

# ON THE INTEGRABILITY OF THE HAMILTONIAN SYSTEMS WITH HOMOGENEOUS POLYNOMIAL POTENTIALS

JAUME LLIBRE<sup>1</sup> AND XIANG ZHANG<sup>2</sup>

ABSTRACT. We summarize the known results on the integrability of the complex Hamiltonian systems with two degrees of freedom defined by the Hamiltonian functions of the form

$$H = \frac{1}{2} \sum_{i=1}^2 p_i^2 + V(q_1, q_2),$$

where  $V(q_1, q_2)$  are homogeneous polynomial potentials of degree  $k$ .

## 1. INTRODUCTION

In the theory of ordinary differential equations and in particular in the theory of Hamiltonian systems the existence of first integrals is important, because they allow to lower the dimension where the Hamiltonian system is defined. Furthermore, if we know a sufficient number of first integrals, these allow to solve the Hamiltonian system explicitly, and we say that the system is integrable. Almost until the end of the 19th century the major part of mathematicians and physicians believe that the equations of classical mechanics were integrable, and that to find their first integrals was mainly a computational problem. Now we know that the integrability is a rare phenomenon, and that in general it is not easy to know when a given Hamiltonian system is or not integrable.

The objective of this paper is to summarize the results that are known on the integrability of the complex Hamiltonian systems defined by the Hamiltonian functions

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2),$$

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where  $V(q_1, q_2)$  are complex homogeneous polynomial potentials of degree  $k$  in the variables  $q_1, q_2$ . That is, we work with the Hamiltonian systems of two degrees of freedom

$$(1) \quad \dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1, 2.$$

For  $\mathbf{p} = (p_1, p_2)$  and  $\mathbf{q} = (q_1, q_2)$  we define the *Poisson bracket* of the functions  $A = A(\mathbf{q}, \mathbf{p})$  and  $B = B(\mathbf{q}, \mathbf{p})$  by

$$\{A, B\} = \sum_{i=1}^2 \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

If  $\{A, B\} = 0$  then we say that the functions  $A$  and  $B$  are *in involution*.

A *first integral* for the Hamiltonian system (1) is a non-locally constant function  $F = F(\mathbf{q}, \mathbf{p})$  in involution with the Hamiltonian function  $H$ , because on the orbits  $(\mathbf{q}(t), \mathbf{p}(t))$  of the Hamiltonian system (1) we have

$$\begin{aligned} \frac{d}{dt} F(\mathbf{q}(t), \mathbf{p}(t)) &= \sum_{i=1}^2 \left( \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \right) \\ &= \sum_{i=1}^2 \left( \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \{H, F\}. \end{aligned}$$

Note that  $H$  itself is always a first integral because  $\{H, H\} = 0$

Two functions  $H$  and  $F$  from  $\mathbb{C}^4$  to  $\mathbb{C}$  are *functionally independent* when their gradients are linearly independent at all points of  $\mathbb{C}^4$  except perhaps in a zero Lebesgue measure set.

Here a Hamiltonian system (1) with two degrees of freedom is *integrable* if it has two functional independent first integrals  $H$  and  $F$ . This definition of integrability restricted to real Hamiltonian systems coincides with the Liouville integrability, see for instance [1, 2].

First we summarize the classification of all complex Hamiltonian systems (1) with homogeneous polynomial potentials of degree  $k \in \{-2, -1, 0, 1, 2, 3, 4\}$ , which are integrable with meromorphic first integrals. As we shall see for all these Hamiltonian systems, except for the ones with potential of degree  $-2$ , the meromorphic first integral independent of the Hamiltonian can be chosen polynomial.

We recall that a *meromorphic function* is defined to be locally a quotient of two holomorphic functions, and that a *holomorphic function* is a complex valued function that is complex differentiable in a neighborhood of every point in its domain.

After we summarize the results on the Hamiltonian systems (1) with homogeneous polynomial potentials of degree  $-3$  which are integrable with polynomial first integrals.

Finally we present the results on the integrability of the Hamiltonian systems (1) with the so called exceptional homogeneous polynomial potentials of degree  $k > 4$ .

As far as we know at this moment it is an open question to provide a complex Hamiltonian system (1) with a homogeneous polynomial potential of degree  $k > 0$  which is integrable with meromorphic first integrals, and such that it has no polynomial first integrals independent of the Hamiltonian.

## 2. EQUIVALENT POTENTIALS

The group of  $2 \times 2$  complex matrices  $A$  satisfying  $AA^T = \alpha \text{Id}$  being  $\text{Id}$  the identity matrix and  $\alpha \in \mathbb{C} \setminus \{0\}$ , is denoted by  $\text{PO}_2(\mathbb{C})$ .

If there is a matrix  $A \in \text{PO}_2(\mathbb{C})$  satisfying  $V_1(\mathbf{q}) = V_2(A\mathbf{q})$ , then we say that the two potentials  $V_1(\mathbf{q})$  and  $V_2(\mathbf{q})$  are *equivalent*. Consequently we can divide the potentials into equivalent classes. From now on a potential means a class of equivalent potentials.

The motivation of this definition of equivalent potentials is due to the following result (for a proof see [10]).

**Proposition 1.** *Let  $V_1$  and  $V_2$  be two equivalent potentials. If the Hamiltonian system (1) with the potential  $V_1$  is integrable, then the Hamiltonian system (1) is also integrable with the potential  $V_2$ .*

## 3. MORALES–RUIZ AND RAMIS RESULTS

The integrable Hamiltonian systems (1) with homogeneous polynomial potentials of degrees 1, 2, 3, 4 and 5 having a second polynomial first integral up to degree 4 in the variables  $p_1$  and  $p_2$  were computed at the beginning of 80's, see [4, 5, 7, 8, 20] and also [9]. The main tools for proving those results were Painlevé test [6] and direct methods [10]. But, of course, the limitation that we only consider first polynomial first integrals and second up to degree 4 in the variables  $p_1$  and  $p_2$ ,

do not guarantee that all the integrable Hamiltonian systems (1) with homogeneous polynomial potentials of degrees 1, 2, 3, 4 and 5 have been obtained.

The first good approach for obtaining all the integrable Hamiltonian systems (1) with homogeneous polynomial potentials was due to Yoshida [22]. Later on his results were improved by Morales–Ruiz and Ramis. In order to present the results of these last authors we need the following definitions.

Let  $V(\mathbf{q})$  be a homogeneous polynomial potential of degree  $k$ , and let  $\mathbf{q}^*$  be a solution of  $\left(\frac{dV(\mathbf{q})}{dq_1}, \frac{dV(\mathbf{q})}{dq_2}\right) = \mathbf{q}$ , and let  $\lambda$  and  $-1$  the eigenvalues of the Hessian of  $V(\mathbf{q})$  at  $\mathbf{q}^*$ . It is known that  $-1$  is always an eigenvalue of that Hessian, see for instance [21].

Morales–Ruiz and Ramis (see the page 100 of the book [19] and the references quoted there) provided the following result on the integrability of the complex Hamiltonian systems with homogeneous polynomial potentials. This result provides the necessary condition for the integrability of such systems being the first integrals meromorphic functions.

**Theorem 2.** *If the Hamiltonian system (1) with the homogeneous potential of degree  $k$  is meromorphically integrable, then the pair  $(k, \lambda)$  belongs to one of the following list:*

$$\begin{aligned} & (k, p + kp(p - 1)/2), & (2, \lambda), \\ & (-2, \lambda), & (-5, 49/50 - 5(1 + 3p)^2/18), \\ & (-5, 49/50 - (2 + 5p)^2/10), & (-4, 9/8 - 2(1 + 3p)^2/9), \\ & (-3, 25/24 - (1 + 3p)^2/6), & (-3, 25/24 - 3(1 + 4p)^2/32), \\ & (-3, 25/24 - 3(1 + 5p)^2/50), & (-3, 25/24 - 3(2 + 5p)^2/50), \\ & (3, -1/24 + (1 + 3p)^2/6), & (3, -1/24 + 3(1 + 4p)^2/32), \\ & (3, -1/24 + 3(1 + 5p)^2/50), & (3, -1/24 + 3(2 + 5p)^2/50), \\ & (4, -1/8 + 2(1 + 3p)^2/9), & (5, -9/40 + 5(1 + 3p)^2/18), \\ & (5, -9/40 + (2 + 5p)^2/10), & (k, ((k - 1)/k + p(p + 1)k)/2), \end{aligned}$$

where  $p$  is an integer.

#### 4. HOMOGENEOUS POLYNOMIAL POTENTIALS OF DEGREE 3

Using Theorem 2 Maciejewski and Przybylska [16] found all Hamiltonian systems (1) with homogeneous polynomial potentials of degree

Case	Potentials of degree 3
$V_1$	$q_1^3$
$V_2$	$q_1^3/3 + cq_2^3/3$
$V_3$	$aq_1^3/3 + q_1^2q_2/2 + q_2^3/6$
$V_4$	$q_1^2q_2/2 + q_2^3$
$V_5$	$\pm 7q_1^3i/15 + q_1^2q_2/2 + q_2^3/15$
$V_6$	$q_1^2q_2/2 + 8q_2^3/3$
$V_7$	$\pm 17\sqrt{14}q_1^3i/90 + q_1^2q_2/2 + q_2^3/45$
$V_8$	$\pm\sqrt{3}q_1^3i/18 + q_1^2q_2/2 + q_2^3$
$V_9$	$\pm 3\sqrt{3}q_1^3i/10 + q_1^2q_2/2 + q_2^3/45$
$V_{10}$	$\pm 11\sqrt{3}q_1^3i/45 + q_1^2q_2/2 + q_2^3/10$

TABLE 1. Non-equivalent integrable homogeneous potentials of degree 3. Here  $i = \sqrt{-1}$ .

3 which could have a meromorphic first integral independent of the Hamiltonian. Then, using the polynomial first integrals found by Hietarinta [9], they noted that each of such Hamiltonian systems had a polynomial first integral independent of the Hamiltonian. Consequently they characterized all Hamiltonian systems (1) with homogeneous polynomial potentials of degree 3 having a meromorphic first integral independent of the Hamiltonian. Their characterization is given in Table 1. This table only provides the non-equivalent homogeneous potentials of degree 3 for which the Hamiltonian systems (1) are integrable with meromorphic first integrals.

Llibre and Valls in [14] using the Kowalevskaya theory of integrability developed by Yoshida [21] recomputed the polynomial first integrals of the potentials of Table 1. These polynomial first integrals are given in Table 2.

## 5. HOMOGENEOUS POLYNOMIAL POTENTIALS OF DEGREE 4

Later on in [17] Maciejewski and Przybylska almost classified the integrable Hamiltonian systems with a homogeneous polynomial potential of degree 4 having a second meromorphic first integral independent of the Hamiltonian. More precisely, they proved that except for

Potential	First integral
$V_0$	$F_0 = p_1 - p_2i,$
$V_1$	$F_1 = 9p_1^2i + 6p_1p_2 + 3p_2^2i - 16aq_1^3 + 24aq_1^2q_2i + 8aq_2^3i,$
$V_2$	$F_2 = -9p_1^2i + 6p_1p_2 - 3p_2^2i - 16aq_1^3 - 24aq_1^2q_2i - 8aq_2^3i,$
$V_3$	$F_3 = p_1 + p_2i,$
$V_4$	$F_4 = p_2,$
$V_5$	$F_5 = 3p_1^2 + 2q_1^3,$
$V_6$	$F_6 = -8p_1p_2q_1 + 8p_1^2q_2 - q_1^2(q_1^2 + 4q_2^2),$
$V_7$	$F_7 = 72p_1^4 - 36p_1p_2q_1^3 - 3q_1^6 + 2(3p_2^2 + 16q_2^3)(3p_2^2 + 6q_1^2q_2 + 16q_2^3) + 12p_1^2(3p_2^2 + 12q_1^2q_2 + 16q_2^3),$
$V_8^\pm$	$F_8^\pm = -q_1^6 \pm 6\sqrt{3}q_1^5q_2i + 27q_1^4q_2^2 \pm 6\sqrt{3}q_1^3(p_1^2 + p_2^2 + 2q_2^3)i + 54q_1^2q_2(p_1^2 + p_2^2 + 2q_2^3) + 27(p_1^2 + p_2^2 + 2q_2^3)^2.$

TABLE 2. All non-equivalent integrable Hamiltonian systems (1) having homogeneous polynomial potentials of degree 3 with their polynomial first integrals independent of the Hamiltonian.

the family of potentials

$$(2) \quad V = \frac{1}{2}aq_1^2(q_1 + iq_2)^2 + \frac{1}{4}(q_1^2 + q_2^2)^2,$$

only the Hamiltonian systems with potentials  $V_i$  for  $i = 0, 1, \dots, 8$  given in Table 3 are the non-equivalent integrable homogeneous potentials of degree 4. As for the potentials of degree 3 they used Theorem 2 for finding the Hamiltonian systems (1) with the potentials  $V_j$  for  $j = 0, 1, \dots, 8$  of Table 3 which could have a meromorphic first integral independent of the Hamiltonian. And they checked in the literature that all such Hamiltonian systems have a polynomial first integral independent of the Hamiltonian, so these systems have a meromorphic first integral independent of the Hamiltonian.

In [11] Llibre, Mahdi and Valls completed the classification of Maciejewski and Przybylska proving that for the family (2) only the potentials  $V_9$  and  $V_{10}$  of Table 3 are integrable, they also find polynomial first integrals for these two potentials independent of the Hamiltonians, see Table 4.

Case	Potential of degree 4
$V_i$	$\alpha(q_2 - iq_1)^i(q_2 + iq_1)^{4-i}$ for $i = 0, 1, 2, 3, 4$ .
$V_5$	$\alpha q_2^4$
$V_6$	$\alpha q_1^4/4 + q_2^4$
$V_7$	$4q_1^4 + 3q_1^2 q_2^2 + q_2^4/4$
$V_8$	$2q_1^4 + 3q_1^2 q_2^2/2 + q_2^4/4$
$V_9$	$(q_1^2 + q_2^2)^2/4$
$V_{10}$	$-q_1^2(q_1 + iq_2)^2 + (q_1^2 + q_2^2)^2/4$

TABLE 3. Non-equivalent integrable homogeneous potentials of degree 4.

Potential	First integral
$V_0$	$F_0 = p_1 - p_2 i,$
$V_1$	$F_1 = 2p_1 p_2 - 2p_2^2 i + \alpha q_1^4 i + 6\alpha q_1^2 q_2^2 i + 8\alpha q_1 q_2^3 - 3\alpha q_2^4 i,$
$V_2$	$F_2 = p_2 q_1 - p_1 q_2,$
$V_3$	$F_3 = 2p_1 p_2 + 2p_2^2 i - \alpha q_1^4 i - 6\alpha q_1^2 q_2^2 i + 8\alpha q_1 q_2^3 + 3\alpha q_2^4 i,$
$V_4$	$F_4 = p_1 + p_2 i,$
$V_5$	$F_5 = p_2^2 + 2\alpha q_2^4,$
$V_6$	$F_6 = p_2^2 + 2q_2^4,$
$V_7$	$F_7 = -p_1 p_2 q_2 + p_2^2 q_1 - 2q_1^3 q_2^2 - q_1 q_2^4,$
$V_8$	$F_8 = p_1^4 + 2p_1^2 p_2^2 + 8p_1^2 q_1^4 + 6p_1^2 q_1^2 q_2^2 + 4p_1 p_2 q_1 q_2^3 + 8p_2^2 q_1^4$ $+ 16q_1^8 + 24q_1^6 q_2^2 + 12q_1^4 q_2^4 + 2q_1^2 q_2^6,$
$V_9$	$F_9 = p_1 q_2 - p_2 q_1,$
$V_{10}$	$F_{10} = p_1^2 + 3p_1 p_2 i - 2p_2^2 + (q_1 - q_2 i)^3 q_2.$

TABLE 4. All non-equivalent integrable Hamiltonian systems (1) having homogeneous polynomial potentials of degree 4 with their polynomial first integral independent of the Hamiltonian.

The authors using the Kowalevskaya theory of integrability have re-computed the polynomial first integrals which appear in Table 4 corresponding to the potentials of Table 3.

6. HOMOGENEOUS POLYNOMIAL POTENTIALS OF DEGREES  $-2$ ,  $-1$ ,  
 $0$ ,  $1$  AND  $2$

In this section we show that all the Hamiltonian systems (1) having homogeneous polynomial potentials of degree  $-2$ ,  $-1$ ,  $0$ ,  $1$  and  $2$  are integrable. Moreover, all except some of the potentials of degree  $-2$  have a polynomial first integral independent of the Hamiltonian. Thus we have that the Hamiltonian systems (1) with homogeneous potentials  $V$  of degrees  $-2$ ,  $-1$ ,  $0$ ,  $1$  and  $2$  have the following first integrals  $F$  independent of the Hamiltonian:

$$\begin{aligned} V &= 1/(aq_1^2 + bq_1q_2 + cq_2^2), & F &= (q_1p_2 - q_2p_1)^2/2 + (q_1^2 + q_2^2)V, \\ V &= 1/(aq_1 + bq_2), & F &= ap_2 - bp_1, \\ V &= a, & F &= p_1, \\ V &= aq_1 + bq_2, & F &= ap_2 - bp_1, \\ V &= aq_1^2 + bq_1q_2 + cq_2^2, & F &= b^2q_1^2 + 4bcq_1q_2 + (b^2 + 4c^2 - 4ac)q_2^2 \\ & & & - 2(a-c)p_2^2 + 2bp_1p_2. \end{aligned}$$

where  $a, b, c \in \mathbb{C}$  and  $V \neq 0$ .

Note that the Hamiltonian systems (1) with potentials of degree  $-1$ ,  $0$ ,  $1$  and  $2$  have a polynomial first integral  $F$  independent of the Hamiltonian. This is not the case in general for the potentials of degree  $-2$ .

To study the integrability of the Hamiltonian systems (1) with the homogeneous potentials of degree  $-2$

$$\frac{1}{aq_1^2 + bq_1q_2 + cq_2^2} \quad \text{with } a, b, \text{ or } c \text{ nonzero,}$$

is equivalent to study the integrability of the Hamiltonian systems (1) with the homogeneous potentials  $1/(aq_1^2 + cq_2^2)$ . Moreover, these last Hamiltonian systems are integrable with a polynomial first integral independent of the Hamiltonian if and only if either  $c = 0$ , or  $c \neq 0$  and  $a \in \{0, c\}$ , and this first integral is  $p_2$  if  $c = 0$ ;  $p_1$  if  $a = 0$  and  $q_1p_2 - q_2p_1$  if  $a = c$ .

**Remark 3.** *Note that the potentials of degree  $-2$  show that there are Hamiltonian systems (1) which are integrable with meromorphic first integrals, but not with two independent polynomial first integrals, as it was the case for the potentials of degree  $-1$ ,  $0$ ,  $1$ ,  $2$ ,  $3$  and  $4$ .*

The rational first integral  $F$  for the potentials of degree  $-2$  can be found in the papers of Borisov, Kilin, and Mamaev [3] and Maciejewski,



Przybylska and Yoshida [18] of the years 2009 and 2010 respectively. The remainder results for the potentials of degree  $-2$ ,  $-1$ ,  $0$ ,  $1$  and  $2$  can be found in the paper of Llibre, Mahdi and Valls in [12] of the year 2011.

## 7. HOMOGENEOUS POLYNOMIAL POTENTIALS OF DEGREES $-3$

In this section we present the results on the integrability of the Hamiltonian systems (1) with homogeneous potentials of degree  $-3$

$$(3) \quad V = \frac{1}{aq_1^3 + bq_1^2q_2 + cq_1q_2^2 + dq_2^3},$$

such that  $aq_1^3 + bq_1^2q_2 + cq_1q_2^2 + dq_2^3 \neq 0$ . These results were obtained by Llibre, Mahdi and Valls in [13].

At this moment the characterization of the integrable Hamiltonian systems (1) with homogeneous polynomial potentials of degree  $-3$  with meromorphic first integrals is unknown. What is done in [13] is the characterization of these Hamiltonian systems having a polynomial first integral independent of the Hamiltonian.

In [13] the authors first reduce the study of the existence or non-existence of polynomial first integrals of the Hamiltonian systems (1) with homogeneous polynomial potential (3) to study the following seven Hamiltonian systems (1) with the potentials

$$\begin{aligned} V_0 &= \frac{1}{q_1^3}; & V_1 &= \frac{1}{aq_1^3 + q_2^3}; & V_{2,3} &= \frac{1}{(q_2^2 + q_1^2)(q_2 \pm iq_1)}; \\ V_{4,5} &= \frac{1}{(q_2 \pm iq_1)^3}; & V_{gen} &= \frac{1}{aq_1^3 + q_1^2q_2 + dq_2^3}. \end{aligned}$$

We say that a Hamiltonian system (1) is *polynomially integrable* if it has a polynomial first integral independent of the Hamiltonian.

The Hamiltonian system (1) with the potential:

- (a)  $V_{gen}$  is not polynomially integrable;
- (b)  $V_0$  is polynomially integrable with the polynomial first integral  $p_2$ ;
- (c)  $V_1$  is polynomially integrable if and only if  $a = 0$ , in which case the polynomial first integral is  $p_1$ ;
- (d)  $V_2$  or  $V_3$  are not polynomially integrable;
- (e)  $V_4$  is polynomially integrable with the polynomial first integral  $p_1 - p_2i$ ; and

- (f)  $V_5$  is polynomially integrable with the polynomial first integral  $p_1 + p_2^i$ .

## 8. THE EXCEPTIONAL HOMOGENEOUS POLYNOMIAL POTENTIALS

The potentials  $V_j$  for  $j = 0, 1, 2, 3, 4$  of Table 3 for homogeneous polynomial potentials of degree 4, can be considered for every homogeneous polynomial potentials of degree  $k > 4$ , i.e. we define the potentials

$$(4) \quad V = V_m = \alpha(q_2 - iq_1)^l(q_2 + iq_1)^{k-l}, \quad m = 0, 1, \dots, k, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

These potentials are called *exceptional*.

Hietarinta proved in [10] that the Hamiltonian systems (1) with the exceptional potentials  $V_0, V_1, V_{k-1}, V_k$  and  $V_{k/2}$  when  $k$  is even are integrable. Thus for these exceptional potentials the polynomial first integrals independent of the Hamiltonian are

$$\begin{aligned} F_0 &= p_1 - ip_2, & F_1 &= k(p_1 - ip_2)^2 - 4\alpha(q_2 + iq_1)^k, \\ F_{k-1} &= k(p_1 + ip_2)^2 - 4\alpha(q_2 - iq_1)^k, & F_k &= p_1 + ip_2, \end{aligned}$$

and when  $k$  is even

$$F_{k/2} = q_2p_1 - q_1p_2.$$

Maciejewski and Przybylska in [17] and Hietarinta in [10] claimed that nothing was known about the integrability of the remaining exceptional potentials. Llibre and Valls in [15] proved that the Hamiltonian systems (1) with exceptional homogeneous polynomial potential  $V_m$ , for  $m = 2, \dots, k/2 - 1, k/2 + 1, \dots, k - 2$ , of degree  $k \geq 6$  even do not admit an analytic first integral independent of Hamiltonian. Consequently, they do not admit a polynomial first integral independent of the Hamiltonian.

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<sup>1</sup> DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, BARCELONA, CATALONIA, SPAIN

*E-mail address:* jllibre@mat.uab.cat

<sup>2</sup> SCHOOL OF MATHEMATICAL SCIENCES, MOE–LSC, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI, 200240, P. R. CHINA

*E-mail address:* xzhang@sjtu.edu.cn