Polynomial Hamiltonian systems of degree 3 with symmetric nilpotent centers

Fabio Scalco Dias

Instituto de Matemática e Computação, Universidade Federal de Itajubá,
Avenida BPS 1303, Pinheirinho, CEP 37.500–903, Itajubá, MG, Brazil.
e–mail: scalco@unifei.edu.br

Jaume Llibre

Departament de Matemàtiques, Universitat Autònoma de Barcelona,
08193 Bellaterra, Barcelona, Spain.
e–mail: jllibre@mat.uab.cat

Claudia Valls

Departamento de Matemàtica, Instituto Superior Técnico, Universidade de Lisboa,
1049-001 Lisboa, Portugal.
e–mail: cvalls@math.ist.utl.pt

Abstract

We provide normal forms and the global phase portraits in the Poincaré disk for all Hamiltonian planar polynomial vector fields of degree 3 symmetric with respect to the $x-$axis having a nilpotent center at the origin.

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1. Introduction and statement of the results

Hamiltonian systems are relevant for many physical studies. Let $H(x, y)$ be a real polynomial in the variables x and y. Then a system of the form

$$x' = H_y(x, y) \quad y' = -H_x(x, y) \tag{1}$$

is called a *polynomial Hamiltonian system*. Here the prime denotes derivative with respect to the independent variable t.

Poincaré in [20] defined a *center* for a vector field on the real plane as a singular point having a neighborhood filled with periodic orbits with the exception of the singular point. Let $p \in \mathbb{R}^2$ be a singular point of an analytic differential system in $\mathbb{R}^2$, and assume that p is a center. Without loss of generality we can assume that p is at the origin of coordinates. Then after a linear change of variables and a rescaling of the time variable (if necessary), the system can be written in one of the following three forms

$$x' = -y + P(x, y), \qquad y' = x + Q(x, y), \qquad (2)$$
$$x' = y + P(x, y), \qquad y' = Q(x, y), \qquad (3)$$
$$x' = P(x, y), \qquad y' = Q(x, y), \qquad (4)$$

where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. In what follows a center of an analytic differential system in $\mathbb{R}^2$ is called *linear type*, *nilpotent* or *degenerate* if after an affine change of variables and a rescaling of the time it can be written as system (2), (3) or (4), respectively

The classification of centers for real planar polynomial differential systems started with the classification of centers for quadratic polynomial differential systems, and these results go back mainly to Dulac [12], Kapteyn [15, 16] and Bautin [4]. In [21] Vulpe provides all the global phase portraits of quadratic polynomial differential systems having a center. There are many partial results for the centers of planar polynomial differential systems of degree larger than two. For instance the linear type centers for cubic systems of the form linear plus homogeneous nonlinearities were characterized by Malkin [19], and by Vulpe and Sibirski [22]. For polynomial differential systems of the form linear plus homogeneous nonlinearities of degree greater than three the centers at the origin are not characterized, but there are partial results for degree four and five for the linear type centers, see for instance Chavarriga and Giné [5, 6]. Some results for higher degree are known see for instance [14]. Recently Colak, Llibre and Valls [7, 8, 9, 10] provided the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center or a nilpotent center at the origin, together with their bifurcation diagrams. The complete classification of the phase portrait of the nilpotent centers in this last case was given in [11]. This has been possible since the classification of the nilpotent centers of system (2) when P and Q are homogeneous cubic polynomial was given in [1]. For a general overview on the centers of planar polynomial differential

systems see [17], and for a classification of the phase portraits of same classes of other centers see [3] and the references quoted there. To know the phase portraits of centers is useful for studying the number of limit cycles which can bifurcate from their periodic orbits when they are perturbed, see for instance [18] and the references cited therein.

In this work we classify the global phase portraits of all Hamiltonian planar polynomial vector fields of degree three symmetric with respect to the $x-$axis having a nilpotent center at the origin. We recall that the differential system (1) is *symmetric with respect to the $x-$axis* if it is invariant under the change of variables $(x, y, t) \to (x, -y, -t)$, sometimes this kind of systems are called *reversible*. The classification will be done using the Poincaré compactification of polynomial vector fields, see section 2. We say that two vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one into the other which sends orbits to orbits preserving or reversing the direction of the flow.

Our main results are the following ones.

Proposition 1. *A Hamiltonian planar polynomial vector field of degree three with a nilpotent center at the origin and symmetric with respect to the $x-$axis, after a linear change of variables and a rescaling of its independent variable can be written as one of the following five classes:*

(I) $x' = y, \quad y' = -x^3$;

(II) $x' = y + \delta y^3, \quad y' = -x^3$;

(III) $x' = y + x^2 y + a y^3, \quad y' = -x^3 - xy^2$;

(IV) $x' = y - x^2 y + a y^3, \quad y' = -x^3 + xy^2$;

(V) $x' = y + 2xy + a x^2 y + b y^3, \quad y' = -x^3 - y^2 - a x y^2$;

where $\delta \in \{-1, 1\}$ and $a, b \in \mathbb{R}$.

Proposition 1 is proved in section 3.

Theorem 2. *The global phase portraits of the five families $(I) - (V)$ in Proposition 1 are topologically equivalent to the phase portraits of Figure 1:*

(a) 1.1 *for systems (I), systems (II) with $\delta = 1$, systems (III) with $a \geq 0$, systems (IV) with $a \geq 1$, systems (V) with $(a, b) \in \tilde{R}_4$, and systems (V) with $a \geq 1$ and $b = 0$;*

(b) 1.2 *for systems (II) with $\delta = -1$, and systems (III) with $a < 0$;*

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(c) 1.3 for systems (IV) with $0 < a < 1$;

(d) 1.4 for systems (IV) with $a = 0$;

(e) 1.5 for systems (IV) with $a < 0$;

(f) 1.6 for systems (V) with $(a, b) \in \tilde{R}_1$;

(g) 1.7 for systems (V) with $(a, b) \in \tilde{R}_2$;

(h) 1.8 for systems (V) with $(a, b) \in \tilde{R}_3$;

(i) 1.9 for systems (V) with $(a, b) \in \tilde{R}_5$;

(j) 1.10 for systems (V) with $a < 0$ and $b = 0$;

(k) 1.11 for systems (V) where $a < 0$, $b < 0$ and $4(a-1)^2(a^3 - a^3 - ab - 8b) - 27b^2 = 0$;

(l) 1.12 for systems (V) with $b = 0$ and $0 \le a < 1$;

(m) 1.13 for systems (V) with $a < 1$, $b > 0$ and $4(a-1)^2(a^3 - a^3 - ab - 8b) - 27b^2 = 0$;

(n) 1.14 for systems (V) with $a < 0$ and $b = a^2$;

(see the proof of Theorem 2 for the definitions of $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_4, \tilde{R}_5$).

In paper [8] there is the classification of the 12 different phase portraits of the cubic Hamiltonian systems with a nilpotent center at the origin and without quadratic terms. So the phase portraits of systems (I)–(IV) are already classified in [8], but it is unknown how many of the 12 different phase portraits can be realized by systems that are symmetric with respect to the x-axis. Theorem 2 shows that only 5 of the 12 phase portraits can be realized by systems (I)–(IV). In fact, the phase portraits 1.1, 1.2, 1.3, 1.4 and 1.5 correspond to the phase portraits 1.1, 1.2, 1.9, 1.3 and 1.6 of [8]. The proof of Theorem 2 is given in section 4.

2. Preliminary results

In this section we summarize the Poincaré compactification that we shall use for describing the global phase portrait of our Hamiltonian systems. For more details on the Poincaré compactification see Chapter 5 of [13]. Let $\mathbb{S}^2$
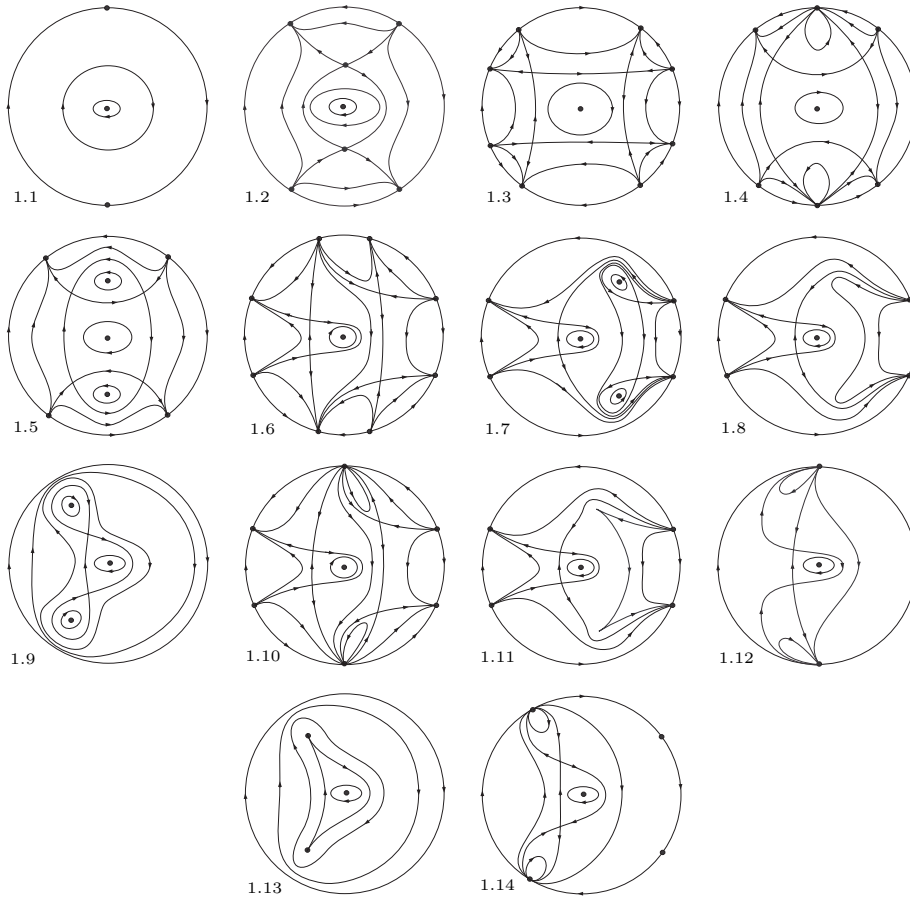
Figure 1: Global phase portraits of the vector fields in Theorem 2.

be the sphere of points $(s_1, s_2, s_3) \in \mathbb{R}^3$ such that $s_1^2 + s_2^2 + s_3^2 = 1$, called the *Poincaré sphere*. Given a polynomial vector field

$$X(x, y) = (x', y') = (P(x, y), Q(x, y))$$

in $\mathbb{R}^2$ of degree d (where d is the maximum of the degrees of the polynomials P and Q) it can be extended analytically to the Poincaré sphere by projecting each point $x \in \mathbb{R}^2$ identified with the point $(x_1, x_2, 1) \in \mathbb{R}^3$ in the Poincaré sphere using the straight line through x and the origin of $\mathbb{R}^3$. The equator $S^1 = \{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 = 0\}$ corresponds to the infinity of $\mathbb{R}^2$. In this way we obtain a vector field $\bar{X}$ in $\mathbb{S}^2 \backslash \mathbb{S}^1$. This vector field $\bar{X}$ is formed by

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two copies of X: one on the northern hemisphere $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 > 0\}$ and another on the southern hemisphere $\{(s_1, s_2, s_3) \in \mathbb{S}^2 : s_3 < 0\}$. The local charts needed for doing the calculations on the Poincaré sphere are

$$U_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i > 0\}, \quad V_i = \{\mathbf{s} \in \mathbb{S}^2 : s_i < 0\},$$

where $\mathbf{s} = (s_1, s_2, s_3)$, with the corresponding local maps

$$\varphi_i(s) : U_i \to \mathbb{R}^2, \quad \psi_i(s) : V_i \to \mathbb{R}^2,$$

such that $\varphi_i(s) = -\psi_i(s) = (s_m/s_i, s_n/s_i) = (u, v)$ for $m < n$ and $m, n \neq i$, for $i = 1, 2, 3$.

We extend $\bar{X}$ to a vector field $p(X)$ at the whole sphere $\mathbb{S}^2$ by taking $p(X) = v^d \bar{X}$. The expression for the corresponding vector field on $\mathbb{S}^2$ in the local chart U_1 is given by

$$u' = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad v' = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right); \quad (5)$$

the expression for U_2 is

$$u' = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \quad v' = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right); \quad (6)$$

and the expression for U_3 is $u' = P(u, v)$, $v' = -Q(u, v)$. The expressions for the charts V_i are those for the charts U_i multiplied by $(-1)^{d-1}$, for $i = 1, 2, 3$. Hence for studying the vector field X it is enough to study its Poincaré compactification restricted to the northern hemisphere plus $\mathbb{S}^1$. To draw the phase portraits we consider the projection, by $\pi(s_1, s_2, s_3) = (s_1, s_2)$, of the closed northern hemisphere into the local disk $\mathbb{D} = \{(s_1, s_2) : s_1^2 + s_2^2 \leq 1\}$, called the *Poincaré disk*.

Finite singular points of X are the singular points $D\pi \circ p(X)$ in the interior of $\mathbb{D}$, and they can be studied using U_3. Infinite singular points of X are the singular points of $D\pi \circ p(X)$ contained in $\mathbb{S}^1$. Note that if $s \in \mathbb{S}^1$ is an infinite singular point, then $-s$ is also an infinite singular point. Hence to study the infinite singular points it suffices to look for them only at $U_{1|v=0}$ and at the origin of U_2.

Finally we mention without getting into too much detail an important result that classifies the finite singular points of Hamiltonian planar polynomial differential systems. For a detailed definition of the (topological) index of a singular point see for instance Chapter 6 of [13], it can be computed

easily using the *Poincaré formula* which takes into account the *parabolic sector*, *hyperbolic sector*, and *elliptic sectors* at a singular point, for details see page 18 of [13]. A vector field is said to have the *finite sectorial decomposition property* at a singular point q if either q is a center, a focus or a node, or it has a neighborhood consisting of a finite union of parabolic, hyperbolic or elliptic sectors. We note that all the isolated singular points of a polynomial differential system satisfy the finite vectorial decomposition property, see [13].

Theorem 3 (Poincaré Formula). *Let q be an isolated singular point having the finite sectorial decomposition property. Let e, h an p denote the number of elliptic, hyperbolic and parabolic sectors of q, respectively. Then the index of q is $(e - h)/2 + 1$.*

From Theorem 3 the following result follows easily.

Corollary 4. *The indices of a saddle, a center and a cusp are $-1, 1$ and 0, respectively.*

To determine the possible number and local phase portraits of the finite singular points of the systems we will use the *Poincaré-Hopf Theorem* for vector fields in the 2−dimensional sphere.

Theorem 5. *For every vector field on the sphere $\mathbb{S}^2$ with a finite number of singular points, the sum of the indices of these singular points is 2.*

We note that singular points with index 0 are more difficult to detect because they do not contribute to the total index of the singular points of the vector fields on the Poincaré sphere. To overcome this difficulty we present the following proposition, but first we make a remark and give some definitions.

If a singular point p of an analytic vector field X has the two real parts of the eigenvalues of $DX(p)$ non-zero then p is *hyperbolic*. If the eigenvalues of $DX(p)$ are purely imaginary, then p is either a center or a focus. If only one eigenvalue of $DX(p)$ is 0, then p is *semi-hyperbolic*. The hyperbolic and semi-hyperbolic singular points are called *elementary*. If both eigenvalues of $DX(p)$ are 0 but $DX(p)$ is not identically zero, then p is *nilpotent*. Finally, if $DX(p)$ is identically zero then p is *linearly zero*. The local phase portraits of hyperbolic, semi-hyperbolic and nilpotent singular points can be studied using, for instance, Theorems 2.15, 2.19 and 3.5 of [13], respectively. The linearly zero singular points must be studied using the changes of variables known as blow-ups, see for instance [2] and [13].

Remark 1. *Nilpotent singular points of Hamiltonian planar polynomial vector fields are either saddles, centers, or cusps (for more details see Theorem 3.5 of [13] and taking into account that Hamiltonian systems cannot have foci).*

We define *energy levels* of a Hamiltonian vector field as the level curves of its Hamiltonian; and a hyperbolic saddle with a loop and a center inside the loop as in Figure 2 will be called a *center–loop*.
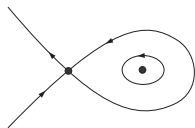

Figure 2: A center–loop.

Proposition 6. *Let X_ε be a real Hamiltonian planar polynomial vector field of degree three. Then X_ε can be written as*

$$
\begin{aligned}
x' &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y \\
&\quad + a_{12}xy^2 + a_{03}y^3, \\
y' &= b_{10}x - a_{10}y + b_{20}x^2 - 2a_{20}xy - \frac{a_{11}y^2}{2} + b_{30}x^3 - 3a_{30}x^2y \\
&\quad - a_{21}xy^2 - \frac{a_{12}y^3}{3} + \varepsilon x.
\end{aligned} \tag{7}
$$

Suppose that p is an isolated singular point of X_ε different from the origin. If $a_{10}^2 + a_{01}b_{10} = 0$ but $a_{01} \neq 0$, then the following statements hold:

(a) *If p is non–elementary, then it is nilpotent.*

(b) *If p is a non–elementary singular point of X_0, then it is an elementary singular point of X_ε with $\varepsilon \neq 0$.*

(c) *If p is a cusp of X_0, then for $\varepsilon \neq 0$ small enough such that $\varepsilon a_{01} < 0$, the origin of X_ε is a linear type center and the local phase portrait of X_ε at p is a center–loop.*

Proof. The proof of these proposition is similar to the one of Lemma 12 in [8] and it will be omitted here. In fact the unique difference with the mentioned Lemma 12 of [8] is that the vector field of that lemma has no quadratic terms, which do not affect the proof. $\square$

3. Proof of Proposition 1

Without loss of generality we can assume that a Hamiltonian system of degree three with a nilpotent center at the origin is given by

$$x' = H_y, \quad y' = -H_x,$$

where

$$H(x,y) = y^2/2 + a_4 x^3 + a_5 x^2 y + a_6 xy^2 + a_7 y^3 + a_8 x^4 + a_9 x^3 y$$
$$+ a_{10} x^2 y^2 + a_{11} xy^3 + a_{12} y^4.$$

Therefore we have the following Hamiltonian system

$$\begin{aligned}
x' &= y + a_5 x^2 + 2a_6 xy + 3a_7 y^2 + a_9 x^3 + 2a_{10} x^2 y + 3a_{11} xy^2 + 4a_{12} y^3, \\
y' &= -(3a_4 x^2 + 2a_5 xy + a_6 y^2 + 4a_8 x^3 + 3a_9 x^2 y + 2a_{10} xy^2 + a_{11} y^3).
\end{aligned} \tag{8}$$

By hypothesis system (8) is invariant under the symmetry $(x, y, t) \to (x, -y, -t)$ and so we have $a_5 = a_7 = a_9 = a_{11} = 0$. Hence system (8) becomes

$$\begin{aligned}
x' &= y + 2a_6 xy + 2a_{10} x^2 y + 4a_{12} y^3, \\
y' &= -(3a_4 x^2 + a_6 y^2 + 4a_8 x^3 + 2a_{10} xy^2).
\end{aligned} \tag{9}$$

Since systems (9) must have a center at the origin, by Theorem 3.5 of [13] we must have $a_4 = 0$ and $a_8 > 0$. Therefore we obtain

$$\begin{aligned}
x' &= y + 2a_6 xy + 2a_{10} x^2 y + 4a_{12} y^3, \\
y' &= -(a_6 y^2 + 4a_8 x^3 + 2a_{10} xy^2).
\end{aligned} \tag{10}$$

Case 1. Assume $a_6 \neq 0$. By the change of coordinates and reparametrization of the time of the form

$$x \to \alpha X, \quad y \to \beta Y, \quad t \to \gamma \tau, \tag{11}$$

with $\alpha = 1/a_6$, $\beta = 2\sqrt{a_8}/a_6^2$ and $\gamma = a_6/2\sqrt{a_8}$, systems (10) can be written as

$$X' = Y + 2XY + 2\frac{a_{10}}{a_6^2} X^2 Y + 16 a_8 \frac{a_{12}}{a_6^4} Y^3,$$

$$Y' = -Y^2 - X^3 - 2\frac{a_{10}}{a_6^2} XY^2.$$

We obtain the normal form (V).

Case 2. Assume $a_6 = 0$ and $a_{10} \neq 0$. By the change of coordinates and reparametrization of the time as in (11) with $\alpha = 1/\sqrt{2|a_{10}|}$, $\beta = -\sqrt{a_8}/a_{10}$,

$\gamma = -\sqrt{|a_{10}|/2a_8}$, systems (10) can be written as

$$X' = Y \pm X^2 Y \pm 4\frac{a_8 a_{12}}{a_{10}^2} Y^3,$$
$$Y' = -X^3 \mp XY^2.$$

We obtain the normal forms (III) and (IV).

Case 3. Assume $a_6 = 0$ and $a_{10} = 0$. By the change of coordinates and reparametrization of the time as in (11) with $\alpha = \beta = \sqrt{1/4a_8}$ and $\gamma = 1$ systems (10) can be written as

$$X' = Y + \frac{a_{12}}{a_8} Y^3,$$
$$Y' = -X^3.$$

If $a_{12} \neq 0$ we can do another change of coordinates and obtain systems (II), otherwise we obtain system (I).

In short we have proved Proposition 1.

4. Proof of Theorem 2

4.1. Global phase portrait of system (I)

Consider system (I)

$$x' = y, \quad y' = -x^3.$$

The origin is the only finite singular point of the system. Using (5) we see that in the local chart U_1 system (I) becomes

$$u' = -1 - u^2 v^2, \quad v' = -uv^3.$$

When $v = 0$ there are no infinite singular points on the local chart U_1. In U_2 we use (6) and we get

$$u' = v^2 + u^4, \quad v' = vu^3. \tag{12}$$

The origin is an infinite singular point of the system, whose linear part is zero. So we need to do blow-ups to describe the local dynamics at this point. We perform the directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u^2$ and we have

$$u' = u^4 + u^4 w^2, \quad w' = -u^3 w(1 + 2w^2). \tag{13}$$

We eliminate the common factor u^3 between u' and w', and get the vector field

$$u' = u(1 + w^2), \quad w' = -w(1 + 2w^2). \tag{14}$$

System (14) has the origin as its unique singular point. The eigenvalues of the linear part at the origin are 1 and -1, so it is a saddle.

Going back through the changes of variables until system (12) as shown in Figure 3, we have that the global phase portrait of system (I) is topologically equivalent to the phase portrait 1.1 of Figure 1.
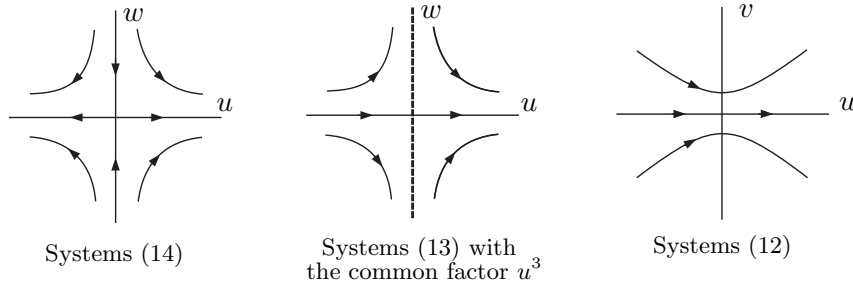

| Systems (14) | Systems (13) with the common factor u^3 | Systems (12) |

Figure 3: Blow-up of the origin of U_2 of system (I).

4.2. Global phase portrait of system (II)

Consider systems (II)

$$x' = y + \delta y^3, \quad y' = -x^3$$

where $\delta \in \{-1, 1\}$. It is easy to see that $(0,0)$ is the unique finite singular point of system when $\delta = 1$. When $\delta = -1$ the finite singular points are $E_0 = (0,0)$ and $E_\pm = (0, \pm 1)$. We will study only the singular point E_+ because the study of the other singular point is analogous. Since the singular point E_+ is nilpotent, using Theorem 3.5 of [13] we obtain that $(0,1)$ is a saddle.

We will now investigate the infinite singular points of systems (II). We distinguish between the cases $\delta = 1$ and $\delta = -1$.

4.2.1. Case $\delta = 1$

In the local chart U_1 system (II) is

$$u' = -1 - u^2 v^2 - u^4, \quad v' = -uv(v^2 + u^2).$$

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When $v = 0$ there are no infinite singular points on U_1. In U_2 system (I) becomes

$$u' = v^2 + 1 + u^4, \quad v' = vu^3.$$

Again when $v = 0$ there are no infinite singular points on the local chart U_2. Therefore the global phase portrait of system (II) (with $\delta = 1$) is topologically equivalent to phase portrait 1.1 of Figure 1.

4.2.2. Case $\delta = -1$

In the local chart U_1 system (I) is

$$u' = -1 - u^2v^2 + u^4, \quad v' = uv(u^2 - v^2).$$

When $v = 0$ the infinite singular points are $P_\pm = (\pm 1, 0)$. The eigenvalues of the linear part at P_- are -1 and -4. So, it is an attracting node. On the other hand, the eigenvalues of the linear part at P_+ are 1 and 4. So it is a repelling node.

Next we should check the origin of U_2. In U_2 system (II) becomes

$$u' = -1 + v^2 + u^4, \quad v' = vu^3.$$

When $v = 0$ the origin is not a singular point.

Now we will determine the global phase portrait according to this local information. The two finite saddles must be on the boundary of the period annulus of the center at the origin due to the symmetry of the system. Therefore the global phase portrait of system (II) (with $\delta = -1$) is topologically equivalent to the phase portrait 1.2 of Figure 1.

4.3. Global phase portrait of systems (III)

Consider systems (III)

$$x' = y + x^2y + ay^3, \quad y' = -x^3 - xy^2.$$

When $a \geq 0$ the only finite singular point is the origin. When $a < 0$ the singular points are the origin and $E_\pm = (0, \pm\sqrt{-1/a})$. The eigenvalues of the linear part at the singular points $E_\pm$ are $\pm\sqrt{-2a}/a$. Hence they are saddles.

We will now investigate the infinite singular points of systems (III). We distinguish between the cases $a > 0$, $a = 0$ and $a < 0$.

4.3.1. Case a > 0

On the local chart U_1 systems (III) become

$$u' = -1 - 2u^2 - u^2v^2 - au^4, \quad v' = -uv(1 + au^2 + v^2).$$

When $v = 0$ there are no infinite singular points on U_1. In U_2, systems (III) can be written as

$$u' = a + 2u^2 + v^2 + u^4, \quad v' = vu(1 + u^2).$$

Again when $v = 0$ there are no infinite singular points on U_2. Therefore the global phase portrait of systems (III) (with $a > 0$) are topologically equivalent to the phase portrait 1.1 of Figure 1.

4.3.2. Case a = 0

Using (5) we see that in the local chart U_1 system (III) becomes

$$u' = -1 - 2u^2 - u^2v^2, \quad v' = -uv(1 + v^2).$$

When $v = 0$ there are no infinite singular points on U_1. In U_2 we use (6) to get

$$u' = 2u^2 + v^2 + u^4, \quad v' = uv(1 + u^2).$$

When $v = 0$ the only infinite singular point is the origin, whose linear part is zero. So we need to do blow-ups to describe the local behavior at this point. We perform the directional blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and we get

$$u' = 2u^2 + u^2w^2 + u^4, \quad w' = -uw(1 + w^2).$$

We eliminate the common factor u between u' and w', obtaining the system

$$u' = u(2 + u^2 + w^2), \quad w' = -w(1 + w^2).$$

The eigenvalues of the linear part at the origin are 2 and -1, hence it is a saddle. Doing a similar analysis to the one done in system (I), we obtain that the global phase portrait of systems (III) (with $a = 0$) is topologically equivalent to the phase portrait 1.1 of Figure 1.

4.3.3. Case a < 0

In the local chart U_1 systems (III) become

$$u' = -1 - 2u^2 - u^2v^2 - au^4, \quad v' = -uv(1 + au^2 + v^2).$$

The infinite singular points are $P_{\pm} = (\pm\sqrt{-a(1+\sqrt{1-a})}/a, 0)$. The eigenvalues of the linear part at P_+ are

$$\frac{\sqrt{-a-a\sqrt{1-a}}\sqrt{1-a}}{a}, \quad \frac{4\sqrt{-a-a\sqrt{1-a}}\sqrt{1-a}}{a},$$

which are negative (because $a < 0$). Hence it is an attracting node. On the other hand, the eigenvalues of the linear part of the systems at P_- are

$$-\frac{\sqrt{-a-a\sqrt{1-a}}\sqrt{1-a}}{a}, \quad -\frac{4\sqrt{-a-a\sqrt{1-a}}\sqrt{1-a}}{a},$$

which are positive (because $a < 0$). Hence it is a repelling node.

In U_2 we use (6) to get

$$u' = a + 2u^2 + v^2 + u^4, \quad v' = vu(1 + u^2).$$

When $v = 0$ the origin is not a singular point because $a < 0$.

A similar analysis to the one done for systems (II) with $\delta = -1$ implies that the global phase portrait of systems (III) with $a < 0$ are topologically equivalent to the phase portrait 1.2 of Figure 1.

4.4. Global phase portrait of system (IV)

Consider the systems (IV), i.e.

$$x' = y - x^2 y + ay^3, \quad y' = -x^3 + xy^2.$$

When $a \geq 1$ the only finite singular point is the origin. When $0 \leq a < 1$, among the origin, we have four new finite singular points:

$$\left(\pm\sqrt{\frac{1}{1-a}}, \pm\sqrt{\frac{1}{1-a}} \right).$$

It is easy to see that all of them are saddles. Finally when $a < 0$ in addition to the previous singular points, we also have the points $\pm(0, \sqrt{-1/a})$ which are centers because the eigenvalues of their linear parts are purely imaginary.

We will now investigate the infinite singular points of systems (IV). We distinguish between the cases $a \geq 1$, $0 \leq a < 1$ and $a < 0$.

4.4.1. Case $a \geq 1$

In the local chart U_1 systems (IV) can be written as

$$u' = -1 + 2u^2 - u^2v^2 - au^4, \quad v' = -uv(-1 + au^2 + v^2). \qquad (15)$$

The infinite singular points are the roots of the polynomial $-1 + 2u^2 - au^4$. Therefore, if $a > 1$ there are no infinite singular points on the local chart U_1. If $a = 1$ the points $(\pm 1, 0)$ are infinite singular points on U_1. Using a blow-up analysis we obtain the same result as in the system (I), hence the points $(\pm 1, 0)$ of U_1 have two hyperbolic sectors, see Figure 3.

In U_2 systems (IV) can be expressed as

$$u' = a - 2u^2 + v^2 + u^4, \quad v' = -vu(1 - u^2). \qquad (16)$$

Again when $v = 0$ there are no singular points on U_2. Therefore the global phase portrait of systems (IV) with $a \geq 1$ are topologically equivalent to the phase portrait 1.1 of Figure 1.

4.4.2. Case $0 \leq a < 1$

In the local chart U_1 systems (IV) are given by (15). Again the infinite singular points are the roots of the polynomial $-1 + 2u^2 - au^4$. Therefore, if $0 < a < 1$ there are the following four infinite singular points on U_1

$$\left(\pm \frac{\sqrt{a(1 + \sqrt{1 - a})}}{a}, 0 \right), \quad \left(\pm \frac{\sqrt{a(1 - \sqrt{1 - a})}}{a}, 0 \right).$$

All these points are nodes. On the other hand, if $a = 0$ there are only two singular points $(\pm\sqrt{2}/2, 0)$ on U_1 and both are nodes.

In U_2 systems (IV) are given by (16). Therefore if $0 < a < 1$ the origin is not a singular point in U_2.

To understand the global phase portrait in this case, let H be the Hamiltonian of systems (IV). We have that the value of H on the four finite singular points is equal. We claim that all four points are on the boundary of the period annulus of the center at the origin. If there were only two saddles on the mentioned boundary, then a straight line passing through the origin and sufficiently close to the saddles which are not on this boundary would have at least six intersection points with the separatrices of these saddles, which are on the same energy level, see Figure 4. But this is impossible because H is only quartic. Hence the claim is proved. Therefore the global phase portrait of systems (IV) with $0 < a < 1$ is topologically equivalent to

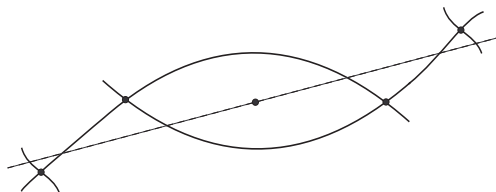the phase portrait 1.3 of Figure 1.

Figure 4: The straight line through the origin intersects the separatrices six times.

If $a = 0$ the origin of the chart U_2 is an infinite singular point of the system, whose linear part is zero, hence we need to do a blow-up to characterize the local dynamics at this point. Doing the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and eliminating the common factor u we get the system

$$u' = u(-2 + u^2 + w^2), \quad w' = -w(w - 1)(w + 1). \qquad (17)$$

When $u = 0$ the singular points of system (17) are the roots of the polynomial $-w(w-1)(w+1)$. Therefore we have three singular points. In this case the origin is a saddle and the other singular points are nodes. Consequently the origin of the local chart U_2 has two elliptic and two parabolic sectors, see Figure 5.
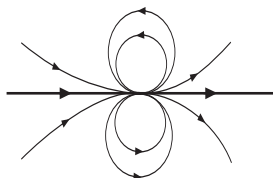

Figure 5: Local phase portrait at the origin of system (17).

Proceeding as in the previous case, using an analogous argument we conclude that the global phase portrait of system (IV) with $a = 0$ is topologically equivalent to the phase portrait 1.4 of Figure 1.

4.4.3. Case $a < 0$

In the local chart U_1 systems (IV) are given by (15). Again the infinite singular points are the roots of the equation $-1 + 2u^2 - au^4 = 0$. Therefore,

if $a < 0$, on U_1 there are the two infinite singular points

$$\left(\pm \frac{\sqrt{a(1 - \sqrt{1 - a})}}{a}, 0 \right).$$

All these points are nodes and the origin of U_2 is not singular. Due to the symmetry of the systems the global phase portrait of system (IV) with $a < 0$ is topologically equivalent to the phase portrait 1.5 of Figure 1.

4.5. Global phase portrait of systems (V)

Consider systems (V)

$$x' = y + 2xy + ax^2y + by^3, \quad y' = -y^2 - x^3 - axy^2. \qquad (18)$$

We now study the infinite singular points of these systems.

4.5.1. Infinite singular points

In U_1 systems (V) become

$$\begin{aligned} u' &= -1 - 2au^2 - 3u^2v - bu^4 - u^2v^2, \\ v' &= -uv(a + 2v + bu^2 + v^2). \end{aligned} \qquad (19)$$

When $v = 0$ the candidates for singular points of systems (V) are the roots of the polynomial $1 + 2au^2 + bu^4$. It is easy to see that when $a = b = 0$, or $b = 0$ and $a > 0$, or $a = 0$ and $b > 0$, there are no singular points on the local chart U_1. When $b = 0$ and $a < 0$, the singular points are $(\pm\sqrt{-1/2a}, 0)$. The linear part of (19) when $v = 0$ is

$$\begin{pmatrix} -4\,au & -3\,u^2 \\ 0 & -au \end{pmatrix}.$$

Thus both of the eigenvalues of two singular points are negative if $u < 0$, and positive if $u > 0$. Hence the points $(-\sqrt{-1/2a}, 0)$ and $(\sqrt{-1/2a}, 0)$ are, attracting and repelling nodes, respectively.

When $a = 0$ and $b < 0$ the singular points are $(\sqrt[4]{-1/b}, 0)$ and $(-\sqrt[4]{-1/b}, 0)$. Proceeding as above we get that they are attracting and repelling nodes, respectively.

When $ab \neq 0$ and $b > a^2$ it is easy to see that there are no singular points on the local chart U_1. So we consider the cases $a = b^2$ and $b < a^2$.

Case 1: $b = a^2$. If $a > 0$ there are no singular points on the local chart U_1. If $a < 0$ the singular points are $(\pm 1/\sqrt{-a}, 0)$. We will study

only the singular point $(-1/\sqrt{-a}, 0)$ because the study of the other singular point is analogous. First we translate $(-1/\sqrt{-a}, 0)$ to the origin. Applying Theorem 3.5 of [13] we obtain that the phase portrait of the singular point $(-1/\sqrt{-a}, 0)$ consists of one hyperbolic and one elliptic sector. Doing blow-ups we get that the local phase portrait of $(-1/\sqrt{-a}, 0)$ is the one described in Figure 6.
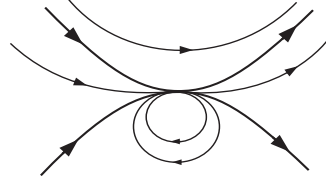

Figure 6: Local phase portrait at the origin of system (19) for $b = a^2$ and $a < 0$.

Case 2: $b < a^2$. If $a > 0$ and $b > 0$ again there are no singular points on U_1. If $a > 0$ and $b < 0$ there are two singular points

$$\left(\pm \frac{\sqrt{-b(-a + \sqrt{a^2 - b})}}{b}, 0 \right).$$

The eigenvalues of the linear part of the system at these two points are

$$4\frac{\sqrt{-b\left(a + \sqrt{a^2 - b}\right)}\sqrt{a^2 - b}}{b}, \quad \frac{\sqrt{-b\left(a + \sqrt{a^2 - b}\right)}\sqrt{a^2 - b}}{b}$$

and

$$-4\frac{\sqrt{-b\left(a + \sqrt{a^2 - b}\right)}\sqrt{a^2 - b}}{b}, \quad -\frac{\sqrt{-b\left(a + \sqrt{a^2 - b}\right)}\sqrt{a^2 - b}}{b},$$

respectively. Hence these points are attracting and repelling nodes respectively. Similarly, if $a < 0$ and $b < 0$ we have two singular points

$$\left(\pm \frac{\sqrt{-b(-a + \sqrt{a^2 - b})}}{b}, 0 \right)$$

and these points are attracting and repelling nodes respectively.

Finally, if $a < 0$ and $b > 0$ there are four singular points

$$P_{1\pm} = \left(\pm \frac{\sqrt{b(-a + \sqrt{a^2 - b})}}{b}, 0 \right), \quad P_{2\pm} = \left(\pm \frac{\sqrt{-b(a + \sqrt{a^2 - b})}}{b}, 0 \right).$$

It is easy to see that the P_{1+} and P_{2-} are attracting nodes and P_{1-} and P_{2+} are repelling nodes.

Now we study the origin of U_2 of systems (V) which in U_2 write as

$$u' = b + 2au^2 + 3uv + v^2 + u^4, \quad v' = v(au + v + u^3).$$

If $b \neq 0$ the origin is not singular. If $b = 0$ we see that the origin is singular and it is degenerate, hence we need blow-up to understand the local behavior at this point. Doing the blow-up $(u, v) \mapsto (u, w)$ with $w = v/u$ and eliminating the common factor u we get the systems

$$u' = u(2a + 3w + u^2 + w^2), \quad w' = -w(a + 2w + w^2). \qquad (20)$$

When $u = 0$ the candidates for singular points of systems (20) are the roots of the polynomial $-w(a + 2w + w^2)$. When $a > 1$ the only singular point of these systems is the origin, and it is a saddle. In summary, going back through the changes of variables the origin of U_2 consists of two hyperbolic sectors, see Figure 3. When $a = 1$, the origin is a singular point of the system, whose linear part is zero. We see that doing the blow-up analysis the origin has two parabolic and two hyperbolic sectors as in Figure 7.
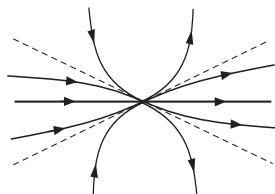

Figure 7: Local phase portraits of the origin of system (20) for $a = 1$.

When $a < 1$ we have three singular points. In addition to the saddle at the origin, the points $(0, -1 \pm \sqrt{1 - a})$ are repelling nodes. Consequently, this time the origin of U_2 has two elliptic sectors and two parabolic sectors, see Figure 5.

In short we have Tables 1 and 2.

Parameters	Infinite singular points in chart U_1
$a = b = 0$ $b = 0$ and $a > 0$ $a = 0$ and $b > 0$	There are no singular points
$b = 0$ and $a < 0$ $a = 0$ and $b < 0$	One attracting node and one repelling node
$b = a^2$ and $a > 0$ $b > a^2$ $b < a^2$ and $a, b > 0$	There are no singular points
$b = a^2$ and $a < 0$	Two singular points with one hyperbolic and one elliptic sector
$b < a^2$, $a > 0$ and $b < 0$ $b < a^2$ and $a, b < 0$	One attracting node and one repelling node
$b < a^2$, $a < 0$ and $b > 0$	Two attracting nodes and two repelling nodes

Table 1: Infinite singular points in the local chart U_1.

Parameters	The origin of the chart U_2
$b \neq 0$	There are no singular points
$b = 0$ and $a > 1$	Two hyperbolic sectors
$b = 0$ and $a = 1$	Two hyperbolic and two parabolic sectors
$b = 0$ and $a < 1$	Two elliptic and two parabolic sectors

Table 2: The origin of the local chart U_2.

4.5.2. Finite singular points

Now we investigate the finite singular points of systems (V). The explicit expressions for the finite singular points of these systems in terms of the parameters a, b are complicated, and therefore it is hard to analyze their existence and their local phase portraits. For this reason we take a different approach. We first find the maximum number of finite singular points allowed by these systems. Then using the Poincaré Formula for the index of a singular point of a planar vector field, we count the indices of the infinite singular points of the systems that we have found in subsection 4.5.1 on the Poincaré sphere. The next step is to determine the possible number and local phase portraits of the finite singular points of these systems using the Poincaré-Hopf Theorem.

Doing the resultant of x' and y' of system (18) with respect to the variable x, eliminating a common factor y^3, and performing the change of variables

$y \mapsto \sqrt{z}$ we get the cubic $\mathcal{C}$

$$
\begin{aligned}
&b(a^2 - b)^2 z^3 + (a^4 - a^3 - 4a^2 b + 10ab + 3b^2)z^2 \\
&\quad + (-2a^2 + 10a + 3b - 8)z + 1 = 0.
\end{aligned} \tag{21}
$$

Assume $(a^2 - b)b$ is not zero. The discriminant of the cubic $\mathcal{C}$ is

$$
\mathcal{D} = -(4(a-1)^2(a^3 - a^2 - ab - 8b) - 27b^2)(-8b + 3a^3 - 4a^2)^2.
$$

When $-8b + 3a^3 - 4a^2 \neq 0$, the sign of $\mathcal{D}$ is the sign of

$$
4(a-1)^2(a^3 - a^2 - ab - 8b) - 27b^2.
$$

We know that if $\mathcal{D} > 0$, then the cubic $\mathcal{C}$ has 3 real roots, if $\mathcal{D} < 0$, it has only 1 real root and if $\mathcal{D} = 0$ it has either 1 triple real root or 1 real and 1 double root. In this way we consider the following curves, see Figure 8.

$$
\begin{aligned}
&\mathcal{C}_1: \ 4(a-1)^2(a^3 - a^2 - ab - 8b) - 27b^2 = 0, \quad (\text{-----}) \\
&\mathcal{C}_2: \ (a^2 - b)b = 0, \quad (\text{———}) \\
&\mathcal{C}_3: -8b + 3a^3 - 4a^2 = 0. \quad (\text{-·-·-·})
\end{aligned}
$$

It is easy to see that in regions $R_1, R_2, R_6, R_8, R_9, R_{12}$ and R_{14}, bounded by curves $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$, the cubic has three real roots and in the other regions the cubic has only one real root, see Figure 8. The sign of these roots follow in Table 3.

Region	Number of positive and negative real roots of $\mathcal{C}$
R_1, R_{12}, R_{14}	one negative and two positive roots
R_2	three positive roots
R_3, R_4, R_5	one positive root
R_6	one positive and two negative roots
$R_7, R_{10}, R_{11}, R_{13}$	one negative root
R_8, R_9	three negative roots

Table 3: Number of positive and negative real roots of $\mathcal{C}$

From Table 3 we obtain the bifurcation diagram shown in Figure 9 for systems (V).

We should also analyze the discriminant of cubic $\mathcal{C}$ when the parameters
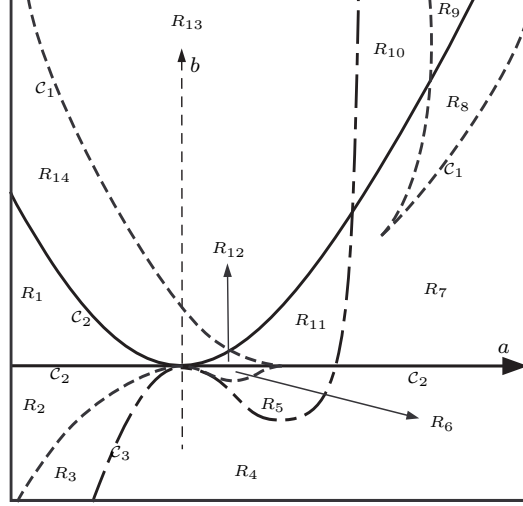
Figure 8: Curves $\mathcal{C}_1 - \mathcal{C}_3$ and regions $R_1 - R_{14}$.

a and b are on the curves

$$L_{12} = \{(a,b) \in \mathbb{R}^2 : b = 0,\ a < 0\},$$
$$L_{34} = \{(a,b) \in \mathbb{R}^2 : b = 0,\ a \geq 1\},$$
$$L_{35} = \{(a,b) \in \mathbb{R}^2 : b = 0,\ 0 \leq a < 1\},$$
$$L_{15} = \{(a,b) \in \mathbb{R}^2 : b = a^2,\ a < 0\},$$
$$L_{23} = \{(a,b) \in \mathcal{C}_1 : a < 0,\ b < 0\},$$
$$L_{45} = \{(a,b) \in \mathcal{C}_1 : a < 1,\ b > 0\}.$$

We will only present the discriminant of the cubic $\mathcal{C}$ on the curve L_{12}. The study of the discriminant of the cubic restricted to the curves L_{15}, L_{34} and L_{35} is performed in a similar way, so we omit it. The results are shown in Table 3. On the curves L_{23} and L_{45} the discriminant is very long so, on these curves we will do a different study by using Proposition 6.

When (a,b) belongs to L_{12} the cubic $\mathcal{C}$ given by (21) becomes

$$a^3(a-1)z^2 - 2(a-1)(a-4)z + 1 = 0. \qquad (22)$$

Now the discriminant is given by $-4(3a-4)^2(a-1)$. Therefore equation (22)
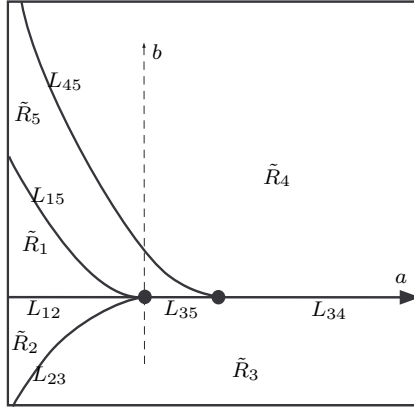
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Figure 9: Bifurcation diagram for System (V).

has two real roots when $a < 1$ $(a \neq 0)$, and has no real roots when $a > 1$. Therefore, on the curve L_{12} there are always two real roots. The number and sign of these roots follow in Table 4.

Curves	Number of positive and negative real roots of $\mathcal{C}$
L_{12}	two positive roots
L_{35}	one positive root and one negative root
L_{34}	there are no roots
L_{15}	one positive root and one negative root

Table 4: Number of positive and negative real roots of $\mathcal{C}$

Proposition 7. *Systems (V) have, among the origin, at most two non-elementary singular points and they exist when the parameters a and b belong to the curve $\mathcal{C}_1$.*

Proof. We need to show that equations (18) and the determinant of the Jacobian matrix on systems (V) cannot vanish simultaneously at more than two points besides the origin. The mentioned determinant is

$$\begin{vmatrix} 2\,y + 2\,axy & 1 + 2\,x + ax^2 + 3\,by^2 \\ -3\,x^2 - ay^2 & -2\,y - 2\,axy \end{vmatrix}.$$

We compute the Gröbner basis for these three polynomials and obtain a set of eight polynomial equations. We equate these eight polynomials to zero,

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eliminating a factor y, and for finding their solutions it is enough to consider the following three equations

$$- 27b^2 + 4(a - 1)^2(a^3 - a^2 - ab - 8b) = 0,$$
$$9b(2a - b)y^2 + 2(-8 + 8a + 2a^2 - 2a^3 + 9b) = 0, \qquad (23)$$
$$2(28 + a(2a - 1 - 2a^2) + 9b + 27x) + 9(2(a^3 - a^2) - 7b - b^2)y^2 = 0.$$

Observe that the first equation of (23) corresponds to the curve $\mathcal{C}_1$. The second equation provides at most two possible solutions for y, and from the last equation for every possible y solution there is at most one x solution. So the proposition is proved. □

In the following subsections we will describe the global phase portraits of systems (V) for the parameters a and b belonging to the regions and curves mentioned above. See Figure 9.

4.5.3. Region $\tilde{R}_1$

In the region $\tilde{R}_1$ the infinite singular points on the Poincaré sphere are four nodes in U_1, see Table 1 (and also the corresponding points on V_1), hence each point has index 1. Moreover, the origin of U_2 is not a singular point (see Table 2). Hence all the infinite singular points are in U_1 and V_1.

Now we analyze the finite singular points of systems (V). Among the finite singular points we only know that the origin of U_3 is a center with index 1. Hence the known singular points have total index 10 on the Poincaré sphere. By Theorem 5, the remaining finite singular points must have total index -8. By Proposition 7 and Table 3 systems (V) have in region $\tilde{R}_1$ the origin (a center) and four finite hyperbolic singularities or centers in U_3. Therefore by Corollary 4 the four hyperbolic singularities must be saddles. Hence we obtain a global phase portrait topologically equivalent to 1.6 of Figure 1.

4.5.4. Region $\tilde{R}_2$

In the region $\tilde{R}_2$ the infinite singular points on the Poincaré sphere are two nodes in U_1 (and also the corresponding points on V_1). Hence each point has index 1. The origin of U_2 is not a singular point. Among the finite singular points we only know that the origin of U_3 is a center with index 1. Hence, the known singular points have total index 6 on the Poincaré sphere. By Theorem 5, the remaining finite singular points must have total index -4. By Proposition 7 and Table 3, systems (V) have in the region $\tilde{R}_2$ the origin as a center and six finite singularities which are either hyperbolic or

centers. By Theorem 5 these six finite singularities are two centers and four saddles. Therefore the global phase portrait is topologically equivalent to 1.7 of Figure 1.

4.5.5. Region $\tilde{R}_3$

With respect to the infinite singular points there is no change between the regions $\tilde{R}_2$ and $\tilde{R}_3$. On the other hand among the origin, there are only two finite singular points, see Table 3. As before these singularities are either hyperbolic, or centers. Therefore the two finite singular points are two saddles. The global phase portrait is topologically equivalent to 1.8 of Figure 1.

4.5.6. Region $\tilde{R}_4$

In the region $\tilde{R}_4$ there are no finite and infinite singular points besides the origin, see Tables 1, 2 and 3. Therefore the global phase portraits are topologically equivalent to 1.1 of Figure 1.

4.5.7. Region $\tilde{R}_5$

In the region $\tilde{R}_5$ there are no infinite singular points, see Tables 1 and 2. Among the finite singular points we only know that the origin of U_3 is a center with index 1. Hence, the known singular points have total index 2 on the Poincaré sphere. By Theorem 5, the remaining finite singular points, if any, must have total index 0. By Proposition 7 and Table 3, systems (V) have in the region $\tilde{R}_5$ the following finite singular points: the origin which is a center and four points which are either hyperbolic, or centers. Therefore these four singular points are two saddles and two centers. Hence the global phase portraits are topologically equivalent to 1.9 of Figure 1.

4.5.8. Curve L_{12}

In the curve L_{12} the infinite singular points on the Poincaré sphere are two nodes in U_1 (and also the corresponding points on V_1), hence each singular point has index 1. The origin of U_2 has two elliptic and two parabolic sectors, so this origin has index 2 by the Poincaré formula. Among the finite singular points we only know that the origin of U_3 is a center with index 1. Hence the known singular points have total index 10 on the Poincaré sphere. By Theorem 5 the remaining finite singular points must have total index -8. On the curve L_{12} the systems have the following singular points: the origin which is a center and four points which are either hyperbolic, or centers, see Table 4 and Proposition 7. So we have four saddles. Hence we obtain that the global phase portraits are topologically equivalent to 1.10 of Figure 1.

4.5.9. Curve L_{23}

In the curve L_{23} the infinite singular points on the Poincaré sphere are two nodes in U_1 (and also the corresponding points on V_1), hence each point has index 1. Moreover, the origin of U_2 is not a singular point, see Table 2. Hence all the infinite singular points are in U_1 and V_1. Among the finite singular points we only know that the origin of U_3 is a center with index 1. Hence, the known singular points have total index 6 on the Poincaré sphere. By Theorem 5 the remaining finite singular points must have total index -4.

We know that systems (V), among the origin, have at most six finite singular points. By Proposition 7 at most two of them are non-elementary (and so nilpotent singular points, due to statement (a) of Proposition 6). Therefore, due to the symmetry of the systems, there are the following possibilities: (i) four saddles and two centers, or (ii) two saddles and two cusps. Due to statement (c) of Proposition 6, only the case (ii) will produce a system compatible with the phase portraits of the regions $\tilde{R}_2$ and $\tilde{R}_3$ after a small perturbation. Consequently, the global phase portraits in this case are topologically equivalent to 1.11 of Figure 1.

4.5.10. Curve L_{34}

On the curve L_{34} there are no infinite singular points in the local chart U_1 and the origin of the local chart U_2 has two hyperbolic sectors when $a > 1$, and two hyperbolic and two parabolic sectors when $a = 1$ (see Tables 1 and 2). By Table 3 systems (V) have no finite singular points except the origin. Consequently, the global phase portraits in this case are topologically equivalent to 1.1 of Figure 1.

4.5.11. Curve L_{35}

On the curve L_{35} there are no infinite singular points in the local chart U_1. The origin of the local chart U_2 has two elliptic and two parabolic sectors, see Tables 1 and 2. So this origin has index 2 by the Poincaré formula. Among the finite singular points we only know that the origin of U_3 is a center. Hence the known singular points have total index 6 on the Poincaré sphere. By Theorem 5, the remaining finite singular points must have total index -4. By Table 3 systems (V) have two finite singular points and by Proposition 7 they are either hyperbolic singularities, or centers. Therefore these two finite singularities are saddles. Hence the global phase portraits are topologically equivalent to 1.12 of Figure 1.

4.5.12. Curve L_{45}

On the curve L_{45} there are no infinite singular points, see Tables 1 and 2. Among the finite singular points we only know that the origin of U_3 is a center. Hence the known singular points have total index 2 on the Poincaré sphere. By Theorem 5 the remaining finite singular points, if any, must have total index 0.

We know that, besides the origin, systems (V) have at most six finite singular points, and by Proposition 7 at most one of them is non-elementary (and so nilpotent due to statement (a) of Proposition 6). Therefore due to the symmetry of the systems, there are the following possibilities: (i) no more finite singular points, (ii) two cusps, (iii) two saddles and two centers, or (iv) two saddles, two centers and two cusps.

Case (iv) cannot occur because by Proposition 6, after a small perturbation, each cusp will produce two singular points, leading to an excess of singular points. For the remaining cases, due to Proposition 6, only the case (ii) will produce a system compatible with the phase portraits of the regions $\tilde{R}_4$ and $\tilde{R}_5$ after a small perturbation. Consequently, the global phase portraits in this case are topologically equivalent to 1.13 of Figure 1.

4.5.13. Curve L_{15}

On the curve L_{15} there are two infinite singular points with one hyperbolic and one elliptic sector in the local chart U_1, so they have index 1 by the Poincaré formula. The origin of the local chart U_2 is not a singular point, see Tables 1 and 2. Among the finite singular points we only know that the origin of U_3 is a center. Hence the known singular points have total index 6 on the Poincaré sphere. By Theorem 5 the remaining finite singular points must have total index -4. By Table 3 systems (V) have two finite singular points and by Proposition 7 they are either hyperbolic, or centers. Therefore these two singularities are saddles and we obtain the global phase portraits topologically equivalent to 1.14 of Figure 1.

Acknowledgements

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