8+\kappa_1(2\kappa_1^2-\kappa_2^2)a_9 = &\quad 0,\end{aligned}$$

We suppose that (41) holds and show that then the origin is a center of system (40). Assume that $a_1a_2 \neq 0$. Then after the transformation

(43)
$$x = a_1X - a_2Y,\ y = a_2X + a_1Y,$$

we get that this system can be written as system (16) for $m = 4$ and with $\kappa_1 = a_1$ and $\kappa_2 = a_2$, and consequently the conditions (42) becomes

$$\begin{aligned}a_1(a_1^2a_7+a_2^2a_8)-a_2(a_2^2a_6+a_1^2a_9) = &\quad 0,\\ 3a_1a_2(a_7a_2-a_6a_1)+a_1(a_1^2-2a_2^2)a_8+a_1(2a_1^2-a_2^2)a_9 = &\quad 0.\end{aligned}$$

By solving these two equations with respect to a_7 and a_9 we get (41). Hence (40) is invariant, after the given linear change (43) is invariant under the transformation $(X,Y,t) \longrightarrow (-X,Y,-t)$, i.e. it is reversible. Thus in view of the Poincaré Theorem we get that the origin is a center of (40) if (41) holds.

Necessity: Now we suppose that the origin is a center of (40) and we prove that (41) holds. Indeed, from Theorem 10 it follows that differential system (40) can be written as

$$\begin{aligned}
&\{H_5,\,x\}+(1+g_1)\{H_4,x\}+(1+g_1+g_2)\{H_3,x\}+(1+g_1+g_2+g_3)\{H_2,x\}\\
=\;& -y+x(a_1x+a_2y+a_4\left(y^2-x^2-\frac{(a_2^2-a_1^2)}{a_1a_2}xy\right)+a_6x^3+a_7y^3+a_8x^2y+a_9xy^2),\\
&\{H_5,\,y\}+(1+g_1)\{H_4,y\}+(1+g_1+g_2)\{H_3,y\}+(1+g_1+g_2+g_3)\{H_2,y\},\\
=\;& x+y(a_1x+a_2y+a_4\left(y^2-x^2-\frac{(a_2^2-a_1^2)}{a_1a_2}xy\right)+a_6x^3+a_7y^3+a_8x^2y+a_9xy^2).
\end{aligned}$$

Hence

$$\begin{aligned}
&\{H_3,x\}+g_1\{H_2,x\}=-y\mu(a_1y-a_2x)+x(a_1x+a_2y)=X_2,\\
&\{H_3,y\}+g_2\{H_2,y\}=x\mu(a_1y-a_2x)+y(a_1x+a_2y)=Y_2,\\
&\{H_4,x\}+g_1\{H_3,x\}+g_2\{H_2,x\}=a_4x\left(y^2-x^2-\frac{(a_2^2-a_1^2)}{a_1a_2}xy\right)\\
=\;& x\Omega_2=X_3,\\
&\{H_4,y\}+g_1\{H_3,y\}+g_2\{H_2,y\}=a_4y\left(y^2-x^2-\frac{(a_2^2-a_1^2)}{a_1a_2}xy\right)\\
=\;& y\Omega_2=Y_3,\\
&\{H_5,\,x\}+g_1\{H_4,x\}+g_2\{H_3,x\}+g_3\{H_2,x\}\\
=\;& x\left(a_6x^3+a_7y^3+a_8x^2y+a_9xy^2\right):=x\Omega_3=X_4,\\
&\{H_5,\,y\}+g_1\{H_4,y\}+g_2\{H_3,y\}+g_3\{H_2,y\}\\
=\;& y\left(a_6x^3+a_7y^3+a_8x^2y+a_9xy^2\right):=y\Omega_3=Y_4,
\end{aligned}\tag{44}$$

The two first equations of (44) are compatible if and only if g_1 satisfies

$$\{H_2,g_1\}=-(\mu-3)(a_1x+a_2y)=\frac{\partial X_2}{\partial x}+\frac{\partial Y_2}{\partial y}.$$

Thus $g_1=-(\mu-3)(a_1y-a_2x)$, and consequently from the first of (44) we obtain that $H_3=-(x^2+y^2)\,(a_1y-a_2x)$

From the third and fourth equations of (44) we get that these equations are compatible if and only if

$$\{H_3,g_1\}+\{H_2,g_2\}=3a_4\left(y^2-x^2-\frac{(a_2^2-a_1^2)}{a_1a_2}xy\right)=\frac{\partial X_3}{\partial x}+\frac{\partial Y_3}{\partial y},$$

and in view of Proposition 7 we get that this equation has the polynomial solution g_2 if and only if

$$\int_0^{2\pi}\left(\{H_3,g_1\}+3a_4\left(y^2-x^2-\frac{(a_2^2-a_1^2)}{a_1a_2}xy\right)\right)\Bigg|_{x=\cos t,\,y=\sin\,t}dt=\quad 0,$$

which holds identically. Thus we obtain the homogenous polynomial

$$\begin{aligned} g_2 = &\left((a_1^2+2a_2^2)(\mu-3a_1a_2)+\frac{(a_1^2-a_2^2)a_4}{a_1a_2}\right)x^2-2(a_1a_2(\mu-3)-2a_4)xy\\ &+\left((a_2^2+2a_1^2)(\mu-3)+\frac{(a_2^2-a_1^2)a_4}{a_1a_2}\right)y^2+c_1H_2, \end{aligned}$$

where c_1 is an arbitrary constant. From (13) with $j=3$ we obtain the homogenous polynomial

$$\begin{aligned} H_4 = &-\frac{1}{4}\left(3g_1H_3+2g_2H_2\right)=c_1H_2^2+\\ &\frac{H_2}{2}\Big(\mu+\frac{1}{a_1a_2}(a_4-3a_1a_2)((a_1-a_2)x+(a_1+a_2)y)((a_1-a_2)y+(a_1+a_2)x)\Big). \end{aligned}$$

From (12) with $j=4$ we compute

$$\{H_4,g_1\}+\{H_3,g_2\}+\{H_2,g_3\}=4\Omega_3=\frac{\partial X_4}{\partial x}+\frac{\partial Y_4}{\partial y}.$$

This last equation has a unique homogenous polynomial solution g_3, which we insert in the expression for H_5 (see (13) with $j=4$) and we obtain

$$H_5=-4g_1H_4/5-3g_2H_3/5-2g_3H_2/5.$$

Hence the homogenous polynomials H_5, H_3, g_1, g_3 are determined and H_4, g_2 are obtained with and arbitrary term of the type $c_k(x^2+y^2)^k$ where $k=1,2$, respectively. On the other hand from (10) with $m=4$ and assuming that $a_1a_2\neq 0$ we get

$$\begin{aligned} I_1 := &\int_0^{2\pi}\left(\{H_5,\,g_1\}+\{H_4,\,g_2\}+\{H_3,\,g_3\}\right)|_{x=\cos t, y=\sin t}\,dt\\ &=3\pi\left(a_2\lambda_1-3a_1\lambda_2\right)=0 \end{aligned}$$

Under this condition the partial differential equation (coming from (12) with $k=5$)

$$\{H_5,\,g_1\}+\{H_4,\,g_2\}+\{H_3,\,g_3\}+\{H_2,g_4\}=0,$$

has a homogenous polynomial solution g_4 which in view of Corollary 6 can be obtained with arbitrary term of the type $c(x^2+y^2)^2$.

The homogenous polynomial H_6 can be determined as follows (see (13) with k=5)

$$H_6=-\frac{5}{6}g_1H_5-\frac{4}{6}g_2H_4-\frac{3}{6}g_3H_3-\frac{2}{6}g_4H_2.$$

Since the integral of the homogenous polynomial of degree 5

$$\int_0^{2\pi}\left(\{H_6,\,g_1\}+\{H_5,\,g_2\}+\{H_4,\,g_3\}+\{H_3,\,g_4\}\right)|_{x=\cos t, y=\sin t}\,dt$$

is zero, we obtain that there is a unique homogenous polynomial g_5 of degree 5 solution of the equation

$$\{H_6,\,g_1\}+\{H_5,\,g_2\}+\{H_4,\,g_3\}+\{H_3,\,g_4\}+\{H_2,g_5\}=0.$$

Calculating the homogenous polynomial of degree 7 (see (13) with $k=6$) we obtain

$$H_7=-\frac{6}{7}g_1H_5-\frac{5}{7}g_2H_5-\frac{4}{7}g_3H_4-\frac{3}{7}g_4H_3-\frac{2}{7}g_5H_2,$$

and inserting it into the integral of the homogenous polynomial of degree 6

$$\begin{aligned} I_2 &:= \int_0^{2\pi} (\{H_7,\, g_1\} + \{H_6,\, g_2\} + \{H_4,\, g_3\} + \{H_3,\, g_4\})|_{x=\cos t, y=\sin t}\, dt \\ &= \pi(\mu-3)\,(\nu_1\lambda_1 + \nu_2\lambda_2) = 0 \end{aligned}$$

where

$$\begin{aligned} \nu_1 &= -\frac{\pi}{42}\Big(\big(4203a_2^3 + 108ca_2\big)\, a_1 - 3255a_1^3a_2 + (157a_1^2 - 489a_2^2)a_4\Big), \\ \nu_2 &= -\frac{\pi}{42}\Big((1401a_2^3 + 36ca_2)a_1 - 2601a_1^3a_2 + (147a_1^2 - 163a_2^2)a_4\Big). \end{aligned}$$

By solving the linear system $I_1 = 0, \quad I_2 = 0$ with respect to λ_1 and λ_2, and by considering that the determinant of the matrix of this system is $\Delta = \dfrac{2\pi^2 a_1^2}{7}(71a_4 - 1137a_1a_2)$. Assuming that $\Delta \neq 0$ we deduce that $\lambda_1 = \lambda_2 = 0$.

The case when $71a_4 - 1137a_1a_2 = 0$ can be analyzed in analogous form. Indeed, by solving $I_j = p_j(\lambda_1, \lambda_2, \lambda_3) = 0$ for $j = 1, 2$ with respect to λ_2, λ_3 we obtain that $\lambda_j = \lambda_j(\lambda_1)$, for $j = 2, 3$ Inserting these expressions into $I_3 = 0$ we get that $\lambda_1\left(e_4\lambda_1^4 + e_3\lambda_1^3 + e_2\lambda_1^2 + e_1\lambda_1 + e_0\right) = 0$, where

$$\begin{aligned} e_4 &= 166446510550a_2^2a_1^2\pi, \\ e3 &= 1048994191680a_1^2a_2^2\pi\, c_1^2 + r_0^{(3)}, \\ e_2 &= 2537102231040a_2^2a_1^2\pi c_1^2 + r_1^{(2)}, \\ e_1 &= 182814295971840a_1^2a_2^4\pi c_1^2 + r_1^{(4)}, \\ e_0 &= 329323217673216a_1^2a_2^6c_1^2 + r_1^{(0)}, \end{aligned}$$

where $r_j^{(n)}$ are convenient polynomials of degree j in the variable c_1. By applying the result given in [16] with D_2 and D_4 given in (35) and choosing the arbitrary constant c_1 conveniently we deduce that the unique real solution of $I_3 = 0$ is $\lambda_1 = 0$. Consequently $\lambda_2 = \lambda_3 = 0$. In short the proposition is proved. □

The following two propositions can be proved in analogous way of the proof of Proposition 19.

Proposition 20. *A polynomial differential system of degree five*

$$\begin{aligned} \dot{x} &= -y(1 + \mu(a_2x - a_1y)) \\ &+x\Big(a_1x + a_2y + +a_4(y^2 - x^2 + \frac{(a_2^2 - a_1^2)}{a_1a_2}xy) + a_6x^3 \\ &+\frac{1}{2a_1^3}((3a_2a_1^2 - a_2^3)a_6 + (a_2^2a_1 - a_1^3)a_8))y^3 + a_8x^2y \\ &+\frac{1}{2a_2a_1^2}(3(a_1^2a_2 - a_2^3)a_6 + (3a_1a_2^2 - a_1^3)a_8))xy^2 \\ &+a_{10}x^4 + a_{11}x^3y + a_{12}x^2y^2 + a_{13}xy^3 + a_{14}y^4\Big), \end{aligned} \tag{45}$$

$$
\begin{aligned}
\dot{y} \ &= x(1+\mu(a_2x-a_1y)) \\
&+y\Big(a_1x+a_2y++a_4(y^2-x^2+\frac{(a_2^2-a_1^2)}{a_1a_2}xy)+a_6x^3 \\
&+\frac{1}{2a_1^3}\left((3a_2a_1^2-a_2^3)a_6+(a_2^2a_1-a_1^3)a_8)\right)y^3+a_8x^2y \\
&+\frac{1}{2a_2a_1^2}\left(3(a_1^2a_2-a_2^3)a_6+(3a_1a_2^2-a_1^3)a_8)\right)xy^2 \\
&+a_{10}x^4+a_{11}x^3y+a_{12}x^2y^2+a_{13}xy^3+a_{14}y^4\Big),
\end{aligned}
$$

where $a_1a_2 \neq 0$ has a weak center at the origin if and only if the following conditions hold

$$
\begin{aligned}
a_{12}+3(a_{10}+a_{14}) = &\ 0, \\
2a_1^3a_2^3a_{13}-\left(a_1^6+7(a_1^2a_2^4-a_1^4a_2^2)\right)a_{10}-\left(a_1^5a_2-4a_1^3a_2^3+a_1a_2^5\right)a_{11} = &\ 0, \\
2a_1^2a_2^2a_{14}-\left(a_1^4-4a_1^2a_2^2+a_2^4\right)a_{10}-(a_1^3a_2-a_1a_2^3)a_{11} = &\ 0.
\end{aligned}
$$

Moreover system (45) under these conditions and after a linear change of variables $(x,y) \longrightarrow (X,Y)$ it is invariant under the transformations $(X,Y,t) \longrightarrow (-X,Y,-t)$.

Proposition 21. *A polynomial differential system of degree six*
(46)

$$
\begin{aligned}
\dot{x} \ &= -y(1+\mu(a_2x-a_1y))+x\Big(a_1x+a_2y++a_4(y^2-x^2+\frac{(a_2^2-a_1^2)}{a_1a_2}xy)+a_6x^3 \\
&+\frac{1}{2a_1^3}((3a_2a_1^2-a_2^3)a_6+(a_2^2a_1-a_1^3)a_8))y^3+a_8x^2y \\
&+\frac{1}{2a_2a_1^2}(3(a_1^2a_2-a_2^3)a_6+(3a_1a_2^2-a_1^3)a_8))xy^2 \\
&+a_{10}x^4+a_{11}x^3y+a_{12}x^2y^2+a_{13}xy^3+a_{14}y^4\Big), \\
\dot{y} \ &= x(1+\mu(a_2x-a_1y))+y\Big(a_1x+a_2y++a_4(y^2-x^2+\frac{(a_2^2-a_1^2)}{a_1a_2}xy)+a_6x^3 \\
&+\frac{1}{2a_1^3}((3a_2a_1^2-a_2^3)a_6+(a_2^2a_1-a_1^3)a_8))y^3+a_8x^2y \\
&+\frac{1}{2a_2a_1^2}(3(a_1^2a_2-a_2^3)a_6+(3a_1a_2^2-a_1^3)a_8))xy^2 \\
&+a_{10}x^4+a_{11}x^3y+a_{12}x^2y^2+a_{13}xy^3+a_{14}y^4\Big),
\end{aligned}
$$

where $a_1a_2 \neq 0$ has a weak center at the origin if and only if the following conditions hold

$$
\begin{aligned}
\lambda_1 = \quad & a_{15}+\frac{1}{8a_1^2a_2^5}\Big(\left(2a_1^5a_2^2-4a_1^3a_2^4+2a_1a_2^6\right)a_{19}-\left(3a_1^7-15a_1^5a_2^2+25a_1^3a_2^4-5a_1a_2^6\right)a_{16} \\
&+\left(a_2^7+11a_1^4a_2^3-9a_1^2a_2^5-3a_1^6a_2\right)a_{20}\Big)=0,
\end{aligned}
$$

$$\begin{aligned}\lambda_2 = \;& a_{17} - \frac{1}{8a_1^3a_2^4}\Big((15a_1^7 - 55a_1^5a_2^2 + 45a_1^3a_2^4 - 5a_1a_2^6)a_{16} + (10a_1^5a_2^2 - 12a_1^3a_2^4 + 2a_1a_2^6)a_{19}\\ & +(-15a_1^6a_2 + 35a_1^4a_2^3 - 13a_1^2a_2^5 + a_2^7)a_{20}\Big) = 0,\\ \lambda_3 = \;& a_{18} + \frac{1}{2a_1^2a_2^3}\Big(- (5a_1^5 - 10a_1^3a_2^2 + 5a_1a_2^4)a_{16}\\ & -(4a_1^3a_2^2 - 2a_1a_2^4)a_{19} - (6a_1^2a_2^3 - 5a_1^4a_2 - a_2^5)a_{20}\Big) = 0.\end{aligned}$$

Moreover system (46) under these conditions and after a linear change of variables $(x, y) \longrightarrow (X, Y)$ it is invariant under the transformations $(X, Y, t) \longrightarrow (-X, Y, -t)$.

Remark 22. *A weak center in general is not invariant with respect to a straight line. Indeed, the cubic Λ-Ω system with a weak center at the origin* [21]

$$\begin{aligned}\dot{x} = \;& -y\left(1 + y + \frac{y^2}{2}\right) + \frac{x}{2}(x - y - y^2),\\ \dot{y} = \;& x\left(1 + y + \frac{y^2}{2}\right) + \frac{y}{2}(x - y - y^2),\end{aligned} \tag{47}$$

is not invariant with respect to the straight line.

□

6. Appendix

The classification of the isochronous centers of Proposition 16 for system (20) has been previously studied in the papers [4] and [2]. But in both papers there are some mistakes.

More precisely, in [4] they write system (20) in in polar coordinates as follows

$$\dot{r} = P_2(\varphi)r^2 + P_3(\varphi)r^3 + P_4(\varphi)r^4, \quad \dot{\varphi} = 1, \tag{48}$$

where

$$\begin{aligned}P_2(\varphi) = \;& R_1\cos\varphi + r_1\sin\varphi,\\ P_3(\varphi) = \;& R_2\cos 2\varphi + r_2\sin 2\varphi,\\ P_4(\varphi) = \;& R_3\cos 3\varphi + r_3\sin 3\varphi + R_4\cos\varphi + r_4\sin\varphi.\end{aligned}$$

We note that in [4] there forgot to write the term $r_1\sin\varphi$. The relations between the parameters of (20) and the parameters of system (48) are

$$\begin{aligned}R_1 = \;& a_1, \quad r_1 = a_2, \quad R_2 = (a_3 - a_4)/2, \quad r_2 = a_5/2,\\ R_0 = \;& (a_3 + a_4)/2, \quad R_3 = (a_6 - a_9)/4, \quad r_3 = (a_8 - a_7)/4,\\ R_4 = \;& (3a_6 + a_9)/4, \quad r_4 = (3a_7 + a_8)/4.\end{aligned}$$

In [2] they write system (20) in complex notation as follows

$$\dot{z} = iz + z\Big(Az + \bar{A}\bar{z} + Bz^2 + 2(b_1 + b_3)z\bar{z} + \bar{B}\bar{z}^2 + Cz^3 + Dz^2\bar{z} + \bar{D}\bar{z}z^2 + \bar{C}\bar{z}^3\Big), \tag{49}$$

being $z = x+iy$, $\bar{z} = x-iy$, $A = (a_1-ia_2)/2$, $B = (b_1+b_3-ib_2)/4$, $C = (d_1-id_2)/8$ and $D = (d_3 - id_4)/8$ where $a_1, a_2, b_1, b_2, b_3, d_1, d_2, d_3, d_4$ are real constants. The relations between the parameters of system (20) and the parameters of system (49) are

$$
\begin{aligned}
a_1 = {} & a_1, \quad a_2 = a_2, \\
a_3 = {} & 5(b_1 + b_3)/2, \quad a_4 = 3(b_1 + b_3)/2, \quad a_5 = b_2, \\
a_6 = {} & (d_3 + d_1)/4, \quad a_7 = (d_4 - d_2)/4, \quad a_8 = (d_4 + 3d_2)/4, \quad a_9 = (d_3 - 3d_1)/4.
\end{aligned}
$$

The following sets of conditions are equivalent

- $r_1 = r_4 = R_0 = R_4 = 0$ and $r_3 \neq 0$ for system (48),
- $a_2 = b_1 + b_3 = d_3 = d_4 = 0$ and $b_2 \neq 0$ for system (49),
- $a_2 = 3a_7 + a_8 = 3a_6 + a_9 = a_3 + a_4 = 0$ and $a_5 \neq 0$ for system (20).

In [4] and [2] they claim that system (20) under the previous conditions has a center, but this is uncorrect because such a system has a week focus due to the fact their Liapunov constants are not all zero. Thus its first non-zero Liapunov constant is $\pi a_1^2 a_3/2$. For more details on Liapunov constants see for instance chapter 5 of [7].

Acknowledgments

The first author is partially supported by a FEDER-MINECO grant MTM2016-77278-P, a MINECO grant MTM2013-40998-P, and an AGAUR grant number 2014SGR-568. The second author was partly supported by the Spanish Ministry of Education through projects TIN2014-57364-C2-1-R, TSI2007-65406-C03-01 "AEGIS".

References

[1] A.Algaba, M.Reyes, T. Ortega and A.Bravo Campos cuárticos con velocidad angular constante, in Actas: XVI CEDYA Congreso de Ecuaciones Diferenciales y Aplicaciones, VI Congreso de matemática Aplicada , Vol. 2. Las Palmas de Gran Canaria 1999, 1341–1348.

[2] A.Algaba and M.Reyes, Computing center conditions for vector fields with constant angular speed, *Journal of Computational and Applied Mathematics,* **154** (2003), 143–159.

[3] J. Artés, J. Itikawa and J. Llibre, Uniform isochronous cubic and quartic centers: Revisited, *Journal of Computational and Applied Mathematics,* **313** (2017), 448-453.

[4] I. J. Chavarriga, I.A.García and J.Giné, On the integrability of differential equations defined by the sum of homogeneous vector fields with degenerate infinity, *International Journal of Bifurcation and Chaos,* 3 (2001), 711–722.

[5] C.B. Collins, Conditions for a centre in a simple class of cubic systems, *Differential and Integral Equations* **10** (1997), 333–356.

[6] R. Conti, Centers of planar polynomial systems. a review, *Le Matematiche*, Vol. LIII, Fasc. II, (1998), 207–240.

[7] F. Dumortier, J. Llibre and J.C. Artés, *Qualitative Theory of Planar Differential Systems*, Universitext, Springer–Verlag, Berlin, 2006.

[8] J. Itikawa and J. Llibre, Phase portraits of uniform isochronous quartic centers, *J. Comp. Appl. Math.* **287** (2015), 98–114.

[9] J. Itikawa and J. Llibre, Global Phase portraits of isochronous centers with quartic homogenous polynomial nonlinearities, *Discrete Contin. Dyn. Syst. Ser. B* **21** (2016), 121–131.

[10] J. Llibre and R. Ramírez, Inverse problems in ordinary differential equations and applications, Progress in Math. **313**, Birkhäuser, 2016.

[11] J. Llibre, R. Ramírez and V. Ramírez, An inverse approach to the center-focus problem for polynomial differential system with homogenous nonlinearities, *J. Differential Equations* **63** (2017), 3327–3369.
[12] J. Llibre, R. Ramírez and V. Ramírez, An inverse approach to the center problem, preprint, (2016)
[13] J. Llibre, R. Ramírez and V. Ramírez, Center problem for $\Lambda - \Omega$ differential systems, preprint, (2017).
[14] M.A. Liapounoff, Problème général de la stabilité du mouvement, Annals of Mathematics Studies **17**, Princeton University Press, 1947.
[15] W.C. Loud Behaviour of the period of solutions of certain of certain autonomous systems near centers, Contributions to Differential equations, **3** (1964), 21–36.
[16] LuYang Recent advances on determining the number of real roots of parametric polynomials, *J. Symbolic Computation* **28** (1999), 225–242.
[17] V.V. Nemytskii and V.V Stepanov, Qualitative Theory of Differential Equations, *Princeton Univ. Press,* 1960.
[18] I.G. Malkin, Stability theory of movements, *Ed. Nauka,* Moscow, (1966) (in Russian).
[19] G. Reeb, Sur certaines propiétés topologiques des variétés feuilletées , W.T. Wu, G. Reeb (Eds.), Sur les espaces fibrés et les variétés feuilletées, Tome XI, in: Actualités Sci. Indust. vol. **1183**, Hermann et Cie, (1952) Paris.
[20] S. Sternberg, Lectures on differential geometry, *Prentice Hall* , 1964.
[21] Z. Zhou, A new method for research on the center-focus problem of differential systems, *Abstract and Applied Analysis* **2014**, (2014), 1–5.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.

E-mail address: jllibre@mat.uab.cat, valentin.ramirez@e-campus.uab.cat

[2] Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Avinguda dels Països Catalans 26, 43007 Tarragona, Catalonia, Spain.

E-mail address: rafaelorlando.ramirez@urv.cat