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## Research Article

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## The Picard group of Brauer-Severi varieties

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**Abstract:** In this paper, we provide explicit generators for the Picard groups of cyclic Brauer-Severi varieties defined over the base field. In particular, we provide such generators for all Brauer-Severi surfaces. To produce these generators we use the theory of twists of smooth plane curves.

**Keywords:** Brauer-Severi varieties, Picard groups, Cyclic Algebras, Theory of Twists, Automorphisms

**MSC:** 11G35, 14J26, 11D41, 14J50, 14J70, 11R34

## 1 Introduction

Let  $B/k$  be a Brauer-Severi variety over a perfect field  $k$ , that is, a projective variety of dimension  $n$  isomorphic over  $\bar{k}$  to  $\mathbb{P}_k^n$ . The Picard  $\text{Pic}(B)$  is known to be isomorphic to  $\mathbb{Z}$ . As far as we know, the first explicit equations defining a non-trivial Brauer-Severi surface in the literature are in [1]. After this, an algorithm to compute these equations for any Brauer-Severi variety is given in [9]. In the appendix, we explain an alternative way to compute them for the case of dimension 2 by using twists of smooth plane curves.

In this note, we show an explicit concrete generator of the Picard group of any Brauer-Severi variety corresponding to a cyclic algebra in its class inside the Brauer group  $\text{Br}(k)$  of  $k$ . In particular, for a Brauer-Severi surface  $B$  and any integer  $r \geq 1$ , we obtain a generator for  $r \text{Pic}(B)$  from twists of a Fermat type smooth plane curve, see Theorem 4.2. Moreover, we can write equations in  $\mathbb{P}^9$  as follows.

**Theorem 1.1.** *Let  $B$  be the Brauer-Severi surface corresponding to a cyclic algebra  $(L/k, \sigma, a)$  of dimension 3<sup>2</sup> as in Theorem 2.5. A smooth model of  $B$  inside  $\mathbb{P}_k^9$  is given by the intersection  $\cap_{\tau \in \text{Gal}(L/k)} \tau X$  where  $X/L$  is the variety in  $\mathbb{P}_L^9$  defined by the set of equations:*

$$\begin{aligned} a^2(l_1\omega_0 + l_2\omega_6 + l_3\omega_9)(l_3\omega_0 + l_1\omega_6 + l_2\omega_9)^2 &= (l_3\omega_1 + l_1\omega_5 + l_2\omega_7)^3 \\ a(l_1\omega_1 + l_2\omega_5 + l_3\omega_7)(l_3\omega_0 + l_1\omega_6 + l_2\omega_9)^2 &= (l_3\omega_1 + l_1\omega_5 + l_2\omega_7)^2(l_3\omega_2 + l_1\omega_3 + l_2\omega_8) \\ a(l_1\omega_2 + l_2\omega_3 + l_3\omega_8)(l_3\omega_0 + l_1\omega_6 + l_2\omega_9)^2 &= (l_3\omega_1 + l_1\omega_5 + l_2\omega_7)^2(l_3\omega_0 + l_1\omega_6 + l_2\omega_9) \\ a(l_2\omega_2 + l_3\omega_3 + l_1\omega_8)(l_3\omega_0 + l_1\omega_6 + l_2\omega_9)^2 &= (l_3\omega_1 + l_1\omega_5 + l_2\omega_7)(l_3\omega_2 + l_1\omega_3 + l_2\omega_8)^2 \\ \omega_4(l_3\omega_0 + l_1\omega_6 + l_2\omega_9)^2 &= (l_3\omega_1 + l_1\omega_5 + l_2\omega_7)(l_3\omega_2 + l_1\omega_3 + l_2\omega_8)(l_3\omega_0 + l_1\omega_6 + l_2\omega_9) \\ a(l_2\omega_0 + l_3\omega_6 + l_1\omega_9)(l_3\omega_0 + l_1\omega_6 + l_2\omega_9)^2 &= (l_3\omega_2 + l_1\omega_3 + l_2\omega_8)^3 \\ (l_2\omega_1 + l_3\omega_5 + l_1\omega_7)(l_3\omega_0 + l_1\omega_6 + l_2\omega_9)^2 &= (l_3\omega_2 + l_1\omega_3 + l_2\omega_8)^2(l_3\omega_0 + l_1\omega_6 + l_2\omega_9), \end{aligned}$$

and  $\{l_1, l_2, l_3\}$  is a non-zero trace (that is, the number  $l_1 + l_2 + l_3 \neq 0$ ) normal basis of  $L$ . Its Picard group  $\text{Pic}(B)$  is generated by the intersection of the hyperplane

$$\omega_0 + \omega_6 + \omega_9 = 0$$

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with  $B$ , which is a genus 1 curve over  $k$ . More generally, for a positive element  $r \in \mathbb{Z} \simeq \text{Pic}(B)$ , we have a generator of  $r \text{Pic}(B)$  given by the intersection of

$$(l_1\omega_0 + l_2\omega_6 + l_3\omega_9)^r + (l_2\omega_0 + l_3\omega_6 + l_1\omega_9)^r + (l_3\omega_0 + l_1\omega_6 + l_2\omega_9)^r = 0,$$

with  $B$ . It defines a curve of genus  $\frac{(3r-1)(3r-2)}{2}$  over  $k$ . More precisely, it is  $k$ -isomorphic to a twist of the Fermat type curve  $X^{3r} + a^r Y^{3r} + a^{2r} Z^{3r} = 0$ .

Several people worked on finding (or trying to find) equations for Brauer-Severi varieties: using ideas of Châtelet (cf. [7, 11]), the Grothendieck descent (cf. [6]), Grassmanians (cf. [5]) and special embeddings in the projective space (cf. [2, §5.2], [9]). All these constructions lack in how to explicitly construct subvarieties of codimension 1 inside them, that is, elements of their Picard group. Accordingly, we are motivated to find curves' equations for generators of the subgroups  $r \text{Pic}(B)$ . This is what we do in Theorem 1.1 when  $B$  is a Brauer-Severi surface, and in Theorem 6.2 for higher dimensional Brauer-Severi varieties (at least for the ones associated to cyclic algebras).

The key idea of this paper is inspired by [1, 10] and the theory of twists, where any fixed twist of a smooth plane curve is embedded into a certain Brauer-Severi surface that becomes trivial ( $k$ -isomorphic to  $\mathbb{P}^2$ ) if and only if that twist has a smooth plane model over  $k$ .

In general, we attach to a cocycle  $\xi \in H^1(k, \text{PGL}_{n+1}(\bar{k}))$  coming from a cyclic algebra, a Brauer-Severi variety of dimension  $n$  together with a codimension 1 subvariety living inside it (in the case  $n = 2$ , this subvariety is a twist of a smooth plane curve).

Here we consider any Brauer-Severi variety  $B$  associated to some cyclic algebra and then we determine a family of smooth hypersurfaces such that some of their twists are embedded into  $B$ . We start by choosing the automorphism group properly (a cyclic group of automorphisms of specified shapes, related to the cyclic algebra we already have). This in turns allows us to conclude that certain twists produce generators for the subgroups  $r \text{Pic}(B)$ .

We obtain explicit equations for the Brauer-Severi varieties  $B$  and for the aforementioned generators in Section 5, 6 and 7. The difference between the approaches in the different sections is the map we use in Galois cohomology transporting the cocycle  $\xi$  to a trivial cocycle in another Galois cohomology set. In Section 5 and 6, we use the Veronese embedding while in Section 7 we use the canonical embedding corresponding to a certain smooth plane curve  $C$  such that we can see  $\xi \in H^1(k, \text{Aut}(C))$ .

## 2 Brauer-Severi varieties

**Definition 2.1.** Let  $V$  be a smooth quasi-projective variety over  $k$ . A variety  $V'$  defined over  $k$  is called a twist of  $V$  over  $k$  if there is a  $\bar{k}$ -isomorphism  $V' \times_k \bar{k} \xrightarrow{\phi} \bar{V} := V \times_k \bar{k}$ . The set of all twists of  $V$  modulo  $k$ -isomorphisms is denoted by  $\text{Twist}_k(V)$ , whereas the set of all twists  $V'$  of  $V$  over  $k$ , such that  $V' \times_k K$  is  $K$ -isomorphic to  $V \times_k K$  is denoted by  $\text{Twist}(V, K/k)$ .

**Theorem 2.2** ([12, Chp. III, §1.3]). Following the above notations, for any Galois extension  $K/k$ , there exists a bijection

$$\begin{aligned} \theta : \text{Twist}(V, K/k) &\rightarrow H^1(\text{Gal}(K/k), \text{Aut}_K(V \times_k K)) \\ V' \times_k K \cong V \times_k K &\mapsto \xi(\tau) := \Phi \circ \tau \Phi^{-1} \end{aligned}$$

where  $\text{Aut}_K(\cdot)$  denotes the group of  $K$ -automorphisms of the object over  $K$ .

For  $K = \bar{k}$ , the right hand side will be denoted by  $H^1(k, \text{Aut}_{\bar{k}}(\bar{V}))$  or simply  $H^1(k, \text{Aut}(\bar{V}))$ .

**Definition 2.3.** A Brauer-Severi variety  $B$  over  $k$  of dimension  $n$  is a twist of  $\mathbb{P}_k^n$ . The set of all isomorphism classes of Brauer-Severi varieties of dimension  $n$  over  $k$  is denoted by  $\text{BS}_n^k$ .

**Corollary 2.4** ([6, Corollary 4.7]). *The set  $BS_n^k$  is in bijection with  $\text{Twist}_k(\mathbb{P}_k^n) = H^1(k, \text{PGL}_{n+1}(\bar{k}))$ .*

### 2.1 Brauer-Severi surfaces

Let  $L/k$  be a Galois cyclic cubic extension and let  $\sigma$  be a fixed generator of the Galois group  $\text{Gal}(L/k)$ . Given  $a \in k^*$ , we may consider a  $k$ -algebra  $(L/k, \sigma, a)$  as follows: As an additive group,  $(L/k, \sigma, a)$  is a 3-dimensional vector space over  $L$  with basis  $1, e, e^2$ :  $(L/k, \sigma, a) := L \oplus Le \oplus Le^2$ . Multiplication is given by the relations:  $e \cdot \lambda = \sigma(\lambda) \cdot e$  for  $\lambda \in L$ , and  $e^3 = a$ . The algebra  $(L/k, \sigma, a)$  is called the *cyclic algebra* associated to  $\sigma$  and the element  $a \in k^*$ .

**Theorem 2.5.** *Any non-trivial Brauer-Severi surface  $B$  over  $k$  corresponds, modulo  $k$ -isomorphism, to a cyclic algebra of dimension 9 of the form  $(L/k, \sigma, a)$ , for some Galois cubic extension  $L/k$  and  $a \in k^*$  which is not a norm of an element of  $L$ . If  $\text{Gal}(L/k) = \langle \sigma \rangle$ , then the image of  $B$  in  $H^1(k, \text{PGL}_3(\bar{k}))$  is given by*

$$\xi(\sigma) = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Moreover, the Brauer-Severi surface attached to  $(L/k, \sigma, a) \in H^1(k, \text{PGL}_3(\bar{k}))$  is trivial if and only if  $a$  is the norm of an element of  $L$ .

The above theorem (Theorem 2.5) can be concluded from the fact that  $H^1(k, \text{PGL}_n(\bar{k}))$  is in correspondence with the set  $Az_n^k$  of central simple algebras of dimension  $n^2$  over  $k$ , modulo  $k$ -isomorphisms [13, Chp. X.5], the fact that  $Az_3^k$  contains only cyclic algebras [15], and the description of cyclic central simple algebras given in [2, Construction 2.5.1 and Proposition 2.5.2] (see also [14, Example 5.5]). For the last statement, we refer to [3, §2.1].

## 3 Smooth plane curves

Fix an algebraic closure  $\bar{k}$  of a perfect field  $k$ . By a smooth plane curve  $C$  over  $k$  of degree  $d \geq 3$ , we mean a curve  $C/k$ , which is  $k$ -isomorphic to the zero-locus in  $\mathbb{P}_k^2$  of a homogenous polynomial equation  $F_C(X, Y, Z) = 0$  of degree  $d$  with coefficients in  $k$ , that has no singularities. In that case, the geometric genus of  $C$  equals  $g = \frac{1}{2}(d - 1)(d - 2)$ . Assuming that  $d \geq 4$ , the base extension  $C \times_k \bar{k}$  admits a unique  $g_d^2$ -linear system up to conjugation in  $\text{Aut}(\mathbb{P}_{\bar{k}}^2) = \text{PGL}_3(\bar{k})$ . It induces a unique embedding  $\gamma_{\bar{k}} : \bar{C} \rightarrow \mathbb{P}_{\bar{k}}^2$ , up to  $\text{PGL}_3(\bar{k})$ -conjugation, giving a  $\text{Gal}(\bar{k}/k)$ -equivariant map  $\text{Aut}(\bar{C}) \rightarrow \text{PGL}_3(\bar{k})$ .

**Theorem 3.1** (Ro e-Xarles, [10]). *Let  $C$  be a curve over  $k$  such that  $\bar{C} = C \times_k \bar{k}$  is a smooth plane curve over  $\bar{k}$  of degree  $d \geq 4$ . Let  $\gamma_{\bar{k}} : \bar{C} \rightarrow \mathbb{P}_{\bar{k}}^2$  be a morphism given by (the unique)  $g_d^2$ -linear system over  $\bar{k}$ , then there exists a Brauer-Severi surface  $B$  defined over  $k$ , together with a  $k$ -morphism  $f : C \rightarrow B$  such that  $f \times_k \bar{k} : \bar{C} \rightarrow \mathbb{P}_{\bar{k}}^2$  is equal to  $\gamma_{\bar{k}}$ .*

In [1] we constructed twists of smooth plane curves over  $k$  not having smooth plane model over  $k$ . These twists happened to be contained in non-trivial Brauer-Severi surfaces as in Theorem 3.1.

**Theorem 3.2** ([1, Theorem 3.1]). *Given a smooth plane curve  $C \subseteq \mathbb{P}_k^2$  over  $k$  of degree  $d \geq 4$ , there exists a natural map*

$$\Sigma : H^1(k, \text{Aut}(C)) \rightarrow H^1(k, \text{PGL}_3(\bar{k})),$$

moreover  $\Sigma^{-1}([\mathbb{P}_k^2])$  is the set of twists for  $C$  that admit a smooth plane model over  $k$ . Here  $[\mathbb{P}_k^2]$  denotes the trivial class associated to the trivial Brauer-Severi surface of the projective plane over  $k$ .

**Remark 3.3** ([1, Remark 3.2]). *We can reinterpret the map  $\Sigma$  in Theorem 3.2 as the map that sends a twist  $C'$  to the Brauer-Severi variety  $B$  in Theorem 3.1.*

These results suggest the opposite question; instead of giving the curve  $C$  and the twist  $C'$ , and then finding the Brauer-Severi surface  $B$ , fix the Brauer-Severi surface  $B$  and try to find the right curve  $C$  and the right twist  $C'$  in order to establish the  $k$ -morphism  $f : C' \rightarrow B$ .

The main idea is to look for smooth plane curves  $C$  of degree divisible by 3, otherwise all their twists are smooth plane curves over  $k$  by [1, Theorem 2.6], and that have an automorphism of the form  $[aZ : X : Y]$ . Next, to consider the twist  $C'$  given by the cocycle defining  $B$ , which sends a certain generator  $\sigma$  of the degree 3 cyclic extension  $L/k$  to the automorphism  $[aZ : X : Y]$ .

**Lemma 3.4.** *For any  $a \in k^*$  and  $r \in \mathbb{Z}_{\geq 1}$ , the equation  $X^{3r} + a^r Y^{3r} + a^{2r} Z^{3r} = 0$ , defines a smooth plane curve  $C_a^r$  over  $k$  of degree  $3r$ , such that  $[aZ : X : Y]$  is an automorphism.*

## 4 The Picard group

**Theorem 4.1** (Lichtenbaum, see [2, Theorem 5.4.10]). *Let  $B$  be a Brauer-Severi variety over  $k$ . Then, there is an exact sequence*

$$0 \rightarrow \text{Pic}(B) \rightarrow \text{Pic}(B \times_k \bar{k}) \xrightarrow{\text{deg}} \mathbb{Z} \xrightarrow{\delta} \text{Br}(k).$$

The map  $\delta$  sends 1 to the Brauer class corresponding to  $B$ .

**Theorem 4.2.** *Let  $B$  be a non-trivial Brauer-Severi surface over  $k$ , associated to a cyclic algebra  $(L/k, \sigma, a)$  of dimension 9 by Theorem 2.5. For any integer  $r \geq 1$ , there is a twist  $C'$  over  $k$  of the smooth plane curve  $C_a^r$ , that lives inside  $B$  and also defines a generator of  $r \text{Pic}(B)$ .*

*Proof.* We conclude by the virtue of Theorem 3.2 and Remark 3.3 that the twist  $C'$  of  $C_a^r$  given by the inflation map of the cocycle

$$\xi(\sigma) = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in H^1(\text{Gal}(L/k), \text{Aut}(\bar{C}_a^r))$$

as in Theorem 2.5 lives inside  $B$  for any integer  $r \geq 2$ . For  $r = 1$ , set  $\text{Aut}_{\text{inv}}(C_a^1)$  for the subgroup of automorphisms of  $C_a^1$ , leaving invariant the equation  $X^3 + aY^3 + a^2Z^3 = 0$ . Therefore, the inclusions  $\text{Aut}_{\text{inv}}(C_a^1) \leq \text{PGL}_3(\bar{k})$  and  $\text{Aut}_{\text{inv}}(C_a^1) \leq \text{Aut}(\bar{C})$  give us the two natural maps  $\text{inv} : H^1(k, \text{Aut}_{\text{inv}}(C_a^1)) \rightarrow H^1(k, \text{Aut}(\bar{C}_a^1))$ , and  $\Sigma : H^1(k, \text{Aut}_{\text{inv}}(C_a^1)) \rightarrow H^1(k, \text{PGL}_3(\bar{k}))$  respectively. Second, compose with the 3-Veronese embedding  $\text{Ver}_3 : \mathbb{P}_k^2 \rightarrow \mathbb{P}_k^9$  to obtain a model of  $C$  inside the trivial Brauer-Severi surface  $\text{Ver}_3(\mathbb{P}_k^2)$ . Because the image of any 1-cocycle by the map  $\tilde{\text{Ver}}_3 : H^1(k, \text{PGL}_3(\bar{k})) \rightarrow H^1(k, \text{PGL}_{10}(\bar{k}))$ , is equivalent to a 1-cocycle with values in  $\text{GL}_{10}(\bar{k})$ , see [9], and since  $H^1(k, \text{GL}_{10}(\bar{k})) = 1$ , then  $\tilde{\text{Ver}}_3([B])$  is given in  $H^1(k, \text{PGL}_{10}(\bar{k}))$  by  $\tau \in \text{Gal}(\bar{k}/k) \mapsto M \circ {}^\tau M^{-1}$ , for some  $M \in \text{GL}_{10}(\bar{k})$ . Consequently,  $(M \circ \text{Ver}_3)(\mathbb{P}_k^2)$  is a model of  $B$  in  $\mathbb{P}_k^9$ , containing  $(M \circ \text{Ver}_3)(C_a^1)$  inside, which is a twist of  $C_a^1$  over  $k$  associated to  $\xi$  by Theorem 2.5.

On the other hand, due to the results of Wedderburn in [15] and Theorem 4.1, the map  $\delta$  sends 1 to the Brauer class  $[B]$  of  $B$  inside the 3-torsion  $\text{Br}(k)[3]$  of the Brauer group  $\text{Br}(k)$  of the field  $k$ . Hence  $[B]$  has exact order 3, being non-trivial, and so  $\text{Pic}(B)$  inside  $\text{Pic}(B \times_k \bar{k} = \mathbb{P}_k^2) \xrightarrow{\text{deg}} \mathbb{Z}$  is isomorphic to  $3\mathbb{Z}$ . Moreover,  $C' \times_k \bar{k} \subseteq \mathbb{P}_k^2$  has degree  $3r$ , hence it corresponds to the ideal  $\langle 3r \rangle \subset \mathbb{Z}$  via the degree map. Consequently, the image of  $C'$  in  $\text{Pic}(B)$  is a generator of  $r \text{Pic}(B)$ . □

### 5 The proof of Theorem 1.1

Let  $B$  be the Brauer-Severi surface corresponding to  $(L/k, \sigma, a)$  as in Theorem 2.5. Then, there is an isomorphism  $\bar{\phi} : B \times_k L \rightarrow \mathbb{P}_L^2$  defined over  $L$  such that

$$\xi(\sigma) = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \bar{\phi} \cdot \sigma \bar{\phi}^{-1}.$$

The results in [9] would be applied to get the equations in the statement of Theorem 1.1 for  $B$  inside  $\mathbb{P}^9$ . We recall that the equations are obtained by twisting the image of  $\mathbb{P}^2$  into  $\mathbb{P}^9$  by the Veronese embedding  $\text{Ver}_3 : \mathbb{P}^2 \rightarrow \mathbb{P}^9$ . Indeed, we can compute following [9]

$$\text{Ver}_3(\bar{\phi}) = \begin{pmatrix} a^2l_1 & 0 & 0 & 0 & 0 & 0 & a^2l_2 & 0 & 0 & a^2l_3 \\ 0 & al_1 & 0 & 0 & 0 & al_2 & 0 & al_3 & 0 & 0 \\ 0 & 0 & al_1 & al_2 & 0 & 0 & 0 & 0 & al_3 & 0 \\ 0 & 0 & al_2 & al_3 & 0 & 0 & 0 & 0 & al_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_3 & 0 & 0 & 0 & l_1 & 0 & l_2 & 0 & 0 \\ al_2 & 0 & 0 & 0 & 0 & 0 & al_3 & 0 & 0 & al_1 \\ 0 & l_2 & 0 & 0 & 0 & l_3 & 0 & l_1 & 0 & 0 \\ 0 & 0 & l_3 & l_1 & 0 & 0 & 0 & 0 & l_2 & 0 \\ l_3 & 0 & 0 & 0 & 0 & 0 & l_1 & 0 & 0 & l_2 \end{pmatrix} : B \times_k L \rightarrow \mathbb{P}_L^2 \subseteq \mathbb{P}^9,$$

where  $L = k(l_1, l_2, l_3)$  with  $\sigma(l_1) = l_2$  and  $\sigma(l_2) = l_3$ .

On the other hand, the twist  $\phi : C' \rightarrow C'_a$  given by the previous cocycle is embedded in  $B$ : we have the  $k$ -morphism  $f : C' \rightarrow B$  arisen from Theorem 3.1 through the  $L$ -morphism  $\bar{\phi}^{-1} \circ \gamma_L \circ \phi \times_k L$ . Composing with  $\text{Ver}_3$  we get the equations of  $C'$  inside  $\mathbb{P}^9$  in the statement of Theorem 1.1.

Finally, the claim about the order of the curves  $C'$  in  $\text{Pic}(B)$  follows from Theorem 4.2.

### 6 Generalizations on Picard group elements for cyclic Brauer-Severi varieties

Let  $L/k$  be a Galois cyclic extension of degree  $n + 1$  and fix a generator  $\sigma$  for  $\text{Gal}(L/k)$ . Given  $a \in k^*$ , one considers a  $k$ -algebra  $(L/k, \sigma, a)$  as follows: As an additive group,  $(L/k, \sigma, a)$  is an  $(n + 1)$ -dimensional vector space over  $L$  with basis  $1, e, \dots, e^n$ :  $(L/k, \sigma, a) := \oplus_{i=0}^n Le^i$  with  $1 = e^0$ . Multiplication is given by the relations:  $e \cdot \lambda = \sigma(\lambda) \cdot e$  for  $\lambda \in L$ , and  $e^{n+1} = a$ . The algebra  $(L/k, \sigma, a)$  is called the *cyclic algebra* associated to  $\sigma$  and the element  $a \in k$ . It is trivial if and only if  $a$  is a norm of certain element of  $L$ . Its class in  $H^1(k, \text{PGL}_{n+1}(\bar{k}))$  corresponds to the inflation of the cocycle in  $H^1(\text{Gal}(L/k), \text{PGL}_{n+1}(L))$  given by

$$\xi(\sigma) = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & a \\ 1 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 \end{pmatrix}.$$

For more details, one may read [2, Construction 2.5.1 and Proposition 2.5.2].

**Lemma 6.1.** *For any  $a \in k^*$  and  $r \in \mathbb{Z}_{\geq 1}$ , the equation*

$$\sum_{i=0}^n a^{ir} X_i^{(n+1)r} = 0$$

defines a non-singular  $k$ -projective model  $X_a^{r,n}$  of degree  $(n + 1)r$  of a smooth projective variety inside  $\mathbb{P}_k^n$ , such that  $A_a := [aX_n : X_0 : \dots : X_{n-1}]$  is leaving invariant  $X_a^{r,n}$ .

**Theorem 6.2.** *Let  $B$  be a Brauer-Severi variety over  $k$ , associated to a cyclic algebra  $(L/k, \sigma, a)$  of dimension  $(n + 1)^2$  and exact order  $n + 1$  in  $\text{Br}(k)$ . For any integer  $r \geq 1$ , there is a twist  $X'$  over  $k$  of  $X_a^{r,n}$ , living inside  $B$  and defining a generator of  $r \text{Pic}(B)$ .*

*Proof.* Set  $m = \binom{2n+1}{n} - 1$  and denote by  $\text{Aut}_{\text{inv}}(X_a^{r,n})$  the subgroup of automorphisms of  $X_a^{r,n}$ , leaving invariant its defining equation. Therefore, the inclusions  $\text{Aut}_{\text{inv}}(X_a^{r,n}) \leq \text{PGL}_{n+1}(\bar{k})$  and  $\text{Aut}_{\text{inv}}(X_a^{r,n}) \leq \text{Aut}(X_a^{r,n} \times_k \bar{k})$  give us the two natural maps  $\text{inv} : H^1(k, \text{Aut}_{\text{inv}}(X_a^{r,n})) \rightarrow H^1(k, \text{Aut}(X_a^{r,n} \times_k \bar{k}))$ , and  $\Sigma : H^1(k, \text{Aut}_{\text{inv}}(X_a^{r,n} \times_k \bar{k})) \rightarrow H^1(k, \text{PGL}_{n+1}(\bar{k}))$  respectively. Compose with the  $n$ -Veronese embedding  $\text{Ver}_n : \mathbb{P}_k^n \rightarrow \mathbb{P}_k^m$ , to obtain a model of  $X_a^{r,n}$  inside the trivial Brauer-Severi variety  $\text{Ver}_n(\mathbb{P}_k^n)$ . Because the image of a 1-cocycle under the map  $\tilde{\text{Ver}}_n : H^1(k, \text{PGL}_{n+1}(\bar{k})) \rightarrow H^1(k, \text{PGL}_{m+1}(\bar{k}))$  is equivalent to a 1-cocycle with coefficients in  $\text{GL}_{m+1}(\bar{k})$  by [9] and since  $H^1(k, \text{GL}_{m+1}(\bar{k})) = 1$ , then  $\tilde{\text{Ver}}_n([B])$  is given in  $H^1(k, \text{PGL}_{m+1}(\bar{k}))$  by  $\tau \in \text{Gal}(\bar{k}/k) \mapsto M \circ {}^\tau M^{-1}$ , for some  $M \in \text{GL}_{m+1}(\bar{k})$ . Consequently,  $(M \circ \text{Ver}_n)(\mathbb{P}_k^n)$  is a model of  $B$  in  $\mathbb{P}_k^m$ , that contains  $(M \circ \text{Ver}_n)(X_a^{r,n})$  inside, which is a twist of  $X_a^{r,n}$  over  $k$  associated to  $\xi : \sigma \mapsto A_a$ .

On the other hand, by Theorem 4.1, the map  $\delta$  sends 1 to the Brauer class  $[B]$  of  $B$  inside the  $(n + 1)$ -torsion  $\text{Br}(k)[n + 1]$  of the Brauer group  $\text{Br}(k)$  of the field  $k$ . Hence  $[B]$  has exact order  $n + 1$ , being non-trivial, and so  $\text{Pic}(B)$  inside  $\text{Pic}(B \times_k \bar{k} = \mathbb{P}_k^m) \cong \mathbb{Z}$  is isomorphic to  $(n + 1)\mathbb{Z}$ . Moreover,  $X' \times_k \bar{k} \subseteq \mathbb{P}_k^m$  has degree  $(n + 1)r$ , hence it corresponds to the ideal  $\langle (n + 1)r \rangle \subset \mathbb{Z}$  via the degree map. Consequently, the image of  $X'$  in  $\text{Pic}(B)$  is a generator of  $r \text{Pic}(B)$ . □

Following the notation of [9, Lemma 3.1], we write  $V_n : \mathbb{P}^n \rightarrow \mathbb{P}^m : (X_0 : \dots : X_n) \mapsto (\omega_0 : \dots : \omega_m)$ , where the  $\omega_k$  are equal to the products  $\omega_{X_0^{\alpha_0} \dots X_n^{\alpha_n}} = \prod_i X_i^{\alpha_i}$  with  $\sum_i \alpha_i = n + 1$  in alphabetical order. The automorphism of  $X_a^{r,n}$  as an automorphism of  $\text{Ver}_n(X_a^{r,n})$  sends  $\omega_{X_i} \mapsto \omega_{X_{i+1}}$  and  $\omega_{X_n} \mapsto a\omega_{X_1}$ .

**Corollary 6.3.** *With the notation above,*

$$\text{Ver}_n(X') : \sum_{i=0}^n \left( \sum_j l_{i+j} \omega_{X_j} \right)^r = 0 \subseteq B$$

*Proof.* By using [8, §3], we find that a matrix  $\phi$  realizing the cocycle  $\xi$ , that is,  $\xi = \phi \circ {}^\sigma \phi^{-1}$ , sends  $\phi(\omega_{X_j}) = a^j \left( \sum_j l_{i+j} \omega_{X_j} \right)$  and  $\phi(\omega_{X_n}) = \sum_j l_{j-1} \omega_{X_j}$ . We plug  $\phi$  into the equation of  $X_a^{r,n}$  and the result follows. □

## 7 Comparison for constructing Brauer-Severi surfaces via canonical embedding of smooth plane curves

The third author shows an algorithm for constructing equations of Brauer-Severi varieties in [9]. Here we show an alternative way for constructing equations of Brauer-Severi surfaces ( $n = 2$ ) by using the theory of twists of plane curves.

Let  $\text{Ver}_n : \mathbb{P}_k^n \hookrightarrow \mathbb{P}_k^{\binom{2n+1}{n}-1}$  be the  $n$ -Veronese embedding. It has been observed by the third author in [9] that the induced map

$$\tilde{\text{Ver}}_n : H^1(k, \text{PGL}_{n+1}(\bar{k})) \rightarrow H^1(k, \text{PGL}_{\binom{2n+1}{n}}(\bar{k})),$$

satisfies that the image of any 1-cocycle is equivalent to a 1-cocycle with values in the lineal group  $\text{GL}_{\binom{2n+1}{n}}(\bar{k})$  and it is well-known that  $H^1(k, \text{GL}_{\binom{2n+1}{n}}(\bar{k}))$  is trivial, by Hilbert 90 Theorem. This fact leads to an algorithm to compute equations for any Brauer-Severi varieties. Here we use the idea coming from the construction in [1] of the equations for a non-trivial Brauer-Severi surface.



**Lemma 7.1.** *Let  $C$  be a smooth plane curve over  $k$  of genus  $g = \frac{1}{2}(d-1)(d-2) \geq 3$ . The canonical embedding of  $C$  is isomorphic to the composition  $\Psi : C \xrightarrow{\iota} \mathbb{P}_k^2 \xrightarrow{\text{Ver}_{d-3}} \mathbb{P}_k^{g-1}$ , where  $\iota$  comes from the (unique)  $g_d^2$ -linear system, all are defined over  $k$ . In particular, fixing a non-singular plane model  $F_C(X, Y, Z) = 0$  in  $\mathbb{P}_k^2$  of  $C$ , one may directly compute its canonical embedding into  $\mathbb{P}_k^{g-1}$  by applying the morphism  $\text{Ver}_{d-3}$ .*

*Proof.* It is fairly well-known that the sheaves  $\Omega^1(C)$  and  $\mathcal{O}(d-3)|_C$  are isomorphic (cf. R. Hartshorne [4, Example 8.20.3]). Hence,  $H^0(\mathbb{P}^2, \mathcal{O}(d-3)) \rightarrow H^0(C, \Omega^1)$  is an isomorphism, and the statement follows.  $\square$

Both maps,  $\iota$  and  $\text{Ver}_{d-3}$ , are  $\text{Gal}(\bar{k}/k)$ -equivariant. Therefore, the natural maps

$$\text{Aut}(\bar{C}) \hookrightarrow \text{Aut}(\mathbb{P}_k^2) = \text{PGL}_3(\bar{k}) \rightarrow \text{Aut}(\mathbb{P}_k^{g-1}) = \text{PGL}_g(\bar{k})$$

are morphisms of  $\text{Gal}(\bar{k}/k)$ -groups.

**Proposition 7.2.** *Given a non-trivial Brauer-Severi surface  $B$  over  $k$ , associated to a cyclic algebra  $(L/k, \sigma, a)$  of dimension 9, a  $k$ -model of  $B$  in  $\mathbb{P}_k^{\frac{9}{2}r(r-1)}$  with  $r \geq 2$  is algorithmically computable by considering any smooth plane curve  $C$  over  $k$  of degree  $3r$  such that  $[aZ : X : Y]$  is an automorphism. In particular, for  $r = 2$  we get a model<sup>1</sup> in  $\mathbb{P}^9$ .*

*Proof.* For non-hyperelliptic curves, see a description in [8], the canonical model gives a natural  $\text{Gal}(\bar{k}/k)$ -inclusion  $\text{Aut}(\bar{C}) \hookrightarrow \text{PGL}_g(\bar{k})$ , but we can go further, the action gives a  $\text{Gal}(\bar{k}/k)$ -inclusion  $\text{Aut}(\bar{C}) \hookrightarrow \text{GL}_g(\bar{k})$ . In this way, the natural map  $H^1(k, \text{Aut}(\bar{C})) \rightarrow H^1(k, \text{PGL}_g(\bar{k}))$ , satisfies that the image of any 1-cocycle is equivalent to a 1-cocycle with values in  $\text{GL}_g(\bar{k})$ , and recall that  $H^1(k, \text{GL}_g(\bar{k}))$  is trivial by applying Hilbert's Theorem 90. This allows us to compute equations for twists via change of variables in  $\text{GL}_g(\bar{k})$  of the canonical model for  $C$ . Now, by Lemma 7.1 and the proof of Theorem 4.2, one could construct a smooth model for the Brauer-Severi surface in  $\mathbb{P}_k^9$  by taking the  $V_{d-3}$  embedding with  $d = 6$  and  $C$  as in the statement.  $\square$

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<sup>1</sup> In [9] the equations for Brauer-Severi surfaces are also obtained in  $\mathbb{P}^9$ .

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