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ALGEBRAIC LIMIT CYCLES FOR QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. We prove that for a quadratic polynomial differential system having three pairs of diametrally opposite equilibrium points at infinity that are positively rationally independent, has at most one algebraic limit cycle. Our result provides a partial positive answer to the following conjecture: *Quadratic polynomial differential systems have at most one algebraic limit cycle.*

1. Introduction and statement of the main results

In this paper we focus on differential systems of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where $P, Q \in \mathbb{R}[x,y]$, being $\mathbb{R}[x,y]$ the ring of all real polynomials in x and y and where the maximum degree of P and Q is two. They are called *quadratic systems* and in what follows will be denoted simply by QS *systems*.

The algebraic curve $g(x,y) = 0$ of $\mathbb{R}^2$ with $g = g(x,y) \in \mathbb{R}[x,y]$ is an *invariant algebraic curve* of the QS system (1) if for some polynomial $K \in \mathbb{R}[x,y]$, we have

$$P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} = Kg.$$

The polynomial K is called *the cofactor* and has degree at most one. Moreover, the algebraic curve $g = 0$ is *invariant* under the flow defined by the solution of the QS system (1). If g is irreducible in $\mathbb{R}[x,y]$ we say that the invariant algebraic curve is *irreducible*. A *limit cycle* of the QS system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1). An *algebraic limit cycle* of system (1) is an oval of a real irreducible invariant algebraic curve which is a limit cycle of system (1). From now on we will denote it by ALC.

The following open conjecture is related with the second part of Hilbert's 16th problem (see [16]): *A QS system has at most one ALC.*

This conjecture has been running from 1958 where the ALC of quadratic systems started to be studied. More concretely, it is proved in [23] that QS systems can have an ALC of degree 2, and that they are unique whenever they exist. In [12, 13, 14] (see also [4, 17, 21, 24] for different proofs) the

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author proved that a QS system does not have ALC of degree 3. In [22] and [15] it is found two different families of ALC of degree 4 inside the QS systems. More recently, two new families have been found and in [7] the authors proved that there are no other families of ALC of degree 4 for QS systems. The uniqueness of these limit cycles was proved in [9]. It is known that there are QS systems having ALC of degrees 5 and 6, see [7], and that this limit cycle is the unique one for these QS systems.

In [19] and [20] the authors proved the conjecture in the cases in which a QS system has one or two pairs of equilibrium points at infinity. Therefore in order to prove the conjecture we can restrict to the cases in which the QS has three pairs of diametrally opposite equilibrium points at infinity. Recall that a QS system has at most three pairs of equilibria at infinity. This problem is still too demanding and so we restrict to the case in which there are two infinite equilibria that are positively rationally independent. An equilibrium point is said to be *positively rationally independent* if the eigenvalues of the Jacobian matrix of the differential equation at this point has two eigenvalues $\lambda \neq 0$, $\mu \neq 0$ such that $\lambda/\mu \notin \mathbb{Q}^+$, where $\mathbb{Q}^+$ denotes the set of positive rational numbers. We shall use the next assumption.

(H1) The three pairs of equilibrium points at infinity are positively rationally independent.

The following is the main result in this paper.

Theorem 1. *A QS system with three pairs of equilibrium points at infinity satisfying the hypothesis* (H1) *has at most one ALC.*

In view of Theorem 1 it remains to prove the conjecture in the case in which a QS has three pairs of equilibrium points at infinity not two of them being positively rationally independent.

The proof of Theorem 1 is divided in two sections 3 and 4. In section 2 we state some known facts about quadratic polynomial differential systems that we shall need.

2. Quadratic systems: Known results

The following results are known. The first one is proved in [9, 4].

Theorem 2. *A QS system having an ALC of degree* 2 *or* 4 *has at most* 1 *ALC.*

In view of Theorem 2 from now on *we will consider ALC of degree* $n \geq 5$.

The next result is proved in [3, 10].

Theorem 3. *A QS system having an invariant straight line has at most* 1 *limit cycle.*

The next result is proved in [18] and [19] (see Theorem 4 of [18] and its proof and Theorem 2 of [19]).

Theorem 4. *If a QS system has an ALC of degree n, then it can be written, through an affine change of variables and a scaling of the time, into one of the following two systems. First*

$$\begin{aligned} \dot{x} &= \xi x - y + ax^2 + bxy, \\ \dot{y} &= x - \xi y + dx^2 + exy + fy^2, \end{aligned} \tag{2}$$

with $d \neq 0$ and $\xi \in (-1, 1)$, and second

$$\begin{aligned} \dot{x} &= -y + ax^2 + bxy + cy^2, \\ \dot{y} &= x + exy + fy^2. \end{aligned} \tag{3}$$

The cofactor of $g = 0$ in both cases is ny. Moreover, if systems (2) or (3) have a limit, it must surround the origin.

The following result is proved in [18] (see Proposition 13).

Proposition 5. *If $yP_2 - xQ_2 \equiv 0$, where P_2 and Q_2 are the homogeneous components of degree two of P and Q in the QS system (1), then it has no limit cycles.*

The following lemma is proved in [6].

Proposition 6. *Assume that a QS system (1) has the invariant algebraic curve $g = 0$ of degree n. Let P_2 and Q_2 be as in Proposition 5 and g_n the homogeneous component of g of degree n. Then the irreducible factors of g_n must be factors of $yP_2 - xQ_2$.*

The proof of the next theorem can be found in [5] (or also in [8] via Theorem A2).

Theorem 7. *A QS system of the form*

$$\dot{x} = -y + a_1x + a_2x^2, \quad \dot{y} = x(1 + a_3x + a_4y),$$

has at most 1 limit cycle surrounding the $(0, 0)$.

The proof of the next theorem is Theorem 2 in [19].

Theorem 8. *A QS system with at most one pair of equilibrium points at infinity has at most 1 ALC.*

The proof of the next theorem can be found in [20] (see Theorem 1).

Theorem 9. *A QS system with at most two pairs of equilibrium points at infinity has at most 1 ALC.*

In view of Theorems 8 and 9 in order to prove Theorem 1 we can restrict ourselves to study the quadratic polynomial differential systems having three pairs of equilibrium points at infinity.

We now give necessary conditions in order that a polynomial be a cofactor K of an invariant algebraic curve. The following result is taken from [1].

Let $g(x, y) = 0$ be an ALC of degree n of a QS system of the form (1). Let (x_0, y_0) be a point such that $g(x_0, y_0) = 0$ and we expand the ALC in powers of $x - x_0$ and $y - y_0$ in the following way

$$g(x, y) = g_s(x, y) + g_{s+1}(x, y) + \cdots + g_n(x, y) \tag{4}$$

where $g_j(x,y)$ are homogeneous polynomials of degree j in powers of $x-x_0$ and $y-y_0$. We denote by s the lowest degree in expansion (4). Note that $s \geq 1$ because $g(x_0,y_0)=0$ we have $s \geq 1$, and since $g_s(x,y)$ is a homogeneous polynomial of degree s it factorizes as follows

$$g_s(x,y) = L_1 L_2 \cdots L_s \quad L_i = a_i(x-x_0)+b_i(y-y_0), \quad a_i, b_i \in C,\ i=1,2\ldots,s.$$

We denote by λ and μ the eigenvalues of the linear approximation of g at this point (x_0,y_0).

Theorem 10. *Consider a QS system* (1) *with* (x_0,y_0) *being one of its equilibrium points and let* $g(x,y)=0$ *be an ALC with cofactor* $K(x,y)$*. If* $g(x_0,y_0) \neq 0$ *then* $K(x_0,y_0)=0$*. Furthermore, if* $g(x_0,y_0)=0$*, with the above notation, we have that* $g_s(x,y) = (L_\lambda)^{s_\lambda}(L_\mu)^{s-s_\lambda}$ *with* $s, s_\lambda \in \mathbb{N}$ *and* $s_\lambda \leq s$ *and* $K(x_0,y_0) = s_\lambda \mu + (s-s_\lambda)\lambda$*. Assume that* $\mu \neq 0$ *and* $\lambda/\mu \notin \mathbb{Q}^+$ *then*

(a) *either* $s=2$, $s_\lambda = 1$ *and* $g_2 = L_\lambda L_\mu$*;*
(b) *or* $s=1$ *and* $g_1 = L_\lambda$*;*
(c) *or* $s=1$ *and* $g_1 = L_\mu$*.*

It is stated in [1] that Theorem 10 is also valid in the local charts U_1 and U_2 of the Poincaré disc. For the definition of Poincaré disc and the local charts for studying it see for instance Chapter 5 in [11].

3. Proof of Theorem 1 for system (2)

In this section we prove the following theorem.

Theorem 11. *Let* $g=0$ *be an invariant algebraic curve of degree* $n \geq 5$ *of a QS system* (2)*. Assume that* $g=0$ *has three pairs of equilibrium points at infinity and that satisfies hypothesis (H1). Then there is at most* 1 *ALC on* $g=0$*.*

Proof. Let $g = \left(\sum_{i=0}^m g_{n-i,i}x^{n-i}y^i\right) + \cdots$, $0 \leq m \leq n$ with $g_{n-m,m} \neq 0$, where the dot denotes the terms of order $n-1$ and lower. The coefficient of the term $x^{n-m}y^{m+1}$ in the expression of $\dot g = nyg$ is equal to

$$g_{n-m,m}((n-m)b + mf - n) = 0.$$

Therefore

$$b = (n-mf)/(n-m) \text{ if } m \neq n, \quad \text{and} \quad f = 1 \text{ if } m = n.$$

From Proposition 6 we have

$$\sum_{i=0}^m g_{n-i,i}x^{n-i}y^i = x^{n-m}(x-x_1y)^k(x-x_2y)^{m-k}, \tag{5}$$

where

$$x_{1,2} = \frac{a-e \pm \sqrt{\Delta}}{2d} \quad \text{with} \quad \Delta = (a-e)^2 + 4d(b-f) > 0$$

are the roots of the polynomial $dx^2 - (a-e)x - (b-f) = 0$. The case in which $\Delta < 0$ was proved in [19] because in that case at infinity there is a unique pair of singular points which are the endpoints of $x = 0$.

On the local chart U_1 in the variables (u, v) the algebraic invariant curve g has the form

$$g(u,v) = u^{n-m}(u-x_1)^{m-k}(u-x_2)^k + vg_{n-1}(u) + v^2 g_{n-2}(u) + \cdots + v^n g_0(u), \tag{6}$$

where $g_j(u)$ is a polynomial of degree j and the cofactor becomes

$$K(u,v) = n. \tag{7}$$

Note that $K(0,0) = K(x_1, 0) = K(x_2, 0) = n \neq 0$ and so in view of Theorem 10 we have $n > m$, $m > k$ and $k > 0$.

We recall that g_n given in (5) must satisfy

$$\Big(ax^2 + \frac{n-mf}{n-m}xy\Big)\frac{\partial g_n}{\partial x} + (dx^2 + exy + fy^2)\frac{\partial g_n}{\partial y} = nyg_n.$$

Doing so we get that three possibilities can hold:

(i) if $f \neq 1$ and $m \neq 2k$ then
$$d = \frac{(n-m)((a-e)k - an)((a-e)(k-m) + an)}{(f-1)(2k-m)^2 n};$$
(ii) If $m = 2k$ and $f \neq 1$ then $a = ek/(k-n)$,
(iii) If $f = 1$, then either $a = ek/(k-n)$ or $a = e(k-m)/(n-m+k)$.

Note that if $f = 1$ then $b = 1$ and $\Delta = (a-e)^2$ which implies that either $x_1 = 0$ or $x_2 = 0$. So, system (2) has only two pairs of equilibrium points at infinity, which is not possible. We will consider the other two cases separately.

For the definition of Poincaré disc and the local charts for studying it, see for instance Chapter 5 in [11].

Case 1*: d as in* (i)*, with $f \neq 1$ and $m \neq 2k$.* In the local chart U_2 system (2) with $b = (n - mf)/(n - m)$ becomes

$$\begin{aligned} \dot{u} &= \frac{(f-1)n}{m-n}u - v + (a-e)u^2 + 2\xi uv - u^2 v - du^3, \\ \dot{v} &= -v\Big(f + eu - \xi v + du^2 + uv\Big), \end{aligned} \tag{8}$$

where d is as in (i). The infinite equilibrium points for system (2) are on $v = 0$ and with coordinates

$$u_1 = 0, \quad u_2 = \frac{(f-1)(2k-m)n}{(n-m)((a-e)k - an)}, \quad u_3 = \frac{(f-1)(2k-m)n}{(n-m)((a-e)(k-m) + an)}.$$

We compute the eigenvalues λ_1 and μ_1 of the Jacobian matrix at the point $(u_1, 0)$ and we get

$$\lambda_1 = -f, \quad \mu_1 = \frac{n(f-1)}{m-n}.$$

Assume that $\lambda_1\mu_1 \neq 0$ and $\lambda_1/\mu_1 \notin \mathbb{Q}^+$. Then it follows from Theorem 10(a) that either $s = 2$ and $r = 1$ in which case $K(u_1, 0) = \lambda_1 + \mu_1$, or $s = 1$ and $r = 0$ in which case $K(u_1, 0) = \lambda_1$, or $s = 1, r = 1$ in which case $K(u_1, 0) = \mu_1$. Since $K(u_1, 0) = n$ we must have either $\lambda_1 + \mu_1 = n$, or $\lambda_1 = n$, or $\mu_1 = n$.

In the first case $\lambda_1 + \mu_1 = n$, solving this last equation in f we get that $f = n(n - m - 1)/(m - 2n)$, which is well-defined because $m < n$. Substituting f in λ_1 and μ_1 we obtain

$$\frac{\lambda_1}{\mu_1} = \frac{m - n - 1}{n + 1} \in \mathbb{Q}^+, \quad \text{if } n \neq m + 1,$$

which is not possible. In the second case $\lambda_1 = n$, solving this equation in f we get $f = -n$ and then

$$\frac{\lambda_1}{\mu_1} = \frac{n - m}{n + 1} \in \mathbb{Q}^+,$$

which is not possible. Finally, in the third case $\mu_1 = n$, solving this equation in f we obtain $f = 1 + m - n$ and

$$\frac{\lambda_1}{\mu_1} = \frac{n - m - 1}{n} \in \mathbb{Q}^+, \quad \text{if } n \neq m + 1,$$

which again is not possible. In view of hypothesis (H1) we must have $n = m + 1$.

By hypothesis (H1) we have that $\lambda_2/\mu_2 \notin \mathbb{Q}^+$ and $\lambda_3/\mu_3 \notin \mathbb{Q}^+$. Now we compute the eigenvalues λ_2 and μ_2 of the Jacobian matrix at the point $(u_2, 0)$ with $n \neq m + 1$ and we get

$$\begin{aligned}\lambda_2 &= \frac{(f - 1)(m + 1)(ma + 2a + em)}{ma + a + ek - ka},\\ \mu_2 &= \frac{2fka - ka + fkma - kma - ma - a - ek - ekm + efkm}{ma + a + ek - ka}.\end{aligned}$$

Since $K(u_i, 0) = n \neq 0$ for $i = 1, 2$, in view of Theorem 10 we must have $g_n(u_2, 0) = g_n(u_3, 0) = 0$. So $m > k > 0$. Since by assumptions we have $\lambda_2/\mu_2 \notin \mathbb{Q}^+$, it follows from Theorem 10(a) that either $s = 2$ and $r = 1$ in which case $K(u_2, 0) = \lambda_2 + \mu_2$, or $s = 1$ and $r = 0$ in which case $K(u_2, 0) = \lambda_2$, or $s = 1, r = 1$ in which case $K(u_2, 0) = \mu_2$. Since $K(u_2, 0) = n$ we must have either $\lambda_2 + \mu_2 = n$, or $\lambda_2 = n$, or $\mu_2 = n$.

In the first case $\lambda_2 + \mu_2 = n$, solving this equation in e we get that

$$e = -\frac{a(m + 2)(f(k + m + 1) - 2(m + 1))}{(f - 1)m(m + 1) + k((f - 2)m - 2)},$$

if $(f - 1)m(m + 1) + k((f - 2)m - 2) \neq 0$. Substituting e in λ_2 and μ_2 we get that

$$\frac{\lambda_2}{\mu_2} = \frac{m + 2}{k - 1} \in \mathbb{Q}^+, \quad \text{if } k \neq 1, \tag{9}$$

which is not possible. If $(f - 1)m(m + 1) + k((f - 2)m - 2) = 0$, solving the equation $\lambda_2 + \mu_2 = n$ we obtain $a = 0$, and again we get (9) which is not possible.

In the second case $\lambda_2 = n$, solving this equation in e we get

$$e = \frac{a(k-2m+f(m+2)-3)}{k-fm+m},$$

if $k-fm+m \neq 0$. Substituting e in λ_2 and μ_2 we get that

$$\frac{\lambda_2}{\mu_2} = \frac{m+1}{k-1} \in \mathbb{Q}^+, \quad \text{if } k \neq 1, \tag{10}$$

which is not possible. If $k-fm+m = 0$, solving the equation $\lambda_2 = n$ we obtain $a = 0$, and again we get (10) which is not possible.

Finally, in the third case $\mu_2 = n$, solving this equation in e we get

$$e = -\frac{a(fk-m-1)(m+2)}{k((f-2)m-2)},$$

if $k((f-2)m-2) \neq 0$. Substituting e it in λ_2 and μ_2 we get

$$\frac{\lambda_2}{\mu_2} = \frac{m+2}{k} \in \mathbb{Q}^+, \tag{11}$$

which is again not possible. If $k((f-2)m-2) = 0$, solving the equation $\mu_2 = n$ we obtain $a = 0$, and again we get (11) which is not possible.

In short, by assumption (H1) we must have $n = m+1$ and $k = 1$.

The eigenvalues λ_3 and μ_3 of the Jacobian matrix at the point $(u_3, 0)$ are

$$\begin{aligned} \lambda_3 &= \frac{(f-1)(m+1)(ma+2a+em)}{2a-e+em}, \\ \mu_3 &= \frac{-am^2-em^2+afm^2+efm^2-am+afm-efm+e-2af}{2a-e+em}. \end{aligned}$$

Since by assumptions we have $\lambda_3/\mu_3 \notin \mathbb{Q}^+$, it follows from Theorem 10(a) that either $\lambda_3 + \mu_3 = n$, or $\lambda_3 = n$, or $\mu_3 = n$.

In the first case $\lambda_3 + \mu_3 = n$, solving this equation in e we get that

$$e = -\frac{2a(m+2)((f-1)m-1)}{m((2f-3)m-1)+2},$$

if $m((2f-3)m-1)+2 \neq 0$. Substituting e in λ_3 and μ_3 we get that

$$\frac{\lambda_3}{\mu_3} = \frac{m+2}{m-2} \in \mathbb{Q}^+, \tag{12}$$

because $n = m+1 \geq 5$ this case is not possible. If $m((2f-3)m-1)+2 = 0$, then $\lambda_3 + \mu_3 = n$ implies that $a = 0$, and we again obtain (12) which is not possible.

In the second case $\lambda_3 = n$, solving this equation in e we get

$$e = \frac{a(m-f(m+2)+4)}{(f-2)m+1},$$

if $(f-2)m+1 \neq 0$. Substituting e in λ_3 and μ_3 we get that

$$\frac{\lambda_3}{\mu_3} = \frac{m+1}{m-2}. \tag{13}$$

Again since $n = m + 1 \geq 5$ this case is not possible. If $(f - 2)m + 1 = 0$, then $\lambda_3 = n$ implies that $a = 0$, and we obtain (13), which is not possible.

Finally, in the third case $\mu_3 = n$, solving this equation in e we get

$$e = -\frac{a(f(m-1) - m - 1)(m+2)}{(m-1)((f-2)m-2)},$$

if $(f - 2)m - 2 \neq 0$. Substituting e it in λ_3 and μ_3 we get

$$\frac{\lambda_3}{\mu_3} = \frac{m+2}{m-1} \in \mathbb{Q}^+, \tag{14}$$

which is again not possible. If $(f - 2)m - 2 = 0$, then $\mu_3 = n$ implies $a = 0$, and we obtain (14) which is not possible.

This concludes the proof of the theorem in Case 1.

Case 2: *$m = 2k$, $a = ek/(k - n)$ and $f \neq 1$.* Clearly $k > 0$. In the local chart U_2 system (2) with $b = (n - mf)/(n - m)$ becomes (8) with $m = 2k$ and $a = ek/(k-n)$, $g(u, v)$ as in (6) and K as in (7). The equilibrium points for system (8) are on $v = 0$ and coordinates

$$u_1 = 0, \quad u_{2,3} = \frac{en(2k-n) \mp \sqrt{S}}{2d(2k^2 - 3kn + n^2)},$$

where

$$S = (2k - n)n(4d(f - 1)(k - n)^2 + e^2(2k - n)n).$$

We compute the eigenvalues λ_1 and μ_1 of the Jacobian matrix at the point $(u_1, 0)$ and we get

$$\lambda_1 = -f, \quad \mu_1 = \frac{n(f-1)}{2k-n}.$$

Note that $g(u_1, 0) = 0$. By assumption $(H1)$ we have that $\lambda_1\mu_1 \neq 0$ and $\lambda_1/\mu_1 \notin \mathbb{Q}^+$. Then it follows from Theorem 10(a) that either $\lambda_1 + \mu_1 = n$, or $\lambda_1 = n$, or $\mu_1 = n$.

In the first case $\lambda_1 + \mu_1 = n$, solving this equation in f we get that $f = (n(n - 2k - 1))/(2(k - n))$ (which is well-defined because $k \neq n$) and

$$\frac{\lambda_1}{\mu_1} = \frac{n-2k-1}{n+1} \in \mathbb{Q}^+, \quad \text{if } n \neq 2k+1,$$

which is not possible. In the second case $\lambda_1 = n$, solving this equation in f we get $f = -n$ and then

$$\frac{\lambda_1}{\mu_1} = \frac{n-2k}{n+1} \in \mathbb{Q}^+,$$

which is not possible. Finally, in the third case $\mu_1 = n$, solving this equation in f we obtain $f = 1 + 2k - n$ and

$$\frac{\lambda_1}{\mu_1} = \frac{n-2k-1}{n} \in \mathbb{Q}^+, \quad \text{if } n \neq 2k+1,$$

which is again not possible.

In short, $n = 2k+1$ and by assumptions (H1) we must have $\lambda_2/\mu_2 \notin \mathbb{Q}^+$ and $\lambda_3/\mu_3 \notin \mathbb{Q}^+$. We compute the eigenvalues $\lambda_{2,3}$ and $\mu_{2,3}$, of the Jacobian matrix at the points $(u_{2,3}, 0)$ respectively, and we get respectively

$$\lambda_{2,3} = \frac{(2k+1)(-2ke^2 - e^2 \mp \sqrt{S}e - 4dk^2 + 4dfk^2 - 4d + 4df - 8dk + 8dfk)}{2d(k+1)^2},$$

$$\mu_{2,3} = \frac{4dfk^3 - 4dk^3 - 2e^2k^2 - 10dk^2 + 8dfk^2 - e^2k - 8dk + 4dfk \mp e\sqrt{S}k - 2d}{2d(k+1)^2},$$

where

$$S = -(2k+1)\left(4d(f-1)(k+1)^2 - e^2(2k+1)\right).$$

Since we are assuming that $\lambda_2/\mu_2 \notin \mathbb{Q}^+$ in view of Theorem 10(a) we have that either $\lambda_2 + \mu_2 = n$, or $\lambda_2 = n$, or $\mu_2 = n$.

In the first case $\lambda_2 + \mu_2 = n$, solving this equation in e we get that

$$e = -\frac{2\sqrt{d}(k+1)(3kf + f - 4k - 2)}{\sqrt{3kf + f - 5k - 3}\sqrt{k(6k+5)+1}},$$

if $3kf + f - 5k - 3 \neq 0$. Substituting e in λ_2 and μ_2 we get

$$\frac{\lambda_2}{\mu_2} = \frac{2(k+1)}{k-1} \in \mathbb{Q}^+,$$

because $n = 2k+1$ and $n \geq 5$. So this case is not possible. If $3kf + f - 5k - 3 = 0$, there is no solution of $\lambda_2 + \mu_2 = n$ because d cannot be zero.

In the second case $\lambda_2 = n$, solving this equation in e we get that

$$e = \frac{\sqrt{d}(3-2f)(k+1)}{\sqrt{f-2}\sqrt{2k+1}},$$

if $f \neq 2$, but then

$$\frac{\lambda_2}{\mu_2} = \frac{2k+1}{k-1} \in \mathbb{Q}^+,$$

because $n = 2k+1$ and $n \geq 5$. So this case is not possible. If $f = 2$, then there is no solution of $\lambda_2 = n$ because d cannot be zero.

Finally, in the third case $\mu_2 = n$, solving this equation in e we get that

$$e = -\frac{2\sqrt{d}(k+1)((f-2)k-1)}{\sqrt{k}\sqrt{2k+1}\sqrt{(f-3)k-2}},$$

if $(f-3)k - 2 \neq 0$. Substituting e in λ_2 and μ_2 we get

$$\frac{\lambda_2}{\mu_2} = \frac{2(k+1)}{k} \in \mathbb{Q}^+,$$

which is again not possible. If $(f-3)k - 2 = 0$, then there is no solution of $\lambda_2 = n$ because d cannot be zero. Hence, this case is not possible and so the proof of the theorem is complete. □

Proof of Theorem 1 for system (2). Theorem 1 for system (2) follows directly from Theorem 11. □

4. Proof of Theorem 1 for system (3)

The proof of Theorem 1 for system (3) will be an immediate consequence of the proof of the following theorem.

Theorem 12. *Let $g = 0$ be an invariant algebraic curve of degree $n \geq 5$ of a QS* (3)*. Assume that $g = 0$ has three pairs of equilibrium points at infinity and that satisfies hypothesis (H1). Then there is at most* 1 *ALC on $g = 0$.*

Proof. Let $g = \left(\sum_{i=0}^{m} g_{n-i,i} y^{n-i} x^i\right) + \cdots$ with $g_{n-m,m} \neq 0$, where the dots indicate terms of degree $n-1$ and lower. The coefficient of the term $y^{n-m}x^{m+1}$ in the expression $\dot{g} - nyg$ is equal to $am + e(n-m) = 0$. Therefore

$$e = \frac{am}{m-n} \quad \text{if } m \neq n, \quad \text{and} \quad a = 0 \text{ if } m = n.$$

We consider different cases: $c = 0$ and $c \neq 0$.

Case 1*:* $c = 0$ In this case system (3) becomes

$$\begin{aligned} \dot{x} &= -y + ax^2 + bxy, \\ \dot{y} &= x + exy + fy^2. \end{aligned} \tag{15}$$

By Propositions 5 and 6 (we are assuming that the line at infinity is not formed by equilibrium points, otherwise the system cannot have a limit cycle), we have that $\sum_{i=0}^{m} g_{n-i,i} y^{n-i} x^i = y^{n-m} x^k ((a-e)x + (b-f)y)^{m-k}$ with $0 \leq k \leq m$. We recall that $b \neq f$ and $e \neq f$, otherwise system (15) would have only two pairs of singular points at infinity which is not possible.

If $n = m$ then $a = 0$. If $b \neq 0$ system (15) has the invariant straight line $x = 1/b$ and in view of Proposition 3 system (3) has at most one limit cycle. So, $b = 0$. Moreover, if $f = 0$, then either the system has no equilibrium points at infinity (if $e \neq 0$) or the line at infinity is formed by equilibrium points (if $e = 0$). In both cases it follows, respectively, from Propositions system (3) has no limit cycles. So, we can assume that $m \neq n$.

On the local chart U_1 we have that $K(u, v) = nu$. Since $K(y_1, 0) = ny_1 \neq 0$, being $y_1 = (a-e)/(b-f)$, we must have $m > k$. Moreover, imposing that $g_n = y^{n-m} x^k (y - y_1 x)^{m-k}$ satisfies

$$(ax^2 + bxy)\frac{\partial g_n}{\partial x} + (exy + fy^2)\frac{\partial g_n}{\partial y} = nyg_n,$$

we get that $e = am/(m-n)$ and $b = (fk + n - fn)/k$. Note that $a \neq 0$ otherwise $e = 0$ which is not possible.

On the local chart U_1 system (15) with e, f as above becomes

$$\begin{aligned} \dot{u} &= \frac{an}{m-n} u + v + \frac{(f-1)n}{k} u^2 + u^2 v, \\ \dot{v} &= -\frac{v}{k}\big(ak + (fk + n - fn)u - kuv\big). \end{aligned} \tag{16}$$

The infinite singular points of system (3) correspond to $v = 0$ in system (16) and coordinates $u_1 = 0$ and $u_2 = ak/((f-1)(n-m))$. We compute the eigenvalues λ_1, μ_1 of the Jacobian matrix at the point $(u_1, 0)$ and we get

$$\lambda_1 = -a, \quad \mu_1 = \frac{an}{m-n}.$$

Note that $\lambda_1/\mu_1 = (n-m)/n \in \mathbb{Q}^+$, so this case is not possible.

Case 2*:* $c \neq 0$. We now consider different cases: $m = n$ and $m < n$.

Case 2.1*:* $m = n$. In this case $a = 0$. Since $c \neq 0$, proceeding as in [18], there are two reasons for having the condition $m = n$. First we simply have chosen the wrong system of coordinates, and there is some other real singular point of the system at infinity through which $g = 0$ passes. In that case the system can be transformed into system (2) with $m < n$.

The second reason for having $m = n$ is that all the branches of $g = 0$ go through non-real equilibrium points of the system at infinity. This means that $y_1 = \overline{y_2} \notin \mathbb{R}$. In that case system (3) would have only a pair of equilibrium points at infinity and in view of Theorem 8 it is proved that in this case the system has at most one limit cycle.

Case 2.2*:* $m < n$. If $a = 0$ then $e = 0$ and it follows from Theorem 7 (making the change $x \to y$ and $y \to x$) that system (3) has at most one limit cycle.

We can thus assume that $a \neq 0$ and so $e = am/(m-n)$. We write $g_n = y^{n-m}(y - y_1 x)^k (y - y_2 x)^{m-k}$ where y_1, y_2 are the simple solutions of $cy^2 + (b-f)y + a - e = 0$ and so $(a-e)^2 - 4c(b-f) > 0$. Moreover $y_1 \neq 0$ and $y_2 \neq 0$ otherwise the system would have only two pairs of infinite singular points which is not possible. Indeed,

$$y_{1,2} = \frac{(b-f) \pm \sqrt{(b-f)^2 - 4acn/(n-m))}}{2c}.$$

On the local chart U_1 system (3) with $e = am/(m-n)$, $a \neq 0$ becomes

$$\begin{aligned} \dot{u} &= \frac{an}{m-n}u + v + (f-b)u^2 - cu^3 + u^2 v, \\ \dot{v} &= -v(a + bu + cu^2 - uv). \end{aligned} \tag{17}$$

The infinite singular points of system (3) correspond to $v = 0$ in system (17) and coordinates $u_1 = 0$ and $u_2 = y_1$ and $u_3 = y_2$. We compute the eigenvalues λ_1, μ_1 of the Jacobian matrix at the point $(u_1, 0) = (0, 0)$ and we get

$$\lambda_1 = -a, \quad \mu_1 = \frac{an}{m-n}.$$

Note that $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1/\mu_1 = (n-m)/n \in \mathbb{Q}^+$, which is not possible. This completes the proof of Theorem 12. $\square$

Proof of Theorem 1 for system (3)*.* Theorem 1 for system (3) follows directly from Theorem 12. $\square$

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