Uniform convergence of Hankel transforms

A. Debernardi*

Centre de Recerca Matemàtica and Universitat Autònoma de Barcelona

08193, Bellaterra, Barcelona, Spain

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Abstract

We investigate necessary and/or sufficient conditions for the pointwise and uniform convergence of the weighted Hankel transforms

$$\mathcal{L}_{\nu,\mu}^{\alpha} f(r) = r^{\mu} \int_0^\infty (rt)^{\nu} f(t) j_\alpha(rt)\, dt, \quad \alpha \geq -1/2, \quad r \geq 0,$$

where $\nu, \mu \in \mathbb{R}$ are such that $0 \leq \mu+\nu \leq \alpha+3/2$. We subdivide these transforms into two classes in such a way that the uniform convergence criteria is remarkably different on each class. In more detail, we have the transforms satisfying $\mu+\nu=0$ (such as the classical Hankel transform), that generalize the cosine transform, and those satisfying $0<\mu+\nu\leq \alpha+3/2$, generalizing the sine transform.

1 Introduction

Given an integral transform

$$Tf(y) = \int_0^\infty f(x)K(x,y)\, dx, \qquad y \in [0,\infty), \tag{1.1}$$

an interesting problem is to study whether Tf converges uniformly. We say that Tf converges uniformly on $E \subset [0,\infty)$ if for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\left| Tf(y) - \int_0^N f(x)K(x,y)\, dx \right| < \varepsilon, \qquad \text{for all } N \geq n,\ y \in E. \tag{1.2}$$

The notion of uniform convergence is of importance in analysis, as it permits to transfer desirable properties from the sequence of functions $T_N f(y) = \int_0^N f(x)K(x,y)\, dx$, $N \in \mathbb{N}$, to the limit function $Tf(y)$, such as continuity or integrability (cf. [24]).

*E-mail: `adebernardipinos@gmail.com`

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Depending on the behaviour of the kernel K, it is natural to consider certain assumptions on f in order to study the uniform convergence of (1.1). For example, if K is bounded then $f \in L^1(0,\infty)$ trivially guarantees the uniform convergence of (1.1). However, the condition $f \in L^1(0,\infty)$ may sometimes not be necessary, and thus it can be replaced by other assumptions. For instance, in the case of the Fourier transform

$$\widehat{F}(y) = \int_{\mathbb{R}^n} F(x) e^{ix\cdot y}\, dx, \tag{1.3}$$

one can guarantee the uniform convergence of $\widehat{F}$ whenever F satisfies certain monotonicity type properties, although $F \notin L^1(\mathbb{R})$ (see, e.g., [3, 8, 12, 23]). In this case the uniform convergence is defined analogously as in (1.2), replacing $\int_0^N$ by $\int_{|x|\le N}$.

Thus, an interesting problem is to study nontrivial necessary and sufficient conditions for the uniform convergence of (1.1). Our goal is to study this problem for a family of (one-dimensional) transforms that arise from Fourier transforms of radial functions. It is known that the Fourier transform of a radial function $F(x) = f_0(|x|)$ is also a radial function [20] given by

$$\widehat{F}(y) = |\mathbb{S}^{n-1}| \int_0^\infty t^{n-1} f_0(t) j_\alpha(2\pi|y|t)\, dt, \quad \alpha = \frac{n}{2} - 1, \quad n \ge 2, \tag{1.4}$$

where $|\mathbb{S}^{n-1}|$ denotes the area of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $j_\alpha(z)$ is the normalized Bessel function

$$j_\alpha(z) = \Gamma(\alpha+1)\left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z), \tag{1.5}$$

and $J_\alpha(z)$ is the classical Bessel function of the first kind of order α. Basic properties of these functions are discussed in Section 2.

For every $\alpha \ge -1/2$, the *Hankel transform of order* α of a function $f : \mathbb{R}_+ \to \mathbb{C}$ is defined as

$$H_\alpha f(r) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty t^{2\alpha+1} f(t) j_\alpha(2\pi r t)\, dt. \tag{1.6}$$

Letting $\alpha = n/2 - 1$ in (1.6), we recover the Fourier transform of a radial function (cf. (1.4)). The Hankel transform belongs to a larger class of operators, introduced by De Carli in [4], namely

$$\overline{\mathcal{L}} = \left\{ \overline{\mathcal{L}}^\alpha_{\nu,\mu} f(r) = r^\mu \int_0^\infty (rt)^\nu f(t) J_\alpha(rt)\, dt \right\}, \quad \alpha \ge -1/2,$$

with $\mu, \nu \in \mathbb{R}$. In particular, operators from $\overline{\mathcal{L}}$ appear in the Fourier transform of a radial function multiplied by a spherical harmonic (cf. [5], [6], [20], [22]). In the present work, we consider operators from $\overline{\mathcal{L}}$, written in terms of the normalized Bessel function j_α, i.e.,

$$\mathcal{L}^\alpha_{\nu,\mu} f(r) = r^\mu \int_0^\infty (rt)^\nu f(t) j_\alpha(rt)\, dt. \tag{1.7}$$

Notice that (1.5) allows us to rewrite (1.7) in terms of an operator from $\overline{\mathcal{L}}$ as

$$\mathcal{L}^\alpha_{\nu,\mu} f = 2^\alpha \Gamma(\alpha+1) \overline{\mathcal{L}}^\alpha_{\nu-\alpha,\mu} f.$$

Unless otherwise specified, the functions f we consider in this paper are complex-valued and defined on $\mathbb{R}_+$.

We list the following examples of classical transforms written in terms of (1.7). Here we denote by F a radial function of n variables, and $F(x) = f_0(|x|)$.

1. Since $j_{-1/2}(z) = \cos z$, the Fourier cosine transform, denoted by $\hat{f}_{\cos}$, corresponds to the transform $\mathcal{L}_{0,0}^{-1/2} f$.
2. The Fourier transform of a radial function for $n \geq 2$ (see (1.4)) satisfies

$$\hat{F}(y) = |\mathbb{S}^{n-1}| \mathcal{L}_{n-1,-(n-1)}^{n/2-1} f_0(2\pi|y|).$$

3. The classical Hankel transform H_α (see (1.6)) can be written as

$$H_\alpha f(y) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)} \mathcal{L}_{2\alpha+1,-(2\alpha+1)}^{\alpha} f(2\pi y).$$

4. If $n \geq 2$ and ψ_k is a solid spherical harmonic of degree k (i.e., the restriction of a harmonic polynomial of degree k to the unit sphere $\mathbb{S}^{n-1}$, cf. [20]), then

$$\widehat{\psi_k F}(y) = \psi_k(y) \cdot 2\pi^{n/2} \left(\frac{\pi}{i}\right)^k \mathcal{L}_{2\alpha+1,-(2\alpha+1)}^{\alpha} f_0(2\pi|y|),$$

with $\alpha = (n + 2k - 2)/2$ (see [20, Ch. IV, Theorem 3.10]).

5. Let $\mathcal{F}_k$ denote the Dunkl transform, defined by means of a root system $R \subset \mathbb{R}^n$, a reflection group $G \subset O(n)$, and a multiplicity function $k : R \to \mathbb{R}$ that is G-invariant. If f is a radial function defined on $\mathbb{R}^n$, then

$$\mathcal{F}_k f = H_{n/2-1+\langle k \rangle} f,$$

where $\langle k \rangle = \frac{1}{2} \sum_{x \in R} k(x)$ (cf. [7, 19] and the references therein). We also refer the reader to [2], where a generalization of the Dunkl transform is introduced, and [10], where uncertainty principle relations are obtained for this new transform.

6. Since $j_{1/2}(z) = \sin z/z$, the Fourier sine transform (denoted by $\hat{f}_{\sin}$) equals $\mathcal{L}_{1,0}^{1/2} f$.

The goal of this paper is to obtain necessary and/or sufficient conditions on f for (1.7) to converge uniformly on $[0, \infty)$, under the restriction $0 \leq \mu + \nu \leq \alpha + 3/2$. Outside this range, we give sufficient conditions for the pointwise convergence of (1.7) and the corresponding ones concerning uniform convergence on subsets of $\mathbb{R}_+$. Following (1.2), by the uniform convergence of $\mathcal{L}_{\nu,\mu}^{\alpha} f$, we mean that the sequence of partial integrals

$$r^\mu \int_0^N (rt)^\nu f(t) j_\alpha(rt)\, dt, \quad N \in \mathbb{N},$$

converges uniformly to $\mathcal{L}_{\nu,\mu}^{\alpha} f$. Equivalently, $\mathcal{L}_{\nu,\mu}^{\alpha} f$ converges uniformly if and only if

$$r^\mu \int_M^N (rt)^\nu f(t) j_\alpha(rt)\, dt \to 0, \quad \text{as } N > M \to \infty, \tag{1.8}$$

uniformly in r. We refer to the integral in (1.8) as the *Cauchy remainder* (cf. [24]).

Let us first make an observation showing a key difference between a general operator (1.7) and the sine and cosine transform. From the fact that

$$|\mathcal{L}^{\alpha}_{\nu,\mu} f(r)| \lesssim r^{\mu+\nu} \int_0^1 t^{\nu} |f(t)|\, dt + r^{\mu+\nu-\alpha-1/2} \int_1^{\infty} t^{\nu-\alpha-1/2} |f(t)|\, dt$$

(see (2.4) in Section 2) we have that the absolute convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ follows from the conditions

$$t^{\nu} f(t) \in L^1(0,1), \quad t^{\nu-\alpha-1/2} f(t) \in L^1(1,\infty). \tag{1.9}$$

However, unlike the case of $\hat{f}_{\sin}$ or $\hat{f}_{\cos}$, the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ does not necessarily follow from the integrability conditions (1.9) that imply its absolute convergence. This is because the kernel

$$K^{\alpha}_{\nu,\mu}(t,r) = r^{\mu} (rt)^{\nu} j_{\alpha}(rt) \tag{1.10}$$

of $\mathcal{L}^{\alpha}_{\nu,\mu}$ need not be uniformly bounded. Indeed, if we consider the choice of parameters $\mu = -1$, $0 < \nu < \alpha + 1/2$ and $f(t) = t^{\mu} = 1/t$, the conditions in (1.9) hold, but the Cauchy remainder

$$r^{\mu} \int_{1/(2r)}^{1/r} (rt)^{\nu} f(t) j_{\alpha}(rt)\, dt \asymp \frac{1}{r} \int_{1/(2r)}^{1/r} t^{-1}\, dt = \frac{\log 2}{r}$$

does not vanish as $r \to 0$ (cf. (2.2) below).

However, there is a special case when the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ follows from (1.9) (cf. Proposition 3.1), namely when

$$\mu + \nu = \alpha + 1/2. \tag{1.11}$$

In particular, the operators representing $\hat{f}_{\sin}$ ($\alpha = 1/2, \nu = 1, \mu = 0$) and $\hat{f}_{\cos}$ ($\alpha = -1/2, \nu = \mu = 0$) satisfy (1.11).

The two main results of the present paper are the following:

Theorem 1.1. *Let $\nu \in \mathbb{R}$ and $\mu = -\nu$. Let f be such that $t^{\nu} f(t) \in L^1(0,1)$, and*

$$|f(M)| = o\big(M^{-\nu-1}\big) \qquad \text{as } M \to \infty, \tag{1.12}$$

$$\int_M^{\infty} t^{\nu-\alpha-1/2} |df(t)| = o\big(M^{-\alpha-3/2}\big) \qquad \text{as } M \to \infty. \tag{1.13}$$

Then, a necessary and sufficient condition for $\mathcal{L}^{\alpha}_{\nu,\mu} f(r)$ to converge uniformly on $\mathbb{R}_+$ is that

$$\left| \int_0^{\infty} t^{\nu} f(t)\, dt \right| < \infty. \tag{1.14}$$

Theorem 1.2. *Let $\nu, \mu \in \mathbb{R}$ be such that $0 < \mu + \nu \le \alpha + 3/2$. Let f be such that $t^{\nu} f(t) \in L^1(0,1)$. If the conditions*

$$|f(M)| = o\big(M^{\mu-1}\big) \qquad \text{as } M \to \infty, \tag{1.15}$$

$$\int_M^{\infty} t^{\nu-\alpha-1/2} |df(t)| = o\big(M^{\mu+\nu-\alpha-3/2}\big) \qquad \text{as } M \to \infty. \tag{1.16}$$

are satisfied, then $\mathcal{L}^{\alpha}_{\nu,\mu} f$ converges uniformly on $\mathbb{R}_+$.

Observe that conditions (1.12) and (1.13) are the same as (1.15) and (1.16), respectively, for the particular case $\mu = -\nu$.

The theorems above generalize the following results obtained by Dyachenko, Liflyand and Tikhonov ([8]).

Theorem A. *Let $f \in L^1(0,1)$ be vanishing at infinity and such that*

$$\int_M^\infty |df(t)| = o\big(M^{-1}\big) \quad \text{as } M \to \infty. \tag{1.17}$$

Then,

$$\int_0^\infty f(t) \cos rt \, dt$$

converges uniformly if and only if $\int_0^\infty f(t)\,dt$ converges.

Theorem B. *Let f be vanishing at infinity and such that $tf(t) \in L^1(0,1)$, and assume* (1.17) *holds. Then,*

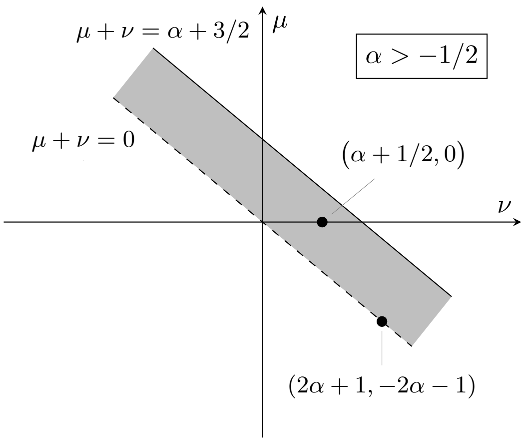
$$\int_0^\infty f(t) \sin rt \, dt$$

converges uniformly.

Note that Theorems A and B are particular cases of Theorems 1.1 and 1.2, whenever $\alpha = -1/2$, $\nu = \mu = 0$ and $\alpha = 1/2$, $\nu = 1$, $\mu = 0$, respectively. Note that if f vanishes at infinity, (1.13) and (1.16) imply (1.12) and (1.15), respectively. In fact, for functions vanishing at infinity, conditions (1.12) and (1.15) may be redundant for certain parameters, thus we present alternative statements to those of Theorems 1.1 and 1.2 (namely, Theorems 4.2 and 5.2, respectively).

In view of the respective relationship of Theorems 1.1 and 1.2 with Theorems A and B, we will call $\mathcal{L}^\alpha_{\nu,\mu} f$ with $\nu = -\mu$ (or simply $\mathcal{L}^\alpha_{\nu,-\nu} f$) *cosine-type* transforms, and $\mathcal{L}^\alpha_{\nu,\mu} f$ with $0 < \mu + \nu \leq \alpha + 3/2$ *sine-type* transforms.

We present a picture showing the range of the parameters μ and ν for which $\mathcal{L}^\alpha_{\nu,\mu}$ is a sine or cosine-type transform, given a fixed $\alpha > -1/2$.



Every point on the dashed line $\mu = -\nu$ corresponds to a cosine-type transform, and the point $(2\alpha + 1, -2\alpha - 1)$ lying on such line represents the Hankel transform of order α. The area between the dashed line $\mu = -\nu$ (not included) and the line $\mu + \nu = \alpha + 3/2$ (included) corresponds to the sine-type transforms.

The extreme case $\alpha = -1/2$ is the only choice of α for which the operator $\mathcal{L}^{\alpha}_{\alpha+1/2,0}$ does not correspond to a sine-type transform, since $\mathcal{L}^{-1/2}_{0,0} f = \hat{f}_{\cos}$.

For every point of the plane outside the grey strip $0 \le \mu + \nu \le \alpha + 3/2$, we give sufficient conditions on f that guarantee the pointwise convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$, as well as the uniform convergence on certain subintervals of $[0, \infty)$ (see Section 3).

Any function f we consider in this work is locally of bounded variation (unless otherwise specified) and locally integrable on $(0, \infty)$. By $f \lesssim g$ and $f \gtrsim g$ we mean that there exist positive constants C, C' such that $f \le Cg$ and $f \ge C'g$, respectively, and we write $f \asymp g$ if $f \lesssim g$ and $f \gtrsim g$ simultaneously.

The paper is organized in the following way. In Section 2 we present the basic concepts that we will use. Subsection 2.1 is devoted to the Bessel functions; first we list several of their known properties, and we obtain estimates of integrals containing j_α. We emphasize that Lemma 2.6 provides the key estimate to be used throughout this work. In Subsection 2.2 we define the class of *general monotone* (GM) functions. To give a flavour, we use the GM property to generalize the following results (that follow from Theorems A and B, respectively; see also [18]):

Theorem A′. *If $f \in GM$ and $f \in L^1(0,1)$, then*

$$\hat{f}_{\cos} \quad \textit{converges uniformly if and only if} \quad \left| \int_0^\infty f(t)\, dt \right| < \infty.$$

Theorem B′. *If $f \in GM$ and $tf(t) \in L^1(0,1)$, then*

$$\hat{f}_{\sin} \quad \textit{converges uniformly if and only if} \quad t|f(t)| \to 0 \quad \textit{as } \to \infty.$$

In Section 3 we obtain sufficient conditions for the pointwise convergence of (1.7) in the whole range of parameters, and for its uniform convergence on subintervals of $\mathbb{R}_+$, using both integrability of the functions and conditions on their variation. In Sections 4 and 5 we study the uniform convergence of cosine-type and sine-type transforms, respectively. The hypotheses used in such sections mainly depend on variation conditions of f. We also give the corresponding statements for GM functions. To conclude Section 5, we give several examples showing the sharpness of the obtained results, and compare the sufficient conditions obtained in Section 3 (namely Corollary 3.6) with those of Theorems 1.2 and 5.2.

2 Preliminary concepts

2.1 Bessel functions

Basic properties. Here we list several properties of the normalized Bessel function $j_\alpha(z)$, which can be found in [9, Chapter VII]. In what follows we will assume $z \in \mathbb{R}_+$. We start with the representation by power series:

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n+\alpha+1)}.$$

Such series converges uniformly and absolutely on any bounded interval. In particular, for $z \le 1$,

$$|1 - j_\alpha(z)| \le Cz^2, \tag{2.1}$$

with $C < 1$, and therefore

$$j_\alpha(z) \asymp 1. \tag{2.2}$$

Moreover, we have the following asymptotic estimate (cf. [20]):

$$j_\alpha(z) = \frac{C_\alpha}{z^{\alpha+1/2}} \cos\left(z - \frac{\pi(\alpha+1/2)}{2}\right) + O\big(z^{-\alpha-3/2}\big), \quad z \to \infty, \tag{2.3}$$

and since $|j_\alpha(z)| \le j_\alpha(0) = 1$ for all $z > 0$, then

$$|j_\alpha(z)| \lesssim \min\left\{1, \frac{1}{z^{\alpha+1/2}}\right\} \quad \text{for all } z > 0. \tag{2.4}$$

Finally, we have the following property concerning the derivatives of j_α:

$$\frac{d}{dz}\Big(z^{2\alpha+2} j_{\alpha+1}(z)\Big) = (2\alpha+2) z^{2\alpha+1} j_\alpha(z), \quad \alpha \ge -1/2, \tag{2.5}$$

from which we deduce

$$\frac{d}{dz} j_{\alpha+1}(z) = \frac{2\alpha+2}{z}\big(j_\alpha(z) - j_{\alpha+1}(z)\big), \quad \alpha \ge -1/2. \tag{2.6}$$

Auxiliary lemmas. We will also need upper estimates for the primitive function of $t^\nu j_\alpha(rt)$. We start by rewriting $\int_M^N t^\nu j_\alpha(rt)\,dt$ in terms of higher order Bessel functions.

Lemma 2.1. *Let $\alpha \ge -1/2$, $r > 0$ and $0 < M < N$. Then, for any $k \ge 1$ and $\nu \in \mathbb{R}$ such that $\nu \ne 2(\alpha+\ell)+1$ with $\ell = 0, \dots, k-1$,*

$$\int_M^N t^\nu j_\alpha(rt)\,dt = \sum_{i=1}^k C_{i,\nu,\alpha}\big(N^{\nu+1} j_{\alpha+i}(rN) - M^{\nu+1} j_{\alpha+i}(rM)\big) + C'_{k,\nu,\alpha} \int_M^N t^\nu j_{\alpha+k}(rt)\,dt, \tag{2.7}$$

where the constants $C_{i,\nu,\alpha}$, $C'_{k,\nu,\alpha}$ are nonzero.

Proof. We prove this statement by induction on k. For $k = 1$, we can rewrite the integral on the left hand side of (2.7) as $\int_M^N t^{\nu-2\alpha-1} t^{2\alpha+1} j_\alpha(rt)\,dt$, and the result follows after integrating by parts together with (2.5). In this case we have $C_{1,\nu,\alpha} = \frac{1}{2\alpha+2}$ and $C'_{1,\nu,\alpha} = -\frac{\nu-2\alpha-1}{2\alpha+2}$.

If (2.7) holds for some $k \ge 1$, since

$$C'_{k,\nu,\alpha} \int_M^N t^\nu j_{\alpha+k}(rt)\,dt = C'_{k,\nu,\alpha} \int_M^N t^{\nu-2(\alpha+k)-1} t^{2(\alpha+k)+1} j_{\alpha+k}(rt)\,dt,$$

the result follows similarly as before, where in this case we obtain $C_{k+1,\nu,\alpha} = \frac{C'_{k,\nu,\alpha}}{2(\alpha+k)+2}$ and $C'_{k+1,\nu,\alpha} = -C'_{k,\nu,\alpha} \frac{\nu-2(\alpha+k)-1}{2(\alpha+k)+2}$. □

Lemma 2.2. *Under the assumptions of Lemma 2.1, we have, for any $\nu \in \mathbb{R}$ such that $\nu = 2(\alpha+\ell)+1$ with some $\ell \in \mathbb{N} \cup \{0\}$,*

$$\int_M^N t^\nu j_\alpha(rt)\,dt = \sum_{i=1}^{\ell+1} C_{i,\nu,\alpha}\big(N^{\nu+1} j_{\alpha+i}(rN) - M^{\nu+1} j_{\alpha+i}(rM)\big),$$

where all the constants $C_{i,\nu,\alpha}$ coincide with those of Lemma 2.1.

Proof. If $\ell = 0$, the result immediately follows from (2.5). If $\ell > 0$, we can apply Lemma 2.1 with $\nu' = 2(\alpha+\ell-1)+1$ in place of ν, and then by (2.5),

$$C'_{\ell,\nu,\alpha} \int_M^N t^{2(\alpha+\ell)+1} j_{\alpha+\ell}(rt)\,dt = C_{\ell+1,\nu,\alpha}\big(N^{\nu+1} j_{\alpha+\ell+1}(rN) - M^{\nu+1} j_{\alpha+\ell+1}(rM)\big),$$

where $C_{\ell+1,\nu,\alpha} = \frac{C'_{\ell,\nu,\alpha}}{2(\alpha+\ell)+1}$. □

Remark 2.3. We can allow $M = 0$ in Lemmas 2.1 and 2.2 whenever $\nu > -1$.

Lemma 2.4. *Let $\alpha \ge -1/2$, $r > 0$ and $0 < M < N$. For any $\nu \in \mathbb{R}$ and any $k \ge 1$ such that $\nu \ne \alpha + k - 1/2$, we have*

$$\left|\int_M^N t^\nu j_\alpha(rt)\,dt\right| \lesssim \frac{1}{r^{\alpha+1/2}} \sum_{i=1}^{k} \frac{1}{r^i}\big(N^{\nu-i-\alpha+1/2} + M^{\nu-i-\alpha+1/2}\big). \tag{2.8}$$

Proof. If ν is as in Lemma 2.2, the estimate follows by just applying (2.4). On the contrary, if ν is as in Lemma 2.1, we estimate the sum of (2.7) in a similar way, whilst since $\nu - \alpha - k - 1/2 \ne -1$,

$$\begin{aligned}\left|\int_M^N t^\nu j_{\alpha+k}(rt)\,dt\right| &\lesssim \frac{1}{r^{\alpha+k+1/2}} \int_M^N t^{\nu-\alpha-k-1/2}\,dt \\ &\lesssim \frac{1}{r^{k+\alpha+1/2}}\big(N^{\nu-k-\alpha+1/2} + M^{\nu-k-\alpha+1/2}\big),\end{aligned}$$

which coincides precisely with the k-th term of the sum in (2.8). □

Since the Bessel function $j_\alpha(z)$ is continuous, if we denote by $g^\nu_{\alpha,r}(t)$ the primitive function of $t^\nu j_\alpha(rt)$, we have, in virtue of the fundamental theorem of calculus,

$$g^\nu_{\alpha,r}(t) = \begin{cases} \displaystyle\int_0^t s^\nu j_\alpha(rs)\,ds, & \text{if } \nu \ge \alpha+1/2 \text{ and } \alpha > -1/2, \text{ or if } \nu > \alpha+1/2, \\ \displaystyle -\int_t^\infty s^\nu j_\alpha(rs)\,ds, & \text{if } \nu < \alpha+1/2, \\ \displaystyle\frac{\sin rt}{r}, & \text{if } \nu = 0 \text{ and } \alpha = -1/2. \end{cases} \tag{2.9}$$

Remark 2.5. Note that

$$\int_0^x t^\nu j_\alpha(rt)\,dt = \frac{x^{\nu+1}}{\nu+1}\, {}_1F_2\Big(\frac{1}{2}(\nu+1); \frac{1}{2}(\nu+3), \alpha+1; -\frac{(rx)^2}{4}\Big), \quad \nu > -1,$$

where ${}_pF_q$ denotes the generalized hypergeometric function (see [17, Ch. 6]).

We are now in a position to obtain the upper bound of (2.9).

Lemma 2.6. *The estimate*

$$|g_{\alpha,r}^{\nu}(t)| \lesssim \frac{t^{\nu-\alpha-1/2}}{r^{\alpha+3/2}}, \quad \alpha \geq -1/2, \quad \nu \in \mathbb{R}, \tag{2.10}$$

holds.

Proof. We distinguish two cases: $\nu \neq \alpha + 1/2$, or $\nu = \alpha + 1/2$. In the first case, estimate (2.10) follows readily applying Lemma 2.4 with $k = 1$ and letting $M \to 0$ or $N \to \infty$ if $\nu > \alpha + 1/2$ or $\nu < \alpha + 1/2$, respectively.

If $\nu = \alpha + 1/2$, and $\alpha = -1/2$, (2.10) follows immediately from (2.9). For $\alpha > -1/2$, we can apply Lemma 2.1 with $k = 2$ (see also Remark 2.3) to obtain

$$\begin{aligned} |g_{\alpha,r}^{\alpha+1/2}(t)| &= \left| C_1 t^{\alpha+3/2} j_{\alpha+1}(rt) + C_2 t^{\alpha+3/2} j_{\alpha+2}(rt) + C_3 \int_0^t s^{\alpha+1/2} j_{\alpha+2}(rs)\, ds \right| \\ &\lesssim \frac{1}{r^{\alpha+3/2}} + \frac{1}{tr^{\alpha+5/2}} + \int_0^t s^{\alpha+1/2} |j_{\alpha+2}(rs)|\, ds \end{aligned}$$

It follows from (2.4) that

$$\int_0^t s^{\alpha+1/2} |j_{\alpha+2}(rs)|\, ds \leq \int_0^{1/r} s^{\alpha+1/2}\, ds + \frac{1}{r^{\alpha+5/2}} \int_{1/r}^{\infty} s^{-2}\, ds \lesssim \frac{1}{r^{\alpha+3/2}}.$$

Collecting the estimates above, we deduce

$$|g_{\alpha,r}^{\alpha+1/2}(t)| \lesssim \frac{1}{r^{\alpha+3/2}} + \frac{1}{tr^{\alpha+5/2}}.$$

In particular, it follows from the latter estimate that (2.10) holds whenever $t \geq 1/r$.

Finally, if $t < 1/r$, using (2.9) together with (2.2) we obtain

$$|g_{\alpha,r}^{\nu}(t)| \asymp \int_0^t s^{\alpha+1/2}\, ds \asymp t^{\alpha+3/2} < \frac{1}{r^{\alpha+3/2}},$$

and the proof is complete. □

2.2 General Monotonicity

It is often useful to consider quantitative characteristics of functions that are locally of bounded variation, such as the so-called *general monotonicity* (cf. [11], [13], [14], [15] and [21]).

Definition 2.7. Let $\beta : \mathbb{R}_+ \to \mathbb{R}_+$. We say that a function f is *β-general monotone*, written $f \in GM(\beta)$, if there exists $C > 0$ such that for every $x > 0$,

$$\int_x^{2x} |df(t)| \leq C\beta(x).$$

In many cases important $GM(\beta)$ classes are those where β depends on the function f itself, rather than on its variation. We restrict ourselves to the concrete choice of β introduced in [14].

Definition 2.8. We say that f is a GM function, written $f \in GM$, if there exist $C, \lambda > 1$ such that for every $x > 0$,

$$\int_x^{2x} |df(t)| \le \frac{C}{x} \int_{x/\lambda}^{\lambda x} |f(t)|\, dt.$$

Note that any monotone function is also a GM function.

We could consider more general $GM(\beta)$ classes, such as the one defined in [8], where

$$\beta(x) = \beta_0(x) = \frac{1}{x} \sup_{s \ge x/\lambda} \int_s^{2s} |f(t)|\, dt.$$

It is known that $GM \subsetneq GM(\beta_0)$. However, the latter class is too wide, and may even give no useful information about the variation of f, if $\int_x^{2x} |f(t)|\, dt$ is not bounded at infinity.

The following holds for any $GM(\beta)$ function (see [13, Lemma 5.2]):

Lemma 2.9. *If $f \in GM(\beta)$ and $x > 0$, then*

$$|f(t)| \le C\beta(x) + \int_t^{2t} \frac{|f(s)|}{s}\, ds \quad \textit{for any } t \in [x, 2x].$$

It follows from Lemma 2.9 that if $f \in GM$ and $\lambda \ge 2$ (which can be assumed without loss of generality), one has

$$|f(x)| \le C \int_{x/\lambda}^{\lambda x} \frac{|f(t)|}{t}\, dt \asymp \frac{1}{x} \int_{x/\lambda}^{\lambda x} |f(t)|\, dt, \quad x > 0. \tag{2.11}$$

Note that the following estimate holds for all $f \in GM$:

$$\begin{aligned} \int_1^\infty t^{\nu-\alpha-1/2} |df(t)| &\lesssim \int_{1/2}^\infty \frac{1}{t} \int_t^{2t} s^{\nu-\alpha-1/2} |df(s)|\, dt \asymp \int_{1/2}^\infty t^{\nu-\alpha-3/2} \int_t^{2t} |df(s)|\, dt \\ &\lesssim \int_{1/2}^\infty t^{\nu-\alpha-5/2} \int_{t/\lambda}^{\lambda t} |f(s)|\, ds\, dt \lesssim \int_{1/(2\lambda)}^\infty s^{\nu-\alpha-3/2} |f(s)|\, ds. \end{aligned} \tag{2.12}$$

We can apply the latter inequality in order to replace the hypotheses on the variation of f by integrability conditions (for instance, compare Proposition 3.3 with Corollary 3.5 below).

3 Pointwise and uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$: first approach

In this section we are interested in finding sufficient conditions on f that guarantee the pointwise convergence of (1.7). We will see that these sufficient conditions also imply the uniform convergence of (1.7) on certain subintervals of $\mathbb{R}_+$. If we assume that $t^\nu f(t) \in L^1(0,1)$, the convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ at $r_0 \in \mathbb{R}_+$ means that

$$\lim_{M\to\infty} \left| r_0^{\mu+\nu} \int_0^M t^\nu f(t) j_\alpha(r_0 t)\, dt \right| < \infty.$$

In contrast with the criteria for uniform convergence (see Theorems 1.1 and 1.2), we do not impose restrictions on the parameters for now. The criterion for convergence at the origin is rather simple:

(i) If $\mu+\nu<0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f(0)$ is not defined.

(ii) If $\mu+\nu=0$, the convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f(0)$ is equivalent to $\left|\int_0^\infty t^\nu f(t)\,dt\right|<\infty$.

(iii) If $\mu+\nu>0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f(0)=0$.

Now we study the pointwise convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}f(r)$ for $r>0$. When possible, we also give sufficient conditions for the uniform convergence on subintervals of $\mathbb{R}_+$. The statements in this section can be subdivided into two categories, depending on their hypotheses. First, we have those relying on integrability of f, and secondly, those involving conditions on the variation of f.

3.1 Integrability conditions

We begin with the statements involving integrability conditions of f. The results in this subsection do not require the assumption of f being locally of bounded variation.

Proposition 3.1. *Let f be such that $t^\nu f(t)\in L^1(0,1)$ and $t^{\nu-\alpha-1/2}f(t)\in L^1(1,\infty)$. Then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges for $r>0$. Moreover,*

1. *if $\mu+\nu-\alpha-1/2<0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on any interval $[\varepsilon,\infty)$ with $\varepsilon>0$;*
2. *if $\mu+\nu-\alpha-1/2>0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on any interval $[0,\varepsilon]$ with $\varepsilon>0$;*
3. *if $\mu+\nu-\alpha-1/2=0$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly on $\mathbb{R}_+$.*

Proof. It is clear that the pointwise convergence of $\mathcal{L}^{\alpha}_{\nu,\mu}$ at $r>0$ is equivalent to

$$\int_M^\infty t^\nu f(t)j_\alpha(rt)\,dt=o(1)\quad\text{as } M\to\infty,$$

which holds by simply applying (2.4) and the fact that $t^{\nu-\alpha-1/2}\in L^1(1,\infty)$.

Let us now prove the statement concerning uniform convergence. For each of the three cases, since $t^{\nu-\alpha-1/2}f(t)\in L^1(1,\infty)$, it follows from (2.4) that

$$r^{\mu+\nu}\int_M^\infty t^\nu f(t)j_\alpha(rt)\,dt\le\varepsilon^{\mu+\nu-\alpha-1/2}\int_M^\infty t^{\nu-\alpha-1/2}|f(t)|\,dt=o(1)\quad\text{as } M\to\infty,$$

that is, the Cauchy remainder vanishes uniformly in r as $M\to\infty$ (in each corresponding interval). □

Proposition 3.1 allows us to derive sufficient conditions for the uniform convergence of $\mathcal{L}^{\alpha}_{\mu,\nu}f$ on $\mathbb{R}_+$ whenever $0\le\mu+\nu\le\alpha+1/2$.

Corollary 3.2. *Let $0\le\mu+\nu\le\alpha+1/2$. If $t^\nu f(t)\in L^1(\mathbb{R}_+)$, then $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly.*

Proof. First, if $0 \le \mu+\nu < \alpha+1/2$, note that since $\alpha \ge -1/2$, $t^\nu f(t) \in L^1(\mathbb{R}_+)$ implies $t^{\nu-\alpha-1/2} f(t) \in L^1(1,\infty)$, so we can apply Proposition 3.1 to deduce that $\mathcal{L}^\alpha_{\nu,\mu} f$ converges uniformly on any interval $[\varepsilon,\infty)$ with $\varepsilon > 0$, whilst the uniform convergence on the interval $[0,\varepsilon]$ follows from

$$r^{\mu+\nu}\int_M^\infty t^\nu f(t) j_\alpha(rt)\,dt \le \varepsilon^{\mu+\nu}\int_M^\infty t^\nu |f(t)|\,dt \to 0 \qquad \text{as } M\to\infty.$$

Secondly, if $\mu+\nu = \alpha+1/2$, then $t^{\nu-\alpha-1/2} f(t) = t^{-\mu} f(t)$, and therefore $t^\nu f(t) \in L^1(\mathbb{R}_+)$ implies $t^{-\mu} f(t) \in L^1(1,\infty)$ (since $\nu \ge -\mu$ for every $\alpha \ge -1/2$), and the result follows by Proposition 3.1. □

3.2 Variational conditions

The statements of this subsection involve conditions on the variation of f. In the case of GM functions, these follow from integrability conditions of f (cf. (2.12)), allowing us to rewrite certain statements. When possible, we also give sufficient conditions for the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ on $\mathbb{R}_+$ that follow after combining the results on the present subsection with those of the previous one.

Proposition 3.3. *Let f be such that $t^\nu f(t) \in L^1(0,1)$ and*

$$\int_1^\infty t^{\nu-\alpha-1/2}|df(t)| < \infty \quad \textit{and} \quad M^{\nu-\alpha-1/2}|f(M)| \to 0 \quad \textit{as } M\to\infty, \tag{3.1}$$

then $\mathcal{L}^\alpha_{\nu,\mu} f(r)$ converges for $r>0$. Moreover, for any $\varepsilon>0$,

1. *if $\mu+\nu-\alpha-3/2 > 0$, the convergence is uniform on any interval $[0,\varepsilon]$;*
2. *if $\mu+\nu-\alpha-3/2 < 0$, the convergence is uniform on any interval $[\varepsilon,\infty)$;*
3. *if $\mu+\nu-\alpha-3/2 = 0$, the convergence is uniform on $\mathbb{R}_+$.*

Remark 3.4. (i) Note that in the extremal case $\mu+\nu = \alpha+3/2$, the conditions (3.1) are equivalent to (1.15) and (1.16).

(ii) In the case $\nu \ge \alpha+1/2$, if f vanishes at infinity the convergence of $\int_1^\infty t^{\nu-\alpha-1/2}|df(t)|$ implies that $M^{\nu-\alpha-1/2} f(M) \to 0$ as $M\to\infty$. Indeed,

$$M^{\nu-\alpha-1/2}|f(M)| \le M^{\nu-\alpha-1/2}\int_M^\infty |df(t)| \le \int_M^\infty t^{\nu-\alpha-1/2}\,|df(t)|,$$

and the right hand side of the latter vanishes as $M\to\infty$. Thus, in this case we only need to assume the convergence of $\int_1^\infty t^{\nu-\alpha-1/2}|df(t)|$ in Proposition 3.3.

For functions satisfying the GM property, we can derive a version of Proposition 3.3 depending on integrability conditions of f, which are less restrictive than those from Proposition 3.1.

Corollary 3.5. *Let $f \in GM$ be such that $t^\nu f(t) \in L^1(0,1)$. If $t^{\nu-\alpha-3/2} f(t) \in L^1(1,\infty)$, all the statements of Proposition 3.3 hold.*

Proof of Proposition 3.3. We fix $r > 0$. Since $t^\nu f(t) \in L^1(0,1)$, the convergence of (1.7) is equivalent to

$$\lim_{M\to\infty} \left| \int_1^M t^\nu f(t) j_\alpha(rt)\, dt \right| < \infty.$$

Note that condition $t^{\nu-\alpha-1/2}|f(t)| \to 0$ as $t \to \infty$ implies that the integrand $t^\nu f(t) j_\alpha(rt)$ vanishes as $t \to \infty$. Integrating by parts, we have

$$\int_1^M t^\nu f(t) j_\alpha(rt)\, dt = g_{\alpha,r}^\nu(M) f(M) - g_{\alpha,r}^\nu(1) f(1) - \int_1^M g_{\alpha,r}^\nu(t)\, df(t),$$

where $g_{\alpha,r}^\nu(t)$ is given by (2.9). Now we estimate each term of the latter expression (note that $g_{\alpha,r}^\nu(1)f(1)$ is bounded). It follows from (2.10) and (3.1) that

$$|g_{\alpha,r}^\nu(M) f(M)| \lesssim \frac{M^{\nu-\alpha-1/2}}{r^{\alpha+3/2}} |f(M)| \to 0 \quad \text{as } M \to \infty.$$

Finally,

$$\int_1^M |g_{\alpha,r}^\nu(t)\, df(t)| \lesssim \frac{1}{r^{\alpha+3/2}} \int_1^M t^{\nu-\alpha-1/2}\, |df(t)|.$$

Thus, the condition $\int_1^\infty t^{\nu-\alpha-1/2}|df(t)| < \infty$ implies that the integral

$$\int_1^\infty |g_{\alpha,r}^\nu(t)\, df(t)|$$

converges, which concludes the part concerning pointwise convergence.

The statement related to uniform convergence is easily proved by simply applying estimates (2.4) and (2.10) to the Cauchy remainder:

$$\begin{aligned} r^\mu \left| \int_M^N (rt)^\nu f(t) j_\alpha(rt)\, dt \right| &= r^{\mu+\nu} \left| g_{\alpha,r}^\nu(N) f(N) - g_{\alpha,r}^\nu(M) f(M) - \int_M^N g_{\alpha,r}^\nu(t)\, df(t) \right| \\ &\lesssim r^{\mu+\nu-\alpha-3/2} \Big(N^{\nu-\alpha-1/2}|f(N)| + M^{\nu-\alpha-1/2}|f(M)| \\ &\qquad + \int_M^N t^{\nu-\alpha-1/2}\, |df(t)| \Big). \end{aligned}$$

Thus, the latter expression vanishes

1. uniformly in $r \in [0, \varepsilon]$ if $\mu + \nu - \alpha - 3/2 > 0$;
2. uniformly in $r \in [\varepsilon, \infty)$ if $\mu + \nu - \alpha - 3/2 < 0$;
3. uniformly in $r \in [0, \infty)$ if $\mu + \nu - \alpha - 3/2 = 0$,

as $N > M \to \infty$. □

Proof of Corollary 3.5. First of all, note that if $f \in GM$, the condition $t^{\nu-\alpha-3/2} f(t) \in L^1(1,\infty)$ implies that $t^{\nu-\alpha-1/2} f(t)$ vanishes at infinity (see (2.11)). Furthermore, by (2.12), we have that all hypotheses of Proposition 3.3 are satisfied, and the result follows. □

Our last statement of this subsection is just a combination of Propositions 3.1 and 3.3.

Corollary 3.6. *Let f be such that $t^\nu f(t) \in L^1(0,1)$. Assume that $\alpha+1/2 \le \mu+\nu < \alpha+3/2$. If the conditions* (3.1) *hold, and if $t^{\nu-\alpha-1/2} f(t) \in L^1(1,\infty)$, then $\mathcal{L}^\alpha_{\nu,\mu} f$ converges uniformly.*

Note that except for the case $\alpha = -1/2$ and $\mu + \nu = 0$, the parameters for which Corollary 3.6 can be applied correspond to sine-type transforms.

3.3 Examples

Let us discuss an application of Proposition 3.3, which is closely related to the following classical result [25, Ch. I, Theorem 2.6] (see also [1, Ch. I, §30]): *Let $\varphi(x)$ be either $\sin x$ or $\cos x$. If $a_n \to 0$ and $\{a_n\} \in BV$, or equivalently,*

$$\sum_{n=N}^{\infty} |a_n - a_{n+1}| = o(1) \quad \text{as } N \to \infty,$$

then $\sum_{n=0}^{\infty} a_n \varphi(nx)$ converges pointwise in $x \in (0, 2\pi)$, and the convergence is uniform on any interval $[\varepsilon, 2\pi - \varepsilon]$, $\varepsilon > 0$.

A version of the latter statement for the sine and cosine transforms follows from Proposition 3.3 (see item 2 of the latter, and note that for the sine and cosine transforms both conditions $\mu + \nu - \alpha - 3/2 < 0$ and $\nu - \alpha - 1/2 = 0$ hold).

Theorem C. *Let f, g be vanishing at infinity and such that $f \in L^1(0,1)$ and $tg(t) \in L^1(0,1)$. Assume that f and g are of bounded variation on $[\delta,\infty)$ for some $\delta > 0$. Then, $\hat{f}_{\cos}(r)$ and $\hat{g}_{\sin}(r)$ converge for every $r > 0$, and the convergence is uniform on every interval $[\varepsilon,\infty)$, with $\varepsilon > 0$.*

Finally, we give an example showing that we cannot guarantee the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ on $\mathbb{R}_+$ outside the range of parameters $0 \le \mu + \nu \le \alpha + 3/2$, whenever f satisfies both conditions from (3.1). The case $\mu + \nu < 0$ is clear, since in this case $\mathcal{L}^\alpha_{\nu,\mu} f(0)$ is not even defined. The case $\mu + \nu > \alpha + 3/2$ is more involved.

Let

$$f(t) = \begin{cases} t^{-\nu}, & \text{if } t < 2, \\ \dfrac{t^{-\nu+\alpha+1/2}}{(\log t)^2}, & \text{if } t \ge 2. \end{cases}$$

On the one hand, since for any $\nu \in \mathbb{R}$ and $\alpha \ge -1/2$ one has

$$f'(t) = (-\nu + \alpha + 1/2)\frac{t^{-\nu-1/2+\alpha}}{(\log t)^2} - 2\frac{t^{-\nu-1/2+\alpha}}{(\log t)^3},$$

it is clear that

$$\int_1^\infty t^{\nu-\alpha-1/2} |df(t)| \le 1 + \int_2^\infty t^{\nu-\alpha-1/2} |f'(t)|\, dt \lesssim \int_2^\infty \frac{1}{t(\log t)^2}\, dt < \infty.$$

On the other hand, for $t \ge 2$

$$t^{\nu-\alpha-1/2} f(t) = \frac{1}{(\log t)^2} \to 0 \qquad \text{as } t \to \infty,$$

and hence f satisfies both conditions from (3.1). Let us now prove that $\mathcal{L}^{\alpha}_{\nu,\mu}f$ does not converge uniformly on $\mathbb{R}_+$. Let $2 < M < N$. Integration by parts along with property (2.5) of j_α yields the following equality:

$$r^{\mu+\nu}\left|\int_M^N t^\nu f(t) j_\alpha(rt)\,dt\right| = r^{\mu+\nu}\left|\frac{1}{2\alpha+2}\left[\frac{t^{\alpha+3/2}}{(\log t)^2}j_{\alpha+1}(rt)\right]_M^N\right.$$
$$+\frac{\alpha+1/2}{2\alpha+2}\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^2}j_{\alpha+1}(rt)\,dt$$
$$\left.+\frac{2}{2\alpha+2}\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^3}j_{\alpha+1}(rt)\,dt\right| =: a_0+b_0+c_0$$

First,

$$a_0 \asymp r^{\mu+\nu}\left|\frac{N^{\alpha+3/2}}{(\log N)^2}j_{\alpha+1}(rN) - \frac{M^{\alpha+3/2}}{(\log M)^2}j_{\alpha+1}(rM)\right|.$$

If we choose $r = (\log M)^{2/(\mu+\nu-\alpha-3/2)}$ and M so that $j_{\alpha+1}(rM) \asymp (rM)^{-\alpha-3/2}$ (such M can be found through (2.3)), we obtain by letting $N \to \infty$,

$$a_0 \asymp \frac{r^{\mu+\nu-\alpha-3/2}}{(\log M)^2} = 1.$$

We now prove that both terms b_0 and c_0 vanish as $N > M \to \infty$ (for this particular choice of r). If we prove such claim, then it follows that $\mathcal{L}^{\alpha}_{\nu,\mu}f$ does not converge uniformly on $\mathbb{R}_+$. Let us proceed to estimate b_0 from above first. Using again integration by parts and (2.5), we obtain

$$r^{\mu+\nu}\left|\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^2}j_{\alpha+1}(rt)\,dt\right| = r^{\mu+\nu}\left|\frac{1}{2\alpha+4}\left[\frac{t^{\alpha+3/2}}{(\log t)^2}j_{\alpha+2}(rt)\right]_M^N\right.$$
$$+\frac{\alpha+5/2}{2\alpha+4}\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^2}j_{\alpha+2}(rt)\,dt$$
$$\left.+\frac{2}{2\alpha+4}\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^3}j_{\alpha+2}(rt)\,dt\right| =: a_1+b_1+c_1.$$

By (2.4), it is clear that

$$a_1 \lesssim r^{\mu+\nu-\alpha-5/2}\left(\frac{1}{M(\log M)^2}+\frac{1}{N(\log N)^2}\right) \lesssim \frac{(\log M)^{2\left|\frac{\mu+\nu-\alpha-5/2}{\mu+\nu-\alpha-3/2}\right|}}{M} \to 0$$

as $N > M \to \infty$, as for b_1, c_1, we note that

$$b_1+c_1 \lesssim r^{\mu+\nu}\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^2}|j_{\alpha+2}(rt)|\,dt \lesssim r^{\mu+\nu-\alpha-5/2}\int_M^N \frac{1}{t^2(\log t)^2}\,dt$$
$$\le \frac{r^{\mu+\nu-\alpha-5/2}}{M} \le \frac{(\log M)^{2\left|\frac{\mu+\nu-\alpha-5/2}{\mu+\nu-\alpha-3/2}\right|}}{M} \to 0$$

as $N > M \to \infty$. Let us now inspect the term c_0. Once again, integration by parts and (2.4) yield

$$r^{\mu+\nu}\left|\int_M^N \frac{t^{\alpha+1/2}}{(\log t)^3}j_{\alpha+1}(rt)\,dt\right| \lesssim r^{\mu+\nu}\left(\left|\left[\frac{t^{\alpha+3/2}}{(\log t)^3}j_{\alpha+2}(rt)\right]_M^N\right| + \int_M^N \frac{t^{\alpha+1/2}}{(\log t)^3}|j_{\alpha+2}(rt)|\,dt\right),$$

and it can be shown similarly as above that the latter vanishes as $N > M \to \infty$. Therefore, we conclude that $\mathcal{L}^\alpha_{\nu,\mu} f$ does not converge uniformly.

4 Uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ with $\mu + \nu = 0$

In the present section we investigate necessary and sufficient conditions for the uniform convergence of the transforms $\mathcal{L}^\alpha_{\nu,\mu} f$ with $\mu+\nu = 0$ (or equivalently, $\mathcal{L}^\alpha_{\nu,-\nu} f$), as for example, the Hankel transform.

4.1 Main Results

Additionally to Theorem 1.1, we have other uniform convergence criteria for cosine-type transforms that will be stated and proved in this section, namely Theorems 4.2 and 4.5. The former is a direct consequence of Theorem 1.1 and relies on hypotheses involving the variation of f, whilst the latter only depends on the continuity of f and its asymptotic behaviour at infinity.

Remark 4.1. (i) In Theorem 1.1 we omit the simple case $f \geq 0$, since the uniform convergence of $\mathcal{L}^\alpha_{\nu,-\nu} f$ is clearly equivalent to $t^\nu f(t) \in L^1(\mathbb{R}_+)$.

(ii) The criterion for the uniform convergence of the Hankel transform can be derived by letting $\nu = 2\alpha + 1$ in Theorem 1.1, i.e., *if* (1.13) *holds, then $H_\alpha f(r)$ converges uniformly if and only if*

$$\left| \int_0^\infty t^{2\alpha+1} f(t)\, dt \right| < \infty.$$

In Theorem 1.1 we do not require that f vanishes at infinity. For functions satisfying the latter property, we have the following simplified statement.

Theorem 4.2. *Let $\nu \in \mathbb{R}$ and $\mu = -\nu$. Let f be vanishing at infinity and such that $t^\nu f(t) \in L^1(0,1)$. Assume that*

$$\int_M^\infty |df(t)| = o\big(M^{-\nu-1}\big) \quad \text{as } M \to \infty, \qquad \text{if } \nu < \alpha + 1/2 \text{ and } \nu > -1, \tag{4.1}$$

$$\int_M^\infty t^{\nu-\alpha-1/2} |df(t)| = o\big(M^{-\alpha-3/2}\big) \quad \text{as } M \to \infty, \qquad \text{if } \nu \geq \alpha + 1/2 \text{ or } \nu \leq -1. \tag{4.2}$$

Then, condition (1.14) *is necessary and sufficient to guarantee the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f(r)$ on $\mathbb{R}_+$.*

We can give an alternative statement to Theorem 1.1 for GM functions.

Corollary 4.3. *Let $\nu \in \mathbb{R}$. Let $f \in GM$ be real-valued and such that $t^\nu f(t) \in L^1(0,1)$. Then $\mathcal{L}^\alpha_{\nu,-\nu} f$ converges uniformly if and only if* (1.14) *holds.*

Remark 4.4. From the proof of the latter it is clear that the same conclusion holds for complex-valued $f \in GM$ if we also assume (1.12).

As mentioned above, we now prove a different criterion that depends on the continuity of f and its behaviour at infinity. Recall that if

$$F_\nu(x) = -\int_x^\infty t^\nu f(t)\, dt,$$

and (1.14) holds, the continuity of f implies that $F'_\nu(x) = x^\nu f(x)$, in virtue of the fundamental theorem of Calculus.

Theorem 4.5. *Let $f \in C(1,\infty)$ be such that $t^\nu f(t) \in L^1(0,1)$. Assume that $\alpha > 1/2$, and that* (1.12) *holds. Then, the transform $\mathcal{L}^\alpha_{\nu,-\nu} f$ converges uniformly if and only if* (1.14) *is satisfied.*

Note that the range of α for which Theorem 4.5 is valid is reduced compared to the one of Theorem 1.1. We also stress that contrarily to Theorems 1.1 and 4.2, Theorem 4.5 does not require any control on the variation of f.

Remark 4.6. Whenever f vanishes at infinity and $\nu > -1$, if $\nu < \alpha + 1/2$ then (4.1) implies (1.12), and if $\nu \geq \alpha + 1/2$, (4.2) implies (1.12). However, the converse is not true. Indeed, consider $f(t) = t^{-\nu-2}\sin t$, with $t > 1$. It is clear that (1.12) holds, and thus $\mathcal{L}^\alpha_{\nu,\mu} f$ converges uniformly, but since $f'(t) = -(\nu+2)t^{-\nu-3}\sin t + t^{-\nu-2}\cos t$, one has

$$M^{\nu+1}\int_M^\infty |f'(t)|\,dt \asymp M^{\nu+1}\int_M^\infty t^{-\nu-2}\,dt \asymp 1,$$
$$M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}|f'(t)|\,dt \asymp M^{\alpha+3/2}\int_M^\infty t^{-\alpha-5/2}\,dt \asymp 1,$$

for $M > 1$, or in other words, neither (4.1) nor (4.2) hold.

4.2 Proofs

Proof of Theorem 1.1. The necessity part follows from the convergence at $r = 0$ and the fact that $j_\alpha(0) = 1$.

In order to prove the sufficiency part, we show that the Cauchy remainder (1.8) vanishes uniformly in r as $N > M \to \infty$.

Let $0 < M < N$. If $r \geq 1/M$, integration by parts yields

$$\int_M^N t^\nu f(t) j_\alpha(rt)\,dt = \big[f(t) g^\nu_{\alpha,r}(t)\big]_M^N - \int_M^N g^\nu_{\alpha,r}(t)\,df(t).$$

It follows from (2.10) that

$$\begin{aligned}\left|\int_M^N t^\nu f(t) j_\alpha(rt)\,dt\right| &\lesssim \frac{1}{r^{\alpha+3/2}}\max_{x\geq M} x^{\nu-\alpha-1/2}|f(x)| + \frac{1}{r^{\alpha+3/2}}\int_M^N t^{\nu-\alpha-1/2}\,|df(t)| \\ &\leq \max_{x\geq M} x^{\nu+1}|f(x)| + M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}\,|df(t)|,\end{aligned}$$

and both terms vanish as $M \to \infty$, by applying (1.12) and (1.13).

If $r < 1/M$, we write

$$\int_M^N t^\nu f(t) j_\alpha(rt)\,dt = \left(\int_M^{1/r} + \int_{1/r}^N\right) t^\nu f(t) j_\alpha(rt)\,dt.$$

The integral $\int_{1/r}^N t^\nu f(t) j_\alpha(rt)\,dt$ can be estimated as above, as for the other integral, we have, by (2.1)

$$\begin{aligned}\left|\int_M^{1/r} t^\nu f(t) j_\alpha(rt)\,dt\right| &\le \max_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + \left|\int_M^{1/r} t^\nu f(t)(1-j_\alpha(rt))\,dt\right| \\ &\le \max_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + r\int_M^{1/r} t^{\nu+1}|f(t)|rt\,dt \\ &\le \max_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + \left(\max_{x\ge M} x^{\nu+1}|f(x)|\right)\int_M^{1/r} r\,dt \\ &\le \max_{x>M}\left|\int_M^x t^\nu f(t)\,dt\right| + \max_{x\ge M} x^{\nu+1}|f(x)|.\end{aligned}$$

The first term of the latter inequality vanishes as $M\to\infty$ by (1.14), whilst the second term also vanishes as $M\to\infty$, by (1.12). □

Proof of Theorem 4.2. Observe that since f is vanishing at infinity, we have that $|f(x)|\le \int_x^\infty |df(t)|$ for all x.

Let us first consider the case $\nu<\alpha+1/2$. On the one hand,

$$M^{\nu+1}|f(M)|\le M^{\nu+1}\int_M^\infty |df(t)|\to 0 \quad \text{as } M\to\infty.$$

On the other hand,

$$M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)|\le M^{\nu+1}\int_M^\infty |df(t)|\to 0 \quad \text{as } M\to\infty,$$

and the result follows, since we are under the conditions of Theorem 1.1.

If $\nu\ge\alpha+1/2$,

$$M^{\nu+1}|f(M)|\le M^{\nu+1}\int_M^\infty |df(t)|\le M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)|\to 0 \quad \text{as } M\to\infty,$$

i.e., we are under the conditions of Theorem 1.1, and the result follows (notice that in this case (1.13) implies (1.12)).

Finally, if $\nu\le -1$, since f vanishes at infinity, condition (1.12) is automatically satisfied, and the result follows, since (4.2) is precisely (1.13). □

Proof of Corollary 4.3. Similarly as above, the necessity follows from the convergence at $r=0$ and $j_\alpha(0)=1$.

In order to prove the sufficiency part, we need the following result whose proof is rather technical and will be shown elsewhere for the sake of brevity:

Lemma 4.7. *Let $g\in GM$ be real-valued and assume that $\int_0^\infty g(t)\,dt$ converges. Then $tg(t)\to 0$ as $t\to\infty$.*

The latter is a generalization of the well known Abel-Olivier's test that deals with non-negative monotone functions (see also [16], where the monotonicity assumption is relaxed). We emphasize that Theorem 4.7 g only needs to be real-valued, instead of nonnegative.

Since $f\in GM$, it follows that $t^\nu f(t)\in GM$ for every $\nu\in\mathbb{R}$. Therefore, by Lemma 4.7 (with $g(t)=t^\nu f(t)$), the convergence of $\int_0^\infty t^\nu f(t)\,dt$ implies that $t^{\nu+1}f(t)\to 0$ as $t\to\infty$, which is precisely condition (1.12).

To conclude the proof, we show that if $f \in GM$, then (1.12) implies (1.13), and the result will follow by Theorem 1.1. Indeed, since $\alpha \geq -1/2$,

$$\begin{aligned}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| &\lesssim \int_{M/2}^\infty \frac{1}{t}\int_t^{2t} s^{\nu-\alpha-1/2}|df(t)| \lesssim \int_{M/2}^\infty t^{\nu-\alpha-5/2}\int_{t/\lambda}^{\lambda t}|f(s)|\,ds \\ &\lesssim \int_{M/2}^\infty \Big(\max_{t/\lambda\leq x\leq \lambda t}|f(x)|\Big)t^{\nu-\alpha-3/2}\,dt \\ &\asymp \int_{M/2}^\infty \Big(\max_{t/\lambda\leq x\leq \lambda t}x^{\nu+1}|f(x)|\Big)t^{-\alpha-5/2}\,dt \\ &\lesssim \Big(\max_{x\geq M/(2\lambda)}x^{\nu+1}|f(x)|\Big)M^{-\alpha-3/2}.\end{aligned}$$

Thus, by (1.12),

$$M^{\alpha+3/2}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| \lesssim \max_{x\geq M/(2\lambda)} x^{\nu+1}|f(x)| \to 0 \qquad \text{as } M\to\infty,$$

i.e., (1.13) holds. This completes the proof. □

Proof of Theorem 4.5. The necessity part is clear, due to the convergence at $r = 0$.

Now we proceed to prove the sufficiency part. Let us denote

$$F_\nu(x) := \int_x^\infty t^\nu f(t)\,dt.$$

First of all, it follows from (2.4) and (2.6) that

$$\left|\frac{d}{dt}j_\alpha(rt)\right| \lesssim \frac{1}{t^{\alpha+1/2}r^{\alpha-1/2}}, \tag{4.3}$$

whenever $rt \geq 1$, or equivalently, $r \geq 1/t$. Now we proceed to estimate the integral

$$\int_M^\infty t^\nu f(t) j_\alpha(rt)\,dt,$$

which is equivalent to estimate the Cauchy remainder (1.8) as $N\to\infty$. On the one hand, if $r \geq 1/M$, we integrate by parts and obtain

$$\begin{aligned}\left|\int_M^\infty t^\nu f(t)j_\alpha(rt)\,dt\right| &\leq \big|j_\alpha(rM)F_\nu(M)\big| + \left|\int_M^\infty F_\nu(t)\Big(\frac{d}{dt}j_\alpha(rt)\Big)\,dt\right| \\ &\leq \max_{N\geq M}|F_\nu(N)| + \max_{N\geq M}|F_\nu(N)|\int_M^\infty\left|\frac{d}{dt}j_\alpha(rt)\right|dt \\ &\lesssim \max_{N\geq M}|F_\nu(N)|\Big(1+\frac{1}{r^{\alpha-1/2}}\int_M^\infty\frac{1}{t^{\alpha+1/2}}\,dt\Big) \\ &\leq \max_{N\geq M}|F_\nu(N)|\Big(1+M^{\alpha-1/2}\int_M^\infty\frac{1}{t^{\alpha+1/2}}\,dt\Big) \\ &\asymp \max_{N\geq M}|F_\nu(N)|,\end{aligned}$$

where we have applied (4.3) and used the fact that $\alpha > 1/2$. Since F_ν vanishes at infinity whenever (1.14) is satisfied, the above estimate vanishes as $M\to\infty$. On the other hand, if $r<1/M$, we write

$$\int_M^\infty t^\nu f(t) j_\alpha(rt)\,dt = \bigg(\int_M^{1/r} + \int_{1/r}^\infty\bigg) t^\nu f(t) j_\alpha(rt)\,dt,$$

and estimate $\int_{1/r}^\infty t^\nu f(t) j_\alpha(rt)\,dt$ as in the previous case. Similarly as in the proof of Theorem 1.1, estimate (2.1) yields

$$\begin{aligned}\bigg|\int_M^{1/r} t^\nu f(t) j_\alpha(rt)\,dt\bigg| &\le \max_{x>M}\bigg|\int_M^x t^\nu f(t)\,dt\bigg| + r\int_M^{1/r} t^{\nu+1}|f(t)|\,dt\\ &\le \max_{x>M}\bigg|\int_M^x t^\nu f(t)\,dt\bigg| + \max_{x\ge M} x^{\nu+1}|f(x)|,\end{aligned}$$

which vanishes as $M\to\infty$. □

5 Uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu}f$ with $0<\mu+\nu\le\alpha+3/2$

In this section we study the uniform convergence of sine-type transforms. We also mention some remarkable facts about the family of operators $\mathcal{L}^\alpha_{\alpha+1/2,0}$, $\alpha>-1/2$.

5.1 Main Results

Additionally to Theorem 1.2, here we give several results involving *GM* functions, and in some cases we can obtain a criterion for the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu}f$. The extremal case $\mu+\nu=0$ is not mentioned here, since it is already treated in Section 4 (see Theorems 1.1 and 4.2).

Remark 5.1. Let us observe an interesting property of the operator $\mathcal{L}^\alpha_{\alpha+1/2,0}$, with $\alpha > -1/2$ (if $\alpha=-1/2$, such operator is the cosine transform). Its kernel $K_\alpha(r,t)=K_\alpha(rt):=(rt)^{\alpha+1/2}j_\alpha(rt)$ is uniformly bounded and does not vanish at infinity in any of the variables r nor t (for any fixed α, this is the only kernel of the type (1.10) with this property). Moreover, K_α vanishes at the origin. Thus, such kernel has a similar behaviour as the kernel $K_{1/2}(rt)=\sin rt$ corresponding to $\hat{f}_{\sin}$. In fact, more than extending the sine transform, the sufficient condition that guarantees the uniform convergence of $\mathcal{L}^\alpha_{\alpha+1/2,0}f$ and $\hat{f}_{\sin}$ is the same, namely (cf. Theorem 1.2)

$$\int_x^\infty |df(t)| = o(1/x) \quad \text{as } x\to\infty.$$

Similarly as for cosine-type transforms, in Theorem 1.2 we do not assume that f vanishes at infinity; for functions satisfying the latter we claim the following:

Theorem 5.2. *Let $\nu,\mu\in\mathbb{R}$ be such that $0<\mu+\nu\le\alpha+3/2$, and let f be vanishing at infinity and such that $t^\nu f(t)\in L^1(0,1)$. Assume that*

$$\int_M^\infty |df(t)| = o\big(M^{\mu-1}\big) \quad \text{as } M\to\infty, \qquad \text{if } \nu<\alpha+1/2 \text{ and } \mu<1, \quad (5.1)$$

$$\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| = o\big(M^{\mu+\nu-\alpha-3/2}\big) \quad \text{as } M\to\infty, \qquad \text{if } \nu\ge\alpha+1/2 \text{ or } \mu\ge 1. \quad (5.2)$$

Then $\mathcal{L}^{\alpha}_{\nu,\mu} f$ converges uniformly on $\mathbb{R}_+$.

We can refine Theorem 1.2 by assuming that $f \in GM$. Furthermore, in this case we can obtain a criterion for non-negative GM functions.

Theorem 5.3. *Let ν, μ be such that $0 < \mu + \nu < \alpha + 3/2$. Let f be a GM function such that $t^{\nu} f(t) \in L^1(0,1)$.*

1. *If*
$$|f(M)| = o\big(M^{\mu-1}\big) \quad \text{as } M \to \infty, \tag{5.3}$$
then $\mathcal{L}^{\alpha}_{\nu,\mu} f$ converges uniformly.

2. *If $f \geq 0$ and $\mathcal{L}^{\alpha}_{\nu,\mu} f$ converges uniformly, then* (5.3) *holds.*

The "if and only if" statement reads as follows:

Corollary 5.4. *Let $f \in GM$ be non-negative, $\alpha \geq -1/2$, and ν, μ be such that $0 < \mu + \nu < \alpha + 3/2$. Then, $\mathcal{L}^{\alpha}_{\nu,\mu} f$ converges uniformly if and only if* (5.3) *holds.*

The GM condition in the sufficiency part of Theorem 5.3 (and therefore also in Corollary 5.4) is sharp, as shown by our next statement.

Proposition 5.5. *Let $0 < \mu + \nu < \alpha + 3/2$. There exists $f \notin GM$ such that condition* (5.3) *does not hold, but $\mathcal{L}^{\alpha}_{\nu,\mu} f$ converges uniformly.*

Note that in Theorem 5.3 we exclude the case $\mu + \nu = \alpha + 3/2$. Actually, the proof of the latter relies on the fact that if $f \in GM$ and $0 < \mu + \nu < \alpha + 3/2$, then (5.3) implies (5.2), and the result follows by Theorem 1.2. This is not the case in the extremal case $\mu + \nu = \alpha + 3/2$.

Proposition 5.6. *Let $\nu \in \mathbb{R}$ and $\mu < 1$ be such that $\mu + \nu = \alpha + 3/2$. If $f \in GM$ vanishes at infinity, then* (5.2) *is equivalent to $t^{-\mu} f(t) \in L^1(1,\infty)$.*

It is clear that if $\mu + \nu = \alpha + 3/2$, then (5.3) does not imply (5.2), since the latter does not imply $t^{-\mu} f(t) \in L^1(1,\infty)$ even for decreasing functions. Moreover, Proposition 5.6 does not hold for $\mu = 1$. Indeed, for f decreasing we have

$$\int_M^{\infty} t^{1-\mu} |df(t)| = \int_M^{\infty} |df(t)| = f(M),$$

and the convergence of $\int_1^{\infty} |df(t)|$ is equivalent to $f(M) \to 0$ as $M \to \infty$. As mentioned above, such condition does not imply that $M^{-1} f(M)$ is integrable.

5.2 Proofs

Proof of Theorem 1.2. Again, we prove that the Cauchy remainder (1.8) vanishes uniformly in r as $N > M \to \infty$. Let $0 < M < N$, and assume that $1/r \leq M$. Integration by parts together with the representation of (2.9), and estimate (2.10) yield

$$
\begin{aligned}
r^{\mu+\nu}\int_M^N t^\nu f(t)j_\alpha(rt)\,dt &= r^{\mu+\nu}\Big(\big[f(t)g^\nu_{\alpha,r}(t)\big]_M^N - \int_M^N g^\nu_{\alpha,r}(t)\,df(t)\Big)\\
&\lesssim r^{\mu+\nu}\Big(\frac{N^{\nu-\alpha-1/2}}{r^{\alpha+3/2}}|f(N)| + \frac{M^{\nu-\alpha-1/2}}{r^{\alpha+3/2}}|f(M)|\\
&\quad + \frac{1}{r^{\alpha+3/2}}\int_M^N t^{\nu-\alpha-1/2}\,|df(t)|\Big)\\
&\lesssim \max_{x\geq M} x^{1-\mu}|f(x)| + M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}\,|df(t)|,
\end{aligned}
$$

which vanishes as $M\to\infty$, by (5.3) and (5.2).

If $1/r > M$, we write

$$
r^{\mu+\nu}\int_M^N t^\nu f(t)j_\alpha(rt)\,dt = r^{\mu+\nu}\Big(\int_M^{1/r}+\int_{1/r}^N\Big)t^\nu f(t)j_\alpha(rt)\,dt,
$$

and estimate the integral $\int_{1/r}^N t^\nu f(t)j_\alpha(rt)\,dt$ as above. Furthermore, since $\mu+\nu>0$, it follows that

$$
\begin{aligned}
\Big|r^\mu\int_M^{1/r}(rt)^\nu f(t)j_\alpha(rt)\,dt\Big| &\leq r^\mu\int_M^{1/r}(rt)^\nu|f(t)|\,dt = r^{\mu+\nu}\int_M^{1/r} t^\nu|f(t)|\,dt\\
&= r^{\mu+\nu}\int_M^{1/r} t^{\mu+\nu-1}t^{1-\mu}|f(t)|\,dt\\
&\leq \Big(\max_{x\geq M} x^{1-\mu}|f(x)|\Big)r^{\mu+\nu}\int_0^{1/r} t^{\mu+\nu-1}\,dt\\
&\asymp \max_{x\geq M} x^{1-\mu}|f(x)|,
\end{aligned}
$$

which vanishes as $M\to\infty$, by (5.3). □

Proof of Theorem 5.2. We will see that our hypotheses imply those of Theorem 1.2, and the result will follow. Consider first the case $\mu<1$ and $\nu<\alpha+1/2$. Then

$$
M^{1-\mu}|f(M)| \leq M^{1-\mu}\int_M^\infty |df(t)| \to 0 \quad \text{as } M\to\infty,
$$

and

$$
M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| \leq M^{1-\mu}\int_M^\infty |df(t)| \to 0 \quad \text{as } M\to\infty.
$$

If $\mu\geq 1$, since f vanishes at infinity, (1.15) holds, and the hypotheses of Theorem 1.2 are met.

Finally, if $\nu\geq\alpha+1/2$,

$$
M^{1-\mu}|f(M)| \leq M^{1-\mu}\int_M^\infty |df(t)| \leq M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| \to 0 \quad \text{as } M\to\infty,
$$

i.e., the hypotheses of Theorem 1.2 hold. □

Proof of Theorem 5.3. Since $f \in GM$, (5.3) implies (5.2) for any choice of the parameters. Indeed, since $\mu+\nu<\alpha+3/2$,

$$\begin{aligned}\int_M^\infty t^{\nu-\alpha-1/2}|df(t)| &\lesssim \int_{M/2}^\infty \frac{1}{t}\int_t^{2t} s^{\nu-\alpha-1/2}|df(s)| \lesssim \int_{M/2}^\infty t^{\nu-\alpha-5/2}\int_{t/\lambda}^{\lambda t}|f(s)|\,ds\\ &\lesssim \int_{M/2}^\infty \Big(\max_{t/\lambda\le x\le \lambda t}|f(x)|\Big)t^{\nu-\alpha-3/2}\,dt\\ &\asymp \int_{M/2}^\infty \Big(\max_{t/\lambda\le x\le \lambda t} x^{1-\mu}|f(x)|\Big)t^{\mu+\nu-\alpha-5/2}\,dt\\ &\lesssim \Big(\max_{x\ge M/(2\lambda)} x^{1-\mu}|f(x)|\Big)M^{\mu+\nu-\alpha-3/2}.\end{aligned}$$

Thus, we deduce that

$$M^{\alpha+3/2-\mu-\nu}\int_M^\infty t^{\nu-\alpha-1/2}\,|df(t)| \lesssim \max_{t\ge M/(2\lambda)} t^{1-\mu}|f(t)| \to 0 \quad \text{as } M\to\infty,$$

so that the result follows by applying Theorem 1.2. This completes the first part of the proof.

For the second part, the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu}f$ implies that the Cauchy remainder

$$r^\mu \int_{1/(\lambda r)}^{\lambda/r} (rt)^\nu f(t) j_\alpha(rt)\,dt \asymp r^\mu \int_{1/(\lambda r)}^{\lambda/r} f(t)\,dt$$

vanishes whenever $r\to 0$, where $\lambda>0$ is the GM constant (cf. Definition 2.8). By (2.11), we have

$$f(1/r) \lesssim r\int_{1/(\lambda r)}^{\lambda/r} f(t)\,dt = r^{1-\mu}r^\mu \int_{1/(\lambda r)}^{\lambda/r} f(t)\,dt,$$

and we deduce that $r^{\mu-1}f(1/r)\to 0$ as $r\to 0$, or equivalently, $t^{1-\mu}f(t)\to 0$ as $t\to\infty$. □

Proof of Proposition 5.5. We construct f in a general setting and then we subdivide the proof into two parts, namely $0<\mu+\nu\le\alpha+1/2$ and $\alpha+1/2<\mu+\nu<\alpha+3/2$.

Let c_n be an increasing nonnegative sequence and $\varepsilon_n>0$ such that $\varepsilon_n<c_{n+1}-c_n$ and $\varepsilon_n\le c_n$ for every n. Define

$$f(t)=\begin{cases} t^{\mu-1}, & \text{if } t\in[c_n,c_n+\varepsilon_n],\ n\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that for such function, $t^{1-\mu}f(t)\not\to 0$ as $t\to\infty$. We are now going to find choices of c_n and ε_n in such a way that $f\notin GM$ and $\mathcal{L}^\alpha_{\nu,\mu}f$ converges uniformly. Also, for any c_n and ε_n, $t^\nu f(t)\in L^1(0,1)$, since $\mu+\nu>0$.

Let us first consider the case $0<\mu+\nu\le\alpha+1/2$. According to Corollary 3.2, the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu}f$ follows from $t^\nu f(t)\in L^1(\mathbb{R}_+)$, which in this case is equivalent to

$$\sum_{n=1}^\infty \varepsilon_n c_n^{\nu+\mu-1}<\infty.$$

Choosing $c_n = 2^n$ and $\varepsilon_n = 2^{-n\beta}$ with $\beta > \nu+\mu-1$, we find that the latter series converges, hence the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ follows. Note also that $f \notin GM$.

Consider now the case $\alpha + 1/2 < \mu + \nu < \alpha + 3/2$. According to Corollary 3.6, the uniform convergence of $\mathcal{L}^\alpha_{\nu,\mu} f$ follows from the conditions

$$t^{\nu-\alpha-1/2} f(t) \to 0 \text{ as } t \to \infty, \qquad t^{\nu-\alpha-1/2} f(t) \in L^1(1,\infty), \qquad \int_1^\infty t^{\nu-\alpha-1/2} |df(t)| < \infty.$$

Since $\mu + \nu < \alpha + 3/2$, $t^{\nu-\alpha-1/2} f(t) \to 0$ as $t \to \infty$. Also,

$$\int_1^\infty t^{\nu-\alpha-1/2} f(t)\, dt = \sum_{n=1}^\infty \varepsilon_n c_n^{\mu+\nu-\alpha-3/2}, \tag{5.4}$$

and

$$\int_1^\infty t^{\nu-\alpha-1/2} |df(t)| \lesssim \sum_{n=1}^\infty \left(c_n^{\mu+\nu-\alpha-3/2} + (c_n + \varepsilon_n)^{\mu+\nu-\alpha-3/2} \right) \lesssim \sum_{n=1}^\infty c_n^{\mu+\nu-\alpha-3/2}. \tag{5.5}$$

Choosing $c_n = 2^n$ and $\varepsilon_n = 1$, we find that series on the right hand sides of (5.4) and (5.5) are convergent, so that $\mathcal{L}^\alpha_{\nu,\mu} f$ converges uniformly by Corollary 3.6, and $f \notin GM$. □

Proof of Proposition 5.6. The proof is similar to that of Proposition 5.4 in [16]. First of all note that since f is locally of bounded variation, condition (5.2) is equivalent to the convergence of $\int_1^\infty t^{1-\mu} |df(t)|$. Since f vanishes at infinity and $\mu < 1$, the estimate

$$\int_1^\infty t^{-\mu} |f(t)|\, dt \le \int_1^\infty t^{-\mu} \int_t^\infty |df(s)|\, dt = \int_1^\infty |df(s)| \int_1^s t^{-\mu}\, dt \lesssim \int_1^\infty t^{1-\mu} |df(t)|$$

proves one direction of the statement, without mention to the GM condition. As for the other direction, we have, since $f \in GM$,

$$\begin{aligned} \int_1^\infty s^{1-\mu} |df(s)| &= \sum_{k=0}^\infty \int_{2^k}^{2^{k+1}} s^{1-\mu} |df(s)| \lesssim \sum_{k=0}^\infty (2^k)^{1-\mu} \frac{1}{2^k} \int_{2^k/\lambda}^{\lambda 2^k} |f(t)|\, dt \\ &\asymp \sum_{k=0}^\infty \int_{2^k/\lambda}^{\lambda 2^k} t^{-\mu} |f(t)|\, dt \lesssim \int_{1/\lambda}^\infty t^{-\mu} |f(t)|\, dt, \end{aligned}$$

as desired. Observe that the latter holds for any μ. □

5.3 Optimality of Theorems 1.2 and 5.2

Sharpness. Here we are interested in studying if the conclusions of Theorems 1.2 and 5.2 hold if we replace o by O in conditions (1.15) and (1.16), or (5.1) and (5.2).

1. Case $\mu < 1$. In this case, we will not discuss sharpness of Theorem 1.2, since condition (1.15) implies that f vanishes at infinity, and therefore we are in the situation of Theorem 5.2. Consider the function $f(t) = t^{1-\mu}$ and $\mu + \nu < \alpha + 3/2$. It is clear that neither (5.1) nor (5.2) hold, but they are satisfied if we replace o by O. Since $\mu + \nu > 0$, we have for any $r > 0$

$$r^{\mu+\nu} \int_{1/(2r)}^{1/r} t^\nu f(t) j_\alpha(rt)\, dt \asymp r^{\mu+\nu} \int_{1/(2r)}^{1/r} t^{\mu+\nu-1}\, dt \asymp 1,$$

i.e., the Cauchy remainder does not vanish as $r \to 0$, and therefore $\mathcal{L}^{\alpha}_{\nu,\mu} f$ does not converge uniformly.

2. Case $\mu = 1$. Note that in this case the statements of Theorems 1.2 and 5.2 are equivalent. If $f(t) = 1$, it is clear that (1.15) does not hold, but holds with O in place of o, whilst (1.16) trivially holds. The Cauchy remainder is the same as in the previous example, substituting $\mu = 1$, and thus $\mathcal{L}^{\alpha}_{\nu,\mu} f$ does not converge uniformly.

3. Case $\mu > 1$. Here the example $f(t) = t^{1-\mu}$ shows that Theorem 1.2 does not hold if we replace o by O in (1.15) and (1.16). The examples $f(t) = t^{\mu-2} \sin t$ and $f(t) = 1$ show that in general, conditions (1.15) and (1.16) do not imply each other.

Finally, we show that the sufficient conditions involving the variation of f that imply the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ (see Theorems 1.2 and 5.2) do not imply neither follow from those integrability conditions that also imply the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ (cf. Corollary 3.6).

Independence of Theorem 1.2 and Corollary 3.6. Let $f(t) = t^{\mu-2} \sin t$ for $t > 1$, and $\alpha + 1/2 \le \mu + \nu < \alpha + 3/2$. Since

$$f'(t) = (\mu - 2)t^{\mu-3} \sin t + t^{\mu-2} \cos t,$$

we have that

$$M^{\alpha+3/2-\mu-\nu} \int_M^\infty t^{\nu-\alpha-1/2} |f'(t)|\, dt \asymp M^{\alpha+3/2-\mu-\nu} \int_M^\infty t^{\mu+\nu-\alpha-5/2}\, dt \asymp 1,$$

or in other words, (1.16) does not hold. Thus, the hypotheses of Theorem 1.2 are not satisfied. Nevertheless, the choice of the parameters implies that $t^{\nu-\alpha-1/2} f(t) \in L^1(0,1)$ (and $t^{-\mu} f(t) \in L^1(1,\infty)$ if $\mu+\nu = \alpha+1/2$), and moreover the conditions (3.1) hold. Hence, the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$ follows, by Corollary 3.6.

On the other hand, let $\alpha = 1/2$, $\nu = 1$ and $\mu = 0$ (recall that $\mathcal{L}^{1/2}_{1,0} f = \hat{f}_{\sin}$). If $f(t) = \dfrac{1}{t \log t}$ for $t > 2$, then clearly $f(t) \notin L^1(2,\infty)$, but (1.16) holds, and the uniform convergence of $\hat{f}_{\sin}$ follows by Theorem 1.2 (or also by Theorem 5.3, since f vanishes at infinity).

Independence of Theorem 5.2 and Corollary 3.6. Let us consider again $f(t) = t^{\mu-2} \sin t$, with $\mu < 2$. We have already seen that $M^{\alpha+3/2-\mu-\nu} \int_M^\infty t^{\nu-\alpha-1/2} |f'(t)|\, dt \asymp 1$, and that additionally to $t^{\nu-\alpha-1/2} f(t) \in L^1(1,\infty)$, the conditions (3.1) hold. Thus, in the case $\nu \ge \alpha + 1/2$ or $\mu \ge 1$, we cannot apply Theorem 5.2, but we can apply Corollary 3.6 instead to deduce the uniform convergence of $\mathcal{L}^{\alpha}_{\nu,\mu} f$. On the other hand, it is easy to see that if $\nu < \alpha + 1/2$ and $\mu < 1$, the hypotheses of Corollary 3.6 hold, and $M^{1-\mu} \int_M^\infty |f'(t)|\, dt \asymp 1$.

Now let $\nu \ge \alpha + 1/2$ and $\alpha + 1/2 - \nu \le \mu < 1$. If $f(t) = \dfrac{t^{\alpha-\nu-1/2}}{\log t}$, then f vanishes at infinity, but $t^{\nu-\alpha-1/2} f(t) = \dfrac{1}{t \log t} \notin L^1(2,\infty)$. However,

$$M^{\alpha+3/2-\mu-\nu} \int_M^\infty t^{\nu-\alpha-1/2} |f'(t)|\, dt \lesssim M \int_M^\infty \frac{1}{t^2 \log t}\, dt \lesssim \frac{1}{\log M} \to 0 \quad \text{as } M \to \infty,$$

so that (5.2) holds. In the case $\mu \ge 1$, note that $\nu < \alpha + 1/2$, and hence the inequality $\mu + \nu \ge \alpha + 1/2$ implies that $\alpha - 1/2 \le \nu$. Thus, the same function f as above vanishes at infinity, and also satisfies (5.2), whilst $t^{\nu-\alpha-1/2}f(t) \notin L^1(2,\infty)$. Finally, consider the case $\nu < \alpha + 1/2$ and $\mu < 1$. Let $f(t) = \dfrac{t^{\mu-1}}{\log t}$. The inequality $\mu + \nu \geq \alpha + 1/2$ implies that

$$t^{\nu-\alpha-1/2}f(t) = \frac{t^{\mu+\nu-\alpha-3/2}}{\log t} \geq \frac{1}{t\log t} \notin L^1(2,\infty),$$

hence f is not under the hypotheses of Corollary 3.6. However, note that since f is monotone,

$$M^{1-\mu}\int_M^\infty |f'(t)|\,dt = M^{1-\mu}f(M) = \frac{1}{\log M} \to 0 \quad \text{as } M \to \infty,$$

and $\mathcal{L}^{\alpha}_{\nu,\mu}f$ converges uniformly, in virtue of Theorem 5.2 (or also by Theorem 5.3).

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