

ON THE MECHANISMS FOR PRODUCING LINEAR TYPE CENTERS IN POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper we study the different mechanisms that give rise to linear type centers for polynomial differential systems. The known mechanisms are the algebraic reversibility and the Liouville integrability. In this paper are discussed such mechanisms and established some open questions. The known mechanisms for the nilpotent and degenerate centers are also summarized.

1. INTRODUCTION AND PRELIMINARY RESULTS

A *center* for a real analytic differential system in the plane

$$(1) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

is a singularity p for which there exist a neighborhood U such that $U \setminus \{p\}$ is filled with periodic orbits.

A center is of *linear type* if the eigenvalues of its linear part are purely imaginary. Determine linear type centers is a classical problem in the qualitative theory of differential equations, see for instance [22, 33, 34, 36, 45, 47]. The linear type centers of the analytic differential systems (1) are characterized by a theorem of Poincaré-Liapunov [34, 45] which says that a center is of linear type if and only if the system has a non-constant analytic first integral defined in a neighborhood of it.

We say that the differential system (1) is *polynomial* when the functions P and Q are polynomials. The *degree* of a polynomial differential system (1) is the maximum degree of the polynomials P and Q .

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We recall that a function V is an *inverse integrating factor associated to the first integral H* of system (1) if

$$\frac{P}{V} = \frac{\partial H}{\partial y}, \quad \text{and} \quad \frac{Q}{V} = -\frac{\partial H}{\partial x}.$$

We say that a polynomial differential system (1) has a *Liouvillian first integral H* if its associated inverse integrating factor is of the form

$$(2) \quad V = \exp(D/E) \prod C_i^{\alpha_i},$$

where D , E and the C_i are polynomials in $\mathbb{C}[x, y]$ and $\alpha_i \in \mathbb{C}$, see [12, 23, 46, 50]. The functions of the form (2) are called *Darboux functions*. We recall that the Darboux functions essentially are due to the existence of invariant algebraic curves and their multiplicities through the exponential factors. We note that the curves $C_i = 0$ are invariant algebraic curves of the polynomial differential system (1), and that the $\exp(D/E)$ is a product of some exponential factors associated to the invariant algebraic curves of the system or to the invariant straight line at infinity, for more details see [7, 14, 15] or Chapter 8 of [17].

For polynomial differential systems it seems natural to think that the analytic first integral, that exists in a neighborhood of a linear type center, must be of algebraic nature attending to the algebraic nature of the polynomial differential system. For an analytic first integral of algebraic nature we first understand that the first integral is a Darboux function. For some classes of polynomial differential systems of lower degree this is true. For instance any linear center perturbed by quadratic or cubic homogeneous polynomials has a Darboux first integral, see for instance [6, 21, 32, 49] and references therein. However, in general, the analytic first integrals of the linear type centers of cubic polynomial differential systems are not Darboux first integrals, and more general mechanisms for producing centers should be introduced, and of course this is also the case for polynomial differential systems of higher degree, see [13, 16, 28].

One of the most studied polynomial differential equations is the polynomial Liénard equation

$$(3) \quad \ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x)$ and $g(x)$ are polynomials and its generalizations, see for instance [8, 9, 10, 11, 13, 16, 24]. Equation (3) can be rewritten as the differential system in the plane

$$(4) \quad \dot{x} = y, \quad \dot{y} = -g(x) - yf(x).$$

These systems arise frequently in the study of various mathematical models of physical, chemical, biological, physiological, economical and other processes, see for instance [31, 35] and references therein.

For the Liénard differential system (4) Cherkas [8] was the first in give necessary and sufficient conditions for the existence of a linear type centers at the origin. His conditions were improved by Christopher [11]. In fact the Liénard differential systems (4) with a center are time-reversible (see below the definition) through an analytic invertible transformation followed by a scaling of time.

The known centers of the polynomial differential systems (4) and its generalizations [25, 26, 28] arise either from the existence of a Liouvillian first integral or from a simple form of algebraic reducibility (see also definition below). Both mechanisms, as we will see, are of algebraic nature.

The first mechanism is the *Liouvillian integrability*, i.e. the existence of a Liouvillian first integral. However there are linear type centers of polynomial differential systems without a Liouville first integral. A simple example is the polynomial Liénard system

$$(5) \quad \dot{x} = y + x^4, \quad \dot{y} = -x,$$

that has neither any invariant algebraic curve, nor an integrating factor of the form (2) and consequently is not Liouvillian integrable, see [18].

Nevertheless, system (5) is invariant by the symmetry $(x, y, t) \rightarrow (-x, y, -t)$ which implies that its phase portrait is symmetric respect to the y axis and this property is called *time-reversibility*. This simple example leads immediately to the second mechanism to produce centers. This mechanism produces centers by pulling back a non-singular differential system via an algebraic map which allows to obtain a symmetric differential system. For system (5) the map is polynomial $(\bar{x}, \bar{y}) \mapsto (x^2, y)$ and we obtain the nonsingular differential system

$$\dot{\bar{x}} = 2(\bar{y} + \bar{x}^2), \quad \dot{\bar{y}} = -1,$$

that is, a differential system without a singular point at the origin

This second mechanism is called *algebraic reducibility*, see [16]. A polynomial differential system is *algebraically reducible* at a singular point p if it is possible to find a map

$$(x, y) \rightarrow (\bar{x}, \bar{y}) = (f(x, y), g(x, y))$$

with f and g analytic functions (real or complex) in the neighborhood of p , which are also algebraic over $\mathbb{C}(x, y)$ such that the differential

equation $Pdy - Qdx = 0$ associated to system (1) is the pull-back of a differential equation $\bar{P}d\bar{y} - \bar{Q}d\bar{x} = 0$ without singularities around the image of p .

In [16] it is mentioned the algebraic reversible mechanism. A system is *algebraic reversible* if there exists an algebraic map that transforms it into a time-reversible system. In [51, 52] it is introduced the notion of *rational reversibility*, which is a particular case of the algebraic reversibility because in this case the map is assumed to be rational. Nevertheless any algebraic reversible system or rational reversible system is also algebraic reducible, see [16].

These two mechanisms for producing centers, the Liouvillian integrability and the algebraic reducibility, explain all the known linear type centers of the polynomial differential systems studied up to now, see for instance [2, 8, 9, 11, 16, 24, 25, 29, 30, 37, 38, 39, 40, 41, 42, 47, 48] and the references therein. As far as we know does not exist a known linear type center produced by a different mechanism than these two explained mechanisms. Moreover both mechanisms are of algebraic nature because in the first case the system has an integrating factor constructed from invariant algebraic curves and exponential factors, and in the second case the map which appears in the definition of algebraic reducibility is algebraic. This allows us to establish the following conjecture.

Conjecture. *Any center of a polynomial differential system is Liouvillian integrable or algebraically reducible.*

This conjecture was first established in [14] but in terms of a generalized symmetry which is equivalent to say that the system is algebraically reversible. If the conjecture is true then any linear type center would be produced by algebraic mechanisms.

The algebraic nature of the algebraic reducibility, or of the Liouvillian integrability allows to find, in general, the transformation to the non-singular differential system, or the Liouvillian first integral. In both cases these mechanisms provide the sufficient conditions for the existence of a linear type center once we have found the necessary conditions for such an existence.

In fact, from a theoretical point of view, both methods can be unified in a unique mechanism as the following result shows for analytic differential systems. The proof of this result can be seen in [28].

Theorem 1. *Any linear type center of an analytic differential system is analytically reducible.*

The key point in the proof of Theorem 1 consists in writing system (1) into its the Poincaré normal form through an analytic change of variables, that we know that exists for any linear type center. However we only can compute some of the terms of the Taylor series of the analytic change that transforms the original system into its Poincaré normal form.

2. SOME OPEN QUESTIONS

When one does a classification of the linear type centers as we have done in the conjecture it is important to know if there are elements in each class that are not in the other one. In the example of system (5) given above, we know that there are algebraic reducible linear type centers that are not Liouvillian integrable. We do not know an example in the converse sense. Therefore we have the following open question.

Open problem 1. *Are there linear type centers of polynomial differential systems which are Liouvillian integrable but are not algebraically reducible?*

System (1) with a singular point at the origin candidate to be a linear type center can be written into the form

$$(6) \quad \dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

where P and Q are analytic functions without constant and linear terms, i.e. $P = \sum_{i=2}^{\infty} P_i(x, y)$ and $Q = \sum_{i=2}^{\infty} Q_i(x, y)$, where P_i and Q_i are homogeneous polynomials of degree i . We note that taking polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$ system (6) takes the form

$$(7) \quad \dot{r} = \sum_{s=2}^{\infty} f_s(\theta) r^s, \quad \dot{\theta} = 1 + \sum_{s=2}^{\infty} g_s(\theta) r^{s-1},$$

where

$$\begin{aligned} f_i(\theta) &= \cos \theta P_{i-1}(\cos \theta, \sin \theta) + \sin \theta Q_{i-1}(\cos \theta, \sin \theta), \\ g_i(\theta) &= \cos \theta Q_{i-1}(\cos \theta, \sin \theta) - \sin \theta P_{i-1}(\cos \theta, \sin \theta). \end{aligned}$$

We remark that f_i and g_i are homogeneous polynomials of degree i in the variables $\cos \theta$ and $\sin \theta$. In the region $\mathcal{R} = \{(r, \theta) : \dot{\theta} > 0\}$ the differential system (7) is equivalent to the differential equation

$$(8) \quad \frac{dr}{d\theta} = \frac{\sum_{s=2}^{\infty} f_s(\theta) r^s}{1 + \sum_{s=2}^{\infty} g_s(\theta) r^{s-1}}.$$

Since P and Q are analytic functions, we can expand the right-hand side of (8) as an analytic series in r to obtain the differential equation

$$(9) \quad \frac{dr}{d\theta} = \sum_{i=1}^{\infty} a_i(\theta)r^{i+1},$$

whose coefficients are trigonometric polynomials. This reduces the center problem for the planar differential system (6) to the center problem for the class of differential equations (9).

An explicit expression for the first return map of the differential equation (9) is given in [3], see also [5]. This expression is given in terms of the following iterated integrals, of order k ,

$$I_{i_1, \dots, i_k}(a) := \int \cdots \int_{0 \leq s_1 \leq \dots \leq s_k \leq 2\pi} a_{i_k}(s_k) \cdots a_{i_1}(s_1) ds_k \cdots ds_1,$$

where, by convention, for $k = 0$ we assume that this equals 1. Let $\rho(\theta; \rho_0; a)$, $\theta \in [0, 2\pi]$, be the Lipschitz solution of the differential equation (9) corresponding to a sequence $a = (a_1, a_2, \dots)$ of parameters of equation (9) with initial value $\rho(0; \rho_0; a) = \rho_0$. Then $P(a)(\rho_0) := \rho(2\pi; \rho_0; a)$ is the first return map of this differential equation, and in [3, 5] it is proved the following:

Theorem 2. *For sufficiently small initial values ρ_0 the first return map $P(a)$ is an absolute convergent power series $P(a)(\rho_0) = \rho_0 + \sum_{n=1}^{\infty} c_n(a)\rho_0^{n+1}$, where*

$$c_n(a) = \sum_{i_1 + \dots + i_k = n} c_{i_1, \dots, i_k} I_{i_1, \dots, i_k}(a), \quad \text{and}$$

$$c_{i_1, \dots, i_k} = (n - i_1 + 1) \cdot (n - i_1 - i_2 + 1) \cdot (n - i_1 - i_2 - i_3 + 1) \cdots 1.$$

The following definition is given in [4]. Equation (9) determines a *universal center* if for all positive integers i_1, \dots, i_k with $k \geq 1$ the iterated integral $I_{i_1, \dots, i_k}(a) = 0$.

In [27] it is proved that equation (9) with all a_i trigonometric polynomials has a universal center if and only if there are a trigonometric polynomial q and polynomials $p_i \in \mathbb{C}[z]$ for $i \geq 1$ such that

$$(10) \quad \tilde{a}_i = p_i \circ q, \quad 1 \leq i \leq n, \quad \tilde{a}_i(\theta) = \int_0^\theta a_i(s) ds.$$

The conditions (10) are called *composition conditions*. Therefore the universality condition for a center of (9) is equivalent to the composition conditions for (9) of all the trigonometric polynomials a_i . The

composition conditions were defined by first time in [1]. In summary we have the following result proved in [27].

Theorem 3. *Any center of the differential equation (9) is universal if and only if the differential equation (9) satisfies the composition conditions (10).*

In [27] it is also proved that any linear type center of the differential system (6) after an analytic change of variables is a universal center. Moreover, also in [27], that any time-reversible center of the differential system (6) is a universal center.

Additionally in several paper, see [27] and references therein, it is proved that the differential equation (9) have centers which are not universal, and consequently the differential system (6) also has no universal centers. In fact these examples are Liouvillian integrable. As far as we know up to now there are no examples of algebraically reducible centers which are not Liouvillian integrable and that do not satisfy the composition condition. In the next section we will provide an example that satisfies these conditions.

The next open problem is about the relations of the universality condition or composition condition with the other two mechanism to have a center for system (1).

Open problem 2. *Are there linear type centers of polynomial differential systems which satisfy the composition condition but are neither Liouvillian integrable nor algebraic reducible?*

If this open question has a positive answer, the conjecture does not hold, and we will have to add the composition condition as a new mechanism for having linear type centers.

In [51, 52] Żoładek established a conjecture about the linear type centers of the cubic polynomial differential systems that said that any linear type cubic center is Liouvillian integrable or rationally reversible. Moreover in these works he gave the classification of the linear type rationally reversible cubic polynomial differential systems with a linear type center.

Recently Nicklason in [43] has presented a particular cubic polynomial differential system which appears to be a counterexample to the conjecture given by Żoładek in [51, 52].

3. THE EXAMPLE OF NICKLASON

Nicklason in [43] uses the properties of the solutions of the Abel differential equations to investigate the integrability of some cubic polynomial differential systems, and he is able to give some sufficient conditions in order that such systems have a linear type center at the origin. Inside these examples he founded the following one

$$(11) \quad \begin{aligned} \dot{x} &= -y - Ax^2 - xy - Ax^3, \\ \dot{y} &= x + x^2 + (2A - 1)xy - \frac{2}{3}y^2 + 2A(1 - 5A)x^3 + \frac{(2A - 1)}{3}x^2y, \end{aligned}$$

where for the values $A = 0, 1/4$ this differential system is solvable in terms of special functions. Thus, when $A = 0$ we have the differential system

$$(12) \quad \dot{x} = -y - xy, \quad \dot{y} = x + x^2 - xy - \frac{x^2y}{3} - \frac{2y^2}{3},$$

which has the associated differential equation

$$(13) \quad \frac{dy}{dx} = \frac{-3x - 3x^2 + 3xy + x^2y + 2y^2}{3(1+x)y}.$$

Equation (13) can be transformed by a complicate transformation of the dependent and independent variables together with a scaling of time into the Abel differential equation $du/dt = u^3 - 2tu^2$ which is solvable in terms of Airy functions. In fact a first integral for system (12) is given by

$$H = \frac{3^{2/3}(3+x)\text{Ai}(s(x)) + 6(1+x)^{1/3}\text{Ai}'(s(x))}{3^{2/3}(3+x)\text{Bi}(s(x)) + 6(1+x)^{1/3}\text{Bi}'(s(x))},$$

where $s(x) = ((3+x)^2 - 4y)/(4 \cdot 3^{2/3}(1+x)^{2/3})$ and $\text{Ai}(z)$ and $\text{Bi}(z)$ is the pair of linearly independent solutions of the Airy differential equation $w'' = zw$, which have the next integral representation

$$\begin{aligned} \text{Ai}(z) &= \frac{1}{\pi} \int_0^\infty \cos[t^3/3 + zt] dt, \\ \text{Bi}(z) &= \frac{1}{\pi} \int_0^\infty \left(\exp[-t^3/3 + zt] + \sin[t^3/3 + zt] \right) dt. \end{aligned}$$

We claim that *system (12) is not Liouvillian integrable* because the inverse integrating factor associated to the first integral H is not a Darboux function. At the end of this section we shall prove this claim.

On the other hand, Nicklason affirms that system (12) is not included in the classification of the rational reversible linear type centers of the

cubic polynomial differential systems given by Żołądek in [51, 52]. We do not know if the classification of Żołądek is complete. If this is the case this would imply that system (12) would be a counterexample to the Żołądek's conjecture. The question that we want to consider now is if system (12) is a counterexample to our conjecture.

System (12) is a Cherkas system, see [24, 25, 28]. Any Cherkas system can be transformed into a Liénard differential system (4). More precisely, the change $y_1 = y\psi = ye^{-\int_0^x P_2/P_3 dx}$ transforms the system

$$(14) \quad \dot{x} = P_3(x)y, \quad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2,$$

into the Liénard differential system

$$(15) \quad \dot{x} = y_1, \quad \dot{y}_1 = \frac{P_0}{P_3}\psi^2 + \frac{P_1}{P_3}\psi y_1.$$

For system (12) we have $P_3 = -(1+x)$ and $P_2 = 2/3$ and we obtain $\psi = 1/(3+3x)^{2/3}$. By the transformation $y_1 = y\psi$ and the scaling of time $dt/d\tau = -3^{2/3}(1+x)^{5/3}$ the differential system (12) becomes

$$(16) \quad \dot{x} = y_1, \quad \dot{y}_1 = -\frac{x}{3 \cdot 3^{1/3}(1+x)^{4/3}} + \frac{x(3+x)}{3 \cdot 3^{2/3}(1+x)^{5/3}} y_1.$$

Now for the Liénard differential system (4) we define $F = \int_0^x f(s)ds$ and $G = \int_0^x g(s)ds$, and the change of variables $y = Y - F(x)$ transforms system (4) into the differential system

$$(17) \quad \dot{x} = Y - F(x), \quad \dot{Y} = -g(x),$$

which are also called *Liénard differential systems*.

For system (16) the system (17) is

$$(18) \quad \dot{x} = Y - \frac{9+6x+x^2-9(1+x)^{2/3}}{4 \cdot 3^{2/3}(1+x)^{2/3}}, \quad \dot{Y} = -\frac{x}{3 \cdot 3^{1/3}(1+x)^{4/3}}.$$

System (18) has a linear type center at the origin, and consequently it must satisfy the following theorem, see [8, 11].

Theorem 4. *System (17) has a center at the origin if and only if $F(x) = \Phi(G(x))$, for some analytic function Φ with $\Phi(0) = 0$.*

For system (18) we have that $F = G^2 - 3^{2/3}G$.

Following [8, 11], since $2G(x) = g'(0)x^2 + \dots$ we introduce the invertible analytic transformation $u = \sqrt{2G(x)} \operatorname{sgn}(x)$ whose inverse is $x = x(u)$, and system (17) takes the form

$$(19) \quad \dot{u} = \frac{g(x(u))}{u} [Y - F(x(u))], \quad \dot{Y} = -g(x(u)).$$

Since $g(x(u))/u = \sqrt{g'(0)} + \dots$ is non-zero doing a scaling of time we get the system

$$(20) \quad \dot{u} = Y - F(x(u)), \quad \dot{Y} = -u.$$

The corresponding system (20) for the system (18) takes the form

$$(21) \quad \dot{u} = Y - \frac{1}{2}3^{2/3}u^2 - \frac{1}{4}u^4, \quad \dot{Y} = -u,$$

which is a time-reversible system, because system (21) is invariant by the symmetry $(u, Y, t) \rightarrow (-u, Y, -t)$. Consequently system (21) is algebraically reducible via the map $(\bar{x}, \bar{y}) \mapsto (u^2, Y)$. Going back through all the changes done in order to arrive to system (21) we obtain that the origin *system (12) is algebraically reducible*. Therefore, system (12) is not a counterexample to our conjecture.

Now we shall prove the claim that system (12) is not Liouvillian integrable. Note that it is sufficient to prove that system (21) that has not an inverse integrating factor of the form (2). From the results obtained by Odani in [44] system (21) does not have invariant algebraic curves. Hence, if system (21) is Liouvillian integrable, then the only possible inverse integrating factor of the form (2) is an exponential factor of the form $\exp(h)$ with $h \in \mathbb{C}[x, y]$, which must come from the multiplicity of the invariant straight line at infinity, see for more details [15]. But from the definition of inverse integrating factor we have that $\mathcal{X}(\exp(h)) = (\operatorname{div}\mathcal{X})\exp(h)$, where \mathcal{X} is the vector field associated to system (21), that is

$$(22) \quad \left(Y - \frac{1}{2}3^{2/3}u^2 - \frac{1}{4}u^4\right)\frac{\partial h}{\partial u} - u\frac{\partial h}{\partial Y} = -3^{2/3}u - u^3,$$

where we have simplified the common factor $\exp(h)$. Let $h(u, Y) = \sum_{i=0}^N h_i(Y)u^i$, where $h_i(Y) \in \mathbb{C}[Y]$ with $h_N(Y) \neq 0$. Equating the highest degree terms in both sides of the equality (22), we obtain that $Nh_N(Y)u^{N+3} = 0$. This implies $N = 0$ which gives a contradiction. Therefore system (21) does not have any Liouvillian first integral. Consequently, taking into account the changes of variables for going from system (12) to system (21), it follows that system (12) has no Liouvillian first integrals, the claim is proved.

Now we take polar coordinates in system (12) and we construct the associated equation (8). Next we expand the right-hand side of (8) as an analytic series in r to obtain equation (9), which for the system (12)

its first three coefficients are

$$\begin{aligned} a_1(\theta) &= \frac{1}{12}(-3 \cos \theta + 3 \cos(3\theta) - 6 \sin \theta + 2 \sin(3\theta)), \\ a_2(\theta) &= \frac{1}{288}(12 \cos(2\theta) - 12 \cos(6\theta) + \sin(2\theta) - 8 \sin(4\theta) + 5 \sin(6\theta)), \\ a_3(\theta) &= \frac{1}{6912}(-234 \cos \theta - 36 \cos(3\theta) + 156 \cos(5\theta) + 105 \cos(7\theta) \\ &\quad + 9 \cos(9\theta) - 100 \sin \theta + 24 \sin(3\theta) + 136 \sin(5\theta) \\ &\quad - 34 \sin(7\theta) - 46 \sin(9\theta)). \end{aligned}$$

Denote by

$$\tilde{a}_i(\theta) := \int_0^\theta a_i(s) ds.$$

Then the iterated integral

$$I_{13}(a) = \int_{0 \leq s_1 \leq s_2 \leq 2\pi} a_1(s_2) a_3(s_1) ds_2 ds_1 = \int_0^{2\pi} \tilde{a}_3(s) a_1(s) ds = \frac{11\pi}{864}.$$

Therefore the center of system (12) is not universal, or which is equivalent, system (12) does not satisfy the composition conditions, see Theorem 3. Summarizing we have proved the following result.

Theorem 5. *The polynomial differential system (12) is algebraically reducible, but it is not Liouvillian integrable, and it does not satisfy the composition conditions.*

In the case that $A = 1/4$ the differential system (11) takes the form

$$(23) \quad \dot{x} = -\frac{x^2}{4} - \frac{x^3}{4} - y - xy, \quad \dot{y} = x + x^2 - \frac{x^3}{8} - \frac{xy}{2} - \frac{x^2y}{6} - \frac{2y^2}{3},$$

In this case we take as a new variable $z = x^2 + 4y$ and system (23) becomes

$$(24) \quad \dot{x} = -\frac{1}{4}(1+x)z, \quad \dot{z} = 4x + 4x^2 - xz - \frac{x^2z}{3} - \frac{z^2}{6}.$$

System (24) is also a Cherkas system that can be transformed into a Liénard system (4). Following the same steps that for the case $A = 0$ we obtain that this case is also algebraically reducible.

4. NILPOTENT AND DEGENERATE CENTERS

The nilpotent and degenerate center problem is much more difficult than the linear type center problem, see [19, 20, 36]. First it is not true that any nilpotent and degenerate center has an analytic first

integral defined around the origin, consequently Theorem 1 is not true for such centers. In any case the two mechanisms described for linear type centers, Liouvillian integrability and algebraic reducibility, can also be applied to obtain nilpotent and degenerate centers. The unique difference is that the Liouvillian first integral can be non-analytic. See for instance some examples in [20] and references therein. However, as far as we know all the known nilpotent and degenerate center satisfy our conjecture. Hence we extend the conjecture to any center of a polynomial differential system (1).

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