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Beltrami equations in the plane and Sobolev regularity

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Abstract

New results regarding the Sobolev regularity of the principal solution of the linear Beltrami equation $\bar{\partial} f = \mu \partial f + \nu \overline{\partial f}$ for discontinuous Beltrami coefficients μ and ν are obtained, using Kato-Ponce commutators, obtaining that $\bar{\partial} f$ belongs to a Sobolev space with the same smoothness as the coefficients but some loss in the integrability parameter. A conjecture on the cases where the limitations of the method do not work is raised.

1 Introduction

Given $\mu, \nu \in L^\infty$ compactly supported satisfying the elliptic condition

$$\| |\mu| + |\nu| \|_{L^\infty} = \kappa < 1, \tag{1.1}$$

the Beltrami equation

$$\bar{\partial} f = \mu \partial f + \nu \overline{\partial f} \tag{1.2}$$

has a unique homeomorphic solution $f \in W^{1,2}_{loc}$ such that $f(z) - z = \mathcal{O}_{z\to\infty}(1/z)$, which we call principal solution to (1.2). The existence of this solution depends deeply on the fact that the Beurling transform

$$\mathcal{B} f = -p.v. \frac{1}{\pi z^2} * f$$

is bounded on L^p spaces for $1 < p < \infty$ and unitary in L^2 (see [AIM09, Chapter 4], for instance).

For each $\kappa < 1$, we define the extremal exponent $p_\kappa := 1 + \frac{1}{\kappa} > 2$. In 1992, Kari Astala published a celebrated theorem on the area distortion of quasiconformal mappings, which implies that every quasiconformal mapping f with Beltrami coefficient $\mu \in L^\infty_c$ such that $\|\mu\|_{L^\infty} = \kappa < 1$ satisfies that

$$\bar{\partial} f \in L^p \qquad \text{whenever } \frac{1}{p_\kappa} < \frac{1}{p} \leqslant 1 \tag{1.3}$$

(see [Ast94, Corollary 1.2]). Some years later, Astala, Iwaniec and Saksman found the following remarkable (and sharp) result.

Theorem 1.1 ([AIS01, Theorem 3]). *Given $\mu, \nu \in L^\infty$ with $\| |\mu| + |\nu| \|_{L^\infty} = \kappa < 1$, the operator*

$$Id - \mu \mathcal{B} - \nu \overline{\mathcal{B}}$$

is invertible on $L^p(\mathbb{C})$ for $\frac{1}{p_\kappa} < \frac{1}{p} < \frac{1}{p'_\kappa}$, with

$$\left\| (Id - \mu \mathcal{B} - \nu \overline{\mathcal{B}})^{-1} \right\|_{L^p \to L^p} \leqslant C_{\kappa,p}. \tag{1.4}$$

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When the coefficients satisfy some extra assumption, we can improve the previous results. We discuss this results in Section 2.3, after a brief introduction of the function spaces we consider. Nevertheless, we sketch here the general facts. Given $0 < s < \infty$ and $1 < p < \infty$, we say that the Sobolev space $W^{s,p}(\mathbb{R}^d)$ (in the sense of Bessel potential spaces, see [Tri83, Section 2.2.2]) is critical if $s - \frac{d}{p} = 0$. If $s - \frac{d}{p} > 0$ it is called supercritical and if, instead, $s - \frac{d}{p} < 0$, then it is called subcritical. Note that so-called differential dimension $s - \frac{d}{p}$ coincides with the homogeneity exponent of the semi-norms of these spaces. The functions in supercritical spaces are continuous, while the functions on critical spaces are just in the space of vanishing mean oscillation functions (VMO), i.e., the closure of C_c^∞ in BMO. For the subcritical spaces we have less self-improvement (see Section 2.1 below).

Roughly speaking, when the coefficients μ and ν are in a supercritical Sobolev space, then $\bar{\partial} f$ inherits the regularity of the Beltrami coefficient. In the critical situation, there is a small loss, and in the subcritical case, there is a bigger gap, in the spirit of (1.3), where μ is in every L^p space but $\bar{\partial} f$ is only p-integrable for a certain range.

The supercritical case is well understood (see [AIM09, Chapter 15] for Hölder spaces and [CFM+09] and [CMO13] for Sobolev, as well as Besov and Triebel-Lizorkin spaces). The critical case is studied in the latter two papers as well as in [BCO17] and [BCG+16], while the literature on the subcritical cases is less complete (see [CFM+09] and [CFR10]). This note is devoted to unify the approaches for the critical and subcritical situations, in the quest to find a complete sharp theory.

Sharp bounds in this theory may lead to a better understanding of the stability of the Calderón inverse problem, as shown in [CFR10]. There, the authors prove that if one knows all the possible pairs of Dirichlet and Neumann data of the solutions to the conductivity equation for conductivities satisfying certain a priori subcritical Sobolev conditions, then the recovery is stable. A crucial step there is to solve a Beltrami equation as (1.2) above: after showing Sobolev regularity of the principal solution to a certain family of equations, the authors show an asymptotic decay of the so-called Complex Geometric Optics Solution. Greater Sobolev regularity of these solutions is translated into higher decay of the solutions and better stability estimates.

We show the following result.

Theorem 1.2. *Let $0 < s < 2$, $1 < p < \infty$, let $\mu, \nu \in W_c^{s,p}(\mathbb{C}) \cap L^\infty$ satisfy (1.1) for $\kappa < 1$ and let f be the principal solution to the Beltrami equation (1.2).*

If $s = \frac{2}{p}$, then

$$\bar{\partial} f \in W^{s,q} \qquad \text{for every } \frac{1}{q} > \frac{1}{p}. \tag{1.5}$$

If $s < \frac{2}{p}$ and $\frac{1}{p} < \frac{1}{p'_\kappa} - \frac{1}{p_\kappa} = \frac{1-\kappa}{1+\kappa}$, then

$$\bar{\partial} f \in W^{s,q} \qquad \text{for every } \frac{1}{q} > \frac{1}{p} + \frac{1}{p_\kappa}. \tag{1.6}$$

However, the restriction $\frac{1}{p} < \frac{1}{p'_\kappa} - \frac{1}{p_\kappa}$ seems rather unnatural (see Section 2.3). Due to this fact, there is only room for p in the conditions for (1.6) if $s < 2\frac{1-\kappa}{1+\kappa}$, which is equivalent to $\kappa < \frac{2-s}{2+s}$. Therefore it is natural to ask whether it can be removed or not (see Conjecture 2.4 below).

Using some embeddings explained in Section 2, we can deduce the following corollary, which covers the case when $\frac{1}{p} \geqslant \frac{1}{p'_\kappa} - \frac{1}{p_\kappa}$ as well.

Corollary 1.3. *Let $0 < s < 2$, $1 < p < \infty$, with $s < \frac{2}{p}$. If $0 < \Theta \leqslant 1$ with $\frac{\Theta}{p} < \frac{1}{p'_\kappa} - \frac{1}{p_\kappa}$, then*

$$\bar{\partial} f \in W^{\theta s,q} \qquad \text{for every } \frac{1}{q} > \frac{\Theta}{p} + \frac{1}{p_\kappa}.$$

The paper is organized in the following way. Section 2 is devoted to making the background of this article clear. In Section 2.1 the definitions and basic properties of the Triebel-Lizorkin and related spaces are given. Section 2.2 specifies some properties of compactly supported Triebel-Lizorkin functions, in order to make a clear picture of the problem and to provide the reader a guide to understand the full scale of Sobolev regularity obtained for the principal mappings in Theorem 1.2. In Section 2.3 there is a discussion on the existing results using the concepts introduced in the former sections. Finally, Section 3 contains the proof of Theorem 1.2.

2 Background

2.1 Definitions and well-known properties of function spaces

First we recall some results on Triebel-Lizorkin spaces.

Let $\{\psi_j\}_{j=0}^{\infty} \subset C_c^{\infty}(\mathbb{R}^d)$ with ψ_0 supported in $\mathbb{D}(0,2)$, ψ_j supported in $\mathbb{D}(0,2^{j+1})\backslash\mathbb{D}(0,2^{j-1})$ for $j \geqslant 1$, such that $\sum_{j=0}^{\infty}\psi_j \equiv 1$ and for every multiindex $\alpha \in \mathbb{N}^d$ there exists a constant c_α such that

$$\|D^\alpha \psi_j\|_\infty \leqslant \frac{c_\alpha}{2^{j(\alpha_1+\cdots+\alpha_d)}} \quad \text{for every } j \geqslant 0.$$

We will use the classical notation $\widehat{f}$ for the Fourier transform of a given Schwartz function,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,dx,$$

and $\check{f}$ will denote its inverse. It is well known that the Fourier transform can be extended to the whole space of tempered distributions by duality and it induces an isometry in L^2 (see for example [Gra08, Chapter 2]).

Definition 2.1. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$. For any tempered distribution $f \in S'(\mathbb{R}^d)$ we define its non-homogeneous Triebel-Lizorkin quasi-norm*

$$\|f\|_{F^s_{p,q}} = \left\| \left\| \left\{ 2^{sj} \left(\psi_j \widehat{f}\right)\check{} \right\} \right\|_{l^q} \right\|_{L^p},$$

and we call $F^s_{p,q} \subset S'$ to the set of tempered distributions such that this quasi-norm is finite.

These quasi-norms (norms when $p, q \geqslant 1$) are equivalent for different choices of $\{\psi_j\}_{j=0}$ (see [Tri83, Section 2.3]). Changing the order of integration and summation above we get the non-homogeneous Besov quasi-norm

$$\|f\|_{B^s_{p,q}} = \left\| \left\{ 2^{sj} \left\| \left(\psi_j \widehat{f}\right)\check{} \right\|_{L^p} \right\} \right\|_{l^q},$$

which makes sense for $0 < p \leqslant \infty$.

For $q = 2$ and $1 < p < \infty$ the Triebel-Lizorkin spaces coincide with the so-called Bessel-potential spaces. In addition, if $s \in \mathbb{N}$ they coincide with the usual Sobolev spaces of functions in L^p with weak derivatives up to order s in L^p, and they coincide with L^p for $s = 0$ ([Tri83, Section 2.5.6]). In the present text, we use the convention

$$W^{s,p} := F^s_{p,2} \qquad \text{for } s \geqslant 0 \text{ and } 1 < p < \infty.$$

and for $s = 0$ we write

$$h^p := F^0_{p,2} \qquad \text{for } 0 < p < \infty,$$

that is, the non-homogeneous hardy space (which coincides with L^p for $1 < p < \infty$). With this convention, complex interpolation between Sobolev spaces is a Sobolev space (see [Tri78, Section 2.4.2, Theorem 1]). For more information on the relation between Triebel-Lizorkin and Besov spaces with other classical spaces we refer the reader to [Tri83, Section 2.2.2] and [RS96, Theorem 2.2.2].

There is a whole structure of embeddings for Triebel-Lizorkin spaces (see [Tri83, Proposition 2.3.2 and Theorem 2.7.1], [RS96, Chapter 1] for end-point cases). For instance, if $-\infty < s < \infty$, $0 < p < \infty$, $0 < q_0, q_1 \leqslant \infty$ and $\varepsilon > 0$, we have that

$$F^s_{p,q_0} \subset F^s_{p,q_0+\varepsilon} \qquad \text{and} \qquad F^{s+\varepsilon}_{p,q_0} \subset F^s_{p,q_1} \tag{2.1}$$

and if $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 < \infty$ satisfy that $s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}$, for $0 < q_0, q_1 \leqslant \infty$ we have that

$$F^{s_0}_{p_0,q_0} \subset F^{s_1}_{p_1,q_1}. \tag{2.2}$$

(see Figure 2.1). Besov spaces present a similar structure.

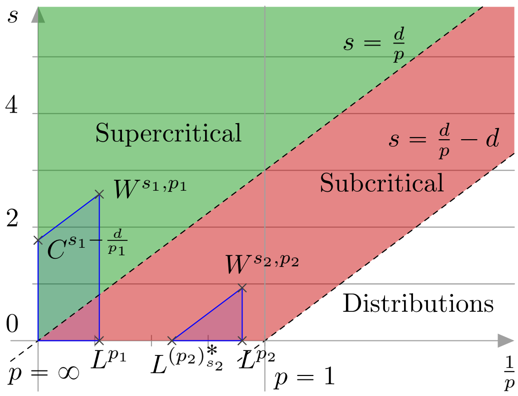


Figure 2.1: General embeddings for Sobolev spaces (and Triebel-Lizorkin spaces with q fixed) in dimension $d = 3$ (see (2.1), (2.2) and subsequent embeddings).

Regarding classical spaces, whenever $0 < p < \infty$ and $0 < q < \infty$, the following holds true:

- If $s > \frac{d}{p}$ (supercritical case) then $F^s_{p,q} \subset \mathcal{C}^{s-\frac{d}{p}} \cap h^p$.
- If $s = \frac{d}{p}$ (critical case), then $F^s_{p,q} \subset VMO \cap h^p$.
- If $\frac{d}{p} - d < s < \frac{d}{p}$ (subcritical case), then $F^s_{p,q} \subset L^{(p)^*_s} \cap h^p$ where $\frac{1}{(p)^*_s} = \frac{1}{p} - \frac{s}{d} < 1$.

2.2 Compactly supported functions

Let us assume that $\mu \in W^{s,p}$ is compactly supported with $p > 1$. Since $\mu \in L^{(p)^*_s}$ (note that one can do the same if $\mu \in F^s_{p,q}$ when $s - \frac{d}{p} > -d$ as we noted above), which coincides with the Hardy space h_p, also $\mu \in h_r$ for $0 < r < p$ (see [Tri83, Section 2.2.2]). By interpolation, we have that $\mu \in W^{\sigma,r}$ as well for $\sigma < s$ and $1 < r \leqslant p$. Combined with the embeddings described in (2.1) and (2.2), this gives us an almost complete picture of the spaces where μ does belong. It remains to see what happens in the endpoint $\sigma = s$ and in the remaining spaces of the scale $F^s_{p,q}$. If s is a natural number, then the derivatives of order s are compactly supported as well and it follows that

$\mu \in W^{s,r}$ for every $0 < r \leqslant p$, but in case s is not a natural number, some extra work needs to be done. Since the author does not know any mention in the literature about this case, it is studied in this section.

We will argue using expressions in terms of differences. Let us write $\Delta_h^1 f(x) := f(x+h) - f(x)$ and, if $M \in \mathbb{N}$ with $M > 1$ we define the M-th iterated difference as $\Delta_h^M f(x) := \Delta_h^1(\Delta_h^{M-1} f)(x) = \sum_{j=0}^M \binom{M}{j} (-1)^{M-j} f(x+jh)$.

Lemma 2.2. *Let $d \in \mathbb{N}$, $0 < s < \infty$, $\frac{d}{d+s} < p_1 \leqslant p_0 < \infty$ and $\frac{d}{d+s} < q \leqslant \infty$. For every compactly supported function $f \in F^s_{p_0,q}$, we have that $f \in F^s_{p_1,q}$.*

Proof. Let $f \in F^s_{p_0,q}$ be given with compact support in $\mathbb{D}_R$. Choose a natural number $M > s$. Clearly

$$d_t^M f(x) := t^{-d} \int_{|h| \leqslant t} |\Delta_h^M f(x)| \, dh,$$

is compactly supported in the disk $\mathbb{D}_{R+M}$ for $t \leqslant 1$, and so is

$$w_q^M f(x) := \left(\int_0^1 \frac{d_t^M f(x)^q}{t^{sq+1}} \, dt \right)^{\frac{1}{q}},$$

(with the usual modifications for $q = \infty$).

By hypothesis we have $\frac{d}{\min\{p_0,p_1,q\}} - d < s$. By [Tri06, Theorem 1.116], writing $\overline{p_j} := \max\{1, p_j\}$ for every locally integrable function g we have that

$$\|g\|_{F^s_{p_j,q}(\mathbb{R}^d)} \approx \|g\|_{L^{p_j}} + \|g\|_{L^{\overline{p_j}}} + \left\|w_q^M g\right\|_{L^{p_j}}. \tag{2.3}$$

Now, the norm in (2.3) and the Hölder inequality grant that

$$\begin{aligned} \|f\|_{F^s_{p_1,q}} &\approx \|f\|_{L^{\overline{p_1}}(\mathbb{D}_R)} + \|f\|_{L^{p_1}(\mathbb{D}_R)} + \left\|w_q^M f\right\|_{L^{p_1}(\mathbb{D}_{R+M})} \\ &\lesssim_R \|f\|_{L^{\overline{p_0}}} + \|f\|_{L^{p_0}} + \left\|w_q^M f\right\|_{L^{p_0}} \approx \|f\|_{F^s_{p_0,q}}. \end{aligned}$$

□

Remark 2.3. *Note that the one can obtain analogous results for Besov spaces using also norms in terms of differences, with the more general assumption $0 < q \leqslant \infty$, $\frac{d}{d+s} < p_1 \leqslant p_0 \leqslant \infty$. The norm in [Tri83, (2.5.12.4)] by replacing $\int_{\mathbb{R}^d}$ by $\int_{|h| \leqslant 1}$, for instance, will do the job.*

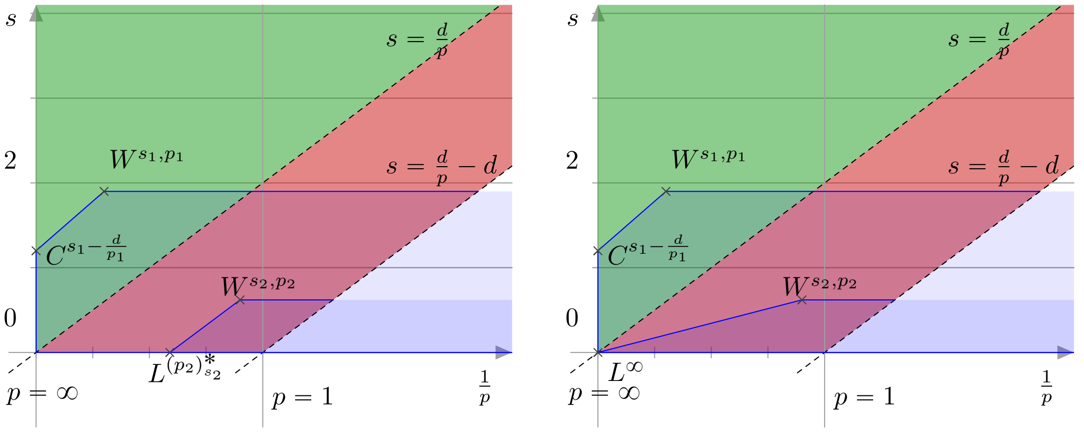
Finally let us compile all the information regarding functions such as the Beltrami coefficients involved in Theorem 1.2 (see Figure 2.2 (b)). In this case, we can interpolate norms with L^∞ by [RS96, Theorem 2.2.5]: Let $0 < s_2 < s < \frac{d}{p}$, $\frac{d}{d+s} < p_1 \leqslant p < \infty$ and $0 < p_2 < \infty$, $0 < q \leqslant q_1 \leqslant \infty$ with $\frac{d}{d+s} < q_1$ and $0 < q_2 \leqslant \infty$. For every compactly supported function $\mu \in F^s_{p,q} \cap L^\infty$ we have that

$$\mu \in F^s_{p_1,q_1} \cap F^{s_2}_{p_2,q_2} \qquad \text{as long as} \qquad s_2 p_2 \leqslant sp. \tag{2.4}$$

2.3 Discussion on the former regularity results

Kari Astala showed in [Ast94] that every quasiconformal mapping f with Beltrami coefficient $\mu \in L^\infty$ compactly supported in the unit disk with $\|\mu\|_{L^\infty} = \kappa < 1$ satisfies that

$$\bar{\partial} f \in L^p \qquad \text{whenever } \frac{1}{p_\kappa} < \frac{1}{p} \leqslant 1,$$



(a) Embeddings for compactly supported functions.

(b) Embeddings for bounded and compactly supported functions.

Figure 2.2: Embeddings for compactly supported or bounded functions.

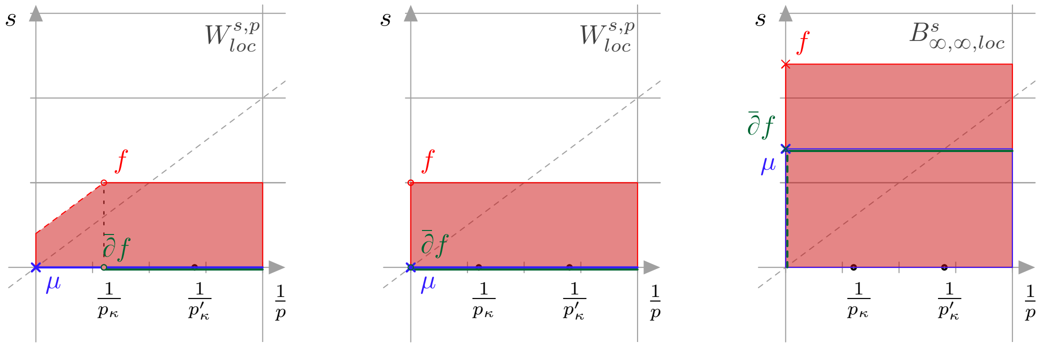
(see Figure 2.3(a)). Moreover, the operator $\mathcal{H} = Id - \mu\mathcal{B} - \nu\overline{\mathcal{B}}$ is invertible in Lebesgue spaces with exponent on the critical range (p_κ', p_κ) as shown in [AIS01]. When the regularity of the coefficients is greater, we can expect the principal solution to (1.2) to have greater regularity as well. The first result in that direction was given by Iwaniec, who could prove the compactness of the commutator $[\mu, \mathcal{B}]$ in every L^p space when μ is a Beltrami coefficient in VMO. By means of a Fredholm theory argument this implies the invertibility of $\mathcal{H}$ in L^p when $\nu = 0$, and the general case is shown by the same argument (see Figure 2.3(b)).

In recent years there has been a great improvement in results. A remarkable one, given by Clop et al. in [CFM$^+$09, Proposition 4], deals with the case of f being the principal solution to the Beltrami equation (1.2) with $\mu \in W^{1,p}_c$ with $1 < p < \infty$ and $\nu \equiv 0$ (see Figure 2.3 (d), (e) and (f)):

- If $\frac{1}{p} < \frac{1}{2}$, then $\bar\partial f \in W^{1,p}$.
- If $\frac{1}{p} = \frac{1}{2}$, then $\bar\partial f \in W^{1,q}$ for every $q < 2$.
- If $\frac{1}{2} < \frac{1}{p} < \frac{1}{p_\kappa'}$, then $\bar\partial f \in W^{1,q}$ for every $\frac{1}{q} > \frac{1}{p} + \frac{1}{p_\kappa}$.

Notice how $W^{1,p}$ being either supercritical, critical or subcritical determines the regularity behavior of f. In the subcritical case, the setting $\nu \neq 0$ was studied in [Bai16, Corolario 3.8]. The author shows that when $\frac{1}{2} < \frac{1}{p} < \frac{1}{p_\kappa'} - \frac{1}{p_\kappa}$, then $\bar\partial f \in W^{1,q}$ for every $\frac{1}{q} > \frac{1}{p} + \frac{1}{p_\kappa}$. Here we observe the same phenomenon as in Theorem 1.2, that is, we need that $\kappa < \frac{1}{3}$ in order to have room for p in the hypothesis.

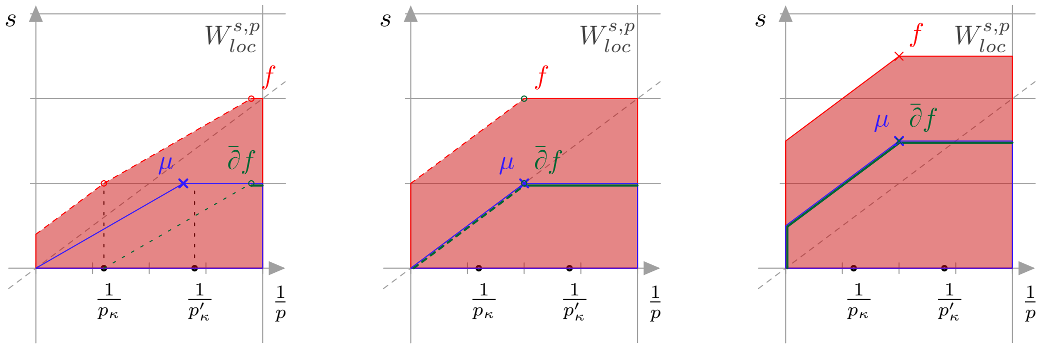
In the supercritical case there is no loss of regularity: if $\mu \in X$, then $\bar\partial f \in X$ as well. Indeed this is the case in the Hölder spaces of fractional order (see [AIM09, Chapter 15], Figure 2.3(c) above) and in the whole Triebel-Lizorkin and Besov scales, as shown by Cruz, Mateu and Orobitg, see [CMO13, Theorem 1], Figure 2.3(f). Here, the authors had to prove the invertibility of $I - \mu\mathcal{B}$ on $F^s_{p,q}$ which they did using Fredholm theory following Iwaniec's scheme, since the boundedness of $\mathcal{B}$ on Triebel-Lizorkin spaces can be deduced from [FTW88, Corollary 3.33], while in Besov spaces it is a consequence of Fourier multipliers theory and interpolation (see [Tri83, Section 2.6], for instance). In the supercritical context but restricted to domains, $\mu \in W^{s,p}(\Omega)$, there are also some positive results in [CMO13] and in [Pra15].



(a) General case from [Ast94], for $q < p_\kappa$, $\|\mu\|_{L^\infty} \leqslant \kappa \implies f \in W^{1,q}_{loc}$.

(b) From [Iwa92], if $1 \leqslant q < \infty$, $\mu \in VMO \implies f \in W^{1,q}_{loc}$.

(c) From [AIM09, Chapter 15]), $\mu \in C^s \implies f \in C^{s+1}_{loc}$.



(d) From [CFM+09], if $p'_\kappa < p < 2$, $\mu \in W^{1,p} \implies \bar\partial f \in W^{1,q}$, when $q << p$.

(e) From [CFM+09], if $p = 2$, $\mu \in W^{1,p} \implies \bar\partial f \in W^{1,q}$ when $q < p$.

(f) From [CMO13], if $sp > 2$, $\mu \in W^{s,p} \implies \bar\partial f \in W^{s,p}$.

Figure 2.3: Regularity of the principal quasiconformal solution to (1.2) when the coefficients satisfy a-priori conditions.

When μ belongs to a critical space $X \subset VMO$, we cannot expect that $\bar\partial f$ is in X, but the loss is minimal. The first result in that spirit is Iwaniec's above. This theorem has been extended to other critical spaces in [CFM+09] for $s = 1$ as commented above, and in [BCO17, Theorem 1] which settles the case $\frac{1}{2} \leqslant s < 1$, finding that

$$\mu \in W^{s,2/s} \implies \log(\partial f) \in W^{s,2/s}.$$

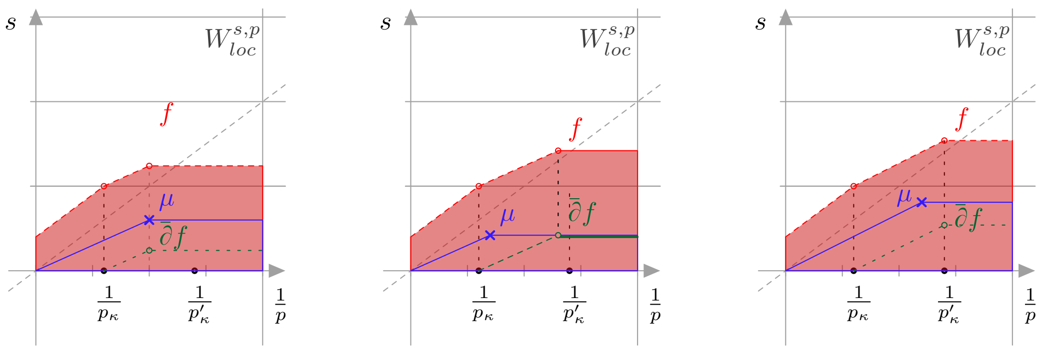
This result implies (1.5) for $\frac{1}{2} \leqslant s < 1$. Thus, the progress in Theorem 1.2 for the critical setting is to cover the whole range $0 < s < 2$. Finally, small improvements on the spaces lead to better results, as shown in [CMO13], where the authors prove that replacing $W^{s,2/s}$ by the Riesz potential of a Lorentz space $I^s(L^{2/s,1})$, then there is no loss. In this case, the functions are continuous, so we can classify these spaces as supercritical.

In the subcritical situation, in addition to the results on classical spaces in [Ast94, Corollary 1.2], [CFM+09, Proposition 4] and [Bai16, Corolario 3.8], the fractional smoothness case $0 < s < 1$ was considered in [CFR10, Theorem 4.3]. In this case, the authors show that $\bar\partial f \in W^{\Theta s,2}$ for every $\Theta < 1 - \frac{2}{p_\kappa}$. Note that in this result there is a loss in "smoothness", that is, in the s parameter. We will show Theorem 1.2 using the same techniques as the authors of that result with some extra care to avoid this loss. We recover their result in Corollary 1.3.

However, when $s=1$ and $\nu=0$ [CFM+09, Proposition 4] is stronger than Theorem 1.2, since the restriction $\frac{1}{p}<\frac{1}{p'_\kappa}-\frac{1}{p_\kappa}$ is replaced by $\frac{1}{p}<\frac{1}{p'_\kappa}$ (compare Figures 2.3(d) and 2.4(c)). Thus, it is natural to ask whether the following conjecture holds.

Conjecture 2.4. *Let $0<s<2$, $1<p<\infty$, with $s<\frac{2}{p}$, let $\mu,\nu\in F^s_{p,q}\cap L^\infty_c$ satisfy (1.1) and let f be the principal solution to the Beltrami equation (1.2). If $\frac{1}{p}<\frac{1}{p'_\kappa}$, then*

$$\bar\partial f\in F^s_{p,q}\qquad \text{for every } \frac{1}{q}>\frac{1}{p}+\frac{1}{p_\kappa}.$$



(a) Behavior for $s<1$ and $p=2$ described in [CFR10, Theorem 4.3].

(b) Behavior described in Theorem 1.2.

(c) Behavior described in Corollary 1.3.

Figure 2.4: Regularity of the principal quasiconformal solution to (1.2) when the coefficients satisfy a-priori conditions.

3 Proof of Theorem 1.2

Definition 3.1. *Given $s\in\mathbb{R}$ and a tempered distribution $f\in\mathcal{S}'(\mathbb{C})$, we define its fractional derivative*

$$D^s f:=\left(|\cdot|^s\hat f(\cdot)\right)^{\vee}$$

in the sense of tempered distributions modulo polynomials (see [Tri83, Sections 5.1.2, 5.2.3]).

The Beurling transform commutes with fractional differentiation of order smaller or equal than s on $W^{s,p}$. Indeed, since the Beurling transform is bounded in $W^{s,p}$ (see [Tri83, Section 2.6.6], for instance), we have that both $D^s\circ\mathcal{B}$ and $\mathcal{B}\circ D^s$ map $W^{s,p}$ into L^p. In the $\mathcal{S}$ class, which is dense, we have that $D^s\mathcal{B}f=\left(|\xi|^s\frac{\bar\xi}{\xi}\hat f(\cdot)\right)^{\vee}=\mathcal{B}D^s f$, so the equality extends to $W^{s,p}$.

Next we define the commutator of the multiplication by a test function with the fractional differentiation.

Definition 3.2. *For every pair $\mu,f\in C^\infty_c$, we define*

$$[\mu,D^s]f:=\mu D^s f-D^s(\mu f).$$

Next we recover Kato-Ponce Leibniz' rule as presented by Hofmann.

Theorem 3.3 (see [Hof98, Corollary 1.2])**.** *Let $0 < s < 1$, $1 < r < \infty$ and $\mu \in C_c^\infty$. The commutator $[\mu, D^s]$ extends to a linear operator with kernel*

$$K_\mu^s(x,y) = c\frac{\mu(x)-\mu(y)}{|x-y|^s},$$

mapping L^p to L^q for every $1 \leqslant q < p \leqslant \infty$, whenever $\frac{1}{r} := \frac{1}{q} - \frac{1}{p} \in (0,1)$. Moreover,

$$\|[\mu, D^s]f\|_{L^q} \lesssim_{p,q} \|D^s\mu\|_{L^r}\|f\|_{L^p}.$$

We will also use the Young inequality. It states that for measurable functions f and g, we have that

$$\|f * g\|_{L^q} \leqslant \|f\|_{L^r}\|g\|_{L^p} \tag{3.1}$$

for $1 \leqslant p, q, r \leqslant \infty$ with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ (see [Ste70, Appendix A2]).

Proof of Theorem 1.2. We will study in one stroke the critical and the subcritical case. To do so, we will use the convention $p_\kappa = \infty$ in the critical situation, that is, whenever $sp = 2$, while we use the standard notation $p_\kappa = 1 + \frac{1}{\kappa}$ in the subcritical situation ($sp < 2$). Thus, $I - \mu\mathcal{B}$ is invertible in L^q for $p_\kappa' < q < p_\kappa$ (see Theorem 1.1 for the subcritical setting and [Iwa92, Section 1] for the critical one).

Without loss of generality, we may assume that $\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu) \subset \mathbb{D}$. Let $\psi \in C_c^\infty(\mathbb{D})$ be a positive radial function such that $\int \psi = 1$ and consider the approximation of the unity $\psi_n(z) := n^2\psi(nz)$ for every $n \in \mathbb{N}$. Consider the families of functions $\mu_n := \psi_n * \mu$ and $\nu_n := \psi_n * \nu$, which are supported in the disk $2\mathbb{D}$. If $\nu_n(z)\mu_n(z) \neq 0$, then

$$|\mu_n(z)| + |\nu_n(z)| = |\mu_n(z)|\left|\frac{|\nu_n(z)|}{|\mu_n(z)|} + 1\right| = \left|\frac{\mu_n(z)}{|\mu_n(z)|}\frac{|\nu_n(z)|}{\nu_n(z)}\nu_n(z) + \mu_n(z)\right| = |(c_z\nu + \mu) * \psi_n(z)|,$$

where c_z stands for the unitary complex number $\frac{\mu_n(z)}{|\mu_n(z)|}\frac{|\nu_n(z)|}{\nu_n(z)}$. This expression, together with the hypothesis (1.1) and the Young inequality (3.1), imply that

$$|\mu_n(z)| + |\nu_n(z)| \leqslant \kappa.$$

On the other hand, being the convolution with ψ_n an approximation of the identity, $D^s\mu_n = D^s\mu * \psi_n$ converges to $D^s\mu$ in L^p if $D^s\mu \in L^p$ for $p < \infty$, and the same can be said about $\{\nu_n\}$. Thus,

$$\|\mu_n - \mu\|_{W^{s,p}} \xrightarrow{n\to\infty} 0 \qquad \text{and} \qquad \|\nu_n - \nu\|_{W^{s,p}} \xrightarrow{n\to\infty} 0. \tag{3.2}$$

Let us define

$$h := (I - \mu\mathcal{B} - \nu\bar{\mathcal{B}})^{-1}(\mu + \nu) \tag{3.3}$$

and

$$h_n := (I - \mu_n\mathcal{B} - \nu_n\bar{\mathcal{B}})^{-1}(\mu_n + \nu_n). \tag{3.4}$$

Then, $f_n(z) := z + \mathcal{C}h_n(z)$ is the principal solution to (1.2) with coefficients μ_n and ν_n ([CFR10, Lemma 4.2]), and we have that

$$\|h_n\|_{L^\ell} \leqslant C_{\kappa,\ell} \tag{3.5}$$

for every $\frac{1}{\ell} > \frac{1}{p_\kappa}$.

The case $s = 1$ is fully covered by [CFM+09, Proposition 4] and [Bai16, Corolario 3.8]. Let us consider first the case $s < 1$. In terms of h_n, equation (1.2) read as

$$h_n = \mu_n\mathcal{B}h_n + \mu_n + \nu_n\overline{\mathcal{B}}h_n + \nu_n.$$

Note that h_n is C_c^∞, and $\mu_n, \nu_n \in C_c^\infty$ as well, while $\mathcal{B}h_n \in C^\infty \cap W^{m,p}$ for every $m \in \mathbb{N}$ and $1 < p < \infty$. Thus, we can take fractional derivatives of order s to get

$$\begin{aligned}D^s h_n &= D^s(\mu_n \mathcal{B}h_n + \nu_n \overline{\mathcal{B}}h_n + \mu_n + \nu_n)\\ &= \mu_n D^s \mathcal{B}h_n - [\mu_n, D^s](\mathcal{B}h_n) + \nu_n D^s \overline{\mathcal{B}}h_n - [\nu_n, D^s](\overline{\mathcal{B}}h_n) + D^s\mu_n + D^s\nu_n,\end{aligned}$$

that is, since the Beurling transform commutes with the fractional derivatives for C_c^∞ functions,

$$D^s h_n - \mu_n \mathcal{B}(D^s h_n) - \nu_n \overline{\mathcal{B}}(D^s h_n) = -[\mu_n, D^s](\mathcal{B}h_n) - [\nu_n, D^s](\overline{\mathcal{B}}h_n) + D^s\mu_n + D^s\nu_n. \tag{3.6}$$

Next, by (3.5) we have that

$$\|\mathcal{B}h_n\|_{L^\ell} \leqslant C_{\kappa,q} \qquad \text{for every } \frac{1}{\ell} > \frac{1}{p_\kappa} \tag{3.7}$$

and, by (3.2), for n big enough

$$\|D^s\mu_n\|_{L^r} + \|D^s\nu_n\|_{L^r} \leqslant C_{\kappa,p,r,\|\mu\|_{W^{s,p}},\|\nu\|_{W^{s,p}}} \qquad \text{for every } \frac{1}{r} \geqslant \frac{1}{p}. \tag{3.8}$$

Given $\frac{1}{q} > \frac{1}{p} + \frac{1}{p_\kappa}$, we can write $\frac{1}{q} = \frac{1}{\ell} + \frac{1}{r}$ satisfying restrictions (3.7) and (3.8), so Theorem 3.3 with (3.5) and (3.7) yields

$$\begin{aligned}\left\|[\mu_n, D^s]\mathcal{B}h_n + [\nu_n, D^s]\overline{\mathcal{B}}h_n\right\|_{L^q} &\lesssim_{\kappa,p,q} \left(\|D^s\mu_n\|_{L^r}\|\mathcal{B}h_n\|_{L^\ell} + \|D^s\nu_n\|_{L^r}\|\overline{\mathcal{B}}h_n\|_{L^\ell}\right)\\ &\leqslant C_{\kappa,p,q,\|\mu\|_{W^{s,p}},\|\nu\|_{W^{s,p}}}.\end{aligned} \tag{3.9}$$

Thus, we have a uniform control on the L^q norm of the right hand side of (3.6). In case

$$\frac{1}{p} + \frac{1}{p_\kappa} < \frac{1}{p'_\kappa}$$

we can find $\frac{1}{q}$ in between where we can invert the operator $I - \mu_n\mathcal{B} - \nu_n\bar{\mathcal{B}}$ and, using (3.6), we can write

$$\begin{aligned}\|D^s h_n\|_{L^q} &= \left\|(I - \mu_n\mathcal{B} - \nu_n\bar{\mathcal{B}})^{-1}([\mu_n, D^s](\mathcal{B}h_n) + [\nu_n, D^s](\overline{\mathcal{B}}h_n) - D^s\mu_n - D^s\nu_n)\right\|_{L^q}\\ &\lesssim \left\|[\mu_n, D^s](\mathcal{B}h_n) + [\nu_n, D^s](\overline{\mathcal{B}}h_n) - D^s\mu_n - D^s\nu_n\right\|_{L^q}\end{aligned}$$

which is uniformly bounded by (1.4), (3.9) and (3.8). Combining these estimates with (3.5), we get

$$\|h_n\|_{W^{s,q}} \lesssim C_{\kappa,p,q,\|\mu\|_{W^{s,p}},\|\nu\|_{W^{s,p}}}. \tag{3.10}$$

The Banach-Alaoglu Theorem shows that there exists a subsequence h_{n_k} converging to $\widetilde{h} \in W^{s,p}$ as tempered distributions, with $\|\widetilde{h}\|_{W^{s,q}} \lesssim C_{\kappa,p,q,\|\mu\|_{W^{s,p}},\|\nu\|_{W^{s,p}}}$.

It only remains to transfer this information to h. First, note that

$$h_n \xrightarrow{n\to\infty} h \qquad \text{in } L^2. \tag{3.11}$$

Indeed, from (3.3) and (3.4), we have that

$$\begin{aligned}|h - h_n| &= |\mu\mathcal{B}h - \mu_n\mathcal{B}h_n + \nu\bar{\mathcal{B}}h - \nu_n\bar{\mathcal{B}}h_n + \mu - \mu_n + \nu - \nu_n|\\ &\leqslant |\mu||\mathcal{B}(h - h_n)| + |\mu - \mu_n||\mathcal{B}h_n| + |\nu||\mathcal{B}(h - h_n)| + |\nu - \nu_n||\mathcal{B}h_n| + |\mu - \mu_n| + |\nu - \nu_n|.\end{aligned}$$

Taking L^2 norms,

$$\|h-h_n\|_{L^2} \leqslant \kappa\|\mathcal{B}(h-h_n)\|_{L^2} + \|(\mu-\mu_n)\mathcal{B}h_n\|_{L^2} + \|(\nu-\nu_n)\mathcal{B}h_n\|_{L^2} + \|\mu-\mu_n\|_{L^2} + \|\nu-\nu_n\|_{L^2}.$$

Since $\kappa < 1$ and the Beurling transform is an isometry in L^2, the left-hand side can absorb the first term in the right-hand side. To deal with the second and the third terms, choose $2 < \widetilde{q} < p_\kappa$, and let $\frac{1}{r} + \frac{1}{\widetilde{q}} = \frac{1}{2}$. Then, by (3.4) and Theorem 1.1, the norm of h_n in $L^{\widetilde{q}}$ is uniformly bounded, and using the L^p version of (3.2) (i.e., Young's inequality) as well we get

$$\|(\mu-\mu_n)\mathcal{B}h_n\|_{L^2} \leqslant \|\mu-\mu_n\|_{L^r}\|\mathcal{B}h_n\|_{L^{\widetilde{q}}} \leqslant C_{\kappa,\widetilde{q},n} \xrightarrow[n\to\infty]{} 0.$$

Using again Young's inequality for the remaining terms, we see that

$$\|h-h_n\|_{L^2} \leqslant \frac{C_{\kappa,n}}{1-\kappa} \xrightarrow[n\to\infty]{} 0,$$

showing (3.11). Since this implies convergence as tempered distributions and $\mathcal{S}'$ is a Hausdorff space, we get that $h = \widetilde{h}$. This finishes the case $s < 1$.

It remains to show that (1.6) holds when $\frac{1}{p} < \frac{1}{p'_\kappa} - \frac{1}{p_\kappa}$ and $s > 1$. If this is the case, then for every $1 < q < \infty$ we have that $\|h\|_{W^{s,q}} \approx \|h\|_{L^q} + \|\partial h\|_{W^{s-1,q}}$ by the lifting property (see [Tri83, Section 2.3.8]) and the boundedness of the Beurling transform in Sobolev spaces. We will assume that $\nu = \nu_n = 0$ to keep a compact notation, leaving the necessary modifications to the reader. Fix $n \geqslant 1$. By assumption, $\mu \in W^{s,p} \cap L^\infty$. By (2.4) we also have that

$$\mu, \mu_n \in W^{1,sp} \cap W^{s-1,\frac{sp}{s-1}} \text{ uniformly in } n. \tag{3.12}$$

As a consequence,

$$\partial h, \partial h_n \in L^r \text{ uniformly in } n \text{ for every given } \frac{1}{r} > \frac{1}{sp} + \frac{1}{p_\kappa} \tag{3.13}$$

by [CFM+09, Proposition 4] and [Bai16, Corolario 3.8], and, as we have proven above,

$$D^{s-1}h, D^{s-1}h_n \in L^r \text{ uniformly in } n \text{ for every given } \frac{1}{r} > \frac{s-1}{sp} + \frac{1}{p_\kappa}. \tag{3.14}$$

Thus, we only need to deal with the homogeneous norm, that is, $\left\|D^{s-1}\partial h\right\|_{L^q}$. From (1.2) we deduce that

$$\partial h_n = \mu_n \partial\mathcal{B}h_n + \partial\mu_n\mathcal{B}h_n + \partial\mu_n$$

and differentiating we get

$$D^{s-1}\partial h_n = \mu_n D^{s-1}\mathcal{B}\partial h_n - [\mu_n, D^{s-1}](\mathcal{B}\partial h_n) + \partial\mu_n D^{s-1}\mathcal{B}h_n - [\partial\mu_n, D^{s-1}](\mathcal{B}h_n) + D^{s-1}\partial\mu_n$$

leading to

$$\begin{aligned}\left\|(I-\mu_n\mathcal{B})(D^{s-1}\partial h_n)\right\|_{L^q} &\leqslant \left\|[\mu_n, D^{s-1}](\mathcal{B}\partial h_n)\right\|_{L^q} + \left\|\partial\mu_n D^{s-1}\mathcal{B}h_n\right\|_{L^q} + \left\|[\partial\mu_n, D^{s-1}](\mathcal{B}h_n)\right\|_{L^q} \\ &\quad + \left\|D^{s-1}\partial\mu_n\right\|_{L^q} = (1) + (2) + (3) + (4).\end{aligned} \tag{3.15}$$

Note that in the previous case, (3.6) had a simpler form, but the essential ideas to control the norm of the right-hand side are the same. We will find $\frac{1}{p_\kappa} < \frac{1}{q} < \frac{1}{p'_\kappa}$ such that $(j) \leqslant C_{\kappa,q}$ for $j \in \{1,2,3,4\}$.

First of all, by Theorem 3.3 we have that

$$\textcircled{1} = \left\|[\mu_n, D^{s-1}](\mathcal{B}\partial h_n)\right\|_{L^q} \leqslant \left\|D^{s-1}\mu_n\right\|_{L^{\frac{sp}{s-1}}} \|\mathcal{B}\partial h_n\|_{L^{r_1}}$$

for $\frac{1}{q} = \frac{s-1}{sp} + \frac{1}{r_1}$. The first term is uniformly bounded by (3.12), and the last one is controlled by (3.13) as long as $\frac{1}{r_1} > \frac{1}{sp} + \frac{1}{p_\kappa}$. It is possible to find such a value for r_2 as long as

$$\frac{1}{q} > \frac{s-1}{sp} + \frac{1}{sp} + \frac{1}{p_\kappa} = \frac{1}{p} + \frac{1}{p_\kappa}.$$

On the other hand, using the commutation of fractional derivatives and the Beurling transform and Hölder's inequality, we have that

$$\textcircled{2} = \left\|\partial\mu_n \mathcal{B} D^{s-1} h_n\right\|_{L^q} \leqslant \|\partial\mu_n\|_{L^{sp}} \left\|\mathcal{B} D^{s-1} h_n\right\|_{L^{r_2}}$$

for $\frac{1}{q} = \frac{1}{sp} + \frac{1}{r_2}$. The first term is uniformly bounded by (3.12), and the last one is controlled by (3.14) as long as $\frac{1}{r_2} > \frac{s-1}{sp} + \frac{1}{p_\kappa}$. It is possible to find such a value for r_2 as long as

$$\frac{1}{q} > \frac{1}{sp} + \frac{s-1}{sp} + \frac{1}{p_\kappa} = \frac{1}{p} + \frac{1}{p_\kappa}.$$

The latter two terms are bounded as before. By Theorem 3.3 we have that

$$\textcircled{3} = \left\|[\partial\mu_n, D^{s-1}](\mathcal{B} h_n)\right\|_{L^q} \leqslant \left\|D^{s-1}\partial\mu_n\right\|_{L^p} \|\mathcal{B} h_n\|_{L^{r_3}}$$

for $\frac{1}{q} = \frac{1}{p} + \frac{1}{r_3}$, the last term being uniformly controlled for $\frac{1}{r_3} > \frac{1}{p_\kappa}$ by Theorem 1.1. It is possible to find such a value for r_3 as long as

$$\frac{1}{q} > \frac{1}{p} + \frac{1}{p_\kappa}.$$

Finally,

$$\textcircled{4} = \left\|D^{s-1}\partial\mu_n\right\|_{L^q} \leqslant C_{\kappa,q} \text{ for every } \frac{1}{q} \geqslant \frac{1}{p}.$$

This facts, together with (3.15) and Theorem 1.1, show that (3.10) holds for $s > 1$ with exactly the same restrictions as when $s < 1$, that is, when $\frac{1}{p} + \frac{1}{p_\kappa} < \frac{1}{q} < \frac{1}{p'_\kappa}$. The theorem follows by the Banach-Alaoglu Theorem again. □

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