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Periodic points of a Landen transformation

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Abstract

We prove the existence of 3-periodic orbits in a dynamical system associated to a Landen transformation previously studied by Boros, Chamberland and Moll, disproving a conjecture on the dynamics of this planar map introduced by the latter author. To this end we present a systematic methodology to determine and locate analytically isolated periodic points of algebraic maps. This approach can be useful to study other discrete dynamical systems with algebraic nature. Complementary results on the dynamics of the map associated with the Landen transformation are also presented.

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1 Introduction

Given a definite integral depending on several parameters, a *Landen transformation* is a map on these parameters that leaves invariant the integral. In [1, 2], G. Boros and V. Moll introduced the dynamical system given by

$$\left\{\begin{array}{l} a_{n+1}=\dfrac{5a_n+5b_n+a_nb_n+9}{(a_n+b_n+2)^{4/3}}, \quad b_{n+1}=\dfrac{a_n+b_n+6}{(a_n+b_n+2)^{2/3}}, \\ \\ c_{n+1}=\dfrac{d_n+e_n+c_n}{(a_n+b_n+2)^{2/3}}, \quad d_{n+1}=\dfrac{(b_n+3)c_n+(a_n+3)e_n+2d_n}{a_n+b_n+2}, \quad e_{n+1}=\dfrac{c_n+e_n}{(a_n+b_n+2)^{1/3}}, \end{array}\right.$$

as a Landen transformation associated to the integral

$$I(a,b,c,d,e)=\int_0^\infty \frac{cx^4+dx^2+e}{x^6+ax^4+bx^2+1}\mathrm{d}x, \tag{1}$$

that is, $I(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}, e_{n+1}) = I(a_n, b_n, c_n, d_n, e_n)$. This dynamical system contains a 2-dimensional uncoupled subsystem. M. Chamberland and V. Moll in [3], related the convergence of the integral (1) with the dynamics given by the iteration of the planar, non invertible map associated to it:

$$G(a,b) := \left(\frac{5a+5b+ab+9}{(a+b+2)^{4/3}}, \frac{a+b+6}{(a+b+2)^{2/3}} \right).$$

In particular they proved that the map G has only three fixed points, characterizing their nature, and they also proved that the region of the (a,b)-plane where the integral (1) converges is the basin of attraction of the fixed point $(3,3)$.

In Section 3 we will give a brief description of the known results about the dynamics of the map G. In Section 4 we prove our main result,

Theorem 1. *Consider the map G.*

(a) It has exactly three fixed points. A super-attracting point in $(3,3)$, an oscillatory saddle in the boundary of the basin of attraction of $(3,3)$ and an unstable focus.

(b) It has not periodic points with minimal period 2.

(c) It has exactly twelve periodic points of minimal period 3, that correspond with four 3-periodic orbits.

For completeness we include in its statement the results about fixed points already proved in [3]. In fact, in that paper it is also proved that there are no periodic points with minimal period 2 above the line $a+b+2=0$. Our statement (b) extends their result to the whole plane.

As we will see, item (c) disproves a conjecture about the dynamics of this map, see [11, Conj. 15.6.3] or Section 3. We will also determine analytically the location of the 3-periodic orbits.

Although it is easy to find 3-periodic points numerically, when trying to prove their existence there appear important computational obstacles. Thus, to prove the existence of 3-periodic points of G, as well as the non-existence of 2-periodic points, we have developed a procedure to determine analytically the number of isolated periodic points of discrete dynamical systems of algebraic nature and locate them with a prescribed precision. This method consists in the following four steps:

- Convert the problem into an algebraic one, characterizing the periodic points as the solutions of a system of polynomial equations.

- Include these solutions into the ones of an uncoupled system of equations given by one-variable polynomials.

- Combine an algorithm based on the *Sturm's method* for isolating the real roots of a one-variable polynomial with a *discard procedure* for systems of polynomial equations in order to efficiently remove those solutions of the later system that do not correspond with the periodic points.

- The application of the *Poincaré-Miranda theorem* to prove that the non discarded solutions are actual solutions of the first system of polynomial equations and, in consequence, give rise to periodic points.

This procedure is explained in detail in next section. Recall that the Poincaré-Miranda theorem is essentially the extension of Bolzano theorem to higher dimensions. It was stated by H. Poincaré in 1883 and 1884, and proved by himself in 1886, [12, 13]. In 1940, C. Miranda re-obtained the result as an equivalent formulation of Brouwer fixed point theorem, [10]. Recent proofs are presented in [8, 15]. We also recall this theorem in Section 2.

As a complement, in Section 5 we characterize the stable set associated to the fixed point of G of saddle type, and we provide an analytic-numeric study that gives evidences of the existence of homoclinic trajectories associated to it, as well as of the existence of some points in the intersection of the unstable set of this fixed point and the non-definition set of the map, which recall that it is formed by all the preimages of the straight line $a + b + 2 = 0$.

2 Determination of periodic points of discrete dynamical systems

We consider a discrete dynamical system defined by a map $F : \mathcal{U} \subseteq \mathbb{R}^k \to \mathcal{U}$ where $\mathcal{U}$ is an open set. Fix $p \in \mathbb{N}$ and assume that it has finitely many p-periodic points. These points are characterized by the real solutions of the system of k equations given by $F^p = \mathrm{Id}$. Let us suppose that the solutions of the above system are in correspondence with the ones of a new system of $n \geq k$ non-trivial *polynomial* equations given by

$$\left\{ \; f_1(\mathbf{x}) = 0,\, f_2(\mathbf{x}) = 0,\, \cdots ,\, f_n(\mathbf{x}) = 0, \right. \tag{2}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ are not necessarily the k-independent variables of F. Suppose also that using some algebraic transformations, like for instance successive resultants between the given equations, we reach an uncoupled polynomial system whose set of solutions *contains* all the solutions of system (2):

$$\left\{ \; q_1(x_1) = 0,\, q_2(x_2) = 0,\, \cdots ,\, q_n(x_n) = 0. \right. \tag{3}$$

To clarify with an example the above situation we sketch here the systems involved in the computation of the 3-periodic points of the map G. The $k = 2$ equations corresponding to $G^3(a, b) = (a, b)$ can be transformed into a new system of $n = 3$ polynomial equations (see system (11)) in the new variables m, n, r given by (10). This new system plays the role of system (2), and its solutions are in correspondence with the periodic points, by forthcoming Lemma 4. Lemma 5 will show that the solutions of system (11) are included in the set of solutions of the uncoupled system $\{d_{17}(m) = d_{17}(n) = d_{17}(r) = 0\}$, where d_{17} is a polynomial of degree 371 introduced in (12). This system plays the role of system (3).

In this setting, the proposed methodology applies in the cases where we do not know how to obtain explicitly the solutions of systems (2) or (3) and follows the next steps:

Step 1: By using an algorithm based on the Sturm's method ([14, Chap. 5.6]) and for each polynomial q_j, it is possible to isolate and count all its real roots by finding intervals with preset maximum length and rational ends, each one of them containing only one isolated root. For each $j = 1, 2, \ldots, n$, let k_j be the number of real roots of q_j, without counting their multiplicities, and denote by $I_{j,m} := [u_{j,m}, v_{j,m}]$, $m = 1, 2, \ldots, k_j$ the found intervals, such that each one of them contains exactly one of these roots. Proceeding in this way we obtain that the set of solutions of system (2) *is contained* in the set formed by $\prod_{j=1}^{n} k_j$ boxes (n-dimensional orthohedrons), of the form

$$\mathcal{I}_{m_1,\ldots,m_n} := I_{1,m_1} \times I_{2,m_2} \times \cdots \times I_{n,m_n},$$

where each $m_j \in \{1, \ldots, k_j\}$, for $j = 1, \ldots, n$.

Step 2: In order to detect those boxes that do not contain any solution of system (2) we apply a *discard procedure* to each box $\mathcal{I}_{m_1,\ldots,m_n}$. This procedure is inspired in a technique used in [4]. To prove that a certain polynomial $P(\mathbf{x})$ has no zeros in a given box $\mathcal{I}_{m_1,\ldots,m_n}$, that for the sake of simplicity we denote as $\mathcal{I}$, we proceed as follows:

- We numerically evaluate P at the center of $\mathcal{I}$. If, compared with the working precision, this value is far from zero, we suspect that P restricted to $\mathcal{I}$ has a given sign. According whether this value is positive or negative we continue with one of next two steps.

- For trying to prove that $P(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{I}$, we search a L such that $0 < L < P(\mathbf{x})$ on $\mathcal{I}$. Write $P(\mathbf{x}) = \sum_\ell M_\ell(\mathbf{x})$ where $M_\ell(\mathbf{x}) = a_\ell x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n}$, we find $\underline{M}_\ell \in \mathbb{R}$ such that $\underline{M}_\ell < M_\ell(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{I}$ (this can be done using the formulas in forthcoming Lemma 2). If the following condition is satisfied: $0 < L := \sum_\ell \underline{M}_\ell < \sum_\ell M_\ell(\mathbf{x}) = P(\mathbf{x})$, then we can discard the box $\mathcal{I}$.

- For trying to prove that $P(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathcal{I}$, we look for $U \in \mathbb{R}$ such that $P(\mathbf{x}) < U < 0$ on $\mathcal{I}$. To do this, similarly than in the previous situation, we find $\overline{M}_\ell \in \mathbb{R}$ such that $M_\ell(\mathbf{x}) < \overline{M}_\ell$ for all $\mathbf{x} \in \mathcal{I}$. If it holds that $P(\mathbf{x}) = \sum_\ell M_\ell(\mathbf{x}) < \sum_\ell \overline{M}_\ell =: U < 0$, then we can discard the box $\mathcal{I}$.

To compute the bounds $\underline{M}_\ell$ and $\overline{M}_\ell$, we use the following straightforward result, which can be easily implemented in any computer algebra software.

Lemma 2. *Consider* $P(\mathbf{x}) = \sum_\ell M_\ell(\mathbf{x})$ *where* $M_\ell(\mathbf{x}) = a_\ell x_1^{\ell_1} x_2^{\ell_2} \cdots x_n^{\ell_n}$*, and a box* $\mathcal{I} = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_n, v_n]$*. Set* $O^+ = \{(x_1, \ldots, x_n)$*, such that* $x_i > 0$ *for all* $i = 1, \ldots, n\}$*. Then,*

(i) If $\mathcal{I} \subset O^+ \subset \mathbb{R}^n$*, then for all* $\mathbf{x} \in \mathcal{I}$*,* $\sum_\ell \underline{M}_\ell \leq P(\mathbf{x}) \leq \sum_\ell \overline{M}_\ell$*, where*

- *(a)* $\underline{M}_\ell = a_\ell\, u_1^{\ell_1} u_2^{\ell_2} \cdots u_n^{\ell_n}$ *and* $\overline{M}_\ell = a_\ell\, v_1^{\ell_1} v_2^{\ell_2} \cdots v_n^{\ell_n}$ *if* $a_\ell > 0$.
- *(b)* $\underline{M}_\ell = a_\ell\, v_1^{\ell_1} v_2^{\ell_2} \cdots v_n^{\ell_n}$ *and* $\overline{M}_\ell = a_\ell\, u_1^{\ell_1} u_2^{\ell_2} \cdots u_n^{\ell_n}$ *if* $a_\ell < 0$.

(ii) If $\mathcal{I} \not\subset O^+$ *we can always take a number* $\xi > 0$*,* $\xi \in \mathbb{Q}$ *such that the new box* $\widetilde{\mathcal{I}} = [u_1 + \xi, v_1 + \xi] \times [u_2 + \xi, v_2 + \xi] \times \cdots \times [u_n + \xi, v_n + \xi] \subset O^+$*, and then find bounds for* P *on* $\mathcal{I}$*, using the bounds given in item (i) for* $\widetilde{P}_\xi(x_1, x_2, \ldots, x_n) := P(x_1 - \xi, x_2 - \xi, \ldots, x_n - \xi)$ *on* $\widetilde{\mathcal{I}}$*.*

We try to apply the discard procedure until the number of remaining boxes coincides with our hopes. These hopes usually came from a previous numerical study of the problem. We start trying to prove that the first function f_1 does not vanish in the given box. It may happen that it is easier to try to prove the same with another f_j. Notice also that sometimes to discard a box we must go to the Step 1 and start with smaller boxes.

Step 3: Once it is achieved an optimized list of non-discarded boxes, we identify those boxes that correspond to either fixed points or periodic points with a period being a divisor of p, which we assume that we already know, and we also discard them.

Step 4: From the non-discarded boxes list obtained in the previous step, we try to show that each box actually contains a solution by applying the Poincaré-Miranda theorem. For completeness, we recall it. As usual, $\overline{B}$ and ∂B denote, respectively, the closure and the boundary of a set $B \subset \mathbb{R}^n$.

Theorem 3 (Poincaré-Miranda)**.** *Set* $\mathcal{I} = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n\ :\ L_i < x_i < U_i, 1 \leq i \leq n\}$*. Suppose that* $f = (f_1, f_2, \ldots, f_n) : \overline{\mathcal{I}} \to R^n$ *is continuous,* $f(\mathbf{x}) \neq \mathbf{0}$ *for all* $\mathbf{x} \in \partial\mathcal{I}$*, and for* $1 \leq i \leq n$*,*

$$f_i(x_1, \ldots, x_{i-1}, L_i, x_{i+1}, \ldots, x_n) \cdot f_i(x_1, \ldots, x_{i-1}, U_i, x_{i+1}, \ldots, x_n) \leq 0,$$

Then, there exists $\mathbf{s} \in \mathcal{I}$ *such that* $f(\mathbf{s}) = \mathbf{0}$*.*

It is clear that when we define f to try to apply Poincaré-Miranda theorem, the order of the components matters. So, sometimes to be under the hypotheses of the theorem it is better to consider $f = (f_{\sigma_1}, f_{\sigma_2}, \ldots, f_{\sigma_n})$ for some permutation σ. In fact, more in general, it is convenient to apply the theorem to $A(f(\mathbf{x}))^t$, where A is a suitable $n \times n$ invertible matrix. When f is differentiable, as we will see it is useful to chose $A = (\mathrm{D}f(\widehat{\mathbf{s}}))^{-1}$, where $\widehat{\mathbf{s}} \in \mathbb{Q}^n$ is a numerical approximation of a zero of f in $\mathcal{I}$.

If we succeed in proving that there is at least a solution in each box, its uniqueness is given by the fact that each of the intervals $I_{j,m}$ contains only a single solution of each polynomial q_j.

Otherwise we can refine boxes, taking them with smaller size, and then repeating the computations in Step 1.

3 An overview of the dynamics of G

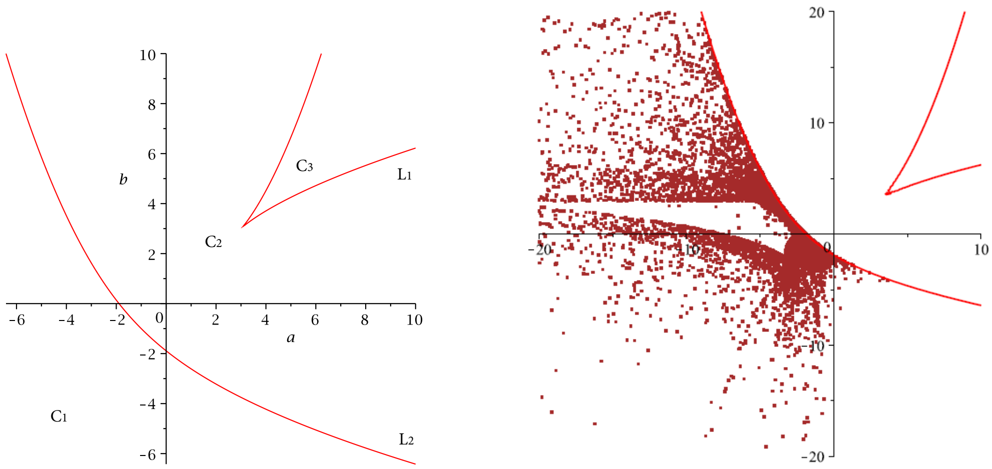
In this section we briefly summarize the known results on the dynamics of the map G and we characterize their invariant sets. We mainly follow the steps in [3]. The rational integral (1) is well-defined and convergent if $P(x) = x^3 + ax^2 + bx + 1$ has not real positive roots. To study the number of real roots of P when a and b vary, we consider

$$R(a, b) := \mathrm{Res}(P, P'; x) = -\Delta_x(P) = -a^2b^2 + 4\,a^3 + 4\,b^3 - 18\,ab + 27,$$

where Δ_x is the discriminant. The curve $R(a, b) = 0$ is known as the *resolvent* one, and after removing the point $(-1, -1)$ it is invariant by G because

$$R(G(a, b)) = \frac{(a - b)^2}{(a + b + 2)^4} R(a, b). \tag{4}$$

The curve has two connected components L_1 i L_2 (see Figure 1 (a)). Note that the fixed point $(3, 3)$ is the cusp of L_1.



(a) (b)

Figure 1. (a) Connected components L_1 and L_2 of the curve $R(a, b) = 0$, and regions C_1, C_2 and C_3 of the plane. (b) 10000 iterates of an orbit with initial condition in C_3.

The resolvent curve defines three open unbounded sets C_1, C_2 and C_3 depicted in Figure 1 (a). By studying the sign of the discriminant of P, and by using the Descartes rule of signs, it is straightforward to obtain that on $L_1 \cup C_2 \cup C_3$, all the real roots of $P(x)$ are negative, so the integral (1) is convergent; and on $C_1 \cup L_2$ there exists at least one positive real root, so the integral diverges.

In [3], the authors proved that G has only three fixed points, namely P_i for $i = 1, 2, 3$, which are described in Theorem 1 and given in Equation (8). One of them, the point $P_1 = (3, 3)$, is a super-attracting one, i.e. both eigenvalues of the jacobian matrix are 0. Their main result states that *the basin of attraction of the fixed point P_1 for the map G, is the region of the (a, b)-plane where the integral (1) converge.* As a consequence, the basin of attraction of P_1 is $L_1 \cup C_2 \cup C_3$.

On the other hand, the connected component $L_2' := L_2 \setminus \{(-1, -1)\}$ is *positively* invariant (as we will see, there are points on C_1 which are mapped into L_2). On L_2 there is one fixed point, P_2, which is a saddle. In Proposition 8 we prove that any orbit with initial condition on L_2' converges to P_2.

In summary, the dynamics of G on the invariant sets $\mathcal{A} := L_1 \cup C_2 \cup C_3$ and L_2 is known and simple. However, there is a poor knowledge of the dynamics of G in the set $\mathcal{B} := C_1 \setminus \mathcal{F}$, where $\mathcal{F} = \{(a, b) \in \mathbb{R}^2 : \exists\, n \geq 0 \, : \, G^n(a, b) \in \{a + b + 2 = 0\}\}$, is the forbidden set of G.

In [11, Conj. 15.6.3], V. Moll established the following conjecture about the dynamics of G in $\mathcal{B}$: *"The orbit of any point below the resolvent curve is dense in the open region below this curve."*

Of course one has to exclude from this conjecture the third fixed point P_3 which is in the set C_1, and the points in the forbidden set $\mathcal{F}$. In Theorem 1, we prove the existence of 3-periodic points in C_1, result that disproves the conjecture. In fact it is not difficult to find *numerically* these orbits as well as other periodic points, however to prove the existence of 3-periodic points is far from being trivial, and it is the main objective of this paper.

In Section 5 we present an analytic-numeric study that evidences the existence of points in the unstable manifold of P_2 which belongs to its stable set, i.e. *homoclinic points*. In the case of diffeomorphisms, by the Smale-Birkhoff homoclinic theorem, the existence of such points implies the existence of a hyperbolic invariant set on which the dynamics is equivalent to a subshift of finite type, see [7]. Similar results are developed in [5, 6] in the non-invertible setting. We also give evidences of the existence of points in the unstable manifold that also belong to the forbidden set $\mathcal{F}$.

4 Proof of Theorem 1

In this section we prove Theorem 1. We split the proof in two subsections. The first one dedicated to the fixed and 2-periodic points, and the second one to study the 3-periodic points.

4.1 Fixed and 2-periodic points

Proof of statements (a) and (b). (a) Following [3] we consider the equations given by $G(a,b) = (a,b)$, and we introduce the auxiliary variable $m^3 = a+b+2$, obtaining

$$\begin{cases} d_1(a,b,m) & := m^3 - a - b - 2 = 0, \\ d_2(a,b,m) & := -am^4 + ab + 5a + 5b + 9 = 0, \\ d_3(a,b,m) & := -bm^2 + a + b + 6 = 0. \end{cases} \tag{5}$$

Isolating a and b from the first and third equations, and substituting the obtained expressions in the second one we get:

$$d_4(m) := -(m-2)(m^2-m+1)(m^2+m+2)(m^3+m^2-m-2)(m^3+m^2+m+2) = 0. \tag{6}$$

The only real roots of the above equation are

$$m_1 = 2, \quad m_2 = \frac{1}{6}\sqrt[3]{A} + \frac{8}{3}\frac{1}{\sqrt[3]{A}} - \frac{1}{3} \simeq 1.20557, \quad m_3 = -\frac{1}{6}\sqrt[3]{B} + \frac{4}{3}\frac{1}{\sqrt[3]{B}} - \frac{1}{3} \simeq -1.35321, \tag{7}$$

where $A = 172 + 12\sqrt{177}$ and $B = 188 + 12\sqrt{249}$. From these values and (5) we obtain

$$\begin{aligned} P_1 &= (3,3), \\ P_2 &= \left(\frac{-43+3\sqrt{177}}{384}A^{2/3} - \frac{1}{6}A^{1/3} - \frac{8}{3}, \frac{13-\sqrt{177}}{48}A^{2/3} + \frac{7+\sqrt{177}}{48}A^{1/3} + \frac{4}{3}\right) \simeq (-4.20557, 3.95774), \\ P_3 &= \left(\frac{-21+\sqrt{249}}{96}B^{2/3} + \frac{15-\sqrt{249}}{12}B^{1/3} - 2, \frac{17-\sqrt{249}}{48}B^{2/3} + \frac{-13+\sqrt{249}}{24}B^{1/3} - \frac{4}{3}\right) \simeq (-5.30914, 0.83118). \end{aligned} \tag{8}$$

A straightforward computation of the differential matrix at these points give that the points are, respectively, a super-attractor (null eigenvalues), an oscillatory saddle, and an unstable focus. Moreover P_2 is in L_2 and P_3 is in C_1.

(b) Again, following [3], we consider c and d such that $G(a,b) = (c,d)$ and $G(c,d) = (a,b)$. By introducing the two auxiliary variables m and n such that $m^3 = a+b+2$ and $n^3 = c+d+2$, we get:

$$\begin{cases} d_1 & := m^3 - a - b - 2 = 0, \quad d_2 := n^3 - c - d - 2 = 0, \\ d_3 & := -cm^4 + ab + 5a + 5b + 9 = 0, \quad d_4 := -dm^2 + a + b + 6 = 0, \\ d_5 & := -an^4 + cd + 5c + 5d + 9 = 0, \quad d_6 := -bn^2 + c + d + 6 = 0. \end{cases}$$

Solving $\{d_1 = 0, d_2 = 0, d_4 = 0, d_6 = 0\}$ we obtain

$$a = \frac{m^3n^2 - n^3 - 2\,n^2 - 4}{n^2}, \quad b = \frac{n^3+4}{n^2}, \quad c = \frac{m^2n^3 - m^3 - 2\,m^2 - 4}{m^2}, \quad d = \frac{m^3+4}{m^2}.$$

By substituting the above result in d_3 and d_5 we reach the following system, which plays the role of system (2) in our methodology:

$$\begin{cases} d_7(m,n) := & -m^4n^7 + m^5n^4 + 2m^4n^4 + m^3n^5 + 5m^3n^4 + 4m^2n^4 - n^6 \\ & +4m^3n^2 - 2n^5 - n^4 - 8n^3 - 8n^2 - 16 = 0, \\ d_8(m,n) := & d_7(n,m) = 0. \end{cases} \tag{9}$$

Now we consider the polynomial

$$
\begin{aligned}
d_9(m) :=& \mathrm{Res}(d_7(m,n), d_8(m,n); n)\\
=& m^4\,(m-2)\,(m+1)^2\left(m^3+m^2+m+2\right)\left(m^3+m^2-m-2\right)P(m),
\end{aligned}
$$

where P is a polynomial of degree 56, without real roots. This is proved by using the Sturm's method and can also be done, for instance, by using the command `realroot` of the computer algebra system Maple. Similarly, $d_{10}(n) := \mathrm{Res}(d_7(m,n), d_8(m,n); m)$. As a consequence of the symmetry we get that $d_{10}(n) = -d_9(n)$ and system $\{d_9(m) = 0,\, d_9(n) = 0\}$ plays the role of system (3) in our methodology. Hence, the only non-zero reals roots of d_9 are $-1, 2, m_1$ and m_2, where these values correspond to the ones associated with the fixed points, because the two degree 3 factors coincide with the ones given in (6).

Hence the 2-periodic points are included in the set with 16 elements $\{-1, 2, m_1, m_2\}^2$. In this particular case, because the real solutions of the uncoupled system are explicit, in Steps 3 and 4 of our approach we have not boxes but points, and the problem is much easier. It is not difficult to check that in this set of points the only solutions of (9) are $(1,1), (m_1, m_1)$ and (m_2, m_2) which correspond to the fixed points of G. In consequence there are not points of minimal period 2 for G ∎

4.2 Proof of Theorem 1 (c): 3-periodic points

We need some preliminary results. Proceeding as in the previous cases we look for a, b, c, d, e, and $f \in \mathbb{R}$, such that $G(a,b) = (c,d)$, $G(c,d) = (e,f)$ and $G(e,f) = (a,b)$, that is

$$
\frac{5a+5b+ab+9}{(a+b+2)^{4/3}} = c, \qquad \frac{a+b+6}{(a+b+2)^{2/3}} = d, \qquad \frac{5c+5d+cd+9}{(c+d+2)^{4/3}} = e,
$$
$$
\frac{c+d+6}{(c+d+2)^{2/3}} = f, \qquad \frac{5e+5f+ef+9}{(e+f+2)^{4/3}} = a, \qquad \frac{e+f+6}{(e+f+2)^{2/3}} = b.
$$

We introduce the auxiliary variables m, n and r, such that $m^3 = a+b+2$, $n^3 = c+d+2$ and $r^3 = e+f+2$. Using this notation we get

$$
\begin{cases}
d_1 := m^3-a-b-2=0, & d_4 := -cm^4+ab+5a+5b+9=0, \quad d_5 := -dm^2+a+b+6=0,\\
d_2 := n^3-c-d-2=0, & d_6 := -en^4+cd+5c+5d+9=0, \quad d_7 := -fn^2+c+d+6=0,\\
d_3 := r^3-e-f-2=0, & d_8 := -ar^4+ef+5e+5f+9=0, \quad d_9 := -br^2+e+f+6=0.
\end{cases}
$$

First we solve the system $\{d_1 = 0, d_2 = 0, d_3 = 0, d_5 = 0, d_7 = 0, d_9 = 0\}$ obtaining:

$$
\begin{aligned}
&a = \frac{m^3r^2 - r^3 - 2\,r^2 - 4}{r^2}, \quad b = \frac{r^3+4}{r^2}, \quad c = \frac{m^2n^3 - m^3 - 2\,m^2 - 4}{m^2},\\
&d = \frac{m^3+4}{m^2}, \quad e = \frac{n^2r^3 - n^3 - 2\,n^2 - 4}{n^2}, \quad f = \frac{n^3+4}{n^2}.
\end{aligned} \tag{10}
$$

Substituting this result in the expressions d_4, d_6, and d_8, we obtain the equations

$$\begin{cases} d_{10}(m,n,r) := & -m^4n^3r^4 + m^5r^4 + 2\,m^4r^4 + m^3r^5 + 5\,m^3r^4 + 4\,m^2r^4 - r^6 + 4\,m^3r^2 \\ & -2\,r^5 - r^4 - 8\,r^3 - 8\,r^2 - 16 = 0, \\ d_{11}(m,n,r) := & d_{10}(n,r,m) = 0, \\ d_{12}(m,n,r) := & d_{10}(r,m,n) = 0. \end{cases} \tag{11}$$

From the equations (10), if (m,n,r) is a real solution of (11) such that $m\cdot n\cdot r \neq 0$, there exists either an orbit with minimal period 3 given by (10) or a fixed point or G. Moreover, if (m_0,n_0,r_0) is a solution of system (11), then so are (n_0,r_0,m_0) and (r_0,m_0,n_0). As a consequence, we obtain

Lemma 4. *Any* 3*-periodic orbit of* G*,* $\{(a,b);(c,d);(e,f)\}$ *with associated parameters* m,n *and* r*, is in correspondence, via* (10)*, with the solutions* (m,n,r)*,* (n,r,m) *and* (r,m,n) *of the system* (11)*.*

The forthcoming Lemma 5 gives a first characterization of the locus where the solutions of system (11) are located. Prior to state this result we introduce the following auxiliary polynomials

$d_{13}(n,r) := \mathrm{Res}(d_{10}(m,n,r), d_{12}(m,n,r); m)$, with degree 37 in n and degree 37 in r,

$d_{14}(n,r) := \mathrm{Res}(d_{11}(m,n,r), d_{12}(m,n,r); m)$, with degree 47 in n and degree 37 in r.

We apply the resultant once again to obtain the polynomials $d_{15}(n) := \mathrm{Res}(d_{13}(n,r), d_{14}(n,r); r)$, and $d_{16}(r) := \mathrm{Res}(d_{13}(n,r), d_{14}(n,r); n)$, where $\deg_n(d_{15}(n)) = 2521$ and $\deg_r(d_{16}(r)) = 1985$. Finally, we introduce the polynomial

$$d_{17}(n) := \gcd\left(d_{15}(n), d_{16}(n)\right)/n^{716}. \tag{12}$$

This polynomial has degree 371, and using once more Sturm's method we get that it has exactly 16 different real non-zero roots.

Lemma 5. *Let* I_i*, with* $i = 1,\ldots,16$*, be disjoint intervals, each one of them containing a unique real root of* d_{17}*. Then, any real solution* (m,n,r) *of system* (11) *is contained in one of the* 16^3 *sets*

$$\mathcal{I}_{i,j,k} := I_i \times I_j \times I_k, \quad i,j,k \in \{1,\ldots,16\}. \tag{13}$$

Proof. Let (m_0,n_0,r_0) be a real solution of (11). We want to show that it is also a solution of $\{d_{17}(m) = 0, d_{17}(n) = 0, d_{17}(r) = 0\}$. Observe that by construction, n_0 must be a root of d_{15}. From Lemma 4, we have that (m_0,n_0) must be also a zero of d_{13} and d_{14}. Hence n_0 must be also a zero of d_{16}, and therefore of $\gcd\left(d_{15}(n), d_{16}(n)\right)$ which is a polynomial of degree 1087 with the factor n^{716}. Since we are interested in its non-zero roots, we remove this factor, obtaining that n_0 must be a root of d_{17}. By using an analogous argument and Lemma 4 again, we can see that m_0 and r_0 are also roots of d_{17}.

Since d_{17} has 16 different real roots, any solution (m,n,r) of system (11) must be contained in a box of the form (13), and each box contains at most one solution. ∎

Now we can prove statement (c) of Theorem 1. We follow the steps explained in Section 2:

Step 1: Recall that d_{17} has 16 non-zero real roots. Two of them are $n=-1$, $n=2$. Although two more explicit roots are $n_i=m_i, i=1,2$ given in (7), we prefer to take 14 intervals with rational ends and length smaller than 10^{-20}, $I_i, i=1,2,\ldots,14$, each one of them containing a unique root. We consider:

$$I_1=\left[-\frac{4308988841618670568853}{1475739525896764129280},-\frac{3447191073294936455082}{1180591620717411303424}\right],\quad I_2=\left[-\frac{3441180573310194930843}{1180591620717411303424},-\frac{1720590286655097465421}{590295810358705651712}\right],$$

$$I_3=\left[-\frac{913839855050902450805}{1180591620717411303424},-\frac{456919927525451225402}{590295810358705651712}\right],\quad I_4=\left[-\frac{441651874085591876219}{590295810358705651712},-\frac{883303748171183752438}{1180591620717411303424}\right],$$

$$I_5=\left[-\frac{994661336537171251825}{295147905179352825856},-\frac{3978645346148685007299}{1180591620717411303424}\right],\quad I_6=\left[-\frac{3977374161031280580629}{1180591620717411303424},-\frac{994343540257820145157}{295147905179352825856}\right],$$

$$I_7=\left[-\frac{197879469664271669175}{73786976294838206464},-\frac{3166071514628346706799}{1180591620717411303424}\right],\quad I_8=\left[-\frac{3144313156826151948503}{1180591620717411303424},-\frac{1572156578413075974251}{590295810358705651712}\right],$$

$$I_9=\left[-\frac{399397086201257638833}{295147905179352825856},-\frac{1597588344805030555331}{1180591620717411303424}\right],\quad I_{10}=\left[-\frac{1053526769518098399097}{4722366482869645213696},-\frac{131690846189762299887}{590295810358705651712}\right],$$

$$I_{11}=\left[-\frac{1064910654630154190265}{9444732965739290427392},-\frac{133113831828769273783}{1180591620717411303424}\right],\quad I_{12}=\left[\frac{1065572958580542810237}{9444732965739290427392},\frac{532786479290271405119}{4722366482869645213696}\right],$$

$$I_{13}=\left[\frac{128535594827653577343}{590295810358705651712},\frac{1028284758621228618745}{4722366482869645213696}\right],\quad I_{14}=\left[\frac{177910645965499912685}{147573952589676412928},\frac{1423285167723999301481}{1180591620717411303424}\right].$$

We also introduce the degenerate intervals $I_{15}=[-1,1]$ and $I_{16}=[2,2]$ containing the exact roots $n=-1$ and $n=2$. By Lemma 5, all the real solutions of system (11) are contained in one of the 16^3 boxes (13), where we also call boxes the ones with some degenerate interval. Recall that if a box $\mathcal{I}_{i,j,k}$ contains a solution of system (11), then this solution is unique.

Step 2: We apply the discard procedure to d_{10} and the 4096 boxes of the form (13) given by the intervals computed before. In Lemma 2 we use the value $\xi=30$ and consider the polynomial $P(m,n,r)=d_{10}(m-\xi,n-\xi,r-\xi)$, which has 224 monomials. The procedure implemented in Maple v.17 took 5.61s of real time in an Intel i7-3770-3.4GHz CPU to discard 4080 boxes. The code is given in [9, Chap. 5]. In short, we obtain that each solution of system (11) must be contained in one of the following 16 non-discarded boxes

$\mathcal{I}_{1,5,11}$	$\mathcal{I}_{2,6,12}$	$\mathcal{I}_{3,7,13}$	$\mathcal{I}_{4,8,10}$	$\mathcal{I}_{5,11,1}$	$\mathcal{I}_{6,12,2}$	$\mathcal{I}_{7,13,3}$	$\mathcal{I}_{8,10,4}$
$\mathcal{I}_{9,9,9}$	$\mathcal{I}_{10,4,8}$	$\mathcal{I}_{11,1,5}$	$\mathcal{I}_{12,2,6}$	$\mathcal{I}_{13,3,7}$	$\mathcal{I}_{14,14,14}$	$\mathcal{I}_{16,16,15}$	$\mathcal{I}_{16,16,16}$

Observe that the degenerated box $\mathcal{I}_{16,16,15}$, which corresponds with $(m,n,r)=(2,2,-1)$, must also be discarded because $d_{10}(2,2,-1)=0$, but $d_{11}(2,2,-1)=2304$.

Step 3: Following similar arguments that in the proof of statement (b) we can discard boxes $\mathcal{I}_{16,16,16}$, $\mathcal{I}_{14,14,14}$ and $\mathcal{I}_{9,9,9}$ because they correspond to the fixed points P_1, P_2 and P_3, respectively.

Step 4: We have obtained 12 non-discarded boxes that, from Lemma 4, if they correspond to periodic points of minimum period 3, they would contain the parameters (m,n,r) corresponding to the periodic points according to the following groupings:

$$\mathcal{O}_1 \subset \mathcal{I}_{1,5,11} \cup \mathcal{I}_{5,11,1} \cup \mathcal{I}_{11,1,5}, \quad \mathcal{O}_2 \subset \mathcal{I}_{2,6,12} \cup \mathcal{I}_{6,12,2} \cup \mathcal{I}_{12,2,6},$$
$$\mathcal{O}_3 \subset \mathcal{I}_{3,7,13} \cup \mathcal{I}_{7,13,3} \cup \mathcal{I}_{13,3,7}, \quad \mathcal{O}_4 \subset \mathcal{I}_{4,8,14} \cup \mathcal{I}_{8,14,4} \cup \mathcal{I}_{14,4,8}. \tag{14}$$

We will prove that the above 12 boxes indeed contain a solution of system (11), which will be unique as reasoned above. To do this, we will apply the Poincaré-Miranda theorem (Theorem 3). Again by Lemma 4 we only need to prove that there is a solution of the system (11) in the boxes: $\mathcal{I}_{1,5,11}$, $\mathcal{I}_{2,6,12}$, $\mathcal{I}_{3,7,13}$, and $\mathcal{I}_{4,8,14}$. For reasons of space we only give details for the first box.

We consider the polynomial map $f(m,n,r) := (d_{10}(m,n,r), d_{11}(m,n,r), d_{12}(m,n,r))$. We denote the ends of the intervals I_1, I_5 and I_{11} respectively: $[\underline{m},\overline{m}] := I_1$, $[\underline{n},\overline{n}] := I_5$, $[\underline{r},\overline{r}] := I_{11}$. Consider also the middle point of $\mathcal{I}_{1,5,11}$, $\widehat{p} = (\widehat{m},\widehat{n},\widehat{r}) = \big((\underline{m}+\overline{m})/2, (\underline{n}+\overline{n})/2, (\underline{r}+\overline{r})/2\big)$.

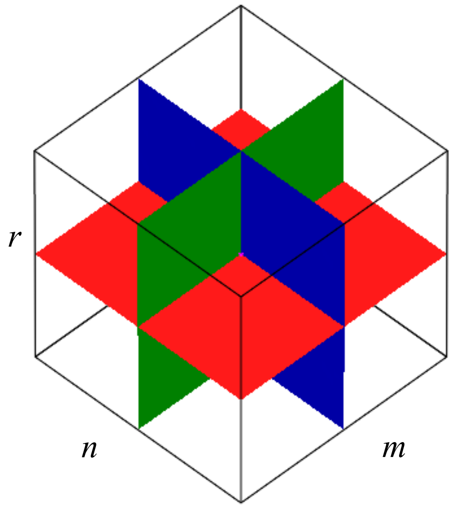


Figure 2. The surfaces $g_1(m,n,r) = 0$, $g_2(m,n,r) = 0$ and $g_3(m,n,r) = 0$ in blue, green and red, respectively, in the box $[\widehat{m}-\varepsilon, \widehat{m}+\varepsilon] \times [\widehat{n}-\varepsilon, \widehat{n}+\varepsilon] \times [\widehat{r}-\varepsilon, \widehat{r}+\varepsilon]$, where $\varepsilon = 10^{-10}$.

The hypothesis of Poincaré-Miranda theorem for f using the box $\mathcal{I}_{1,5,11}$ are not satisfied: for instance, at the points $(\underline{m},\widehat{n},\widehat{r})$ and $(\overline{m},\widehat{n},\widehat{r})$ none of the functions d_{10}, d_{11} and d_{12} changes sign. So in order to rectify the level 0 surfaces of the components of f, we consider the new function

$$g(m,n,r) = (g_1(m,n,r), g_2(m,n,r), g_3(m,n,r)) := (\mathrm{D}f(\widehat{p}))^{-1}(f(m,n,r))^t.$$

We omit here the expressions of $(\mathrm{D}f(\widehat{p}))^{-1}$ and g since they involve huge rational numbers with numerators and denominators with hundreds of digits. Notice that since $\det(\mathrm{D}f(\widehat{p})) \neq 0$ the point (m_0,n_0,r_0) is a zero of g if and only if it is a zero of f.

Observe that $g(m,n,r) = g(\widehat{p}) + (m-\widehat{m}, n-\widehat{n}, r-\widehat{r}) + O(||(m-\widehat{m}, n-\widehat{n}, r-\widehat{r})||^2)$. Since $g(\widehat{p}) \simeq 0$, near $\widehat{p}$ it holds that $g(m,n,r) \simeq (m-\widehat{m}, n-\widehat{n}, r-\widehat{r})$ and so, a small enough box centered at $\widehat{p}$ should be under the hypotheses of Poincaré-Miranda theorem, see Figure 2. Now we will check that, indeed, this is the situation for the function g in the box $\mathcal{I}_{1,5,11}$.

In order to prove that the components of the function g have no roots, and alternate signs at the faces of $\mathcal{I}_{1,5,11}$ we will apply repeatedly the following technical result, that is a simplified version adapted to our interests of a result given in [4]:

Lemma 6. *Let $G_\alpha(x) = g_n(\alpha)x^n + g_{n-1}(\alpha)x^{n-1} + \cdots + g_1(\alpha)x + g_0(\alpha)$ be a family of real polynomials that depend continuously on a real parameter $\alpha \in \Lambda = [\alpha_1, \alpha_2] \subset \mathbb{R}$. Fix $J = [a, b] \subset \mathbb{R}$ and assume that:*

(i) There exists $\alpha_0 \in \Lambda$ such that $G_{\alpha_0}(x)$ has no real roots in J.

(ii) For all $\alpha \in \Lambda$, $G_\alpha(a) \cdot G_\alpha(b) \cdot \Delta_x(G_\alpha) \neq 0$, where $\Delta_x(G_\alpha)$ is the discriminant of G_α with respect to x.

Then for all $\alpha \in \Lambda$, $G_\alpha(x)$ has no real roots in J.

We will prove that the first component of g has no roots, and alternates signs at the faces $m = \underline{m}$ and $m = \overline{m}$ of the box $\mathcal{I}_{1,5,11}$. Consider the function $G_n(r) = g_1(\underline{m}, n, r) \cdot g_1(\overline{m}, n, r)$. We will prove that $G_n(r) < 0$ for all $(n, r) \in I_5 \times I_{11}$ using Lemma 6 with $\Lambda = I_5$ and $J = I_{11}$.

By the Sturm's method it can be seen that the polynomial $G_{\widehat{n}}(r)$ has only 6 different real roots and that none of them is in the interval I_{11}. Hence the hypothesis (i) is satisfied. Moreover $G_{\widehat{n}}(r)$ restricted to I_{11} is negative.

Proceeding in an analogous way, we obtain that $G_n(\underline{r}) \cdot G_n(\overline{r})$ has only 4 different real roots and none of them belongs to I_5. Hence $G_n(\underline{r}) \cdot G_n(\overline{r}) \neq 0$ for all $n \in I_5$. We also check that the discriminant $\Delta_r(G_n(r))$, which is a polynomial of degree 192 in n, has 37 different real roots. Again, we prove that they are not in I_5 and so, we are under the hypothesis (ii) of Lemma 6. Hence by this lemma we get that $G_n(r) < 0$ for all $(n, r) \in I_5 \times I_{11}$, as we wanted to prove.

Doing similar arguments and computations, we obtain that the second and third component of g do not vanish, and alternate signs on the faces $n = \underline{n}$ and $n = \overline{n}$, and $r = \underline{r}$ and $r = \overline{r}$ of $\mathcal{I}_{1,5,11}$, respectively. See [9, Chap. 5] for more details. Thus $g(m, n, r)$ verifies the hypothesis of the Poincaré-Miranda theorem in $\mathcal{I}_{1,5,11}$. Hence the function g, and therefore the function f, have at least one zero in this box, which is unique by construction.

4.3 Analytic location of the 3-periodic points

In this section we use that the parameters m, n and r associated to each periodic point are located in the 12 boxes given in (14), to obtain an analytic location of them in the (a, b)-plane.

Lemma 7. *Let $m \in [\underline{m}, \overline{m}]$ and $r \in [\underline{r}, \overline{r}]$, and the functions $a(m, r)$ and $b(r)$ given by (10), then:*

(i) If $0 < \underline{r} \leq \overline{r}$ then $\underline{a} := \underline{m}^3 - \overline{r} - 2 - 4/\underline{r}^2 \leq a(m, r) \leq \overline{m}^3 - \underline{r} - 2 - 4/\overline{r}^2 =: \overline{a}$ and $\underline{b} := \underline{r} + 4/\overline{r}^2 \leq b(r) \leq \overline{r} + 4/\underline{r}^2 =: \overline{b}$

(ii) If $\underline{r} \leq \overline{r} < 0$ *then* $\underline{a} := \underline{m}^3 - \overline{r} - 2 - 4/\overline{r}^2 \leq a(m,r) \leq \overline{m}^3 - \underline{r} - 2 - 4/\underline{r}^2 =: \overline{a}$ *and* $\underline{b} := \underline{r} + 4/\underline{r}^2 \leq b(r) \leq \overline{r} + 4/\overline{r}^2 =: \overline{b}$.

Proof. (i) From (10) we get that $a(m,r) = m^3 - r - 2 - 4/r^2$ and $b(r) = r + 4/r^2$. Notice that if $0 < \underline{r} \leq r \leq \overline{r}$ then $-\overline{r} \leq -r \leq -\underline{r} < 0$ and $-\frac{4}{\underline{r}^2} \leq -\frac{4}{r^2} \leq -\frac{4}{\overline{r}^2} < 0$. Moreover, $\underline{m}^3 \leq m^3 \leq \overline{m}^3$. By combining all these chains of inequalities we get statement (i). The statement (ii) follows similarly. ∎

By using the inequalities in *(ii)* of Lemma 7 we obtain, for example, that the 3-periodic point (a,b) of G, associated to the parameters $(m,n,r) \in \mathcal{I}_{1,5,11}$, satisfies $a \in [\underline{a}, \overline{a}]$ and $b \in [\underline{b}, \overline{b}]$ where

$$\underline{a} = -\frac{1435686715756812113129131753291751212473714621389705932746390847605145815709035232062993533718832495489341}{56947609584619278435915236206283183709714097978506070511694763452312581699417401160811385506316156928},$$
$$\overline{a} = -\frac{47044582301919219323098597682011430719430330620984084471100755414697990442772197382375529104298060913286874119}{1866059270868804515791575090678155019012542140400238364480469557193740712623709418953041434224970065510400}$$

and

$$\underline{b} = \frac{3368785687756582636246263551756811406295236320753178521304454421527}{10710654937528498667637446691242283113536911386660380934878003200},$$
$$\overline{b} = \frac{6579659546399575461418490144259606329620802274396204966850496409}{20919247924860348960190099800217926294342327605140376818548736}.$$

By using the decimal approximation we get,

$$a \in [\underline{a}, \overline{a}] \simeq [-25210.658115921519312682, -25210.658115921519312679],$$
$$b \in [\underline{b}, \overline{b}] \simeq [314.5265819322469464743, 314.5265819322469464749],$$

where we observe that $\max(\overline{a} - \underline{a}, \overline{b} - \underline{b}) \simeq 2.8 \times 10^{-18}$.

Applying Lemma 7 to each of the 12 boxes (14), we obtain rational bounds for the components of each periodic point of minimal period 3, that are summarized in the following tables, where only the decimal expression of some significative digits is given. In all the cases the maximum length of the interval localizing the 3-periodic points is smaller than 10^{-17}, so the given expression of both ends of the intervals coincide.

$\mathcal{O}_1$	a	b
	-25210.658115921519313	314.52658193224694647
	-11.080089229288244821	-29.194152462502174029
	1.0164106270635353803	-3.0178440371837045505

$\mathcal{O}_2$	a	b
	-25080.503857555317449	314.36115078061939834
	-11.094342178650567807	-29.143225143670723223
	1.0179782228602330827	-3.0165421366176918413

$\mathcal{O}_3$	a	b
	-550.35997876621370288	84.580855473468510676
	-13.613164340185764400	-7.6737642167841728949
	0.13590789992610542444	-2.1255835876361107899

$\mathcal{O}_4$	a	b
	-500.96942815695686889	80.145842594816842809
	-13.481597649423988848	-7.4104176831057891201
	0.088325991394389446424	-2.0994294342645985249

5 Dynamics associated to the saddle point P_2

In this section we study the invariant sets of the saddle point P_2. First, Proposition 8 characterizes the stable set of P_2. We will also give numerical evidences of the existence of homoclinic orbits, that is, initial conditions on the local unstable manifold, whose orbit converges to P_2. Finally, we will provide numerical evidences of the existence of points on the local unstable manifold, whose orbits end in the non-definition set.

5.1 The stable set of P_2

We denote the *stable set* of the fixed point P_2 as $W^s(P_2) = \{(a,b) \in \mathbb{R}^2 : \lim_{n\to\infty} G^n(a,b) = P_2\}$. This set is not necessarily a manifold. For hyperbolic points, like P_2 there is also the so called *local stable manifold* $W^s_{\text{loc}}(P_2) \subset W^s(P_2)$, that is an actual manifold and is only defined in a small neighborhood of the fixed point. Our first result characterizes totally the stable set of P_2 and its local stable manifold.

Proposition 8. *It holds that*

$$W^s(P_2) = \big(L_2 \cup \{(a,b) \in \mathbb{R}^2 : \exists\, n \geq 0 : G^n(a,b) \in R_1 \cup C_1\}\big) \setminus \{(-1,-1)\},$$

where $R_1 = \{a - b = 0\}$. Moreover, $W^s_{loc}(P_2)$ is contained in L_2.

Proof. Observe that Equation (4) implies that the only initial conditions mapped by G to the resolvent curve are the points of the straight line R_1, except $(-1,-1)$. Hence, to prove the proposition it suffices to show that $L_2' = L_2 \setminus \{(-1,-1)\} \subset W^s(P_2)$. Let us prove this inclusion.

Recall that $L_2 \subset \{R(a,b) = 0\}$. The resolvent curve $R(a,b) = 0$ is algebraic and has genus 0, so it admits *rational parametrizations*. For instance, if we define $P(t) = (P_1(t), P_2(t)) = \left(\frac{t^3+4}{t^2}, \frac{t^3+16}{4t}\right)$ it holds that $R(P_1(t), P_2(t)) \equiv 0$. This parametrization has been already was also used in [3, Thms 3 and 4]. The component L_2 corresponds with $t \in (-\infty, 0)$, and L_1 with $t \in (0,\infty)$. Some computations give $P^{-1}(a,b) = \frac{4\,(a^2-3\,b)}{a^2b-4\,b^2+3\,a}$. Hence, to study the dynamics of G on the component L_2 we need to study the one-dimensional map

$$g(t) = P^{-1} \circ G \circ P(t) = \sqrt[3]{4}\,\frac{t}{(t+2)^2}\left(\frac{(t^2+4)(t+2)^2}{t^2}\right)^{2/3}, \text{ for } t \in \mathcal{I} := (-\infty, 0) \setminus \{-2\},$$

see also [3, Thm 4]. Observe that $t = -2$ corresponds with $(a,b) = (-1,-1)$ which belongs to the non-definition line $\{a + b + 2 = 0\}$ and is excluded in our statement. The map $g(t)$ has a unique fixed point in $\mathcal{I}$

$$p = -\frac{1}{3}\,\frac{4\,C + \sqrt[3]{2}C^2 + 8\,2^{2/3}}{C} \simeq -4.4111, \text{ where } C = \sqrt[3]{86 + 6\,\sqrt{177}}.$$

Our objective is to prove that this fixed point is a global attractor of $g(t)$ in $\mathcal{I}$.

First we summarize some features of $g(t)$ in $\mathcal{I}$ that we will need (see Figure 3): (i) It has only two relative extremes (maximum) in $\mathcal{I}$ given by $t = -4 \mp 2\sqrt{3}$ ($t \simeq -7.4641$ and $t \simeq -0.5359$ respectively), and such that $g(-4 \mp 2\sqrt{3}) = -4$. We denote $m := -4 - 2\sqrt{3}$. (ii) It holds that $\lim\limits_{t\to -2^{\pm}} g(t) = \lim\limits_{t\to 0^-} g(t) = -\infty$. (iii) It also holds that $\lim\limits_{t\to-\infty} g(t) = -\infty$. (iv) For all $t \in (-\infty, p)$, we have $g(t) > t$. (v) The map g has not 2-periodic points as a consequence of Theorem 1 (b).

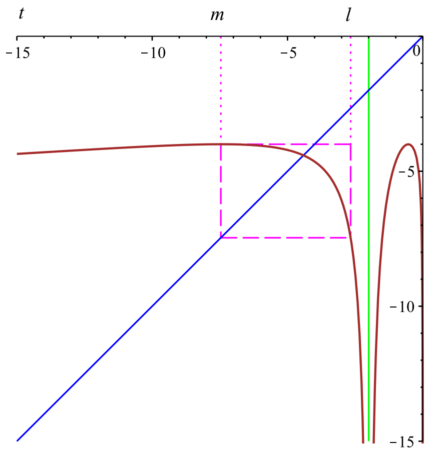


Figure 3. Graph of the function $g(t)$ in $\mathcal{I}$.

The proof has three steps, namely (A)–(C): (A) From the properties (i) and (ii), we conclude $g\left((-2,0)\right)=(-\infty,-4]$.
(B) Using (i) and (ii) again, we conclude that $g\left((-\infty,-2)\right)=(-\infty,-4]$, hence the interval $(-\infty,-2)$ is invariant by g. We will study the dynamics in this interval.

Let $\ell\in(p,-2)$ be the unique value in this interval such that $g(\ell)=m$ ($\ell\simeq-2.6675$). By using the monotony of g, the interval $[m,\ell]$ is invariant. Indeed, $g([m,\ell])=[g(\ell),g(m)]=[m,-4]\subset[m,\ell]$, see again Figure 3. Now we claim that *for all* $t\in(-\infty,m)\cup(\ell,-2)$ *there exists* $n>0$ *such that* $t_n=g^n(t)\in[m,\ell]$.

Indeed, by the monotonicity of g in $(\ell,-2)$ we have that $g((\ell,-2))=(-\infty,m)$. Since for all $t\in(-\infty,m)$, we have $g(t)<-4<\ell$, then $g(t)\notin(\ell,-2)$, hence $g(t)\in(-\infty,\ell)$. We only need, therefore to prove the claim in $t\in(-\infty,m)$. We proceed by contradiction. Consider $t_0\in(-\infty,m)$ and suppose that none iterate $t_n\in[m,\ell]$, so that for all $n>0$ we have $t_n\in(-\infty,m)$. From (iv), the sequence $\{t_n\}$ is increasing, and as we are assuming that it is bounded from above by m, the sequence must have a limit that, by continuity, must be a fixed point, which is a contradiction because there is no fixed point in $(-\infty,m]$. Hence the claim is proved, and we only have to study the dynamics of g in $[m,\ell]$.
(C) We study now the dynamics on the interval $[m,\ell]$. We denote $m_0:=m$ and $\ell_0:=\ell$, and consider the sequences

$$\ell_k=g(m_{k-1})=g^2(\ell_{k-1})\ \text{ and } m_k=g(\ell_k)=g^2(m_{k-1}). \tag{15}$$

Observe that since g is strictly decreasing in $[m,\ell]$, for $k\geq1$ we obtain

$$[m_{k-1},\ell_k]:=g^{2k-1}\left([m,\ell]\right)\ \text{ and }\ [m_k,\ell_k]:=g^{2k}\left([m,\ell]\right).$$

We will prove that $\{m_k\}$ and $\{\ell_k\}$ are increasing and decreasing sequences, respectively, that converge to the fixed point p, thus proving the result.

Some computations show that $\ell_1 = g(m) = -4 < \ell = \ell_0$, and that $m_0 = m < m_1 = g(\ell_1) = g(-4)$. We proceed by induction, assuming that $m_{k-1} < m_k$ and $\ell_k < \ell_{k-1}$. Using (15), as g is decreasing and $m_{k-1} < m_k$, we have $\ell_k = g(m_{k-1}) > g(m_k) = \ell_{k+1}$. Likewise, since $\ell_{k+1} < \ell_k$ we have $m_{k+1} = g(\ell_{k+1}) > g(\ell_k) = m_k$. Therefore, the sequences $\{m_k\}$ and $\{\ell_k\}$ are monotonous increasing and decreasing, respectively. Since both sequences are bounded, and using the expressions in (15), we have that both converge to a fixed point of g^2. But since there are not 2-periodic points, except the fixed point p, we have $\lim\limits_{k\to\infty} m_k = \lim\limits_{k\to\infty} \ell_k = p$. ∎

5.2 Local expression of the unstable manifold

In order to search numerically the homoclinic points associated to P_2, we compute an approximation of the local unstable manifold of the saddle point $P_2 = (a_2, b_2)$. We consider the change $u = a - a_2$ and $v = b - b_2$, which brings P_2 to the origin $(0,0)$. We also consider the map $\widetilde{G}(u,v) = G(u + a_2, v + b_2) - (a_2, b_2)$ which is conjugate with G, and the linear map given by $H(r,s) = L \cdot (r,s)^t$, where L is the matrix formed by the eigenvectors of $\mathrm{D}G(P_2)$. Hence

$$L^{-1} \cdot \mathrm{D}G(P_2) \cdot L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1 and λ_2 are the eigenvalues of $\mathrm{D}G(P_2)$, given by

$$\lambda_1 = \frac{-1}{384}\left(\left(7\sqrt{177} - 111\right) A^{2/3} + \left(8\sqrt{177} - 264\right) A^{1/3} - 768\right) \simeq 7.0701,$$
$$\lambda_2 = \frac{-1}{1152}\left(\left(\sqrt{177} - 25\right) A^{2/3} + \left(8\sqrt{177} - 136\right) A^{1/3} + 1280\right) \simeq -0.4470,$$

where $A := 172 + 12\sqrt{177}$.

We compute the Taylor development of the unstable manifold associated to the origin of the map $F(r,s) = H^{-1} \circ \widetilde{G} \circ H(r,s) = \left(\lambda_1 r + O(||(r,s)||^2), \lambda_2 s + O(||(r,s)||^2\right)$. *The expression of the local unstable manifold* $W^u_{loc}(0,0)$ *of* F *is* $s = w(r) = w_2 r^2 + w_3 r^3 + w_4 r^4 + w_5 r^5 + O(r^6)$, *where*

$$w_2 \simeq -0.00259107002218996975513519324145, \qquad w_3 \simeq -0.000132205296506666505558465802906,$$
$$w_4 \simeq -0.0000088987035667484756038434860 1, \qquad w_5 \simeq -0.000000693748122744413434736913 30.$$

These coefficients have been computed using the formulas in Lemma 9 of the Appendix, by using floating-point arithmetic with 60 digits in the mantissa. Observe that we can parametrize $W^u_{\mathrm{loc}}(P_2)$ using the function $s = w(r)$, by considering

$$r \longrightarrow H(r, w(r)) + P_2, \quad \text{for } r \simeq 0. \tag{16}$$

We use this parametrization to obtain Figures 4 and 5.

Finally, from the expression of the local unstable manifold of the origin for the map F, we obtain that the points $(a,b) \in W^u_{\mathrm{loc}}(P_2)$ satisfy $w(H_1^{-1}(a - a_2, b - b_2)) - H_2^{-1}(a - a_2, b - b_2) = 0$, that can be approximated by

$$D_1(a,b) := \sum_{i=2}^{5} w_i \big(H_1^{-1}(a-a_2, b-b_2)\big)^i - H_2^{-1}(a-a_2, b-b_2) = 0,$$

where $D_1(a,b)$ is a polynomial of degree 5 that we do not explicite for the sake of shortness, see [9, Chap. 5] for more details.

5.3 Computation of the homoclinic point

Previous to find a homoclinic point we remember that, by Proposition 8, any point (a,b) such that there exists $k \in \mathbb{N}$ verifying $G^k(a,b) \in R_1 = \{a-b=0\} \cap C_1 \setminus \{(-1,-1)\}$ belongs to the stable set of P_2, since $G^{k+1}(a,b) \in L_2$. In this sense, we have graphically observed that, except for the point P_2, there is no intersection of $W^u_{\rm loc}(P_2)$ with the curve L_2. Also we have observed neither intersections of $W^u_{\rm loc}(P_2)$ with $R_1 = \{a-b=0\}$ at the region C_1, nor points $(a,b) \in W^u_{\rm loc}(P_2)$ such that $G(a,b) \in R_1 \cup C_1$, but we have seen the existence of at least one point such that $G^2(a,b) \in R_1 \cup C_1$. See Figure 4.

Imposing $G_1(a,b) - G_2(a,b) = 0$, we find that the points (a,b) such that $G(a,b) \in R_1$ satisfy:

$$D_2(a,b) := (ab + 5\,a + 5\,b + 9)^3 - (a+b+6)^3\,(a+b+2)^2 = 0.$$

Hence, the points such that $G^2(a,b) \in R_1$ are those satisfying $D_2(G(a,b)) = 0$, or equivalently $D_3(a,b) := \text{numer}(D_2(G(a,b))) = 0$, where $D_3(a,b)$ is a polynomial of degree 10 in the variable $m = (a+b+2)^{2/3}$ with 22 terms, that we omit here.

Therefore, the homoclinic point P must verify the system $\{D_1(a,b) = 0, D_3(a,b) = 0\}$. We solve it numerically, using floating-point arithmetic with 60 digits in the mantissa, and we get a solution in $[-6,-5] \times [3.5,5]$, given by $P = (p_1, p_2)$ where

$$p_1 \simeq -5.67750144031789435343891174392876990152177028290023619512062,$$
$$p_2 \simeq 4.10574868714920935493626045239900450809925741194290963919902.$$

By using the parametrization of $W^u_{\rm loc}(P_2)$ given by (16), we find that the point P corresponds with the parameter $r \simeq -1.48202152087749433523$.

By construction, $G^3(P)$ which must lie on L_2. A computation shows that the absolute error when we evaluate $R(a,b)$ on this point, is $\big|R\big(G^3(p)\big)\big| \simeq 10^{-58}$. Accordingly, the point P exhibits, numerically, a homoclinic behavior.

As can be seen in Figure 4, there exists another solution of $\{D_1(a,b) = 0, D_3(a,b) = 0\}$ in $[-8,-6] \times [3.5,5]$, given by $\tilde{P} = (\tilde{p}_1, \tilde{p}_2)$ where

$$\tilde{p}_1 \simeq -7.32664831286596004531700787733138125161658087249633041273728,$$
$$\tilde{p}_2 \simeq 4.26205920129322448141657538934356322617112224124511704493689.$$

The point $\tilde{P}$ corresponds to the parameter value $r \simeq -3.14702449177907104545$.

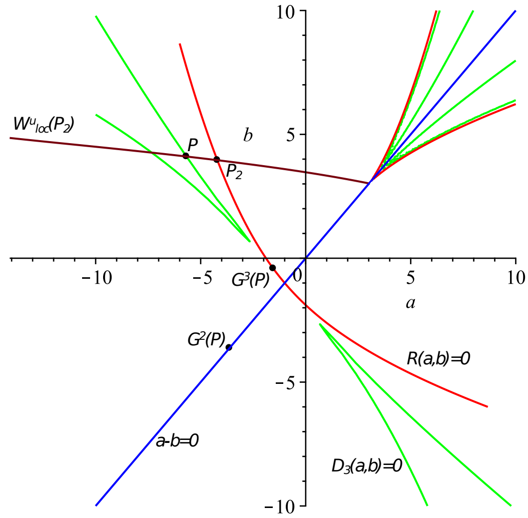


Figure 4. Location of the points $P, G^2(P)$ and $G^3(P)$ on $W^u_{\rm loc}(P_2)$ (brown), the curve $D_3(a,b) = 0$ (green), the diagonal R_1 (blue) and the resolvent curve (red), respectively. The point $G(P)$ is outside the image.

5.4 Computation of points in $W^u_{\rm loc}(P_2) \cap \mathcal{F}$

To find a point in $W^u_{\rm loc}(P_2) \cap \mathcal{F}$, we solve numerically the system $\{D_1(a,b) = 0, a + b + 2 = 0\}$, obtaining the point $Q = (q_1, q_2)$, where

$$q_1 \simeq -6.1516301702919311427053988329227669955805787623398035072028 2, \quad q_2 = -2 - q_1,$$

This point corresponds with the parameter $r \simeq -1.96025815386161687597$.

To find another point with a parameter value closer to zero (hence giving a better evidence of really being in $W^u_{\rm loc}(P_2)$), we find a point Q_{-1} such that $G(Q_{-1}) = Q$. The points (a,b) such that $G(a,b) \in \{a + b + 2 = 0\}$, verify

$$D_4(a,b) := 5\,a + 5\,b + ab + 9 + (a + b + 6)\,\sqrt[3]{(a+b+2)^2} + 2\,\sqrt[3]{(a+b+2)^4} = 0.$$

By solving numerically the system $\{D_1(a,b) = 0, D_4(a,b) = 0\}$, we find $Q_{-1} := (z_1, z_2)$ where

$$z_1 \simeq -4.4393173395192730671391497614676155081075004857947832775890 4,$$
$$z_2 \simeq 3.9818528436589958997246709557856460056942884880182583684838 4.$$

The point Q_{-1} has an associated parameter value $r \simeq -0.23505956788542861108$. The location of the above points is shown in Figure 5.

Observe that the parameters of the points Q, P and Q_{-1} are interspersed, so the points are also interspersed in $W^u_{\rm loc}(P_2)$. An analytic proof of this fact would show that arbitrarily near of P_2 there are homoclinic points and points in $\mathcal{F}$.

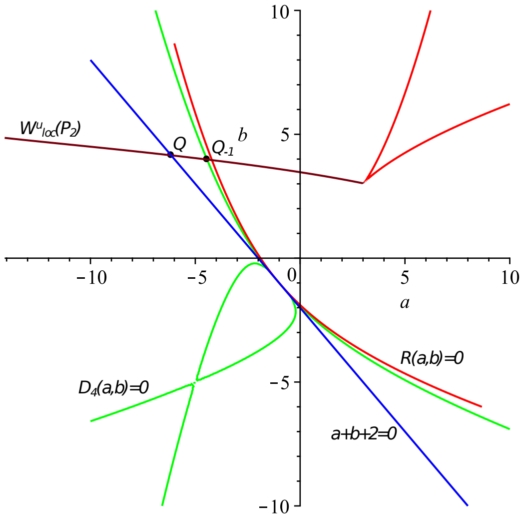


Figure 5. Location of the points Q and Q_{-1} in $W^u_{\text{loc}}(P_2) \cap \mathcal{F}$; the curve $D_4(a,b) = 0$ (green); the line $a+b+2=0$ (blue); and the resolvent curve (red), respectively.

Appendix: Local unstable manifold near a hyperbolic saddle point

Lemma 9. *Consider the smooth map, defined in a neighborhood of the origin $\mathcal{U}$:*

$$F(x,y) = \left(\lambda x + \sum_{i+j=2}^{5} f_{i,j} x^i y^j + O(||(x,y)||^6), \mu y + \sum_{i+j=2}^{5} g_{i,j} x^i y^j + O(||(x,y)||^6)\right)$$

where $|\lambda| > 1 > |\mu|$, so that the origin is a hyperbolic saddle. Let $y = w(x) = \sum_{k=2}^{5} w_k x^k + O(x^6)$ be the expression of the local unstable manifold in a neighborhood of the origin. Then:

$$w_2 = \frac{g_{2,0}}{\lambda^2 - \mu}, \quad w_3 = \frac{\lambda^2 g_{3,0} - 2\,\lambda\, f_{2,0} g_{2,0} - \mu\, g_{3,0} + g_{1,1} g_{2,0}}{(\lambda^2 - \mu)(\lambda^3 - \mu)}, \quad w_4 = \frac{W_4}{(\lambda^2 - \mu)^2 (\lambda^3 - \mu)(\lambda^4 - \mu)},$$

where

$$\begin{aligned} W_4 = {} & g_{4,0}\lambda^7 + (-3\, f_{2,0} g_{3,0} - 2\, f_{3,0} g_{2,0})\,\lambda^6 + \left(5\, f_{2,0}^2 g_{2,0} - 2\, g_{4,0}\mu + g_{2,0} g_{2,1}\right)\lambda^5 \\ & + \left((6\, f_{2,0} g_{3,0} + 2\, f_{3,0} g_{2,0} - g_{4,0})\,\mu - 2\, f_{1,1} g_{2,0}^2 - 3\, f_{2,0} g_{1,1} g_{2,0} + g_{1,1} g_{3,0}\right)\lambda^4 \\ & + \left(g_{4,0}\mu^2 + \left(-5\, f_{2,0}^2 g_{2,0} + 2\, f_{3,0} g_{2,0} - g_{2,0} g_{2,1}\right)\mu - 2\, f_{2,0} g_{1,1} g_{2,0} + g_{0,2} g_{2,0}^2\right)\lambda^3 \\ & + \left((-3\, f_{2,0} g_{3,0} + 2\, g_{4,0})\,\mu^2 + \left(f_{2,0}^2 g_{2,0} + 3\, f_{2,0} g_{1,1} g_{2,0} - 2\, g_{1,1} g_{3,0} - g_{2,0} g_{2,1}\right)\mu \right. \\ & \left. + g_{1,1}^2 g_{2,0}\right)\lambda^2 + \left(-2\, f_{3,0} g_{2,0}\mu^2 + \left(2\, f_{1,1} g_{2,0}^2 + 2\, f_{2,0} g_{1,1} g_{2,0}\right)\mu\right)\lambda \\ & - g_{4,0}\mu^3 + \left(-f_{2,0}^2 g_{2,0} + g_{1,1} g_{3,0} + g_{2,0} g_{2,1}\right)\mu^2 + \left(-g_{0,2} g_{2,0}^2 - g_{1,1}^2 g_{2,0}\right)\mu \end{aligned}$$

and

$$w_5 = \frac{\sum_{i=0}^{13} p_i\, \lambda^i}{(\lambda^2 - \mu)^3 (\lambda^3 - \mu)(\lambda^4 - \mu)(\lambda^5 - \mu)},$$

where $p_i, i = 1, 2, \ldots, 13$ are polynomials in the other variables of F that we skip, although we have used, for the sake of shortness (they are given in [9, Chapter 5]).

Proof. Due to the particular form of the linear part of F, the local unstable manifold $W^u_{\rm loc}(0,0)$ is given by a smooth function of the form $y = w(x) = w_2x^2 + w_3x^3 + w_4x^4 + w_5x^5 + O(x^6)$, that is, a point is on the local stable manifold if it is of the form $(x, w(x))$. Imposing that $F(x, w(x)) = (F_1(x, w(x)), F_2(x, w(x)))$ is also on this curve we get that the points on the local unstable manifold must satisfy $F_2(x, w(x)) = w\big(F_1(x, w(x))\big)$. The result follows by comparing the terms in the Taylor development of both members of the last equation. ∎

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