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PERIODIC ORBITS BIFURCATING FROM A NON–ISOLATED ZERO–HOPF EQUILIBRIUM OF THREE-DIMENSIONAL DIFFERENTIAL SYSTEMS REVISITED

Murilo R. Cândido

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra
Barcelona, 08193, Spain
candidomr@mat.uab.cat

Jaume Llibre

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra
Barcelona, 08193, Spain
jllibre@mat.uab.cat

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In this paper we study the periodic solutions bifurcating from a non-isolated zero-Hopf equilibrium in a polynomial differential system of degree two in $\mathbb{R}^3$. More specifically, we use recent results of averaging theory to improve the conditions for the existence of one or two periodic solutions bifurcating from such a zero–Hopf equilibrium. This new result is applied for studying the periodic solutions of differential systems in $\mathbb{R}^3$ having n-scroll chaotic attractors.

Keywords: averaging theory, periodic solutions, polynomial differential systems, zero–Hopf bifurcation, zero–Hopf equilibrium.

1. Introduction and Statement of the Main Result

In this paper we study the periodic orbits bifurcating from a non-isolated zero–Hopf equilibrium of a three-dimensional autonomous differential system, it means that the differential system has a continuum of equilibria containing a point with a zero eigenvalue and a pair of purely imaginary eigenvalues. The zero-Hopf bifurcation is an interesting subject in differential systems and have been studied by Guckenheimer [1981]; Guckenheimer & Holmes [2013]; Scheurle & Marsden [1984]; Kuznetsov [2013] and many other authors. Usually the zero–Hopf bifurcation is a two-parameter unfolding of a three–dimensional autonomous differential system with an isolated zero–Hopf equilibrium. It is known that some complicated invariant sets can bifurcate from an isolated zero-Hopf equilibrium, for instance zero-Hopf bifurcation can imply local birth of “chaos”, see for instance Scheurle & Marsden [1984].

Most papers study isolated zero–Hopf equilibrium. One of the few works about non-isolated zero–Hopf equilibrium was done in 2012 by Llibre & Xiao [2014]. The authors studied the periodic orbits bifurcating from the non-isolated zero-Hopf equilibrium located at the origin of the following family of polynomial differential systems of degree two

$$
\begin{aligned}
\frac{dU}{dt} =& \varepsilon\lambda U - \omega V + \sum_{i+j+k=2} a_{ijk}(\varepsilon)U^iV^jW^k,\\
\frac{dV}{dt} =& \omega U + \varepsilon\lambda V + \sum_{i+j+k=2} b_{ijk}(\varepsilon)U^iV^jW^k,\\
\frac{dW}{dt} =& \varepsilon\mu W + \sum_{i+j+k=2} c_{ijk}(\varepsilon)U^iV^jW^k,
\end{aligned} \tag{1}
$$

with $\mu = 1$ and where the coefficients functions are of the form

$$
\begin{aligned}
a_{ijk}(\varepsilon) &= \sum_{l=0}^{1} a_{ijkl}\varepsilon^l + \mathcal{O}(\varepsilon^2),\\
b_{ijk}(\varepsilon) &= \sum_{l=0}^{1} b_{ijkl}\varepsilon^l + \mathcal{O}(\varepsilon^2),\\
c_{ijk}(\varepsilon) &= \sum_{l=0}^{1} c_{ijkl}\varepsilon^l + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{2}
$$

When $\varepsilon = 0$, this system has a continuum of equilibria which fill a segment, or a half-straight line. As shown by Llibre & Xiao [2014, Proposition 2.2] this continuum of equilibria will have a (non-isolated) zero-Hopf equilibrium at the origin if and only if system (1) satisfies the following hypothesis

$$(H_0): \; a_{0020} = b_{0020} = c_{0020} = 0.$$

In a small neighbourhood of the origin the authors reduced system (1) to a 2π-periodic differential system using a kind of cylindrical coordinates and a scaling of the variables. Then second order averaging theory was used for providing explicit conditions for the existence of one or two periodic orbits bifurcating from the non-isolated zero–Hopf equilibrium, see Llibre & Xiao [2014, Theorem 2.4].

We shall use recent results obtained in the averaging theory to weak the hypotheses of Llibre & Xiao [2014, Theorem 2.4] improving those results. Mainly their result was obtained assuming the following hypothesis

$$(H_1): c_{2000} = c_{1100} = c_{0200} = 0.$$

We extend these results giving sufficient conditions for the existence of $\varepsilon-$families of periodic orbits bifurcating from the origin of system (1) using the weaker hypothesis

$$(H_1'): c_{2000} = -c_{0200}.$$

This will be our main result, see Theorem 1. In section 2 we formulate the averaging theorem (see Theorem 2) used for proving our main result. We use averaging theory in this paper, although the same kind of study could be done by Melnikov method. In particular, the Melnikov functions recently developed by Tian & Han [2017] could also be applied here. In section 3 we shall apply Theorem 1 to study three polynomial differential systems of degree two that cannot be studied using Llibre & Xiao [2014, Theorem 2.4]. More precisely these three polynomial differential systems are system (11) proposed by Li [2008], system (12) provided by Pan *et al.* [2010] and system (10) proposed by Elhadj & Sprott [2013]. For certain coefficients values all these systems present n-scroll chaotic attractors. They also have non-isolated zero-Hopf equilibria, and we shall use Theorem 1 to give sufficient conditions for the existence of periodic orbits for these systems.

Our main result is the following.

Theorem 1. *Assume that (H_0) holds. Then system (1) has a non-isolated zero–Hopf equilibrium at the origin when $\varepsilon = 0$. Furthermore if the assumption (H_1') holds, there is ε^* such that $0 < \varepsilon^* \ll 1$ and for any ε and $0 < \varepsilon < \varepsilon^*$, the following statements hold.*

(i) *System (1) has one family of periodic orbits bifurcating from the origin if one of the following conditions holds:*

(a) $a_{1010} + b_{0110} = 0$, $\lambda B_1 < 0$ *and* $\lambda(B_1 + B_4) \neq B_1\mu$,
(b) $a_{1010} + b_{0110} \neq 0$, $\lambda\mu = 0$ *and* $Q_1Q_2 < 0$,
(c) $a_{1010} + b_{0110} \neq 0$, $Q_1 = 0$ *and* $Q_2\lambda\mu < 0$,
(d) $a_{1010} + b_{0110} \neq 0$, $Q_2 = 0$, $\mu\lambda > 0$ *and* $Q_1 < 0$,
(e) $a_{1010} + b_{0110} \neq 0$, $3Q_2^2 - 4Q_1\lambda\mu \neq 0$, $Q_2 \neq 0$, $\mu\lambda > 0$ *and* $Q_1 < 0$,
(f) $a_{1010} + b_{0110} \neq 0$, $3Q_2^2 - 4Q_1\lambda\mu \neq 0$, $Q_2 \neq 0$, $\mu\lambda < 0$ *and* $Q_1 > 0$.

(ii) *System (1) has two families of periodic orbits bifurcating from the origin if one of the following condition holds:*

(g) $a_{1010} + b_{0110} \neq 0$, $3Q_2^2 - 4Q_1\lambda\mu \neq 0$, $Q_2 < 0$, $\lambda\mu > 0$ *and* $0 < Q_1 < 3Q_2^2/(4\lambda\mu)$,
(h) $a_{1010} + b_{0110} \neq 0$, $3Q_2^2 - 4Q_1\lambda\mu \neq 0$, $Q_2 > 0$, $\lambda\mu < 0$ *and* $3Q_2^2/(4\lambda\mu) < Q_1 < 0$,

where

$$
\begin{aligned}
B_1 =& 4a_{2000}\left(b_{2000} + 2c_{0110}\right) - 2a_{0200}(a_{1100} + 2b_{0200} - 4c_{0110}) \\
& - c_{1100}(a_{1010} + 11b_{0110} - 8c_{0020}) - 10c_{2000}\left(a_{0110} + b_{1010}\right) \\
& - 2a_{1100}a_{2000} + 2\left(b_{0200} + b_{2000}\right)\left(b_{1100} - 4c_{1010}\right), \\
B_2 =& 2a_{0200}\left(181a_{2000} + 67b_{1100} - 80c_{1010}\right) - 2a_{1100}(61b_{0200} \\
& + 67b_{2000} - 8c_{0110}) - 16c_{1010}(8a_{2000} + b_{1100}) + 72c_{1100}(a_{0110} \\
& + b_{1010}) + 122a_{2000}b_{1100} + 235a_{0200}^2 + 5a_{1100}^2 + 123a_{2000}^2 + 32(4b_{0200} \\
& + 5b_{2000})c_{0110} - 123b_{0200}^2 - 5b_{1100}^2 - 235b_{2000}^2 - 362b_{0200}b_{2000}, \\
B_3 =& a_{0200}\left(a_{1100} - 233b_{0200} - 47b_{2000} + 112c_{0110}\right) + a_{1100}(7a_{2000} \\
& + b_{1100} - 8c_{1010}) - 303a_{2000}b_{0200} - 137a_{2000}b_{2000} + 128a_{2000}c_{0110} \\
& - 8b_{1100}c_{0110} + 16\left(2b_{0200} + b_{2000}\right)c_{1010} + 36c_{1100}\left(c_{0020} - 2b_{0110}\right) \\
& + 55b_{1100}b_{0200} + 49b_{1100}b_{2000}, \\
B_4 =& 4a_{2000}\left(b_{2000} + 2c_{0110}\right) - 2a_{0200}\left(a_{1100} + 2b_{0200} - 4c_{0110}\right) \\
& - c_{1100}\left(a_{1010} + 11b_{0110} - 8c_{0020}\right) - 10c_{2000}\left(a_{0110} + b_{1010}\right) \\
& - 2a_{1100}a_{2000} + 2\left(b_{0200} + b_{2000}\right)\left(b_{1100} - 4c_{1010}\right), \\
Q_1 =& \left(a_{1010} + b_{0110}\right)\left(c_{2000}\left(48c_{2000}\left(a_{1010} + b_{0110}\right) + B_3\right) - B_2c_{1100}\right) \\
& + 3B_1(B_1 + B_4), \\
Q_2 =& -2\left(a_{1010} + b_{0110}\right)\left(c_{1100}(\mu - 2\lambda) + 2\omega\left(c_{0,2,0,1} + c_{2,0,0,1}\right)\right) \\
& + B_1(\lambda + \mu) + B_4\lambda.
\end{aligned}
$$

2. Averaging Theory and Proof of Theorem 1

To find the periodic orbits bifurcating from a non-isolated zero–Hopf equibilbrium of the differential system (1) we use averaging theory. The averaging theory has a long history and for a modern exposition of this topic the reader is addressed to Murdock *et al.* [2007].

We are interested in the formulation of the averaging theory for systems with non-trivial unperturbed part, i.e., we consider the differential system

$$\dot{x}(t) = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{3}$$

where $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 1, 2$, and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable and D is an open subset of $\mathbb{R}^n$. There are several recent works developing and improving the averaging theory for this kind of systems, see for instance [Cândido *et al.*, 2016; Coll *et al.*, 2012; Llibre *et al.*, 2014; Giné *et al.*, 2016]. For the convenience of the reader we present here the second order averaging theory for system (3). Let $\Phi(\cdot, z) : [0, t_z] \to \mathbb{R}^n$ be the solution of the unperturbed system, $\dot{x}(t) = F_0(t, x)$, such that $\Phi(0, z) = z$. For $i = 1, 2$ we define the following averaged functions $g_i : D \to \mathbb{R}^n$ of order i as

$$g_i(z) = M(T,z)^{-1} \frac{y_i(T,z)}{i!}, \tag{4}$$

where

$$\begin{aligned} y_1(t,z) =& M(t,z) \int_0^t M(\tau,z)^{-1} F_1(\tau, \Phi(\tau,z)) d\tau, \\ y_2(t,z) =& M(t,z) \int_0^t M(\tau,z)^{-1} \left[2F_2(\tau,\Phi(\tau,z)) + \frac{dF_1}{dz}(\tau,\Phi(\tau,z)).y_1(\tau,z) \right. \\ &\left. + \frac{d^2F_0}{dx^2}(\tau,\Phi(\tau,z)) y_1(\tau,z) d\tau \right], \end{aligned}$$

and $M(t, z)$ is the fundamental matrix of the variational differential equation of the unperturbed system along the solution $\Phi(t, z)$, such that $M(0, z)$ is the identity matrix. The next theorem provides the second order averaging theory, for a proof see Llibre *et al.* [2014].

Theorem 2. *Assume the following conditions.*

(i) *There exists an open subset W of D such that for any $z \in \overline{W}$, $\Phi(t, z)$ is T-periodic in the variable t.*
(ii) *Assume that $g_1(z) \equiv 0$, and that there exists $a^* \in W$ with $g_2(a^*) = 0$ and $\det(Dg_2(a^*)) \neq 0$.*

Then for $|\varepsilon| \neq 0$ sufficiently small there exists a T-periodic solution $x(t, z(\varepsilon), \varepsilon)$ of system (3) such that $|z(\varepsilon) - a^*| = \mathcal{O}(\varepsilon)$.

Now we shall use Theorem 2 for proving our main result.

Proof. [Proof of Theorem 1] Assuming hypotheses (H_0) and (H_1'), we start by writing system (1) into the normal form for applying the averaging Theorem 2. We use the change of coordinates $U = R\cos\theta$, $V = R\sin\theta$ and $W = RZ$. Then we scale the system taking $R = \sqrt{\varepsilon} r$ and $Z = \sqrt{\varepsilon} z$ with $\varepsilon > 0$ a small parameter. These are exactly the same steps used to obtain the equation (2.8) of [Llibre & Xiao, 2014]. Taking θ as the new independent variable we obtain the following differential system

$$\left(\frac{dr}{d\theta}, \frac{dz}{d\theta} \right) = F_0(\theta, r, z) + \sqrt{\varepsilon} F_1(\theta, r, z) + \varepsilon F_2(\theta, r, z) + \mathcal{O}(\varepsilon^{\frac{3}{2}}), \tag{5}$$

where $F_i(\theta, r, z) = \left(F_i^1(\theta, r, z), F_i^2(\theta, r, z)\right)$ are $2\pi-$periodic in the variable θ for $i = 0, 1, 2$, and are given by

$$\begin{aligned} F_0^1(\theta,r,z) =& 0, \\ F_0^2(\theta,r,z) =& \frac{r}{\omega} \left(c_{0200} \sin^2\theta + \cos\theta \left(c_{1100} \sin\theta + c_{2000} \cos\theta \right) \right), \\ F_1^1(\theta,r,z) =& \frac{r^2}{\omega} \left(a_{2000} \cos^3\theta + a_{1100} \sin\theta \cos^2\theta + a_{0200} \sin^2\theta \cos\theta \right. \\ &\left. + b_{0200} \sin^3\theta + b_{2000} \sin\theta \cos^2\theta + b_{1100} \sin^2\theta \cos\theta \right), \end{aligned}$$

$$
\begin{aligned}
F_1^2(\theta,r,z) =& \frac{r}{\omega}\Big(r\left(c_{2000}\cos(2\theta)+c_{1100}\sin(\theta)\cos(\theta)\right)\left(a_{0200}\sin^3(\theta)\right.\\
&-\cos(\theta)\left(-a_{1100}\sin^2(\theta)-a_{2000}\sin(\theta)\cos(\theta)\right.\\
&\left.\left.+b_{0200}\sin^2(\theta)+b_{2000}\cos^2(\theta)+b_{1100}\sin(\theta)\cos(\theta)\right)\right)\\
&-\omega z\left(a_{2000}\cos^3(\theta)+a_{1100}\sin(\theta)\cos^2(\theta)+a_{0200}\sin^2(\theta)\cos(\theta)\right.\\
&+b_{0200}\sin^3(\theta)+b_{2000}\sin(\theta)\cos^2(\theta)+b_{1100}\sin^2(\theta)\cos(\theta)\\
&\left.-c_{0110}\sin(\theta)-c_{1010}\cos(\theta)\right)\Big)\\
&\Big(c_{0200}\sin^2\theta+\cos\theta\left(c_{1100}\sin\theta+c_{2000}\cos\theta\right)\Big)\Big),\\
F_2^1(\theta,r,z) =& \frac{r}{\omega}\Big(\lambda+rz\left(a_{1010}\cos^2\theta+a_{0110}\sin\theta\cos\theta+\sin\theta\right.\\
&\left.(b_{0110}\sin\theta+b_{1010}\cos\theta)\right)+\frac{r^2}{\omega}\Big(\sin\theta\cos\theta(\sin\theta(a_{1100}\\
&-b_{0200})+\cos\theta\left(a_{2000}-b_{1100}\right))+a_{0200}\sin^3\theta-b_{2000}\cos^3\theta\Big)\\
&\Big(a_{2000}\cos^3\theta+a_{1100}\sin\theta\cos^2\theta+a_{0200}\sin^2\theta\cos\theta\\
&+b_{0200}\sin^3\theta+b_{2000}\sin\theta\cos^2\theta+b_{1100}\sin^2\theta\cos\theta\Big)\Big),\\
F_2^2(\theta,r,z) =& \frac{z(\mu-\lambda)}{\omega}+\frac{r^3}{\omega^2}(\sin\theta\cos\theta\left(\sin\theta(a_{1100}-b_{0200})+\cos\theta\right.\\
&\left.(a_{2000}-b_{1100})\right)+a_{0200}\sin^3\theta-b_{2000}\cos^3\theta)^2(c_{2000}\cos(2\theta)\\
&+c_{1100}\sin\theta\cos\theta)+\frac{r^2z}{\omega^2}((\sin\theta\cos\theta\left(\sin\theta\left(b_{0200}-a_{1100}\right)\right.\\
&\left.+\cos\theta\left(b_{1100}-a_{2000}\right)\right)-a_{0200}\sin^3\theta+b_{2000}\cos^3\theta)\\
&(a_{2000}\cos^3\theta+a_{1100}\sin\theta\cos^2\theta+a_{0200}\sin^2\theta\cos\theta\\
&+b_{0200}\sin^3\theta+b_{2000}\sin\theta\cos^2\theta+b_{1100}\sin^2\theta\cos\theta-c_{0110}\sin\theta\\
&-c_{1010}\cos\theta)+a_{0110}\sin^2\theta\left(c_{2000}\cos(2\theta)+c_{1100}\sin\theta\cos\theta\right)\\
&+a_{1010}\sin\theta\cos\theta\left(c_{2000}\cos(2\theta)+c_{1100}\sin\theta\cos\theta\right)\\
&-b_{1010}\cos^2\theta\left(c_{2000}\cos(2\theta)+c_{1100}\sin\theta\cos\theta\right)\\
&-b_{0110}\sin\theta\cos\theta\left(c_{2000}\cos(2\theta)+c_{1100}\sin\theta\cos\theta\right))\\
&-\frac{rz^2}{\omega}\Big((a_{1010}\cos^2\theta+a_{0110}\sin\theta\cos\theta+\sin\theta(b_{0110}\sin\theta\\
&+b_{1010}\cos\theta))+\frac{c_{0201}\sin^2\theta+\cos\theta\left(c_{1101}\sin\theta+c_{2001}\cos\theta\right)}{\omega}\Big).
\end{aligned}
$$

Consider the unperturbed system corresponding to (5), i. e.

$$
\begin{aligned}
\frac{dr}{d\theta} =& 0,\\
\frac{dz}{d\theta} =& \frac{r}{\omega}\left(c_{0200}\sin^2\theta+\cos\theta\left(c_{1100}\sin\theta+c_{2000}\cos\theta\right)\right).
\end{aligned}
$$

Taking $(\overline{r},\overline{z})\in\mathbb{R}^+\times\mathbb{R}$ as initial conditions, the unperturbed system has the solution

$$
\Phi(\theta,\overline{r},\overline{z}) = \left(\overline{r},\ \frac{\overline{r}c_{1100}\sin^2\theta+\overline{r}c_{2000}\sin(2\theta)+2\omega\overline{z}}{2\omega}\right).
$$

Thus the solution $\phi(\theta,\overline{r},\overline{z})$ is $2\pi-$periodic for all $(\overline{r},\overline{z})\in\mathbb{R}^+\times\mathbb{R}$, then system (5) satisfies the hypotheses (i) of Theorem 2. Furthermore, the fundamental matrix associated to the variational equation of the solution

$\phi(\theta, \overline{r}, \overline{z})$ is

$$M(\theta) = \begin{pmatrix} 1 & 0 \\ \dfrac{c_{1100} - c_{1100}\cos^2\theta + c_{2000}\sin(2\theta)}{2\omega} & 1 \end{pmatrix}.$$

Thus using (4) the averaged functions of system (5) are $g_1(\overline{r}, \overline{z}) = (0, 0)$ and

$$\begin{aligned} g_2(\overline{r}, \overline{z}) = \Bigg(& \frac{2\pi\overline{r}^2\overline{z}\,(a_{1010} + b_{0110})}{\omega} + \frac{\pi B_1 \overline{r}^3}{4\omega^2} + \frac{4\pi\lambda\overline{r}}{\omega}, \frac{\pi\overline{r}^3}{96\omega}(B_2 c_{1100} \\ & - B_3 c_{2000} - 48 c_{2000}^2\,(a_{1010} + b_{0110})\,) - \overline{r}\left(\frac{2\pi\overline{z}^2\,(a_{1010} + b_{0110})}{\omega}\right. \\ & \left. - \frac{\pi\left((\mu - 2\lambda)c_{1100} + 2\omega\,(c_{0201} + c_{2001})\right)}{\omega^2}\right) + \frac{\pi B_4 \overline{r}^2 \overline{z}}{4\omega^2} + \frac{4\pi\overline{z}(\mu - \lambda)}{\omega}\Bigg). \end{aligned} \tag{6}$$

Now we have to find the simple zeros of the system

$$g_2(\overline{r}, \overline{z}) = (0, 0). \tag{7}$$

We are interest only in the solutions $(r_0, z_0) \in \mathbb{R}^2$ such that $g_2(r_0, z_0) = 0$ and $r_0 > 0$. We divide the study of these solutions in the following cases:

Case 1: $a_{1010} + b_{0110} = 0$. If $\lambda B_1 < 0$ and $\lambda(B_1 + B_4) \neq B_1\mu$, then system (7) has the solution

$$\begin{aligned} (r_0, z_0) = \Bigg(4\sqrt{\frac{-\lambda\omega}{B_1}}, & \frac{r_0}{6\omega(B_1\mu - \lambda(B_1 + B_4))}\Big(c_{1100}(6B_1\mu - \lambda(12B_1 + B_3)) \\ & + 12B_1\omega c_{0201} + 12B_1\omega c_{2001} + B_2\lambda c_{2000}\Big)\Bigg). \end{aligned}$$

Furthermore we have

$$\det\left(Dg_2(r_0, z_0)\right) = \frac{32\pi^2\lambda((B_1 + B_4)\lambda - \mu B_1)}{\omega^2 B_1}.$$

Consequently (r_0, z_0) is a simple zero of system (7).

Case 2: $a_{1010} + b_{0110} \neq 0$. Solving the first equation of system (7) with respect to $\overline{z}$ and $\overline{r} > 0$ we have that $\overline{z} = -(B_1\overline{r}^2 + 16\lambda\omega)/(8\overline{r}\omega(a_{1010} + b_{0110}))$. Eliminating $\overline{z}$ in the system (7) we obtain the polynomial,

$$P(r) = Q_1\overline{r}^4 + 48Q_2\overline{r}^2\omega + 768\lambda\mu\omega^2. \tag{8}$$

The bi-quadratic polynomial (8) may have one or two real positive roots. Thus we use its discriminant

$$\mathbf{D} : 7247757312 Q_1\lambda\mu\left(3Q_2^2 - 4Q_1\lambda\mu\right)^2\omega^6, \tag{9}$$

to study such roots and then verify when them provide simple zeros of system (7). The discriminant vanishes if $Q_1\lambda\mu = 0$ or $3Q_2^2 - 4Q_1\lambda\mu = 0$, in this case one can verify the following subcases

Subcase 2.1: If $\lambda\mu = 0$ and $Q_1 Q_2 < 0$, system (7) has the zero

$$(r_1, z_1) = \left(4\sqrt{\frac{-3\omega Q_2}{Q_1}}, \frac{B_1 r_1^2}{8\omega r_1(a_{1010} + b_{0110})}\right).$$

It is a simple zero because $\det\left(Dg_2(r_1, z_1)\right) = \dfrac{96\pi^2 Q_2^2}{Q_1\omega^2} \neq 0$.

Subcase 2.2: If $Q_1 = 0$ and $Q_2\lambda\mu < 0$, system (7) has the zero

$$(r_2, z_2) = \left(4\sqrt{\frac{-\lambda\mu\omega}{Q_2}}, \frac{B_1 r_2^2 + 16\lambda\omega}{8\omega r_2(a_{1010} + b_{0110})}\right).$$

It is also a simple zero since $\det\left(Dg_2(r_1,z_1)\right) = -\dfrac{32\pi^2\lambda\mu}{\omega^2} \neq 0$.

Subcase 2.3: If $3Q_2^2 - 4Q_1\lambda\mu = 0$, system (7) has no simple zeros.

When the discriminant (9) is non-zero the polynomial (8) has 4 distinct solutions. These solutions may present zero, one or two positive real values for $\overline{r}$. Then assuming $Q_1\lambda\mu\left(3Q_2^2 - 4Q_1\lambda\mu\right) \neq 0$ we have the following subcases.

Subcase 2.4: If $Q_2 < 0$, $\lambda\mu > 0$ and $0 < Q_1 < 3Q_2^2/(4\lambda\mu)$; or $Q_2 > 0$, $\lambda\mu < 0$ and $Q_2^2/(4\lambda\mu) < Q_1 < 0$ system (7) has two zeros

$$(r_3, z_3) = \left(2\sqrt{\frac{-\omega\left(6Q_2 + 2\sqrt{9Q_2^2 - 12Q_1\lambda\mu}\right)}{Q_1}}, \frac{B_1 r_3^2 + 16\lambda\omega}{8\omega r_3(a_{1010} + b_{0110})}\right),$$

$$(r_4, z_4) = \left(2\sqrt{\frac{\omega\left(6Q_2 - 2\sqrt{9Q_2^2 - 12Q_1\lambda\mu}\right)}{Q_1}}, \frac{B_1 r_4^2 + 16\lambda\omega}{8\omega r_4(a_{1010} + b_{0110})}\right).$$

Furthermore for $i = 3, 4$ the Jacobian determinant of system (7) is

$$\det\left(Dg_2(r_i, z_i)\right) = -\frac{64\pi^2\lambda\mu}{\omega^2} - \frac{2\pi^2 Q_2 r_i^2}{\omega^3} \neq 0,$$

consequently these zeros are simple.

Subcase 2.5: If $Q_2 = 0$, $\mu\lambda > 0$ and $Q_1 < 0$ system (7) has the zero

$$(r_5, z_5) = \left(2\sqrt{\frac{-\omega\left(2\sqrt{-12Q_1\lambda\mu}\right)}{Q_1}}, \frac{B_1 r_5^2 + 16\lambda\omega}{8\omega r_5(a_{1010} + b_{0110})}\right),$$

and its Jacobian determinant is $\det\left(Dg_2(r_i, z_i)\right) = -\dfrac{64\pi^2\lambda\mu}{\omega^2} \neq 0$.

Subcase 2.6: If $Q_2 \neq 0$, $\mu\lambda > 0$ and $Q_1 < 0$ system (7) has the simple zero

$$(r_6, z_6) = \left(2\sqrt{\frac{\omega\left(6Q_2 - 2\sqrt{9Q_2^2 - 12Q_1\lambda\mu}\right)}{Q_1}}, \frac{B_1 r_6^2 + 16\lambda\omega}{8\omega r_6(a_{1010} + b_{0110})}\right),$$

Subcase 2.7: If $Q_2 \neq 0$, $\mu\lambda < 0$ and $Q_1 > 0$ system (7) has the simple zero

$$(r_7, z_7) = \left(2\sqrt{\frac{\omega\left(-6Q_2 - 2\sqrt{9Q_2^2 - 12Q_1\lambda\mu}\right)}{Q_1}}, \frac{B_1 r_7^2 + 16\lambda\omega}{8\omega r_7(a_{1010} + b_{0110})}\right),$$

for $i = 6, 7$ the Jacobian determinant is $\det\left(Dg_2(r_i, z_i)\right) = -\dfrac{64\pi^2\lambda\mu}{\omega^2} - \dfrac{2\pi^2 Q_2 r_i^2}{\omega^3} \neq 0$.

The result follows by applying Theorem 2 for each $a_1^* = (r_i, z_i)$ with $i = 0, 1, \ldots, 7$. This completes the proof of Theorem 1. ■

3. Applications

The existence of differential systems with only zero, one or fewer than n equilibrium points generating $n-$*scroll* chaotic attractors is an important open problem whose solution is not easy. For more information about $n-$ *scroll* chaotic attractors see Lü & Chen [2006]. This type of system has several real word applications, for instance in engineering and secure communication. Elhadj & Sprott [2013] have shown that the simplest family of systems displaying n–scroll chaotic attractors is given by the quadratic polynomial differential system

$$\begin{aligned}\dot{x} =&a_1x + a_2y + a_3z + a_9yz,\\ \dot{y} =&b_1x + b_2y + b_3z + b_8xz + b_9yz,\\ \dot{z} =&c_1x + c_2y + c_3z + c_7xy + c_8xz + c_9yz.\end{aligned} \tag{10}$$

Using Theorem 1 we can find conditions in order that the differential system (10) has two periodic orbits.

Theorem 3. *Consider system* (10) *with the coefficients*

$$\begin{array}{lll} a_1 = \varepsilon^2, & a_2 = \dfrac{\varepsilon(\varepsilon-\mu)k_1}{k_3}, & a_3 = -k_1,\\ a_9 = -k_2, & b_1 = \dfrac{\varepsilon(\varepsilon-\mu)k_3}{k_1}, & b_2 = \varepsilon^2,\\ b_3 = -k_3, & b_8 = \varepsilon^2, & b_9 = -\dfrac{k_2(3239k_1^2+5616k_3^2)}{4992k_1k_3},\\ c_1 = \dfrac{\varepsilon^4+\omega^2}{k_1}, & c_2 = \dfrac{\varepsilon^2(\varepsilon+25\mu)(3\varepsilon+25\mu)}{k_3}, & c_3 = -\varepsilon(2\varepsilon+51\mu),\\ c_7 = -\dfrac{79\omega^2k_2}{24k_3^2}, & c_8 = \dfrac{k_1k_4+k_3k_5}{k_1}, & c_9 = -k_5. \end{array}$$

Assume that $\omega > 0$, $\mu > 0$ and $k_i > 0$ for all $i = 1, \ldots, 5$. Then for $\varepsilon > 0$ sufficiently small system (10) *has two periodic orbits bifurcating from the zero-Hopf equilibrium point localized at the origin.*

Furthermore we also shall use Theorem 1 for proving the existence of ε-families of periodic orbits in the following two differential systems

$$\begin{aligned}\dot{x} =&a(y-x) + dxz,\\ \dot{y} =&\rho x - xz + fy,\\ \dot{z} =&-ex^2 + xy + cz,\end{aligned} \tag{11}$$

$$\begin{aligned}\dot{x} =&a(y-x) + dxz,\\ \dot{y} =&(f_0-a)x - xz + f_1y,\\ \dot{z} =&-ex^2 + xy + cz + m,\end{aligned} \tag{12}$$

System (11) was derived from the classical Lorenz system by Li [2008]. This system exhibits a three–scroll chaotic attractor, with two scrolls symmetric with respect to the $z-$axis as in the Lorenz attractor, and the third scroll is around the $z-$axis. System (12) were provided by Pan *et al.* [2010] with $f_0 = f_1$. It was derived from the Chen system and also presents a three–scroll chaotic attractor. The authors show that the parameter m works as a control parameter that can dramatically change the dynamics of the system. Theorem 5 will reveal that the parameter m is also important for the existence of periodic orbits in system (12).

In the next results we used Theorem 1 to provide sufficient conditions for the existence of periodic orbits in systems (11) and (12) respectively.

Theorem 4. *Let $c = \varepsilon\mu$, $f = a - 2\varepsilon\lambda$, $\rho = -\left(\omega^2 + (a-\varepsilon\lambda)^2\right)/a$ and $e = 1 + \varepsilon\lambda/a$. If $ad\mu(\mu - 4\lambda)\lambda > 0$ then system* (11) *has a family of periodic solutions bifurcating from the zero–Hopf equilibrium localized at the origin.*

Theorem 5. *Let $m = \varepsilon m_1$, $c = -\varepsilon\mu$, $d = (a-f_1-2\varepsilon\lambda)\mu/m_1$ and $f_0 = (m_1 a+a^2\mu-(f_1+\varepsilon\lambda)^2\mu-\mu\omega^2)/(a\mu)$. If $a(f_1-a)m_1\lambda(8\lambda-\mu) < 0$, then system (12) has a family of periodic solutions bifurcating from the zero-Hopf equilibrium $p_0 = \left(0, 0, \dfrac{m_1}{\mu}\right)$.*

3.1. *Proofs of Theorems 3, 4 and 5*

Proof. [Proof of Theorem 3] Using the linear change of variables

$$
\begin{aligned}
U =& \frac{zk_1\varepsilon^2\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)}{k_3\left(\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2-\omega^2\right)}+x,\\
V =& \frac{xk_3}{k_1}+z,\\
W =& \frac{x\varepsilon(25\mu+2\varepsilon)+y\omega}{k_3}+\frac{z\varepsilon(\varepsilon-\mu)\left(2\varepsilon^4+100\mu\varepsilon^3+625\mu^2\varepsilon^2-\omega^2\right)}{k_3\left(\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2-\omega^2\right)},
\end{aligned}
$$

system (10) becomes

$$
\begin{aligned}
\dot U =& -\varepsilon 26\mu U-\omega V+\sum_{i+j+k=2} a_{ijk}(\varepsilon)U^iV^jW^k,\\
\dot V =& \omega U-\varepsilon 26\mu V+\sum_{i+j+k=2} b_{ijk}(\varepsilon)U^iV^jW^k,\\
\dot W =& \varepsilon\mu W+\sum_{i+j+k=2} c_{ijk}(\varepsilon)U^iV^jW^k.
\end{aligned}
\tag{13}
$$

Where the functions $a_{ijk}(\varepsilon)$, $b_{ijk}(\varepsilon)$ and $c_{ijk}(\varepsilon)$ are defined in the appendix. We note that when $\varepsilon = 0$ the origin of system (13) is a non-isolated zero–Hopf equilibrium point. Since

$$
c_{1100} = -\frac{\omega\left(3239k_1^2k_2+624k_2k_3^2\right)}{4992k_1^3} \neq 0,
$$

system (13) does not satisfy the hypothesis (H_1) of [Llibre & Xiao, 2014, Theorem 2.4]. However, we have $a_{0020} = b_{0020} = c_{0020} = 0$ and $c_{2000} = -c_{0200} = 0$ thus we can apply Theorem 1. From the coefficients of system (13) we have $\lambda = -26\mu$ and

$$
\begin{aligned}
a_{1010} =& 0,\\
b_{0110} =& -k_5,\\
Q_1 =& -\frac{395k_2^2k_4k_5\omega^2\left(3239k_1^2+624k_3^2\right)}{1038336k_1^3k_3},\\
Q_2 =& \frac{k_2\mu\omega\left(32864k_1^3k_4+16195k_1^2k_3k_5+3120k_3^3k_5\right)}{4992k_1^3k_3},\\
3Q_2^2-4\lambda\mu Q_1 =& \frac{32448k_3^2Q_1^2\mu^2}{6241k_2^2k_4^2\omega^2}+\frac{6241k_2^2k_4^2\omega^2\mu^2}{192k_3^2}.
\end{aligned}
\tag{14}
$$

By equations (14) we have that $a_{1010}-b_{0110} \neq 0$ and $3Q_2^2-4\mu\lambda Q_1 > 0$. Furthermore if $\mu > 0$ we have $Q_2 > 0$, $\lambda\mu = -26\mu^2 < 0$ and $3Q_2^2/(4\lambda\mu) < Q_1 < 0$, then system (10) satisfies statement (h) of Theorem 1. Then the theorem is proved. ■

Proof. [Proof of Theorem 4] Under the hypothesis of Theorem 4 system (11) has three equilibrium points. Mainly, the origin and $p_\pm = (\pm x_\varepsilon, \pm y_\varepsilon, z_\varepsilon)$ where

$$
x_\varepsilon = \sqrt{\frac{a\mu\varepsilon(\omega^2+\varepsilon^2\lambda^2)}{d\omega-\lambda\varepsilon(a+ad+d\varepsilon\lambda)}},
$$

$$
\begin{aligned}
y_\varepsilon =& x_\varepsilon \frac{a^2(d+1)+2ad\lambda\varepsilon+d\left(\omega^2+\lambda^2\varepsilon^2\right)}{a(ad+a+2d\lambda\varepsilon)},\\
z_\varepsilon =& \frac{\omega^2+\lambda^2\varepsilon^2}{ad+a+2d\lambda\varepsilon}.
\end{aligned}
$$

The eigenvalues at the origin are $\varepsilon\mu$, $\varepsilon\lambda - i\omega$ and $\varepsilon\lambda + i\omega$. Applying the change of variables $(x,y,z) = (U, (Ua - U\varepsilon\lambda - V\omega)/\omega, W)$ the differential system (11) becomes

$$
\begin{aligned}
\dot{U} =& \varepsilon\lambda U - \omega V + a_{101}(\varepsilon)UW,\\
\dot{V} =& \omega U + \varepsilon\lambda V + b_{101}(\varepsilon)UW,\\
\dot{W} =& \varepsilon\mu W + c_{110}(\varepsilon)UV,
\end{aligned}
\tag{15}
$$

where $a_{101}(\varepsilon) = d$, $b_{101}(\varepsilon) = \dfrac{a+ad+\varepsilon d\lambda}{\omega}$ and $c_{110}(\varepsilon) = \dfrac{\omega}{a}$.

We note that when $\varepsilon = 0$ the origin of system (15) is a non-isolated zero–Hopf equilibrium point. Consequently we have $a_{1010} - b_{0110} = d$, $Q_1 = 0$ and $Q_2 = d\omega(\mu - 4\lambda)/a$. Thus by hypotheses system (15) satisfies the conditions (c) of Theorem 1. Hence the theorem is proved. ■

Proof. [Proof of Theorem 5] The point $p_0 = (0, 0, m_1/\mu)$ is an equilibrium of system (12). We translate p_0 to the origin doing the following change of variables

$$
(x,y,z) = \left(U, \frac{f_1 U + \varepsilon\lambda U - \omega V}{a}, W + \frac{m_1}{\mu}\right).
$$

$$
\begin{aligned}
\dot{U} =& \varepsilon\lambda U - \omega V + a_{101}(\varepsilon)UW,\\
\dot{V} =& \omega U + \varepsilon\lambda V + b_{101}(\varepsilon)UW,\\
\dot{W} =& \varepsilon\mu W + c_{200}(\varepsilon)U^2 + c_{110}(\varepsilon)UV.
\end{aligned}
\tag{16}
$$

Where the functions (2) vanishing except for

$$
\begin{aligned}
a_{101}(\varepsilon) &= \frac{(a - f_1 - 2\varepsilon\lambda)}{m_1}, & b_{101}(\varepsilon) &= \frac{a\, m_1 + (a - f_1 - 2\varepsilon\lambda)(f_1 + \varepsilon\lambda)\mu}{m_1\omega},\\
c_{200}(\varepsilon) &= \varepsilon\frac{\lambda}{a}, & c_{110}(\varepsilon) &= \frac{\omega}{a}.
\end{aligned}
$$

We note that when $\varepsilon = 0$ the origin of system (16) is a non-isolated zero–Hopf equilibrium point. Thus we have

$$
a_{1010} - b_{0110} = \frac{\mu(a - f_1)}{m_1}, \qquad Q_1 = 0, \qquad Q_2 = \frac{(f_1 - a)(8\lambda - \mu)\mu\omega}{am_1}.
$$

Consequently system (15) satisfies the conditions (c) of Theorem 1 since we have by hypothesis that

$$
\lambda\mu Q_2 = \frac{\lambda(f_1 - a)(8\lambda - \mu)\mu^2\omega}{am_1} < 0.
$$

This concludes the proof of the theorem. ■

4. Appendix

Here we write the coefficients of system (13).

$$
a_{200}(\varepsilon) = \frac{\varepsilon(25\mu + 2\varepsilon)\left(k_1^2\varepsilon^2\left(4992\varepsilon^2 - 3239k_2\right)\left(677\mu^2 + \varepsilon^2 + 50\mu\varepsilon\right)\right)}{4992k_1^2k_3\left(\omega^2 + 729\mu^2\varepsilon^2\right)}
$$

$$
\begin{aligned}
&-\frac{\varepsilon(25\mu+2\varepsilon)\left(624k_2k_3^2\left(8\omega^2+\varepsilon^4+50\mu\varepsilon^3+6509\mu^2\varepsilon^2\right)\right)}{4992k_1^2k_3\left(\omega^2+729\mu^2\varepsilon^2\right)},\\
a_{020}(\varepsilon)=&0,\\
a_{002}(\varepsilon)=&\frac{\varepsilon(\varepsilon-\mu)\left(-\omega^2+2\varepsilon^4+100\mu\varepsilon^3+625\mu^2\varepsilon^2\right)}{4992k_3^3\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2\right)^2\left(\omega^2+729\mu^2\varepsilon^2\right)}\\
&\times\Big(k_1^2\varepsilon^2\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)\left(4992\varepsilon^4\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)\right.\\
&\left.-3239k_2\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2\right)\right)-624k_2k_3^2(-8\omega^4\\
&+\varepsilon^8+100\mu\varepsilon^7+8957\mu^2\varepsilon^6+322850\mu^3\varepsilon^5+\varepsilon^4\left(7\omega^2-338468\mu^4\right)\\
&+350\mu\omega^2\varepsilon^3-6925\mu^2\omega^2\varepsilon^2)\Big),\\
a_{110}(\varepsilon)=&-\frac{\omega}{4992k_1^2k_3\left(\omega^2+729\mu^2\varepsilon^2\right)}\left(k_1^2\varepsilon^2\left(3239k_2-4992\varepsilon^2\right)\right.\\
&\left.\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)+624k_2k_3^2\left(8\omega^2+\varepsilon^4+50\mu\varepsilon^3+6509\mu^2\varepsilon^2\right)\right),\\
a_{101}(\varepsilon)=&\frac{\varepsilon}{4992k_1k_3^2\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2\right)\left(\omega^2+729\mu^2\varepsilon^2\right)}\\
&\left(k_1^2\varepsilon^2\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)\left(3239k_2\left(3\omega^2(8\mu+\varepsilon)\right.\right.\right.\\
&\left.\left.-\varepsilon^2\left(-1925\mu^3+4\varepsilon^3+223\mu\varepsilon^2+1671\mu^2\varepsilon\right)\right)+4992\left(\omega^2\varepsilon^2(\mu-\varepsilon)\right.\right.\\
&\left.\left.\left.+\varepsilon^4\left(16300\mu^3+4\varepsilon^3+223\mu\varepsilon^2+3129\mu^2\varepsilon\right)\right)\right)-624k_2k_3^2\left(8\omega^2\right.\\
&\left.+\varepsilon^4+50\mu\varepsilon^3+6509\mu^2\varepsilon^2\right)\left(\varepsilon^2\left(-1925\mu^3+4\varepsilon^3+223\mu\varepsilon^2+1671\mu^2\varepsilon\right)\right.\\
&\left.\left.-3\omega^2(8\mu+\varepsilon)\right)\right),\\
a_{011}(\varepsilon)=&\frac{\omega}{4992k_1}\left(\frac{\varepsilon^2\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)\left(k_1^2\left(4992\varepsilon^2-3239k_2\right)-624k_2k_3^2\right)}{k_3^2\left(\omega^2+729\mu^2\varepsilon^2\right)}\right.\\
&\left.+\frac{4992k_1^2\varepsilon^4\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)}{k_3^2\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2\right)}-4992k_2\right)\\
b_{200}(\varepsilon)=&\frac{1}{4992k_1^2k_3\omega(729\varepsilon^2\mu^2+\omega^2)}\Big(k_1^2\left(79k_2\left(-208\omega^4+\omega^2\varepsilon^2\left(-152657\mu^2\right.\right.\right.\\
&\left.\left.+82\varepsilon^2+943\mu\varepsilon\right)+1107\mu\varepsilon^4(25\mu+2\varepsilon)\left(650\mu^2+\varepsilon^2+77\mu\varepsilon\right)\right)\\
&+4992\varepsilon(25\mu+2\varepsilon)\left(\omega^2\left(k_3k_4+\varepsilon^3(\mu-\varepsilon)\right)-27\mu\varepsilon^2\left(-27k_3k_4\mu+\varepsilon^5\right.\right.\\
&\left.\left.\left.+77\mu\varepsilon^4+650\mu^2\varepsilon^3\right)\right)\right)+624k_2k_3^2\varepsilon^2(25\mu+2\varepsilon)\left(27\mu\varepsilon^2\left(6050\mu^2+\varepsilon^2\right.\right.\\
&\left.\left.+509\mu\varepsilon\right)+\omega^2(199\mu+17\varepsilon)\right)\Big),\\
b_{020}(\varepsilon)=&0,\\
b_{002}(\varepsilon)=&\frac{\varepsilon}{4992k_3^3\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2\right)^2\left(\omega^3+729\mu^2\omega\varepsilon^2\right)}\\
&\Big(k_1^2\varepsilon\left(79k_2\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2\right)\left(-\omega^4\left(140857\mu^2+249\varepsilon^2\right.\right.\right.\\
&\left.+10318\mu\varepsilon\right)+1107\mu\varepsilon^4(\varepsilon-\mu)\left(625\mu^2+2\varepsilon^2+100\mu\varepsilon\right)\left(650\mu^2+\varepsilon^2\right.\\
&\left.+77\mu\varepsilon\right)+\omega^2\varepsilon^2\left(-101909689\mu^4+82\varepsilon^4+2829\mu\varepsilon^3-218257\mu^2\varepsilon^2\right.\\
&\left.\left.-8263061\mu^3\varepsilon\right)\right)-4992\varepsilon(\varepsilon-\mu)\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)\left(-\omega^2+2\varepsilon^4\right.\\
&\left.+100\mu\varepsilon^3+625\mu^2\varepsilon^2\right)\left(\omega^2\left(-k_3k_4+\varepsilon^4-\mu\varepsilon^3\right)+27\mu\varepsilon^2\left(-27k_3k_4\mu\right.\right.\\
&\left.\left.\left.+\varepsilon^5+77\mu\varepsilon^4+650\mu^2\varepsilon^3\right)\right)\right)+4992k_1k_3^2k_5(\varepsilon-\mu)\left(-\omega^2+2\varepsilon^4\right.
\end{aligned}
$$

$$+100\mu\varepsilon^3+625\mu^2\varepsilon^2\right)\left(\omega^2+729\mu^2\varepsilon^2\right)^2+624k_2k_3^2\varepsilon(\varepsilon-\mu)\left(-\omega^2\right.$$
$$\left.+2\varepsilon^4+100\mu\varepsilon^3+625\mu^2\varepsilon^2\right)\left(\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2-\omega^2\right)$$
$$\left(27\mu\varepsilon^2\left(6050\mu^2+\varepsilon^2+509\mu\varepsilon\right)+\omega^2(199\mu+17\varepsilon)\right)\Big)\Big),$$

$$\begin{aligned}
b_{110}(\varepsilon) =&\frac{k_1^2}{4992k_1^2k_3\left(\omega^2+729\mu^2\varepsilon^2\right)}\left(27\mu\varepsilon^2\left(\varepsilon\left(3239k_2-4992\varepsilon^2\right)\left(650\mu^2+\varepsilon^2\right.\right.\right.\\
&\left.+77\mu\varepsilon\right)+134784k_3k_4\mu\Big)+\omega^2\left(\varepsilon\left(3239k_2-4992\varepsilon^2\right)(\varepsilon-\mu)\right.\\
&\left.+4992k_3k_4\right)\Big)+624k_2k_3^2\varepsilon\left(27\mu\varepsilon^2\left(6050\mu^2+\varepsilon^2+509\mu\varepsilon\right)\right.\\
&\left.+\omega^2(199\mu+17\varepsilon)\right),\\
b_{101}(\varepsilon) =&\frac{k_1^2}{4992k_1k_3^2\omega\left(\omega^2+729\mu^2\varepsilon^2\right)\left(\omega^2-\varepsilon^2\left(-52\mu^2+\varepsilon^2+50\mu\varepsilon\right)\right)}\\
&\left(79k_2\left(-208\omega^6+\omega^4\varepsilon^2\left(-22616\mu^2+539\varepsilon^2+21661\mu\varepsilon\right)\right.\right.\\
&-1107\mu\varepsilon^6\left(650\mu^2+\varepsilon^2+77\mu\varepsilon\right)\left(-1925\mu^3+4\varepsilon^3+223\mu\varepsilon^2\right.\\
&\left.+1671\mu^2\varepsilon\right)+\omega^2\varepsilon^4\left(111960275\mu^4-164\varepsilon^4-5658\mu\varepsilon^3+526181\mu^2\varepsilon^2\right.\\
&\left.\left.+19515022\mu^3\varepsilon\right)\right)+4992\varepsilon\left(\mu\omega^2+4\varepsilon^5+223\mu\varepsilon^4+3129\mu^2\varepsilon^3\right.\\
&\left.+16300\mu^3\varepsilon^2-\omega^2\varepsilon\right)\left(\omega^2\left(-k_3k_4+\varepsilon^4-\mu\varepsilon^3\right)+27\mu\varepsilon^2\left(-27k_3k_4\mu\right.\right.\\
&\left.\left.\left.+\varepsilon^5+77\mu\varepsilon^4+650\mu^2\varepsilon^3\right)\right)\right)-4992k_1k_3^2k_5\varepsilon(25\mu+2\varepsilon)\\
&\left(\omega^2+729\mu^2\varepsilon^2\right)^2-624k_2k_3^2\varepsilon^2\left(\varepsilon^2\left(-1925\mu^3+4\varepsilon^3+223\mu\varepsilon^2\right.\right.\\
&\left.\left.+1671\mu^2\varepsilon\right)-3\omega^2(8\mu+\varepsilon)\right)\left(27\mu\varepsilon^2\left(6050\mu^2+\varepsilon^2+509\mu\varepsilon\right)\right.\\
&\left.+\omega^2(199\mu+17\varepsilon)\right),\\
b_{011}(\varepsilon) =&0,\\
c_{200}(\varepsilon) =&\frac{\varepsilon(25\mu+2\varepsilon)}{4992k_1^3\left(\omega^2+729\mu^2\varepsilon^2\right)}\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2\right)\\
&\left(k_1^2\left(3239k_2-4992\varepsilon^2\right)+624k_2k_3^2\right),\\
c_{020}(\varepsilon) =&0,\\
c_{002}(\varepsilon) =&-\frac{\varepsilon(\varepsilon-\mu)\left(-\omega^2+2\varepsilon^4+100\mu\varepsilon^3+625\mu^2\varepsilon^2\right)}{4992k_1k_3^2\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2\right)\left(\omega^2+729\mu^2\varepsilon^2\right)}\\
&\left(k_1^2\left(4992\varepsilon^4\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)-3239k_2\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3\right.\right.\right.\\
&\left.\left.-52\mu^2\varepsilon^2\right)\right)+624k_2k_3^2\left(\omega^2-\varepsilon^2\left(-52\mu^2+\varepsilon^2+50\mu\varepsilon\right)\right)\Big),\\
c_{110}(\varepsilon) =&\frac{\omega\left(-\omega^2+\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2\right)}{4992k_1^3\left(\omega^2+729\mu^2\varepsilon^2\right)}\\
&\left(k_1^2\left(3239k_2-4992\varepsilon^2\right)+624k_2k_3^2\right),\\
c_{101}(\varepsilon) =&\frac{1}{4992k_1^2k_3\left(\omega^2+729\mu^2\varepsilon^2\right)}\Big(k_1^2\varepsilon\left(3239k_2\left(\varepsilon^2\left(-1925\mu^3\right.\right.\right.\\
&\left.+4\varepsilon^3+223\mu\varepsilon^2+1671\mu^2\varepsilon\right)-3\omega^2(8\mu+\varepsilon)\Big)-4992\varepsilon^2\left(\mu\omega^2+4\varepsilon^5\right.\\
&\left.\left.+223\mu\varepsilon^4+3129\mu^2\varepsilon^3+16300\mu^3\varepsilon^2-\omega^2\varepsilon\right)\right)\\
&+624k_2k_3^2\varepsilon\left(\varepsilon^2\left(-1925\mu^3+4\varepsilon^3+223\mu\varepsilon^2+1671\mu^2\varepsilon\right)\right.\\
&\left.-3\omega^2(8\mu+\varepsilon)\right)\Big),\\
c_{011}(\varepsilon) =&-\frac{\omega}{4992k_1^2k_3\left(\omega^2+729\mu^2\varepsilon^2\right)}\left(k_1^2\left(4992\varepsilon^4\left(677\mu^2+\varepsilon^2+50\mu\varepsilon\right)\right.\right.
\end{aligned}$$

$$-3239k_2\left(\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2-\omega^2\right)\big)$$
$$-624k_2k_3^2\left(\varepsilon^4+50\mu\varepsilon^3-52\mu^2\varepsilon^2-\omega^2\right)\big).$$

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