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LIMIT CYCLES BIFURCATING FROM THE PERIODIC ORBITS OF THE WEIGHT-HOMOGENEOUS POLYNOMIAL CENTERS OF WEIGHT-DEGREE 3

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Abstract. In this article we obtain two explicit polynomials, whose simple positive real roots provide the limit cycles which bifurcate from the periodic orbits of a family of polynomial differential centers of order 5, when this family is perturbed inside the class of all polynomial differential systems of order 5, whose average function of first order is not zero. Then the maximum number of limit cycles that bifurcate from these periodic orbits is 6 and it is reached.

This family of of centers completes the study of the limit cycles which can bifurcate from periodic orbits of all centers of the weight-homogeneous polynomial differential systems of weight-degree 3 when perturbed in the class of all polynomial differential systems having the same degree and whose average function of first order is not zero.

1. Introduction and statement of the main results

One of the main goals in the qualitative theory of real planar polynomial differential systems is the determination of their limit cycles. Studying the number of limit cycles of a polynomial differential system is strongly motivated by Hilbert's 16-th problem (1900). For more details see [8] and [14].

Many authors have studied the number of limit cycles which may bifurcate from the periodic orbits of a center of a polynomial differential system when it is perturbed up to first order in the parameter of the perturbation. This problem is known as the *weak Hilbert's problem*. See for example [1, 3].

Among the many tools for studying the maximum number of limit cycles that may bifurcate from the periodic annulus of a center we have the Poincaré return map, the Poincaré-Melnikov integrals, the Abelian integrals, and the averaging theory. The last three methods are equivalent at first order, see for instance [7]. For studies on the weak Hilbert's problem see, for example, the second part of [6] and the hundreds of references quoted therein.

Here we consider the *polynomial differential systems*

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1.1}$$

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where P and Q are polynomials with real coefficients. The *degree* of the system is the maximum of the degrees of the polynomials P and Q .

We say that system (1.1) is *weight-homogeneous* if there exist $(s_1, s_2) \in \mathbb{N}^2$ and $d \in \mathbb{N}$ such that for any $\lambda \in \mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$ we have

$$P(\lambda^{s_1}x, \lambda^{s_2}y) = \lambda^{s_1-1+d}P(x,y), \quad Q(\lambda^{s_1}x, \lambda^{s_2}y) = \lambda^{s_2-1+d}Q(x,y).$$

The vector (s_1, s_2) is called the *weight-exponent* of system (1.1) and d is called *weight-degree* with respect to the weight-exponent (s_1, s_2).

Our main goal is to solve the weak Hilbert's problem for the weight-homogeneous polynomial differential systems of weight-degree 3.

The classification of all centers of a planar *weight-homogeneous* polynomial differential systems up to weight-degree 4 is found in [11]. In [11] two families of weight-homogeneous polynomial differential systems having centers with weight-degree 3 are provided. The first family can be written as

$$\begin{aligned} \dot{x} &= ax^3 + (b - 3\alpha\mu)x^2y - axy^2 - \alpha y^3, \\ \dot{y} &= \alpha x^3 + ax^2y + (b + 3\alpha\mu)xy^2 - ay^3, \end{aligned} \tag{1.2}$$

with $\alpha \in \{-1, 1\}$, $a, b, \mu \in \mathbb{R}$ and $\mu > -1/3$, after doing an affine change of variables and a rescaling of the time. The weight-exponent of this family is $(s_1, s_2) = (1, 1)$.

The second family is

$$\begin{aligned} \dot{x} &= ax^3 + by = P(x,y), \\ \dot{y} &= cx^5 + dx^2y = Q(x,y), \end{aligned} \tag{1.3}$$

with $bc \neq 0$, $3a + d = 0$ and $12(bc - ad) < 0$. The weight-exponent of this family is $(s_1, s_2) = (1, 3)$.

In [10] we provide a polynomial whose real positive simple zeros give exactly the number of limit cycles that bifurcate from the periodic orbits of (1.2) when perturbed in the class of all polynomial differential systems of degree n and nonzero first order average function is. In particular the maximum number of limit cycles obtained is exactly $[(n-1)/2]$, where $[x]$ denotes the integer part of x.

In this article we give two explicit polynomials whose real positive simple zeros provide the number of limit cycles which bifurcate from the periodic orbits of the center of the weight-homogeneous polynomial differential system (1.3), when the first order average function is non-zero. The maximum number of limit cycles is reached when the parameter $d \neq 0$.

More precisely here we consider the polynomial differential system

$$\begin{aligned} \dot{x} &= -\frac{d}{3}x^3 + by + \varepsilon p(x,y), \\ \dot{y} &= cx^5 + dx^2y + \varepsilon q(x,y), \end{aligned} \tag{1.4}$$

where

$$p(x,y) = \sum_{k=0}^{5} p_k(x,y), \quad q(x,y) = \sum_{i=0}^{5} q_k(x,y), \tag{1.5}$$

$$p_k(x,y) = \sum_{i=0}^{k} c_{k-i\,i}\ x^{k-i}y^i, \quad q_k(x,y) = \sum_{i=0}^{k} d_{k-i\,i}\ x^{k-i}y^i,$$

and ε is a small parameter.

In [9] the authors found an upper bound for the maximum number of limit cycles of system (1.4). Looking at statement (c) of Theorem A of [9] with $n = 5$, $p = 3$ and $q = 1$ an upper bound for the number of limit cycles of system (1.4) with ε sufficiently small coming from the periodic orbits of the center (1.3) is 8. *Here we prove that the maximum number of limit cycles that system* (1.4) *can have is exactly 6 and it is reached.* See Corollary 1.3. See [12, Theorem 2] for another study on the limit cycles that may bifurcate from the periodic orbits surrounding a non-Hamiltonian center using averaging theory.

In what follows we state our main results where the functions $f_i(\theta)$, for $i = 0, 1, \dots, 9$, $W(\theta)$, $g_1(\theta)$ and $k(\theta)$ are given in Section 3 and the functions $f_i^*(\theta)$, for $i = 0, 1, \dots, 9$, $W^*(\theta)$, $g_1^*(\theta)$ and $k^*(\theta)$ are given in Section 4. We do not provide these functions here due to their length.

Theorem 1.1. *Suppose that $d \neq 0$. Let $r_{0,s}$ be a positive simple root of the polynomial*

$$r_0\mathcal{F}(r_0) = \frac{1}{2\pi}\sum_{k=1}^{7} r_0^{2k-2}\int_0^{2\pi} A_{2k+1}(\theta)d\theta, \tag{1.6}$$

where

$$A_i(\theta) = \frac{W(\theta)f_i(\theta)k(\theta)^{i-5}}{g_1(\theta)^2}.$$

Then for $|\varepsilon| > 0$ sufficiently small parameter the perturbed systems (1.4) *have a limit cycle bifurcating from the periodic orbit $r(\theta, r_{0,s}) = k(\theta)r_{0,s}$ of the period annulus of the center* (1.3) *if the first order average function is non-zero. In particular, the polynomial* (1.6) *has at most* 6 *positive simple real roots and they are reached.*

Theorem 1.2. *Suppose that $d = 0$. Let $r_{0,s}$ be a positive simple root of the polynomial*

$$r_0\mathcal{F}^*(r_0) = \frac{1}{2\pi}\Big(\sum_{k=1}^{5} r_0^{2k-2}\int_0^{2\pi} A_{2k+1}^*(\theta)d\theta + r_0^{11}\int_0^{2\pi} A_{15}^*(\theta)d\theta\Big), \tag{1.7}$$

where

$$A_i^*(\theta) = \frac{W^*(\theta)f_i^*(\theta)k^*(\theta)^{i-5}}{g_1^*(\theta)^2}.$$

Then for $|\varepsilon| > 0$ sufficiently small the perturbed systems (1.4) *have a limit cycle bifurcating from the periodic orbit $r(\theta, r_{0,s}) = k^*(\theta)r_{0,s}$ of the period annulus of the center* (1.3) *if the first order average function is non-zero. In particular, the polynomial* (1.7) *has at most* 5 *positive simple real roots and they are reached.*

Theorem 1.1 is proved in Section 3 and Theorem 1.2 is proved in Section 4. In Section 5 we provide two examples that illustrate Theorems 1.1 and 1.2 with the maximum number of limit cycles.

From Theorems 1.1 and 1.2 the next result follows.

Corollary 1.3. *Applying the averaging theory of first order to the perturbed system* (1.4) *when $|\varepsilon|$ is sufficiently small we can obtain at most* 6 *limit cycles bifurcating from the periodic orbits of the center* (1.3)*, and we have systems where these* 6 *limit cycles are reached.*

2. Preliminaries

In this section we give some well known results that we shall need for proving Theorem 1.1. Consider the system

$$\dot{\mathrm{x}} = F_0(t, \mathrm{x}) + \varepsilon F_1(t, \mathrm{x}) + \mathcal{O}(\varepsilon^2), \tag{2.1}$$

where $\varepsilon \neq 0$ is sufficiently small and the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, $T-$periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. We suppose that the unperturbed system

$$\dot{\mathrm{x}} = F_0(t, \mathrm{x}) \tag{2.2}$$

has a submanifold of periodic solutions of dimension n.

Let $x(t, z, \varepsilon)$ be the solution of system (2.2) such that $\mathrm{x}(0, \mathrm{z}, \varepsilon) = z$. The linearization of the unperturbed system along a periodic solution $\mathrm{x}(t, \mathrm{z}, 0)$ is

$$\dot{\mathrm{y}} = D_{\mathrm{x}} F_0(t, \mathrm{x}(t, \mathrm{z}, 0))\mathrm{y}. \tag{2.3}$$

In what follows we denote by $M_{\mathrm{z}}(t)$ the fundamental matrix solution of the linearized system (2.3) such that $M_{\mathrm{z}}(0)$ is the identity matrix.

We assume that there is an open set U with $\mathrm{Cl}(U) \subset \Omega$ such that for each $z \in \mathrm{Cl}(U)$, $\mathrm{x}(t, \mathrm{z}, 0)$ is $T-$periodic, where $\mathrm{x}(t, z, 0)$ denotes the solution of the unperturbed system (2.2), and $\mathrm{Cl}(U)$ the closure of U. The set $\mathrm{Cl}(U)$ is *isochronous* for system (2.2), i.e. it is formed only by periodic orbits with period T.

The following result is the a version of averaging theorem for studying the bifurcation of $T-$periodic solutions of system (2.1) from the periodic solutions $\mathrm{x}(t, \mathrm{z}, 0)$ contained in $\mathrm{Cl}(U)$ of system (2.2) when $|\varepsilon| > 0$ is sufficiently small. See [4] for a proof. For more details on the averaging theory see [5] and [13].

Theorem 2.1 (Perturbations of an isochronous set)**.** *We assume that there exists an open and bounded set U with $\mathrm{Cl}(U) \subset \Omega$ such that for each $\mathrm{z} \in \mathrm{Cl}(U)$, the solution $\mathrm{x}(r, \mathrm{z}, 0)$ is $T-$periodic. Consider the function $\mathcal{F} : \mathrm{Cl}(U) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathrm{z}) = \frac{1}{T} \int_0^T M_{\mathrm{z}}^{-1}(t) F_1(t, \mathrm{x}(t, \mathrm{z}, 0)) dt. \tag{2.4}$$

Then if there exists $\mathbf{a} \in U$ with $\mathcal{F}(\mathbf{a}) = 0$ and $\det((\partial \mathcal{F}/\partial \mathrm{z})(\mathbf{a})) \neq 0$ then there exists a $T-$periodic solution $\mathrm{x}(t, \varepsilon)$ of system (2.1) such that $\mathrm{x}(0, \varepsilon) \to \mathbf{a}$ when $\varepsilon \to 0$.

In fact, if $\mathrm{x}(t, \mathrm{z}, \varepsilon)$ denotes the solution of the differential system (2.1) such that $\mathrm{x}(0, \mathrm{z}, \varepsilon) = \mathrm{z}$, then the average function satisfies that $\mathrm{x}(T, \mathrm{z}, \varepsilon) - \mathrm{z} = \varepsilon \mathcal{F}(\mathrm{z}) + O(\varepsilon^2)$, see for more details [4, 7]. Then, by the Implicit Function Theorem it follows that if $\mathcal{F}(\mathrm{z}) \neq 0$, then the simple zeros of the function $\mathcal{F}(\mathrm{z})$ provide limit cycles of the differential system (2.1).

The following result is the generalized Descartes Theorem about the number of zeros of a real polynomial. See [2] for a proof.

Theorem 2.2. *Consider the real polynomial $p(x) = a_{i_1} x^{i_1} + a_{i_2} x^{i_2} + \cdots + a_{i_r} x^{i_r}$ with $0 \le i_1 < i_2 < \cdots < i_r$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \ldots, r\}$. When $a_{i_j} a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r - 1$ positive real roots.*

3. Proof of Theorem 1.1

Suppose that $d \neq 0$. We apply the affine change of variables
$$\tilde{x} = \alpha x, \quad \tilde{y} = \frac{\alpha^3 b}{d} y, \quad \tilde{t} = \frac{d}{\alpha^2} t,$$
with $\alpha \neq 0$ and system (1.3) becomes
$$\begin{aligned} \dot{x} &= P(x,y) = -\frac{1}{3}x^3 + y, \\ \dot{y} &= Q(x,y) = a_1 x^5 + x^2 y, \end{aligned}$$
where $a_1 = -(4+b^2)/12$ and $b \neq 0$. In the case $b = 0$ working in a similar way we also can reach the previous differential system. The perturbed system corresponding to the previous system is
$$\begin{aligned} \dot{x} &= -\frac{1}{3}x^3 + y + \varepsilon p(x,y), \\ \dot{y} &= a_1 x^5 + x^2 y + \varepsilon q(x,y)\,. \end{aligned} \tag{3.1}$$
We write system (3.1) in the generalized polar coordinates $x = r\cos\theta$, $y = r^3 \sin\theta$, and we obtain the differential equation
$$\frac{dr}{d\theta} = F_0(r,\theta) + \varepsilon F_1(r,\theta) + \mathcal{O}(\varepsilon^2), \tag{3.2}$$
in the standard form for applying the averaging theory of first order described in Section 2, where
$$\begin{gathered} F_0(r,\theta) = \frac{h_1(\theta)}{g_1(\theta)} r, \\ \begin{aligned} F_1(r,\theta) = \frac{144(\cos^2\theta + 3\sin^2\theta)}{r^7 g_1(\theta)^2} \Big(& Q(r\cos\theta, r^3\sin\theta) p(r\cos\theta, r^3\sin\theta) \\ & - P(r\cos\theta, r^3\sin\theta) q(r\cos\theta, r^3\sin\theta) \Big), \end{aligned} \\ h_1(\theta) = \cos\theta\left(\left(b^2+4\right)\sin\theta\cos^4\theta - 6\sin\theta(\sin(2\theta)+2) + 4\cos^3\theta\right), \\ g_1(\theta) = (4+b^2)\cos^6\theta - 24\cos^3\theta\sin\theta + 36\sin^2\theta. \end{gathered}$$

Note that the differential equation (3.2) satisfies the assumptions of Theorem 2.1. Consider $r(\theta, r_0)$ the periodic solution of the differential equation
$$\frac{dr}{d\theta} = r\,\frac{h_1(\theta)}{g_1(\theta)},$$
such that $r(0, r_0) = r_0$. By solving the previous differential equation we get
$$r(\theta, r_0) = k(\theta) r_0,$$
where
$$k(\theta) = \frac{2^{5/6}(4+b^2)^{1/6}}{B(\theta)},$$
with
$$\begin{aligned} B(\theta) = \Big(& 3\left(5b^2 - 172\right)\cos(2\theta) + 6\left(b^2+4\right)\cos(4\theta) + \left(b^2+4\right)\cos(6\theta) + 10b^2 \\ & - 192\sin(2\theta) - 96\sin(4\theta) + 616\Big)^{1/6}. \end{aligned}$$

Solving the variational equation (2.3) for the differential equation (3.2) we see that the fundamental matrix solution $M(\theta)$ is $k(\theta)$. Using the polynomials p and q given in (1.5) and system (1.3) we have that the integrant of the integral (2.4) for the differential equation (3.2) is

$$\begin{aligned}
M^{-1}(\theta)F_1(\theta,r(\theta,r_0)) &= \sum_{i=0}^{17}\frac{W(\theta)f_i(\theta)}{g_1(\theta)^2M(\theta)}r(\theta,r_0)^{i-4}\\
&= \sum_{i=0}^{17} r_0^{i-4}\frac{W(\theta)f_i(\theta)k(\theta)^{i-5}}{g_1(\theta)^2}\\
&= \sum_{i=0}^{17} r_0^{i-4}A_i(\theta),
\end{aligned}$$

where

$$\begin{aligned}
f_0(\theta) &= 4d_{00}\cos^3\theta - 12d_{00}\sin\theta,\\
f_1(\theta) &= 4d_{10}\cos^4\theta - 12d_{10}\sin\theta\cos\theta,\\
f_2(\theta) &= \cos^5\theta\left(-b^2c_{00}-4c_{00}+4d_{20}\right)+12(c_{00}-d_{20})\sin\theta\cos^2\theta,\\
f_3(\theta) &= \cos^6\theta\left(-b^2c_{10}-4c_{10}+4d_{30}\right)+4\sin\theta\cos^3\theta(3c_{10}+d_{01}-3d_{30})\\
&\quad -12d_{01}\sin^2\theta,\\
f_4(\theta) &= \cos^7\theta\left(-b^2c_{20}-4c_{20}+4d_{40}\right)+4\sin\theta\cos^4\theta(3c_{20}+d_{11}-3d_{40})\\
&\quad -12d_{11}\sin^2\theta\cos\theta,\\
f_5(\theta) &= \sin\theta\cos^5\theta\left(-b^2c_{01}-4c_{01}+12c_{30}+4d_{21}-12d_{50}\right)\\
&\quad +\cos^8\theta\left(-b^2c_{30}-4c_{30}+4d_{50}\right)+12(c_{01}-d_{21})\sin^2\theta\cos^2\theta,\\
f_6(\theta) &= \sin\theta\cos^6\theta\left(-b^2c_{11}-4c_{11}+12c_{40}+4d_{31}\right)-\left(b^2+4\right)c_{40}\cos^9\theta\\
&\quad +4\sin^2\theta\cos^3\theta(3c_{11}+d_{02}-3d_{31})-12d_{02}\sin^3\theta,\\
f_7(\theta) &= \sin\theta\cos^7\theta\left(-b^2c_{21}-4c_{21}+12c_{50}+4d_{41}\right)-\left(b^2+4\right)c_{50}\cos^{10}\theta\\
&\quad +4\sin^2\theta\cos^4\theta(3c_{21}+d_{12}-3d_{41})-12d_{12}\sin^3\theta\cos\theta,\\
f_8(\theta) &= \sin^2\theta\cos^5\theta\left(-b^2c_{02}-4c_{02}+12c_{31}+4d_{22}\right)\\
&\quad -\left(b^2+4\right)c_{31}\sin\theta\cos^8\theta+12(c_{02}-d_{22})\sin^3\theta\cos^2\theta,\\
f_9(\theta) &= \sin^2\theta\cos^6\theta\left(-b^2c_{12}-4c_{12}+12c_{41}+4d_{32}\right)\\
&\quad -\left(b^2+4\right)c_{41}\sin\theta\cos^9\theta+4\sin^3\theta\cos^3\theta(3c_{12}+d_{03}-3d_{32})\\
&\quad -12d_{03}\sin^4\theta,\\
f_{10}(\theta) &= -\left(b^2+4\right)c_{22}\sin^2\theta\cos^7\theta+4(3c_{22}+d_{13})\sin^3\theta\cos^4\theta\\
&\quad -12d_{13}\sin^4\theta\cos\theta,\\
f_{11}(\theta) &= \sin^3\theta\cos^5\theta\left(-b^2c_{03}-4c_{03}+12c_{32}+4d_{23}\right)\\
&\quad -\left(b^2+4\right)c_{32}\sin^2\theta\cos^8\theta+12(c_{03}-d_{23})\sin^4\theta\cos^2\theta,\\
f_{12}(\theta) &= -\left(b^2+4\right)c_{13}\sin^3\theta\cos^6\theta+4(3c_{13}+d_{04})\sin^4\theta\cos^3\theta\\
&\quad -12d_{04}\sin^5\theta,
\end{aligned}$$

$$
\begin{aligned}
f_{13}(\theta) &= -\left(b^2+4\right) c_{23}\sin^3\theta\cos^7\theta + 4(3c_{23}+d_{14})\sin^4\theta\cos^4\theta \\
&\quad - 12d_{14}\sin^5\theta\cos\theta, \\
f_{14}(\theta) &= 12c_{04}\sin^5\theta\cos^2\theta - \left(b^2+4\right) c_{04}\sin^4\theta\cos^5\theta, \\
f_{15}(\theta) &= -\left(b^2+4\right) c_{14}\sin^4\theta\cos^6\theta + 4(3c_{14}+d_{05})\sin^5\theta\cos^3\theta \\
&\quad - 12d_{05}\sin^6\theta, \\
f_{16}(\theta) &= 0, \\
f_{17}(\theta) &= 12c_{05}\sin^6\theta\cos^2\theta - \left(b^2+4\right) c_{05}\sin^5\theta\cos^5\theta, \\
W(\theta) &= 12\left(3\sin^2\theta + \cos^2\theta\right).
\end{aligned}
$$

Computing integral (2.4) we obtain

$$
\mathcal{F}(r_0) = \frac{1}{2\pi}\int_0^{2\pi} M^{-1}(\theta)F_1(\theta, r(\theta,r_0))d\theta = \frac{1}{2\pi}\sum_{i=0}^{17} r_0^{i-4}\int_0^{2\pi} A_i(\theta)d\theta,
$$

where the function $A_i(\theta)$ is defined in the statement of Theorem 1.1.

If i is even then it is easy to check that $f_i(\theta) = -f_i(\theta+\pi)$, for $i = 0,\dots,17$, and $\theta \in [\pi, 3\pi/2] \cup [3\pi/2, \pi]$. Since that $k(\theta) = k(\theta+\pi)$, $g_1(\theta) = g_1(\theta+\pi)$ and $W(\theta) = W(\theta+\pi)$, for $\theta \in [\pi, 3\pi/2] \cup [3\pi/2, \pi]$ we can easily show that

$$
\begin{aligned}
\int_\pi^{\frac{3\pi}{2}} A_i(\theta)d\theta &= \int_\pi^{\frac{3\pi}{2}} \frac{f_i(\theta)W(\theta)k(\theta)^{i-5}}{g_1(\theta)^2}d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{f_i(\theta+\pi)W(\theta+\pi)k(\theta+\pi)^{i-5}}{g_1(\theta+\pi)^2}d\theta \\
&= \int_0^{\frac{\pi}{2}} -\frac{f_i(\theta)W(\theta)k(\theta)^{i-5}}{g_1(\theta)^2}d\theta \\
&= -\int_0^{\frac{\pi}{2}} A_i(\theta)d\theta,
\end{aligned}
$$

$$
\begin{aligned}
\int_{\frac{3\pi}{2}}^{2\pi} A_i(\theta)d\theta &= \int_{\frac{3\pi}{2}}^{2\pi} \frac{f_i(\theta)W(\theta)k(\theta)^{i-4}}{g_1(\theta)^2M(\theta)}d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} \frac{f_i(\theta+\pi)W(\theta+\pi)k(\theta+\pi)^{i-4}}{g_1(\theta+\pi)^2M(\theta+\pi)}d\theta \\
&= \int_{\frac{\pi}{2}}^{\pi} -\frac{f_i(\theta)W(\theta)k(\theta)^{i-4}}{g_1(\theta)^2M(\theta)}d\theta \\
&= -\int_{\frac{\pi}{2}}^{\pi} A_i(\theta)d\theta.
\end{aligned}
$$

Thus if i is even we conclude that

$$
\int_0^{2\pi} A_i(\theta)d\theta = 0.
$$

The coefficients A_1 and A_{17} are

$$
A_1 = [768\ 2^{2/3}d_{10}\cos\theta(\cos(2\theta)-2)(-12\sin\theta+3\cos\theta+\cos(3\theta))]/L,
$$

$$A_{17} = [3 \times 2^{22} \left(b^2+4\right)^2 c_{05} \sin^5\theta \cos^2\theta (\cos(2\theta) - 2)M]/N,$$

where

$$\begin{aligned} L &= \left(b^2+4\right)^{2/3} \Big(3\left(5b^2-172\right)\cos(2\theta) + 6\left(b^2+4\right)\cos(4\theta) \\ &\quad + \left(b^2+4\right)\cos(6\theta) + 10b^2 - 192\sin(2\theta) - 96\sin(4\theta) + 616\Big)^{4/3}, \\ M &= \left(b^2+4\right)\cos^3\theta - 12\sin\theta, \\ N &= \Big(3\left(5b^2-172\right)\cos(2\theta) + 6\left(b^2+4\right)\cos(4\theta) + \left(b^2+4\right)\cos(6\theta) \\ &\quad + 10b^2 - 192\sin(2\theta) - 96\sin(4\theta) + 616\Big)^4. \end{aligned}$$

Computing the integrals of the coefficients A_1 and A_{17} in the variable θ, in the interval $[0, 2\pi]$ we obtain that both are zero.

Claim: For $i = 3, 5, 7, 9, 11, 13$ or 15 we can choose the parameters that appear in A_i such that $\int_0^{2\pi} A_i(\theta) d\theta \neq 0$.

The proof of this claim follows from Example 5.1. In summary the function $\mathcal{F}$ defined in (2.4) can be written as

$$\mathcal{F}(r_0) = \frac{1}{2\pi} \sum_{k=1}^{7} r_0^{2k-3} \int_0^{2\pi} A_{2k+1}(\theta) d\theta. \tag{3.3}$$

Note that the coefficients $A_{2k+1}(\theta)$ in (3.3) are linearly independent for $k = 1, .., 7$. Thus by the generalized Descartes Theorem, the average function $\mathcal{F}$ has at most 6 positive simple zeros which provide limit cycles of system (1.4), when the average function is non-zero.

4. Proof of Theorem 1.2

Suppose that $d = 0$. We take the affine change of coordinates

$$\tilde{x} = x, \quad \tilde{y} = y\sqrt{-b/c}, \quad \tilde{t} = t\sqrt{-bc},$$

and system (1.3) becomes $\dot{x} = -y$, $\dot{y} = x^5$. We write system (1.4) in the generalized polar coordinates $x = r\cos\theta$, $y = r^3 \sin\theta$, and we obtain the differential equation

$$\frac{dr}{d\theta} = F_0(r, \theta) + \varepsilon F_1(r, \theta) + \mathcal{O}(\varepsilon^2), \tag{4.1}$$

in the standard form for applying the averaging theory of first order described in Section 2, where

$$\begin{aligned} F_0(r,\theta) &= \frac{r\left(\sin\theta\cos^5\theta - \sin\theta\cos\theta\right)}{\cos^6\theta - 3\cos^2\theta + 3}, \\ F_1(r,\theta) &= -\frac{\cos(2\theta) - 2}{r^4\left(3\sin^2\theta + \cos^6\theta\right)^2} \Big(r^2\cos^5\theta\, p\left(r\cos\theta, r^3\sin\theta\right) \\ &\quad + \sin\theta\, q\left(r\cos\theta, r^3\sin\theta\right)\Big). \end{aligned}$$

Denote by $g_1^*(\theta) = \cos^6\theta - 3\cos^2\theta + 3$. Note that the differential equation (4.1) satisfies the assumptions of Theorem 2.1. Consider $r(\theta, r_0)$ the periodic solution of the differential equation $\dot{r} = F_0(r, \theta)$ such that $r(0, r_0) = r_0$. For solving this differential equation we take $z = \cos^2\theta$ in $g_1^*(\theta)$, and we obtain a polynomial of degree 3 in z which can be factorized in the form

$$g_1^*(z) = (z - z_1)(z - z_2)(z - z_3),$$

where the coefficients of $g_2^*(z) = (z - z_1)$ and $g_3^*(z) = (z - z_2)(z - z_3)$ are reals, and z_i are the roots of g_1^*, for $i = 1, \dots, 3$ given by

$$z_1 = -\frac{2 + \sqrt[3]{2}\left(3 - \sqrt{5}\right)^{2/3}}{2^{2/3}\sqrt[3]{3 - \sqrt{5}}},$$
$$z_{2,3} = \frac{2\sqrt[3]{2}\left(1 \mp i\sqrt{3}\right) + \left(1 \pm i\sqrt{3}\right)\left(6 - 2\sqrt{5}\right)^{2/3}}{4\sqrt[3]{3 - \sqrt{5}}}.$$

Thus the differential equation (4.1) with $\varepsilon = 0$ can be rewritten in the form

$$\frac{dr}{d\theta} = r\Big(C_1\frac{\cos\theta\sin\theta}{-z_3 + \cos^2\theta} + C_2\frac{\cos\theta\sin\theta}{-z_1 + \cos^2\theta} + C_3\frac{\cos\theta\sin\theta}{-z_2 + \cos^2\theta}\Big), \tag{4.2}$$

where

$$C_1 = \frac{z_3^2 - 1}{(z_3 - z_1)(z_3 - z_2)}, \quad C_2 = \frac{z_1^2 - 1}{(z_1 - z_2)(z_1 - z_3)}, \quad C_3 = \frac{z_2^2 - 1}{(z_2 - z_1)(z_2 - z_3)}.$$

The solution of differential equation (4.2) with initial condition $r(0, r_0) = r_0$ is

$$r(\theta, r_0) = r_0 k^*(\theta),$$

where

$$\begin{aligned} k^*(\theta) = r_0(1 - z_3)^{C_1/2}(1 - z_1)^{C_2/2}(1 - z_2)^{C_3/2}\left(\cos^2\theta - z_3\right)^{-C_1/2} \\ \times \left(\cos^2\theta - z_1\right)^{-C_2/2}\left(\cos^2\theta - z_2\right)^{-C_3/2}. \end{aligned}$$

Solving the variational equation (2.3) for our differential equation (4.1) we get that the fundamental matrix is the function $M^*(\theta) = k^*(\theta)$. Note that $M^*(\theta)$ does not depend on r_0. Using the polynomials p and q given in (1.5) and system (1.3) we have that the integrant of the integral (2.4) for the differential equation (4.1) is

$$\begin{aligned} M^{*-1}(\theta)F_1(\theta, r(\theta, r_0)) &= \sum_{i=0}^{17} \frac{W^*(\theta)f_i^*(\theta)}{g_1^*(\theta)^2 M^*(\theta)} r(\theta, r_0)^{i-4} \\ &= \sum_{i=0}^{17} r_0^{i-4}\frac{W^*(\theta)f_i^*(\theta)k^*(\theta)^{i-5}}{g_1^*(\theta)^2} \\ &= \sum_{i=0}^{17} r_0^{i-4} A_i^*(\theta), \end{aligned}$$

where

$$\begin{aligned} f_0^*(\theta) &= d_{00}\sin\theta, \\ f_1^*(\theta) &= d_{10}\sin\theta\cos\theta, \\ f_2^*(\theta) &= c_{00}\cos^5\theta + d_{20}\sin\theta\cos^2\theta, \\ f_3^*(\theta) &= c_{10}\cos^6\theta + d_{01}\sin^2\theta + d_{30}\sin\theta\cos^3\theta, \\ f_4^*(\theta) &= c_{20}\cos^7\theta + d_{11}\sin^2\theta\cos\theta + d_{40}\sin\theta\cos^4\theta, \\ f_5^*(\theta) &= (c_{01} + d_{50})\sin\theta\cos^5\theta + c_{30}\cos^8\theta + d_{21}\sin^2\theta\cos^2\theta, \end{aligned}$$

$$
\begin{aligned}
f_6^*(\theta) &= c_{11}\sin\theta\cos^6\theta + c_{40}\cos^9\theta + d_{02}\sin^3\theta + d_{31}\sin^2\theta\cos^3\theta,\\
f_7^*(\theta) &= c_{21}\sin\theta\cos^7\theta + c_{50}\cos^{10}\theta + d_{12}\sin^3\theta\cos\theta + d_{41}\sin^2\theta\cos^4\theta,\\
f_8^*(\theta) &= c_{02}\sin^2\theta\cos^5\theta + c_{31}\sin\theta\cos^8\theta + d_{22}\sin^3\theta\cos^2\theta,\\
f_9^*(\theta) &= c_{12}\sin^2\theta\cos^6\theta + c_{41}\sin\theta\cos^9\theta + d_{03}\sin^4\theta + d_{32}\sin^3\theta\cos^3\theta,\\
f_{10}^*(\theta) &= c_{22}\sin^2\theta\cos^7\theta + d_{13}\sin^4\theta\cos\theta,\\
f_{11}^*(\theta) &= c_{03}\sin^3\theta\cos^5\theta + c_{32}\sin^2\theta\cos^8\theta + d_{23}\sin^4\theta\cos^2\theta,\\
f_{12}^*(\theta) &= c_{13}\sin^3\theta\cos^6\theta + d_{04}\sin^5\theta,\\
f_{13}^*(\theta) &= c_{23}\sin^3\theta\cos^7\theta + d_{14}\sin^5\theta\cos\theta,\\
f_{14}^*(\theta) &= c_{04}\sin^4\theta\cos^5\theta,\\
f_{15}^*(\theta) &= c_{14}\sin^4\theta\cos^6\theta + d_{05}\sin^6\theta,\\
f_{16}^*(\theta) &= 0,\\
f_{17}^*(\theta) &= c_{05}\sin^5\theta\cos^5\theta,\\
W^*(\theta) &= 3\sin^2\theta + \cos^2\theta.
\end{aligned}
$$

Computing the integral (2.4) we obtain

$$
\mathcal{F}^*(r_0) = \frac{1}{2\pi}\int_0^{2\pi}(M^*)^{-1}(\theta)F_1(\theta, r(\theta, r_0))d\theta = \frac{1}{2\pi}\sum_{i=0}^{17} r_0^{i-4}\int_0^{2\pi} A_i^*(\theta)d\theta,
$$

where the function $A_i^*(\theta)$ is defined in the statement of Theorem 2.1.

Analogously as in the proof of Theorem 1.1 we can show that if i is even then

$$
\int_0^{2\pi} A_i^*(\theta)d\theta = 0.
$$

The coefficients A_1^*, A_{13}^* and A_{17}^* are given by

$$
\begin{aligned}
A_1^* &= -d_{10}\frac{2^{\frac{55}{9}}\left(7-3\sqrt{5}\right)^{4/9}\sin(2\theta)(\cos(2\theta)-2)}{\left(3-\sqrt{5}\right)^{8/9}(-33\cos(2\theta)+6\cos(4\theta)+\cos(6\theta)+58)^{4/3}},\\
A_{13}^* &= -\frac{2^{\frac{4}{3}}\left(2+\sqrt{5}\right)^{4/9}\sin^3\theta\cos\theta(\cos(2\theta)-2)\left(c_{23}\cos^6\theta + d_{14}\sin^2\theta\right)}{\left(1+\sqrt{5}\right)^{4/3}(\cos^6\theta - 3\cos^2\theta+3)^{4/3}\left(3\sin^2\theta+\cos^6\theta\right)^2},\\
A_{17}^* &= -c_{05}\frac{32\sin^5\theta\cos^5\theta(\cos(2\theta)-2)}{R(\theta)},
\end{aligned}
$$

where

$$
\begin{aligned}
R(\theta) = \sqrt[3]{47+21\sqrt{5}}\Big(&2\sqrt[3]{2}\cos^4\theta + 4\sqrt[3]{3\Big(2+2^{2/3}\sqrt[3]{3-\sqrt{5}}+2^{2/3}\sqrt[3]{3+\sqrt{5}}\Big)}\\
&\times\cos^2\theta + \Big(2(3+\sqrt{5})\Big)^{2/3} + (6-2\sqrt{5})^{2/3} + 4\sqrt[3]{2}\Big)\Big(-2(3-\sqrt{5})^{2/3}\\
&\times\cos^4\theta + \Big(2\sqrt[3]{6-2\sqrt{5}} - 2^{2/3}(\sqrt{5}-3)\Big)\cos^2\theta + \sqrt[3]{30\sqrt{5}-50}\\
&+2(3-\sqrt{5})^{2/3} - 3\sqrt[3]{6-2\sqrt{5}} - 2\ 2^{2/3}\Big)^2\left(3\sin^2\theta+\cos^6\theta\right)^2.
\end{aligned}
$$

The integrals of the coefficients A_1^*, A_{13}^* and A_{17}^* in the variable θ, in the interval $[0, 2\pi]$ are zero because A_1^*, A_{13}^* and A_{17}^* are odd functions.
Claim: For $i = 3, 5, 7, 9, 11$ or 15 we can choose the parameters that appear in A_i^* such that $\int_0^{2\pi} A_i^*(\theta)d\theta \neq 0$.

The proof of this claim follows from Example 5.2.

In short the function $\mathcal{F}$ defined in (2.4) can be written as

$$\mathcal{F}^*(r_0) = \frac{1}{2\pi}\Big(\sum_{k=1}^{5} r_0^{2k-3}\int_0^{2\pi} A_{2k+1}^*(\theta)d\theta + r_0^{11}\int_0^{2\pi} A_{15}^*(\theta)d\theta\Big). \tag{4.3}$$

Note that the coefficients $A_{2k+1}^*(\theta)$ in (4.3) are linearly independent for $k = 1, 2, 3, 4, 5, 7$. Thus by the generalized Descartes Theorem, the average function $\mathcal{F}^*$ has at most 5 positive simple zeros which provide limit cycles of system (1.4), when the average function is non-zero.

5. Examples

Example 5.1. Consider the quintic polynomial differential system with a center at the origin

$$\dot{x} = -\frac{1}{3}x^3 + y, \quad \dot{y} = -\frac{5}{12}x^5 + x^2 y,$$

with the perturbation

$$\dot{x} = -\frac{1}{3}x^3 + y, \quad \dot{y} = -\frac{5}{12}x^5 + x^2 y + \varepsilon q(x, y), \tag{5.1}$$

where

$$q(x, y) = d_{01}y + d_{21}x^2 y + d_{12}xy^2 + d_{03}y^3 + d_{23}x^2 y^3 + d_{14}xy^4 + d_{05}y^5.$$

Writing system (5.1) in the coordinates $x = r\cos\theta$ and $y = r^3\sin\theta$ and taking the quotient $\dot{r}/\dot{\theta}$ we get the following system in the standard form of Theorem 2.1 for applying the averaging theory

$$\frac{dr}{d\theta} = F_0(r, \theta) + \varepsilon F_1(r, \theta) + \mathcal{O}(\varepsilon^2), \tag{5.2}$$

where

$$\begin{aligned} F_0(r, \theta) &= \frac{r\cos\theta\left(-6\sin\theta(\sin(2\theta) + 2) + 4\cos^3\theta + 5\sin\theta\cos^4\theta\right)}{36\sin^2\theta + 5\cos^6\theta - 24\sin\theta\cos^3\theta}, \\ F_1(r, \theta) &= -48\, C(\theta)E(\theta)\frac{q\left(r\cos\theta, r^3\sin\theta\right)}{r^4\left(36\sin^2\theta + 5\cos^6\theta - 24\sin\theta\cos^3\theta\right)^2}, \end{aligned}$$

with $C(\theta) = \cos(2\theta) - 2$, and $E(\theta) = \cos^3\theta - 3\sin\theta$. Thus for system (5.2) we have $M(\theta) = k(\theta) = (160/G(\theta))^{1/6}$, where

$$G(\theta) = -192\sin(2\theta) - 96\sin(4\theta) - 501\cos(2\theta) + 30\cos(4\theta) + 5\cos(6\theta) + 626,$$

and the integrant of the integral (2.4) of system (5.2) is

$$\sum_{1}^{7} r_0^{2k-3} A_{2k+1}(\theta),$$

with

$$A_3(\theta) = -d_{01}\frac{3072\sqrt[3]{\frac{2}{5}}\sin\theta\, C(\theta)(-12\sin\theta + 3\cos\theta + \cos(3\theta))}{T(\theta)^{5/3}},$$

$$\begin{aligned}
A_5(\theta) &= -d_{21}\frac{48\sin\theta\cos^2\theta\, C(\theta)E(\theta)}{\left(36\sin^2\theta+5\cos^6\theta-24\sin\theta\cos^3\theta\right)^2},\\
A_7(\theta) &= -d_{12}\frac{98304\ 2^{2/3}\sqrt[3]{5}\sin^2\theta\cos\theta\, C(\theta)E(\theta)}{T(\theta)^{7/3}},\\
A_9(\theta) &= -d_{03}\frac{393216\sqrt[3]{2}\,5^{2/3}\sin^3\theta\, C(\theta)E(\theta)}{T(\theta)^{8/3}},\\
A_{11}(\theta) &= -d_{23}\frac{7864320\sin^3\theta\cos^2\theta\, C(\theta)E(\theta)}{T(\theta)^3},\\
A_{13}(\theta) &= -d_{14}\frac{15728640\ 2^{2/3}\sqrt[3]{5}\sin^4\theta\cos\theta\, C(\theta)E(\theta)}{T(\theta)^{10/3}},\\
A_{15}(\theta) &= -d_{05}\frac{62914560\sqrt[3]{2}5^{2/3}\sin^5\theta\, C(\theta)E(\theta)}{T(\theta)^{11/3}},
\end{aligned}$$

and

$$T(\theta)=-192\sin(2\theta)-96\sin(4\theta)-501\cos(2\theta)+30\cos(4\theta)+5\cos(6\theta)+626.$$

Computing numerically the integral (2.4) for system (5.2) we obtain

$$\begin{aligned}
\mathcal{F}(r_0)=\frac{1}{r_0}\Big(&-4.2608..\,d_{01}-2.0944..\,d_{21}r_0^2-1.2770..\,d_{12}r_0^4-1.2427..\,d_{03}r_0^6\\
&-1.0908..\,d_{23}r_0^8-0.7348..\,d_{14}r_0^{10}-0.5419..\,d_{05}r_0^{12}\Big).
\end{aligned}$$

Taking

$$\begin{gathered}
d_{01}=-\frac{720}{4.2608..},\quad d_{21}=\frac{1764}{2.0944..},\quad d_{12}=-\frac{1624}{1.2770..},\\
d_{03}=\frac{735}{1.2427..},\quad d_{23}=-\frac{175}{1.0908..},\quad d_{14}=\frac{21}{0.7348..},\quad d_{05}=-\frac{1}{0.5419..}.
\end{gathered}$$

The function $\mathcal{F}$ becomes

$$\mathcal{F}(r_0)=\frac{r_0^{12}-21r_0^{10}+175r_0^8-735r_0^6+1624r_0^4-1764r_0^2+720}{r_0}=\frac{1}{r_0}\prod_{i=1}^{6}(r_0^2-i).$$

Thus we have that $\mathcal{F}$ has 6 positive simple zeros given by $r_{0,i}=\sqrt{i}$, for $i=1,\dots,6$ which by Theorem 2.1, provide 6 limit cycles of the perturbed system (5.1) for $\varepsilon\neq 0$ sufficiently small.

Example 5.2. Consider the quintic polynomial differential system with a center at the origin

$$\dot{x}=-y,\quad \dot{y}=x^5,$$

with the perturbation

$$\dot{x}=-y+\varepsilon p(x,y),\quad \dot{y}=x^5+\varepsilon q(x,y), \tag{5.3}$$

where

$$\begin{aligned}
p(x,y)&=c_{30}x^3+c_{50}x^5+c_{14}xy^4,\\
q(x,y)&=d_{01}y+d_{03}y^3+d_{23}x^2y^3.
\end{aligned}$$

Writing system (5.3) in the coordinates $x = r\cos\theta$ and $y = r^3\sin\theta$ and taking the quotient $\dot r/\dot\theta$ we obtain the following system in the standard form of Theorem 2.1 for applying the averaging theory

$$\frac{dr}{d\theta} = F_0(r,\theta) + \varepsilon F_1(r,\theta) + \mathcal{O}(\varepsilon^2), \tag{5.4}$$

with $F_0(r,\theta)$ given in the proof of Theorem 1.1 and

$$\begin{aligned} F_1(r,\theta) = -\frac{\cos(2\theta)-2}{r\left(3\sin^2\theta+\cos^6\theta\right)^2}\Big[& r^6\sin^4\theta\left(c_{14}r^6\cos^6\theta + d_{03} + d_{23}r^2\cos^2\theta\right) \\ &+ r^2\cos^8\theta\left(c_{30}+c_{50}r^2\cos^2\theta\right) + d_{01}\sin^2\theta\Big]. \end{aligned}$$

The functions $k^*(\theta)$ and $M^*(\theta)$ for system (5.4) are given also in the proof of Theorem 1.1 and the integrant of the integral (2.4) of system (5.4) is

$$\sum_{k=1}^{5} r_0^{2k-3}A^*_{2k+1}(\theta) + r_0^{11}A^*_{15}(\theta),$$

where

$$\begin{aligned}
A_3^*(\theta) &= -d_{01}\frac{256\left(6-2\sqrt{5}\right)^{2/9}\sin^2\theta(\cos(2\theta)-2)}{\sqrt[9]{7-3\sqrt{5}}(-33\cos(2\theta)+6\cos(4\theta)+\cos(6\theta)+58)^{5/3}},\\
A_5^*(\theta) &= -c_{30}\frac{\cos^8\theta(\cos(2\theta)-2)}{\left(3\sin^2\theta+\cos^6\theta\right)^2},\\
A_7^*(\theta) &= -c_{50}\frac{\cos^{10}\theta(\cos(2\theta)-2)}{\sqrt[3]{\cos^6\theta-3\cos^2\theta+3}\left(3\sin^2\theta+\cos^6\theta\right)^2},\\
A_9^*(\theta) &= -d_{03}\frac{2^{\frac{40}{3}}\sin^4\theta(\cos(2\theta)-2)}{(-33\cos(2\theta)+6\cos(4\theta)+\cos(6\theta)+58)^{8/3}},\\
A_{11}^*(\theta) &= -d_{23}\frac{2^{\frac{43}{3}}\left(\sqrt{5}-1\right)^{4/3}\sin^4\theta\cos^2\theta(\cos(2\theta)-2)}{\left(3-\sqrt{5}\right)^{2/3}(-33\cos(2\theta)+6\cos(4\theta)+\cos(6\theta)+58)^3},\\
A_{15}^*(\theta) &= -c_{14}\frac{2^{\frac{55}{3}}\sin^4(\theta)\cos^6\theta(\cos(2\theta)-2)}{(-33\cos(2\theta)+6\cos(4\theta)+\cos(6\theta)+58)^{11/3}},
\end{aligned}$$

Computing numerically the integral (2.4) for system (5.4) we obtain

$$\begin{aligned} \mathcal{F}^*(r_0) = \frac{1}{r_0}\Big(& 2.1033..\, d_{01} + 1.8138..\, c_{30}r_0^2 + 1.6169..\, c_{50}r_0^4 + 0.6310..\, d_{03}r_0^6 \\ &+ 0.1512..\, d_{23}r_0^8 + 0.0394..\, c_{14}r_0^{12}\Big). \end{aligned}$$

Taking

$$\begin{gathered} d_{01} = -\frac{1800}{2.1033..}, \quad c_{30} = \frac{3990}{1.8138..}, \quad c_{50} = -\frac{3101}{1.6169..}, \\ d_{03} = \frac{1050}{0.6310..}, \quad d_{23} = -\frac{140}{0.1512..}, \quad c_{14} = \frac{1}{0.0394..}. \end{gathered}$$

The function $\mathcal{F}^*$ is now given by

$$\mathcal{F}^*(r_0) = \frac{r_0^{12} - 140r_0^8 + 1050r_0^6 - 3101r_0^4 + 3990r_0^2 - 1800}{r_0}$$

$$= \frac{r_0^2 + 15}{r_0} \prod_{i=1}^{5} (r_0^2 - i).$$

Thus we have that $\mathcal{F}^*$ has 5 positive simple zeros given by $r_{0,i} = \sqrt{i}$, for $i = 1, \dots, 5$ which by Theorem 2.1, provide 5 limit cycles of the perturbed system (5.3) with $\varepsilon \neq 0$ sufficiently small.

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