

## AN INVERSE APPROACH TO THE CENTER PROBLEM

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ABSTRACT. We consider analytic or polynomial vector fields of the form

$$\mathcal{X} = (-y + X) \frac{\partial}{\partial x} + (x + Y) \frac{\partial}{\partial y},$$

where  $X = X(x, y)$  and  $Y = Y(x, y)$  start at least with terms of second order. It is well-known that  $\mathcal{X}$  has a center at the origin if and only if  $\mathcal{X}$  has a Liapunov-Poincaré local analytic first integral of the form  $H = \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j$ , where  $H_j = H_j(x, y)$  is a homogenous polynomial of degree  $j$ .

The classical center-focus problem already studied by Poincaré consists in distinguishing when the origin of  $\mathcal{X}$  is either a center or a focus. In this paper we study the inverse center problem, i.e. for a given analytic function  $H$  of the previous form defined in a neighborhood of the origin, we determine the analytic or polynomial vector field  $\mathcal{X}$  for which  $H$  is a first integral. Moreover, given an analytic function  $V = 1 + \sum_{j=1}^{\infty} V_j$  in a neighborhood of the origin, where  $V_j$  is a homogenous polynomial of degree  $j$ , we determine the analytic or polynomial vector field  $\mathcal{X}$  for which  $V$  is a Reeb inverse integrating factor.

We study the particular case of centers which have a local analytic first integral of the form  $H = \frac{1}{2}(x^2 + y^2) \left( 1 + \sum_{j=1}^{\infty} \Upsilon_j \right)$ , in a neighborhood of the origin, where  $\Upsilon_j$  is a homogenous polynomial of degree  $j$  for  $j \geq 1$ . These centers are called weak centers, they contain the uniform isochronous centers and the isochronous holomorphic centers, but they do not coincide with the class of isochronous centers.

We extended to analytic or polynomial differential systems the weak conditions of a center given by Alwash and Lloyd for linear centers with homogeneous polynomial nonlinearities. Furthermore the centers satisfying these weak conditions are weak centers. Finally as an application we obtain the necessary and sufficient conditions for the existence of a weak center in a class of polynomial differential systems of degree four.

### 1. INTRODUCTION

Let

$$(1) \quad \mathcal{X} = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y},$$

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be the real planar polynomial vector field associated to the real planar polynomial differential system

$$(2) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

where the dot denotes derivative with respect to an independent variable here called the time  $t$ , and  $P$  and  $Q$  are real coprime polynomials in  $\mathbb{R}[x, y]$ . We say that the polynomial differential system (2) has *degree*  $m = \max \{\deg P, \deg Q\}$ .

In what follows we assume that origin  $O := (0, 0)$  is a singular point, i.e.  $P(0, 0) = Q(0, 0) = 0$ .

The singular point  $O$  is a *center* if there exists an open neighborhood  $U$  of  $O$  where all the orbits contained in  $U \setminus \{O\}$  are periodic.

The study of the centers of analytical or polynomial differential systems (2) has a long history. The first works are due to Poincaré [23] and Dulac [11]. Later on were developed by Bendixson [4], Frommer [12], Liapunov [20] and many others.

Assume that the origin of the analytic or polynomial differential system (2) is a center. It is well-known that, after a linear change of variables and a constant scaling of the time variable (if necessary), system (2) can be written in one of the next three forms:

$$(3) \quad \begin{aligned} \dot{x} &= -y + X(x, y), & \dot{y} &= x + Y(x, y), \\ \dot{x} &= y + X(x, y), & \dot{y} &= Y(x, y), \\ \dot{x} &= X(x, y), & \dot{y} &= Y(x, y), \end{aligned}$$

where  $X(x, y)$  and  $Y(x, y)$  are analytic or polynomials without constant and linear terms defined in a neighborhood of the origin. Then the origin  $O$  of the analytical or polynomial differential system (2) is called *linear type*, *nilpotent* or *degenerate* if after a linear change of variables and a scaling of the time it can be written as the first, second and third system of (3), respectively.

In this paper we shall study the differential system of the linear type

$$(4) \quad \dot{x} = -y + X, \quad \dot{y} = x + Y,$$

where  $X = X(x, y)$  and  $Y = Y(x, y)$  are real analytic functions in an open neighborhood of  $O$  whose Taylor expansions at  $O$  do not contain constant and linear terms. For  $X, Y$  polynomials of a given degree, the *Poincaré center-focus problem* asks about conditions on the coefficients of  $X$  and  $Y$  under which  $O$  is a center.

In the study of the center-focus problem the following theorems play a very important role (see for instance [20, 23, 25])

**Theorem 1.** *For the analytic differential system (4) there exists a formal power series*

$$W = \sum_{n=2}^{\infty} W_n := \frac{1}{2}(x^2 + y^2) + \sum_{n=3}^{\infty} W_n(x, y),$$

where  $W_j = W_j(x, y)$  is a homogenous polynomial of degree  $j$  such that

$$\begin{aligned} \frac{dW}{dt} &= \left( x + \frac{\partial W_3}{\partial x} + \frac{\partial W_4}{\partial x} + \dots \right) (-y + X(x, y)) \\ &\quad + \left( y + \frac{\partial W_3}{\partial y} + \frac{\partial W_4}{\partial y} + \dots \right) (x + Y(x, y)) \\ &= \sum_{j=1}^{\infty} v_j (x^2 + y^2)^{j+1}, \end{aligned}$$

where  $v_j$  are the Poincaré-Liapunov constants.

Assume that the formal power series  $W$  converges. If the constants  $v_j = 0$  for  $j \in \mathbb{N}$  then there exists a first integral  $H := \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} W_j$ , and consequently the origin is a center. If there exists a first non-zero Liapunov constant  $v_j$ , then the origin is a stable focus if  $v_j < 0$  and unstable if  $v_j > 0$ .

Poincaré and Liapunov proved the next two results, see for instance [23, 20, 13, 26].

**Theorem 2.** *A planar polynomial differential system*

$$(5) \quad \dot{x} = -y + \sum_{j=2}^m X_j(x, y), \quad \dot{y} = x + \sum_{j=2}^m Y_j(x, y),$$

of degree  $m$  has a center at the origin if and only if it has a first integral of the form

$$(6) \quad H = \sum_{j=2}^{\infty} H_j(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j(x, y),$$

where  $X_j, Y_j$  and  $H_j$  are homogenous polynomials of degree  $j$ .

The analytic function (6) is called the Poincaré-Liapunov local first integral.

**Theorem 3.** *An analytic planar differential system*

$$(7) \quad \dot{x} = -y + \sum_{j=2}^{\infty} X_j(x, y), \quad \dot{y} = x + \sum_{j=2}^{\infty} Y_j(x, y),$$

has a center at the origin if and only if it has a first integral of the form (6).

Theorem 2 is due to Poincaré, and Theorem 3 is due to Liapunov.

From Theorems 1, 2 and 3 it is clear that an analytic or polynomial differential system (4) has a center at the origin if and only if the Poincaré-Liapunov constants  $v_k = 0$  for  $k \geq 1$  (*Poincaré's criterion*). Moreover, the  $v_k$ 's are polynomials over  $\mathbb{Q}$  in the coefficients of the polynomial differential system. A necessary and sufficient condition to have a center is then the annihilation of all these constants. In view of the Hilbert's basis theorem this occurs if and only if for a finite number of  $k$ ,  $k < j$  and  $j$  sufficiently large,  $v_k = 0$ . Unfortunately, trying to solve the center problem computing the Poincaré-Liapunov constants is in general not possible due to the huge computations.

Although we have an algorithm for computing the Poincaré-Liapunov constants for linear type center, we have no algorithm to determine how many of them need to be zero to imply that all of them are zero for cubic or higher degree polynomial differential systems. Bautin [2] showed in 1939 that for a quadratic polynomial differential system, to annihilate all  $v_k$ 's it suffices to have  $v_k = 0$  for  $i = 1, 2, 3$ . So the problem of the center is solved for quadratic systems. This problem was solved for the cubic differential systems with homogenous nonlinearities (see for instance [25, 31, 32]).

We recall the following definition. Let  $U$  be an open and dense set in  $\mathbb{R}^2$ . We say that a non-constant  $C^r$  with  $r \geq 1$  function  $F: U \rightarrow \mathbb{R}$  is a *first integral* of the analytic or polynomial vector field  $\mathcal{X}$  on  $U$ , if  $F(x(t), y(t))$  is constant for all values of  $t$  for which the solution  $(x(t), y(t))$  of  $\mathcal{X}$  is defined on  $U$ . Clearly  $F$  is a first integral of  $\mathcal{X}$  on  $U$  if and only if  $\mathcal{X}F = 0$  on  $U$ .

Now we shall introduce another criterion for solving the center problem due to Reeb.

We need the following definitions and notions. A function  $V = V(x, y)$  is an *inverse inverse integrating factor* of system (2) in an open subset  $U \subset \mathbb{R}^2$  if  $V \in C^1(U)$ ,  $V \neq 0$  in  $U$  and

$$\frac{\partial \left( \frac{P}{V} \right)}{\partial x} + \frac{\partial \left( \frac{Q}{V} \right)}{\partial y} = 0 \iff P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y} = V \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right).$$

The first integral  $F$  associated to the inverse inverse integrating factor  $V$  is given by the line integral or path integral

$$F(x, y) = \int_{\gamma} \left( -\frac{P}{V} dy + \frac{Q}{V} dx \right),$$

We note that  $\{V = 0\}$  is formed by orbits of system (2). The function  $1/V$  defines an inverse integrating factor in  $U \setminus \{V = 0\}$  of system (2) which allows to compute a first integral for (2) in  $U \setminus \{V = 0\}$ .

We consider now the relation between the existence of a center and that of an inverse integrating factor for analytic or polynomial vector fields. The main result is given by the following theorem which is analogous to Theorems 2 and 3.

**Theorem 4.** [Reeb 's criterion] (see for instance [29]). *The analytic differential system (7) has a center at the origin if and only if there is a local nonzero analytic inverse integrating factor of the form  $V = 1 + h.o.t.$  in a neighborhood of the origin.*

An analytic inverse integrating factor having the Taylor expansion at the origin  $V = 1 + h.o.t.$  is called a *Reeb inverse integrating factor*.

Darboux gave his geometric method of integration in his seminal work [8] of 1878. The geometric method of Darboux uses algebraic invariant curves of a polynomial differential system for computing a first integral of the system. There were numerous publications on the problem of the center using the Darboux method during the last part of the 20th century and the beginning of the 21st century (see for instance [6, 16, 34]). In fact there is the following conjecture due to Zoladek [34], see also [6]. See these papers for more details on this conjecture.

**Conjecture 5.** *Suppose that the polynomial differential system (4) has a center at the origin. Then this system has a Darboux first integral or an algebraic symmetry.*

To show that a singular point is a center for system (4) we have two basic mechanisms: we either apply Poincaré–Liapunov Theorem and we show that we have a local analytic first integral, or we apply the Reeb inverse integrating factor. Another mechanism for detecting centers has been given by Mikonenko see [28].

The main objective of the present paper is to analyze the center problem from the inverse point of view (see for instance [17, 30]). Indeed, either given an analytic function  $H$  of the form (6) we shall determine the analytic functions  $X$  and  $Y$  in (4) in such a way that the function  $H$  is a first integral of the differential system (4), or given an analytic function  $1 + \sum_{j=1}^{\infty} V_j$  in a neighborhood of the origin we shall determine the analytic functions  $X$  and  $Y$  in (4) in such a way that the analytic differential system (4) has the function  $V$  as a Reeb inverse integrating factor.

We say that a center at the origin of an analytic differential system is a *weak center* if in a neighborhood of the origin it has an analytic first integral of the form  $H = \frac{1}{2}(x^2 + y^2) \left( 1 + \sum_{j=1}^{\infty} \Upsilon_j \right)$ , where  $\Upsilon_j$  is a homogenous polynomial of degree  $j$ . We have characterized the expression of an analytic or polynomial differential system having a weak center at the origin, see Theorem 15. Moreover we prove that the uniform isochronous centers and isochronous holomorphic centers are weak centers.

We have extended the weak conditions of a center given by Alwash and Lloyd in [1] for linear center with homogenous polynomial nonlinearities (see Proposition 10), to a general analytic and polynomial differential system see Theorem 25. Furthermore the centers satisfying the generalized weak conditions of a centers are weak centers. Finally as an application we obtain the necessary and sufficient conditions for the existence of a weak center in a class of polynomial differential systems of degree four.

## 2. PRELIMINARY CONCEPTS AND RESULTS

In the proofs of the results that we provide in this paper it plays an important role the following results.

As usual the *Poisson bracket* of the functions  $f(x, y)$  and  $g(x, y)$  is defined as

$$\{f, g\} := \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}.$$

We will need the following result.

**Proposition 6.** *The next relation holds*

$$\int_0^{2\pi} \{H_2, \Psi\}|_{x=\cos t, y=\sin t} dt = 0$$

for arbitrary  $C^1$  function  $\Psi = \Psi(x, y)$  defined in the interval  $[0, 2\pi]$ .

*Proof.* Indeed, if we change  $x = \cos t$ ,  $y = \sin t$  then it is easy to show that

$$\{H_2, \Psi\}|_{x=\cos t, y=\sin t} = x \frac{\partial \Psi}{\partial y} - y \frac{\partial \Psi}{\partial x} \Big|_{x=\cos t, y=\sin t} = \frac{d\Psi(\cos t, \sin t)}{dt}.$$

Hence,

$$\int_0^{2\pi} \{H_2, \Psi\}|_{x=\cos t, y=\sin t} dt = \Psi(\cos t, \sin t) \Big|_{t=0}^{t=2\pi} = 0.$$

□

The following result is due to Liapunov (see Theorem 1, page 276 of [20]).

**Theorem 7.** *If all the roots  $\lambda_1, \dots, \lambda_n$  of the equation*

$$\begin{vmatrix} p_{11} - \lambda & p_{21} & \dots & p_{n1} \\ p_{12} & p_{22} - \lambda & \dots & p_{n2} \\ \dots & \dots & \dots & \dots \\ p_{1n} & p_{2n} & \dots & p_{nn} - \lambda \end{vmatrix}$$

*are such that the relation  $\lambda = m_1 \lambda_1 + \dots + m_n \lambda_n$ , is not vanishing for an arbitrary non-negative integers  $m_1, \dots, m_n$  linked by the expression  $m = m_1 + \dots + m_n \neq 0$ . Then for an arbitrary given homogenous polynomial  $U = U(x_1, \dots, x_n)$  of degree  $m$  there exists a unique homogenous polynomial  $V = V(x_1, \dots, x_n)$  of degree  $m$  which is a solution of the equation*

$$\sum_{j=1}^n (p_{j1} x_1 + \dots + p_{jn} x_n) \frac{\partial V}{\partial x_j} = U.$$

*In particular, for  $n = 2$  the partial differential equation*

$$(8) \quad x \frac{\partial V}{\partial y} - y \frac{\partial V}{\partial x} := \{H_2, V\} = U,$$

*has a unique solution  $V$  if and only if*

$$\lambda_1 m_1 + \lambda_2 m_2 = i(m_1 - m_2) \neq 0 \quad \text{with} \quad m = m_1 + m_2 \neq 0.$$

As a simple consequence of Theorem 7 we have the next result.

**Corollary 8.** *Let  $U = U(x, y)$  be a homogenous polynomial of degree  $m$ . The linear partial differential equation (8) has a unique homogenous polynomial solution  $V$  of degree  $m$  if  $m$  is odd; and if  $V$  is a homogenous polynomial solution when  $m$  is even then any other homogenous polynomial solution is of the form  $V + c(x^2 + y^2)^{m/2}$  with  $c \in \mathbb{R}$ . Moreover, for  $m$  even these solutions exist if and only if*

$$\int_0^{2\pi} U(x, y)|_{x=\cos t, y=\sin t} dt = 0.$$

In what follows some examples of planar vector fields having a center are studied.

**2.1. Hamiltonian system.** When system (4) is Hamiltonian, i.e. there exists a function  $F = F(x, y)$  such that

$$-y + X(x, y) = -\frac{\partial F(x, y)}{\partial y}, \quad x + Y(x, y) = \frac{\partial F(x, y)}{\partial x}.$$

Hence  $F = \frac{1}{2}(x^2 + y^2) + h.o.t.$  is a first integral.

**2.2. Reversible system.** Besides Hamiltonian systems there is another class of systems (4) for which the origin is a center, namely the reversible systems satisfying the following definition.

We say that system (4) is *reversible with respect to the straight line  $l$*  through the origin if it is invariant with respect to reversion about  $l$  and a reversion of time  $t$  (see for instance [7]).

The following criterion goes back to Poincaré see for instance [24], p.122.

**Theorem 9.** *The origin of system (4) is a center if the system is reversible.*

In particular this theorem is applied for the case when (4) is invariant under the transformations  $(x, y, t) \rightarrow (-x, y, -t)$  or  $(x, y, t) \rightarrow (x, -y, -t)$ .

**2.3. Weak condition for a center.** The following condition weak condition for a center was due to Alwash and Lloyd [1, 18], see also [18].

**Proposition 10.** *The origin is a center of a polynomial differential system of the form*

$$(9) \quad \dot{x} = -y + X_m, \quad \dot{y} = x + Y_m,$$

where  $X_m$  and  $Y_m$  are homogenous polynomial of degree  $m$ , if there exists  $\mu \in \mathbb{R}$  such that

$$(x^2 + y^2) \left( \frac{\partial X_m}{\partial x} + \frac{\partial Y_m}{\partial y} \right) = \mu (xX_m + yY_m),$$

and either  $m = 2k$  is even; or  $m = 2k - 1$  is odd and  $\mu \neq 2k$ ; or  $m = 2k - 1$  is odd,  $\mu = 2k$  and

$$\int_0^{2\pi} \left( \frac{\partial X_m}{\partial x} + \frac{\partial Y_m}{\partial y} \right) \Big|_{x=\cos t, y=\sin t} dt = 0.$$

In [9] the author proved that if  $\mu = 2m$  then system (9) has the rational first integral

$$\frac{x^2 + y^2 - 2(xY_m - yX_m)}{(x^2 + y^2)^m}.$$

**2.4. Cauchy-Riemann condition for a center.** Another particular case of differential systems with a center are the systems satisfying the Cauchy-Riemann conditions (see for instance [7]).

**Proposition 11.** *(Cauchy-Riemann condition for a center) Let  $O$  be a center of (2). Then  $O$  is isochronous center if  $P$  and  $Q$  satisfy the Cauchy-Riemann equations*

$$(10) \quad \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

A center of system (4) for which (10) holds is called a *holomorphic center*, which is also an isochronous center, see for more details [21] and [22]. We recall that a center of system (4) located at the origin is an *isochronous center* if all the periodic solutions in a neighborhood of the origin have the same period.

## 3. STATEMENT OF THE MAIN RESULTS

The main results are stated in the following four subsections.

**3.1. Analytic and polynomial vector fields with a linear type center.** We state and solve the following inverse problems for the centers of analytic and polynomial vector fields.

**Inverse Poincaré-Liapunov's Problem** *Determine the analytic (polynomial) planar vector fields*

$$(11) \quad \mathcal{X} = \left(-y + \sum_{j=2}^k X_j\right) \frac{\partial}{\partial x} + \left(x + \sum_{j=2}^k Y_j\right) \frac{\partial}{\partial y}, \quad \text{for } k \leq \infty,$$

for which the given function (6) is a local analytic first integral where  $X_j = X_j(x, y)$ ,  $Y_j = Y_j(x, y)$  for  $j \geq 2$  are homogenous polynomials of degree  $j$ .

**Inverse Reeb Problem** *Determine the analytic (polynomial) planar vector fields*

(11) for which the  $V = 1 + \sum_{j=1}^{\infty} V_j$  is the Reeb inverse integrating factor, i.e.

$$(12) \quad \mathcal{X}(x) \frac{\partial V}{\partial x} + \mathcal{X}(y) \frac{\partial V}{\partial y} = V \left( \frac{\partial \mathcal{X}(x)}{\partial x} + \frac{\partial \mathcal{X}(y)}{\partial y} \right)$$

The inverse Poincaré-Liapunov's problem and inverse Reeb problem for the analytic ( $k = \infty$ ) planar vector fields has been solved in the following theorem which provides the expressions of the analytic differential systems (7) in function of its first integral (6) or in function of its Reeb inverse integrating factor.

**Theorem 12.** *Consider the analytic vector field  $\mathcal{X}$ . Then this vector field has a Poincaré-Liapunov local first integral if and only if it has a Reeb inverse integrating factor. Moreover,*

(i) *the analytic differential system associated to the vector field  $\mathcal{X}_{\infty}$  for which  $H = (x^2 + y^2)/2 + h.o.t.$  is a local first integral can be written as*

$$(13) \quad \begin{aligned} \dot{x} &= -y + \sum_{j=2}^{\infty} X_j = -y + \sum_{j=1}^{\infty} (\{H_{j+1}, x\} + g_1 \{H_j, x\} + \dots + g_{j-1} \{H_2, x\}), \\ &= \left( 1 + \sum_{j=1}^{\infty} g_j \right) \{H, x\} \\ \dot{y} &= x + \sum_{j=2}^{\infty} Y_j = x + \sum_{j=1}^{\infty} (\{H_{j+1}, y\} + g_1 \{H_j, y\} + \dots + g_{j-1} \{H_2, y\}), \\ &= \left( 1 + \sum_{j=1}^{\infty} g_j \right) \{H, y\} \end{aligned}$$



where  $g_j = g_j(x, y)$  is an arbitrary homogenous polynomial of degree  $j$  which we choose in such a way that the series  $\sum_{j=1}^{\infty} g_j$  converge in the neighborhood of the origin.

- (ii) The differential system associated to the vector field  $\mathcal{X}_{\infty}$  for which  $V = 1 + \sum_{j=1}^{\infty} V_j$  is a Reeb integrating factor can be written as

$$\dot{x} = \left(1 + \sum_{j=1}^{\infty} V_j\right) \{F, x\}, \quad \dot{y} = \left(1 + \sum_{j=1}^{\infty} V_j\right) \{F, y\},$$

where  $F = \sum_{j=2}^{\infty} F_j$  and  $F_2 = (x^2 + y^2)/2$ ,  $F_j = F_j(x, y)$  for  $j > 2$  is an arbitrary homogenous polynomial of degree  $j$  which we choose in such a way that  $\sum_{j=2}^{\infty} F_j$  converges, i.e.  $F$  is an arbitrary Poincaré-Liapunov local first integral.

In fact, in the proof of Theorem 12 we provide the expression of the vector fields having a given Poincaré-Liapunov local first integral and the expression of the vector fields having a given Reeb inverse integrating factor.

The inverse Poincaré-Liapunov's problem and inverse Reeb problem for the polynomial planar vector fields ( $k = m < \infty$ ) has been solved in the following theorem which provides the expressions of the analytic differential systems (7) in function of its first integral (6) or in function of its Reeb integrating factor.

**Theorem 13.** *Consider the polynomial vector field  $\mathcal{X}_m$ . Then this polynomial vector field has a Poincaré-Liapunov local first integral if and only if it has a Reeb inverse integrating factor. Moreover, the differential system associated to the vector field  $\mathcal{X}_m$  for which  $H = (x^2 + y^2)/2 + h.o.t.$  is a local first integral can be written as*

$$\begin{aligned} \dot{x} &= \left(1 + \sum_{j=1}^{\infty} g_j\right) \{H, x\} \\ &= \{H_{m+1}, x\} + (1 + g_1)\{H_m, x\} + \dots + (1 + g_1 + \dots + g_{m-1})\{H_2, x\}, \\ \dot{y} &= \left(1 + \sum_{j=1}^{\infty} g_j\right) \{H, y\} \\ &= \{H_{m+1}, y\} + (1 + g_1)\{H_m, y\} + \dots + (1 + g_1 + \dots + g_{m-1})\{H_2, y\}. \end{aligned} \tag{14}$$

where

$$(15) \quad \begin{aligned} H &= \frac{1}{2}(x^2 + y^2) + \sum_{j=2}^{\infty} H_j = \tau_1 H_{m+1} + \tau_2 H_m + \dots + \tau_m H_2 \\ &= \int_{\gamma} \Omega \left( dH_{m+1} + (1 + g_1)dH_m + \dots + (1 + g_1 + \dots + g_{m-1})dH_2 \right) \end{aligned}$$

where  $\Omega := \left( 1 + \sum_{j=1}^{\infty} g_j \right)^{-1}$ ,  $\gamma$  is an oriented curve (see for instance [33]),  $\tau_j =$

$\tau_j(x, y)$  is a convenient analytic function in the neighborhood of the origin such that  $\tau_j(0, 0) = 1$ , and  $g_j = g_j(x, y)$  is an arbitrary homogenous polynomial of degree  $j$  which we choose in such a way that  $\Omega$  is the inverse Reeb inverse integrating factor which satisfies the first order partial differential equation

$$(16) \quad \{H_{m+1}, \Omega\} + \{H_m, (1 + g_1)\Omega\} + \dots + \{H_2, (1 + g_1 + \dots + g_{m-1})\Omega\} = 0.$$

**Remark 14.** From the proof of Theorem 13 it follows that (16) is equivalent to the infinite number of first order partial differential equations

$$(17) \quad \begin{aligned} 0 &= \{H_{m+1}, g_1\} + \{H_m, g_2\} + \dots + \{H_3, g_{m-1}\} + \{H_2, g_m\}, \\ 0 &= \{H_{m+1}, g_1^2 - g_2\} + \{H_m, g_1 g_2 - g_3\} + \dots + \{H_3, g_1 g_{m-1} - g_m\} \\ &\quad + \{H_2, g_1 g_m - g_{m+1}\}, \\ &\quad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \end{aligned}$$

with unknowns the homogenous polynomials  $g_j$  of degree  $j \geq m$ . Hence by Corollary 6 we obtain the conditions

$$\begin{aligned} \int_0^{2\pi} (\{H_{m+1}, g_1\} + \{H_m, g_2\} + \dots + \{H_3, g_{m-1}\})|_{x=\cos t, y=\sin t} dt &= 0, \\ \int_0^{2\pi} (\{H_{m+1}, g_1^2 - g_2\} + \{H_m, g_1 g_2 - g_3\} + \dots + \{H_3, g_1 g_{m-1} - g_m\})|_{x=\cos t, y=\sin t} dt &= 0, \\ &\qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \end{aligned}$$

The first condition, by Corollary 8 guarantees the existence of the solution  $g_m$  of first equation of (17), the second condition, again by Corollary 8, guarantees the existence of the solution  $g_{m+1}$  of the second equation of (17), and so on.

**3.2. Analytic and polynomial vector fields with local analytic first integral of the form  $H = \frac{1}{2}(x^2 + y^2)(1 + h.o.t.)$ .** We say that a differential system (4) has a *weak center* at the origin if it has a local analytic first integral of the form

$$H = \frac{1}{2}(x^2 + y^2) \left( 1 + \sum_{j=1}^{\infty} \Upsilon_j(x, y) \right) := H_2 \Phi(x, y),$$

where  $\Upsilon_j$  is a convenient homogenous polynomial of degree  $j$ .

The aim of this section is to study the weak centers for analytic and polynomial differential systems.

In the study of the weak centers plays a fundamental role the differential systems of the form

$$(18) \quad \dot{x} = -y(1 + \Lambda) + x\varphi, \quad \dot{y} = x(1 + \Lambda) + y\varphi,$$

where  $\Lambda = \Lambda(x, y)$  and  $\varphi = \varphi(x, y)$  are convenient analytic functions, as we can show from the following theorem.

**Theorem 15.** *An analytic differential system (7) has a weak center at the origin if and only if this system can be written as*

$$(19) \quad \begin{aligned} \dot{x} &= -y \left( 1 + \sum_{j=2}^{\infty} \left( \frac{j+1}{2} \Upsilon_{j-1} + \frac{j}{2} g_1 \Upsilon_{j-2} + \dots + \frac{3}{2} g_{j-2} \Upsilon_1 + g_{j-1} \right) \right) \\ &\quad + \frac{x}{2} \sum_{j=2}^{\infty} \left( \{\Upsilon_{j-1}, H_2\} + g_1 \{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2} \{\Upsilon_1, H_2\} \right) \\ &:= -y(1 + \Lambda) + x\varphi \\ \dot{y} &= x \left( 1 + \sum_{j=2}^{\infty} \left( \frac{j+1}{2} \Upsilon_{j-1} + \frac{j}{2} g_1 \Upsilon_{j-2} + \dots + \frac{3}{2} g_{j-2} \Upsilon_1 + g_{j-1} \right) \right) \\ &\quad + \frac{y}{2} \sum_{j=2}^{\infty} \left( \{\Upsilon_{j-1}, H_2\} + g_1 \{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2} \{\Upsilon_1, H_2\} \right) \\ &:= x(1 + \Lambda) + y\varphi, \end{aligned}$$

where  $\Upsilon_0 = 1$ ,  $g_0 = 1$ ,  $g_j$  and  $\Upsilon_j$  are homogenous polynomial of degree  $j$  for  $j \geq 1$  and has the first integral  $H = H_2 \left( 1 + \sum_{j=2}^{\infty} \Upsilon_j \right)$ . Moreover assuming that

$$\begin{aligned} \frac{j+1}{2} \Upsilon_{j-1} + \frac{j}{2} g_1 \Upsilon_{j-2} + \dots + \frac{3}{2} g_{j-2} \Upsilon_1 + g_{j-1} &= 0, \\ \{\Upsilon_{j-1}, H_2\} + g_1 \{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2} \{\Upsilon_1, H_2\} &= 0, \end{aligned}$$

for  $j \geq m+1$ , we obtain necessary and sufficient conditions under which the polynomial differential system (19) of degree  $m$  and has the first integral

$$(20) \quad H = H_2 \Phi = H_2 (1 + \mu_1 \Upsilon_1 + \dots + \mu_{m-1} \Upsilon_{m-1}),$$

where  $\mu_j = \mu_j(x, y)$  is a convenient analytic function in the neighborhood of the origin for  $j = 1, \dots, m-1$ .

The singular point of system (4) located at the origin is an *isochronous center* if all the periodic solutions in a neighborhood of it has the same period.

**Corollary 16.** *The weak center of a polynomial differential system (18) is an isochronous center if and only if*

$$(21) \quad \int_0^{2\pi} \frac{d\theta}{1 + \Lambda(r \cos \theta, r \sin \theta)} = 2\pi,$$

where  $(r, \theta)$  are the polar coordinates, and  $r$  satisfies that

$$H(r \cos \theta, r \sin \theta) = r^2/2 \left( 1 + \sum_{j=1}^{\infty} r^j \Upsilon_j(\cos \theta, \sin \theta) \right)$$

is a constant on any periodic solution surrounding the isochronous center.

A center  $O$  of system (2) is a *uniform isochronous center* if the equality  $x\dot{y} - y\dot{x} = \kappa(x^2 + y^2)$  holds for a nonzero constant  $\kappa$ ; or equivalently in polar coordinates  $(r, \theta)$  such that  $x = r \cos \theta$ ,  $y = r \sin \theta$ , we have that  $\dot{\theta} = \kappa$ .

**Corollary 17.** *The weak center of an analytic differential system (18) is a uniform isochronous center if and only if*

$$(22) \quad \begin{aligned} \dot{x} &= -y + x \sum_{j=2}^{\infty} \frac{1}{j+1} \left( \{H_j, g_1\} + \dots + \{H_2, g_{j-1}\} \right), \\ \dot{y} &= x + y \sum_{j=2}^{\infty} \frac{1}{j+1} \left( \{H_j, g_1\} + \dots + \{H_2, g_{j-1}\} \right). \end{aligned}$$

Moreover the weak center of polynomial differential system of degree  $m$  (18) is a uniform isochronous center if and only if (22) holds and

$$\{H_j, g_1\} + \dots + \{H_2, g_{j-1}\} = 0,$$

holds for  $j \geq m+1$ . In particular for quasi-homogenous differential system (9) we have that (22) becomes

$$\begin{aligned} \dot{x} &= -y + \frac{x}{m+1} \{H_2, g_{m-1}\}, \\ \dot{y} &= x + \frac{y}{m+1} \{H_2, g_{m-1}\}, \end{aligned}$$

and has the Poincaré-Liapunov first integral  $F = H_2 \left( 1 + \frac{m-1}{m+1} g_{m-1} \right)^{2/(1-m)}$ .

The inverse approach to study the uniform isochronous center was given in [19].

Theorem 15 has the following additional corollary.

**Corollary 18.** *Assume that the planar differential system (5) has a center at the origin. Then this center is a holomorphic isochronous center if and only if system (5) can be written as (18), i.e. is a weak center, with the function  $\Lambda$  and  $\varphi$  satisfying the Cauchy-Riemann conditions*

$$\frac{\partial \varphi}{\partial x} - \frac{\partial \Lambda}{\partial y} = 0, \quad \frac{\partial \varphi}{\partial y} + \frac{\partial \Lambda}{\partial x} = 0.$$

Hence  $\varphi + i(1 + \Lambda) = f(z)$  where  $z = x + iy$ , and  $f = f(z)$  is a holomorphic function on  $\mathbb{C}$ . Moreover, a polynomial differential system (18) with a holomorphic center at the origin is Darboux integrable.

**Remark 19.** *From Corollaries 18 and 17 it follows that all the uniform isochronous centers and all the holomorphic isochronous centers for polynomial differential systems are always weak centers.*

It is important to observe that there is not a relation between isochronous centers and weak centers, i.e. there exist isochronous centers which are not weak centers and weak centers which are not isochronous centers. Then for instance the quadratic isochronous center

$$\dot{x} = -y - \frac{4x^2}{3}, \quad \dot{y} = x\left(1 - \frac{16y}{3}\right),$$

is not a weak center because it has the first integral  $H = (9 - 24y + 32x^2)^2 / (3 - 16y)$  for more details see [5]. On the other hand the quadratic system

$$\dot{x} = -y - x^2 - 3y^2, \quad \dot{y} = x + 2xy,$$

has a weak center at the origin because it has the first integral  $H = (1 + 2y)(x^2 + y^2)$  but it is not isochronous see [5]. In fact in [18] we provide all the quadratic system with weak centers.

We observe that any linear type center after an analytic change of variables is locally a weak center. This follows from the following theorem which goes back to Poincaré and Liapunov, see [23, 20, 27].

**Theorem 20** (Poincaré normal form of a nondegenerate center). *For a polynomial differential system (5) with a center at the origin, there exists a local analytic change of coordinates*

$$(23) \quad u = x + h.o.t., \quad v = y + h.o.t.,$$

and an analytic function  $\Psi = \Psi(u^2 + v^2)$  such that the coordinate change (23) transforms system (5) into the form

$$\dot{u} = -\frac{\partial H}{\partial v}, \quad \dot{v} = \frac{\partial H}{\partial u},$$

where  $H = \frac{1}{2} \int (1 + \Psi(u^2 + v^2)) d(u^2 + v^2)$ . Without loss of generality we can assume that  $\Psi(0, 0) = 0$ .

Now we introduce the following definitions and notations.

Let  $\mathbb{R}[x, y]$  be the ring of all real polynomials in the variables  $x$  and  $y$ , and let  $\mathcal{X}$  be the polynomial vector field (2) of degree  $m$ . Let  $g = g(x, y) \in \mathbb{R}[x, y] \setminus \mathbb{R}$ . Then  $g = 0$  is an *invariant algebraic curve* of  $\mathcal{X}$  if

$$\mathcal{X}g = P \frac{\partial g}{\partial x} + Q \frac{\partial g}{\partial y} = Kg,$$

where  $K = K(x, y)$  is a polynomial of degree at most  $m - 1$ , which is called the *cofactor* of  $g = 0$ . A function  $g = g(x, y)$  satisfying that  $g = 0$  is an *invariant curve* (i.e. formed by orbits of the vector field  $\mathcal{X}$ ) is called *partial integral*. If  $g \in \mathbb{R}[x, y] \setminus \mathbb{R}$  then  $g$  is called a *polynomial partial integral* or a *Darboux polynomial*. If the polynomial  $g$  is irreducible in  $\mathbb{R}[x, y]$ , then we say that the invariant algebraic curve  $g = 0$  is *irreducible*, and that its *degree* is the degree of the polynomial  $g$ . A first integral  $F$  of the polynomial vector field (1) is called *Darboux* if

$$F = e^{k(x,y)/h(x,y)} g_1^{\lambda_1}(x, y) \dots g_r^{\lambda_r}(x, y),$$

where  $k, h, g_1, \dots, g_r$  are polynomials and  $\lambda_1, \dots, \lambda_r$  are complex constants. For more details on the so-called Darboux theory of integrability see for instance Chapter 8 of [10].

We introduce the following definition. We say that a polynomial vector field  $\mathcal{X}$  of degree  $m$  is *quasi-Darboux integrable* if there exist  $r$  polynomial partial integrals  $g_1, \dots, g_r$  and  $s$  non-polynomial partial integrals  $f_1, \dots, f_s$  analytic in  $D \subseteq \mathbb{R}^2$  satisfying

$$\mathcal{X}(f_j) = P \frac{\partial f_j}{\partial x} + Q \frac{\partial f_j}{\partial y} = K_j f_j,$$

where  $K_j = K_j(x, y)$  is a convenient polynomials of degree  $m - 1$ , for  $j = 1, \dots, s$  such that the function

$$F = e^{k(x,y)/h(x,y)} g_1^{\lambda_1}(x, y) \dots g_r^{\lambda_r}(x, y) f_1^{\kappa_1}(x, y) \dots f_s^{\kappa_s}(x, y),$$

is a first integral, where  $k = k(x, y)$ ,  $h = h(x, y)$  are polynomials, and  $\lambda_1, \dots, \lambda_r, \kappa_1, \dots, \kappa_s$ , are complex constants. We observe that a generalization of the Darboux theory was developed in the paper [14], which evidently contains the above definition with another name, but for our aim we shall use the name of quasi-Darboux integrable.

We have the following conjecture.

**Conjecture 21.** *A polynomial differential system (18) having a weak center at the origin is quasi-Darboux integrable.*

This conjecture is supported by several facts which we give below.

**Proposition 22.** *A polynomial differential system (18) with a weak center at the origin is quasi-Darboux integrable in a neighborhood of the origin with the first integral*

$$H = \frac{1}{2}(x^2 + y^2) \left( 1 + \sum_{j=1}^{m+1} \tau_j(x, y) \Upsilon_j(x, y) \right) := H_2 f(x, y),$$

where  $H_2 = 0$  is an invariant algebraic curve and  $f = 0$  is an analytic (non polynomial) invariant curve with cofactor  $2\varphi$  and  $-2\varphi$  respectively.

**3.3. Center problem for analytic or polynomial vector fields with a generalized weak condition of a center.** First we prove the following two propositions.

**Proposition 23.** *Assume that a differential system (4) satisfies the relation*

$$(24) \quad (x^2 + y^2) \left( \frac{\partial(-y + X)}{\partial x} + \frac{\partial(x + Y)}{\partial y} \right) = \mu(x(-y + X) + y(x + Y)),$$

with  $\mu \in \mathbb{R} \setminus \{0\}$ . Then the system can be written as in (18) with

$$\varphi(x, y) = \frac{2}{\mu} \left( \frac{\partial(-y + X)}{\partial x} + \frac{\partial(x + Y)}{\partial y} \right),$$

and  $\Lambda = \Lambda(x, y)$  an arbitrary analytic function in a neighborhood of the origin. Moreover system (4) has the inverse integrating factor  $(x^2 + y^2)^{\mu/2}$ , and it can be written as

$$(25) \quad \dot{x} = (x^2 + y^2)^{\mu/2} \{F, x\} \quad \dot{y} = (x^2 + y^2)^{\mu/2} \{F, y\},$$

with

$$(26) \quad \begin{aligned} F &= \int_{\gamma} \left( \frac{-Xdy + Ydx}{(x^2 + y^2)^{\mu/2}} + \frac{d(x^2 + y^2)}{2(x^2 + y^2)^{\mu/2}} \right) \\ &= \begin{cases} \frac{1}{2 - \mu} (x^2 + y^2)^{(\mu-2)/2} + \int_{\gamma} \frac{-Xdy + Ydx}{(x^2 + y^2)^{\mu/2}} & \text{if } \mu \neq 2, \\ \log \sqrt{x^2 + y^2} + \int_{\gamma} \frac{-Xdy + Ydx}{(x^2 + y^2)}, & \text{if } \mu = 2. \end{cases} \end{aligned}$$

Note that if in (24) we have that  $\mu = 0$ , then system (4) is a Hamiltonian system.

**Proposition 24.** *Consider the polynomial differential system (2) of degree  $m$  which satisfy the relations*

$$(27) \quad \int_0^{2\pi} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Big|_{x=\cos t, y=\sin t} dt = 0.$$

Then there exist polynomials  $\tilde{H} = \sum_{j=3}^{m+1} H_j$  and  $G = \sum_{j=1}^{m-1} G_j$  of degree  $m+1$  and  $m-1$  respectively such that system (2) can be written as

$$(28) \quad \begin{aligned} \dot{x} &= P = \{\tilde{H}, x\} + (1 + G)\{H_2, x\}, \\ \dot{y} &= Q = \{\tilde{H}, y\} + (1 + G)\{H_2, y\}. \end{aligned}$$

Note that we have extended the definition of “weak condition for a center” given in subsection 2.5 for a quasi-homogenous polynomial differential system to a general analytic differential system. Proposition 10 can be generalized as follows.

**Theorem 25.** *[Generalized weak condition of a center of an analytic (polynomial) differential systems] We consider an analytic (polynomial) differential system (7). Then the origin is a weak center if there exists  $\mu \in \mathbb{R} \setminus \{0\}$  such that (61) hold. Moreover this differential system can be written as (25) with the first integral:*

$$F = \int_{\gamma} \left( \frac{(1 + (1 - 1/\lambda)\Upsilon + q(H_2))dH_2 + H_2d\Upsilon}{H_2^{1/\lambda}} \right),$$

with  $\lambda = 2/\mu$  and  $\Upsilon = \Upsilon(x, y)$  and  $q = q(H_2)$  are a convenient analytic functions, or which is equivalent

$$(29) \quad \begin{aligned} \dot{x} &= -y \left( 1 + q(H_2) + (1 - 1/\lambda)\Upsilon + \frac{1}{2} \left( x \frac{\partial \Upsilon}{\partial x} + y \frac{\partial \Upsilon}{\partial y} \right) \right) + \frac{x}{2} \{\Upsilon, H_2\}, \\ \dot{y} &= x \left( 1 + q(H_2) + (1 - 1/\lambda)\Upsilon + \frac{1}{2} \left( x \frac{\partial \Upsilon}{\partial x} + y \frac{\partial \Upsilon}{\partial y} \right) \right) + \frac{y}{2} \{\Upsilon, H_2\}. \end{aligned}$$

Moreover, if (29) is a polynomial differential system of degree  $m$ , i.e.  $\Upsilon = \Upsilon(x, y)$

is a polynomials of degree  $m-1$  and  $q(H_2) = \sum_{j=1}^{[(m-1)/2]} \alpha_j H_2^{j-1}$ , here  $[(m-1)/2]$  is

the integer part of  $(m-1)/2$ ,  $\alpha_j$  is a constant for  $j = 1, \dots, [(m-1)/2]$  such that

$$1 + \alpha_1 + \frac{\lambda - 1}{\lambda} \Upsilon(0, 0) \neq 0.$$

then the system (29) is quasi Darboux-integrable with the first integral  $F$  which is given in what follows

(i) If  $\lambda \neq 1$  and  $\prod_{n=2}^{[m/2]} (n - 1/\lambda) \neq 0$ , then

$$(30) \quad F = \frac{H_2}{\left( \Upsilon + \frac{1 + \alpha_1}{1 - 1/\lambda} + \frac{\alpha_2 H_2}{2 - 1/\lambda} + \dots + \frac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2] - 1/\lambda} \right)^{\lambda/(\lambda-1)}}.$$

The algebraic curves  $H_2 = 0$  and

$$g = \Upsilon + \frac{1 + \alpha_1}{1 - 1/\lambda} + \frac{\alpha_2 H_2}{2 - 1/\lambda} + \dots + \frac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2] - 1/\lambda} = 0$$

are invariant curves with cofactors  $\{g, H_2\}$  and  $(1 - 1/\lambda)\{g, H_2\}$ , respectively.

(ii) If  $\lambda = 1$  and  $1 + \alpha_1 \neq 0$ , then

$$(31) \quad F = H_2 e^{-\left( \Upsilon + \alpha_2 H_2 + \alpha_3 H_2^2/2 + \dots + \alpha_m H_2^{[(m-1)/2]-1} / ([[(m-1)/2]-1]) \right) / (1 + \alpha_1)}.$$

The algebraic curves  $H_2 = 0$  is invariant with cofactor  $\{H_2, \Upsilon\}$ .

(iii) If  $1/[m/2] \leq \lambda = 1/k < 1$  and  $\left( \alpha_k \prod_{n=2, n \neq k}^{[(m-1)/2]} (n - k) \right) \neq 0$ , then

$$(32) \quad F = \frac{H_2}{\left( \Upsilon + \frac{1 + \alpha_1}{1 - k} + \sum_{j=2, j \neq k}^{[(m-1)/2]} \frac{\alpha_j}{j - k} H_2^{j-1} + \alpha_k H_2^{k-1} \log H_2 \right)^{1/(k-1)}}.$$

The algebraic curve  $H_2 = 0$  and non-polynomial curve

$$f = \Upsilon + \frac{1 + \alpha_1}{1 - k} + \sum_{j=2, j \neq k}^{[(m-1)/2]} \frac{\alpha_j}{j - k} H_2^{j-1} + \alpha_k H_2^{k-1} \log H_2 = 0,$$

are invariant curves with cofactors  $\{\Upsilon, H_2\}$  and  $(1 - k)\{\Upsilon, H_2\}$ , respectively.

We observe that  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ .

(iv) If  $1/[(m-1)/2] \leq \lambda = 1/k < 1$ ,  $\alpha_k = 0$  and  $\prod_{n=2, n \neq k}^{[(m-1)/2]} (n - k) \neq 0$ , then

$$(33) \quad F = \frac{H_2}{\left( \Upsilon + \frac{1 + \alpha_1}{1 - k} + \sum_{j=2, j \neq k}^{[(m-1)/2]} \frac{\alpha_j}{j - k} H_2^{j-1} \right)^{1/(k-1)}}.$$

The algebraic curves  $H_2 = 0$  and

$$g = \Upsilon + \frac{1 + \alpha_1}{1 - k} + \sum_{j=2, j \neq k}^{[(m-1)/2]} \frac{\alpha_j}{j - k} H_2^{j-1} = 0$$

are invariant algebraic curves with cofactors  $\{g, H_2\}$  and  $(1 - k)\{g, H_2\}$  respectively.



The given first integrals has the following Taylor extension at the origin  $F = H_2(1 + h.o.t.)$  Consequently the origin is a weak center.

In an analogous way we can study the analytic case.

**3.4. Linear centers with degenerate infinity.** We shall study the following class of differential systems

$$(34) \quad \dot{x} = -y + \sum_{j=2}^{m-1} X_j + xR_{m-1}, \quad \dot{y} = x + \sum_{j=2}^{m-1} Y_j + yR_{m-1},$$

where  $R_{m-1} = R_{m-1}(x, y)$  is a convenient nonzero homogenous polynomial of degree  $m - 1$ . Such system are polynomial differential systems with a *degenerate infinity*. This name is due to the fact that in the Poincaré compactification of (34) the line at infinity is filled with singular points.

**Proposition 26.** *Assume that a polynomial differential system (5) has a center at the origin with a first integral  $H$  given in (6). Then this system has a degenerate infinity if it can be written as*

$$\begin{aligned} \dot{x} &= \sum_{j=2}^m g_{m+1-j} \{\bar{\Psi}_j, x\} + \frac{x}{m+1} \sum_{j=1}^{m-1} \{H_{m+1-j}, g_j\}, \\ \dot{y} &= \sum_{j=2}^m g_{m+1-j} \{\bar{\Psi}_j, y\} + \frac{y}{m+1} \sum_{j=1}^{m-1} \{H_{m+1-j}, g_j\}, \end{aligned}$$

where  $\bar{\Psi}_j = \sum_{k=2}^{j-1} H_k$ .

Proposition 26 characterizes the polynomial differential systems having a degenerate infinity and a linear type center at the origin.

**Proposition 27.** *Polynomial differential system (18) has a degenerate infinity if it can be written as*

$$(35) \quad \begin{aligned} \dot{x} &= -y(1 + \sum_{j=2}^{m-2} \Lambda_j) + x \sum_{j=2}^{m-2} \varphi_j + x\varphi_{m-1}, \\ \dot{y} &= x(1 + \sum_{j=2}^{m-2} \Lambda_j) + y \sum_{j=2}^{m-2} \varphi_j + y\varphi_{m-1}, \end{aligned}$$

i.e.  $\Lambda_{m-1} = 0$ .

All the results of this subsection are proved in section 7.

#### 4. THE PROOFS OF SUBSECTION 3.1

*Proof of Theorem 12 .* First we prove the “only if part”. Assume that the analytic differential system (7) has a Poincaré-Liapunov local first integral. Then we shall see that it can be written as (13).

Consider a general analytic vector field with a singular point at the origin. Then it can be written as that we write as

$$\mathcal{X} = \left( \sum_{j=1}^{\infty} X_j(x, y) \right) \frac{\partial}{\partial x} + \left( \sum_{j=1}^{\infty} Y_j(x, y) \right) \frac{\partial}{\partial y},$$

where  $X_j$  and  $Y_j$  for  $j = 0, 1, \dots$  are homogenous polynomials of degree  $j$ . Since the analytic first integral  $H$  starts with  $H_2 = (x^2 + y^2)/2$ , without loss of generality this implies that  $X_1(x, y) = -y$  and  $Y_1(x, y) = x$ . Hence the following infinite number of equations must be satisfied

$$\begin{aligned} 0 &= \frac{dH}{dt} = \left( x + \frac{\partial H_3}{\partial x} + \dots \right) (-y + X_2 + X_3 + \dots) \\ &\quad + \left( y + \frac{\partial H_3}{\partial y} + \dots \right) (x + Y_2 + Y_3 + \dots) \\ &= xX_2 + yY_2 + \{H_2, H_3\} \\ &\quad + xX_3 + yY_3 + \frac{\partial H_3}{\partial x} X_2 + \frac{\partial H_3}{\partial y} Y_2 + \{H_2, H_4\} \\ &\quad + xX_4 + yY_4 + \frac{\partial H_3}{\partial x} X_3 + \frac{\partial H_3}{\partial y} Y_3 + \frac{\partial H_4}{\partial x} X_2 + \frac{\partial H_4}{\partial y} Y_2 + \{H_2, H_5\} + \dots \\ &\quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ &\quad + xX_n + yY_n + \frac{\partial H_3}{\partial x} X_{n-1} + \frac{\partial H_3}{\partial y} Y_{n-1} + \dots + \frac{\partial H_n}{\partial x} X_2 + \frac{\partial H_n}{\partial y} Y_2 + \{H_2, H_{n+1}\} \\ &\quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \end{aligned}$$

Consequently

$$(36) \quad \begin{aligned} xX_2 + yY_2 + \{H_2, H_3\} &= 0, \\ xX_3 + yY_3 + \frac{\partial H_3}{\partial x} X_2 + \frac{\partial H_3}{\partial y} Y_2 + \{H_2, H_4\} &= 0, \\ xX_4 + yY_4 + \frac{\partial H_3}{\partial x} X_3 + \frac{\partial H_3}{\partial y} Y_3 + \frac{\partial H_4}{\partial x} X_2 + \frac{\partial H_4}{\partial y} Y_2 + \{H_2, H_5\} &= 0, \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots & \\ xX_n + yY_n + \frac{\partial H_3}{\partial x} X_{n-1} + \dots + \frac{\partial H_n}{\partial x} X_2 + \frac{\partial H_n}{\partial y} Y_2 + \{H_2, H_{n+1}\} &= 0, \\ xX_{n+1} + yY_{n+1} + \frac{\partial H_3}{\partial x} X_n + \dots + \frac{\partial H_{n+1}}{\partial x} X_2 + \frac{\partial H_{n+1}}{\partial y} Y_2 + \{H_2, H_{n+2}\} &= 0, \\ \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots & \end{aligned}$$

First, we introduce the notations

$$\mathcal{X}_0 = \sum_{j=2}^k \mathcal{X}_j = \sum_{j=2}^k (\{H_{j+1}, \cdot\} + g_1 \{H_j, \cdot\} + \dots + g_{j-1} \{H_2, \cdot\}),$$

where  $k \leq \infty$  and  $g_j = g_j(x, y)$  is a homogenous polynomial in the variables  $x$  and  $y$  of degree  $j$ , for  $j = 1, 2, \dots, k$ .

The first equation of (36) can be rewritten as follows

$$x \left( X_2 + \frac{\partial H_3}{\partial y} \right) + y \left( Y_2 - \frac{\partial H_3}{\partial x} \right) = 0.$$

Solving it with respect to  $X_2$  and  $Y_2$  we obtain

$$\begin{aligned} X_2 &= -\frac{\partial H_3}{\partial y} - yg_1 = \{H_3, x\} + g_1\{H_2, x\} := \mathcal{X}_2(x), \\ Y_2 &= \frac{\partial H_3}{\partial x} + xg_1 = \{H_3, y\} + g_1\{H_2, y\} := \mathcal{X}_2(y), \end{aligned}$$

where  $g_1 = g_1(x, y)$  is an arbitrary homogenous polynomial of degree one. By substituting these polynomials into the second equation of (36) we get

$$x \left( X_3 + \frac{\partial H_4}{\partial y} + g_1 \frac{\partial H_3}{\partial y} \right) + y \left( Y_3 - \frac{\partial H_4}{\partial x} - g_1 \frac{\partial H_3}{\partial x} \right) = 0.$$

By solving this equation with respect to  $X_3$  and  $Y_3$  we have

$$\begin{aligned} X_3 &= -\frac{\partial H_4}{\partial y} - g_1 \frac{\partial H_3}{\partial y} - yg_2 = \{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2, x\} := \mathcal{X}_3(x), \\ Y_3 &= \frac{\partial H_4}{\partial x} + g_1 \frac{\partial H_3}{\partial x} + xg_2 = \{H_4, y\} + g_1\{H_3, y\} + g_2\{H_2, y\} := \mathcal{X}_3(y), \end{aligned}$$

where  $g_2 = g_2(x, y)$  is an arbitrary homogenous polynomial of degree two. By continuing this process we obtain  $X_4, Y_4, \dots, X_n, Y_n$ , i.e.

$$(37) \quad \begin{aligned} X_n &= \{H_{n+1}, x\} + g_1\{H_n, x\} + \dots + g_{n-1}\{H_2, x\} := \mathcal{X}_n(x), \\ Y_n &= \{H_{n+1}, y\} + g_1\{H_n, y\} + \dots + g_{n-1}\{H_2, y\} := \mathcal{X}_n(y), \end{aligned}$$

where  $g_n = g_n(x, y)$  is an arbitrary homogenous polynomial of degree  $n$ . Hence, since  $\sum_{j=1}^{\infty} g_j$  converges in a neighborhood of the origin, we get that

$$\begin{aligned} \dot{x} &= -y + X_2 + X_3 + \dots + X_j + \dots = -y + \mathcal{X}(x) = -y + \sum_{j=2}^{\infty} \mathcal{X}_j(x) \\ &= \left( 1 + \sum_{j=1}^{\infty} g_j \right) \{H, x\}, \\ \dot{y} &= x + Y_2 + Y_3 + \dots + Y_j + \dots = x + \mathcal{X}(y) = x + \sum_{j=2}^{\infty} \mathcal{X}_j(y) \\ &= \left( 1 + \sum_{j=1}^{\infty} g_j \right) \{H, y\}. \end{aligned}$$

Note that the function  $1 + \sum_{j=1}^{\infty} g_j$  is an analytic integrating factor of the differential system (13) i.e. it is a Reeb inverse integrating factor.

Now we prove the “if” part. We assume that system (5) has a Reeb inverse integrating factor. From the equation (12), i.e.

$$\begin{aligned}
& (X_1 + X_2 + X_3 + \dots) \left( \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial x} + \frac{\partial V_3}{\partial x} + \dots \right) \\
& + (Y_1 + Y_2 + Y_3 + \dots) \left( \frac{\partial V_1}{\partial y} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial y} + \dots \right) \\
= & (1 + V_1 + V_2 + \dots) \left( \frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y} + \frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y} + \frac{\partial X_3}{\partial x} + \frac{\partial Y_3}{\partial y} + \dots \right)
\end{aligned}$$

it follows that

$$\begin{aligned}
(38) \quad & 0 = \frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}, \\
& Y_1 \frac{\partial V_1}{\partial y} + X_1 \frac{\partial V_1}{\partial x} = \frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y}, \\
& Y_1 \frac{\partial V_2}{\partial y} + X_1 \frac{\partial V_2}{\partial x} + X_2 \frac{\partial V_1}{\partial x} + Y_2 \frac{\partial V_1}{\partial y} = \frac{\partial X_3}{\partial x} + \frac{\partial Y_3}{\partial y} + V_1 \left( \frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y} \right), \\
& \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots
\end{aligned}$$

From the first equation of (38) we get that

$$X_1 = -\frac{\partial F_2}{\partial y}, \quad Y_1 = \frac{\partial F_2}{\partial x},$$

where  $F_2 = F_2(x, y)$  is an arbitrary homogenous polynomial of degree 2. From the second equation of (38) we obtain

$$\frac{\partial}{\partial x} \left( X_2 + V_1 \frac{\partial F_2}{\partial y} \right) + \frac{\partial}{\partial y} \left( Y_2 - V_1 \frac{\partial F_2}{\partial x} \right) = 0,$$

hence

$$(39) \quad X_2 = -\frac{\partial F_3}{\partial y} - V_1 \frac{\partial F_2}{\partial y}, \quad Y_2 = \frac{\partial F_3}{\partial x} + V_1 \frac{\partial F_2}{\partial x},$$

where  $F_3 = F_3(x, y)$  is an arbitrary homogenous polynomial of degree 3. From the third equation of (38) we obtain

$$\frac{\partial}{\partial x} \left( X_3 + V_1 \frac{\partial F_3}{\partial x} + V_2 \frac{\partial F_2}{\partial y} \right) + \frac{\partial}{\partial y} \left( Y_3 - V_1 \frac{\partial F_3}{\partial y} - V_2 \frac{\partial F_2}{\partial x} \right) = 0,$$

thus

$$X_3 = -\frac{\partial F_4}{\partial y} - V_1 \frac{\partial F_3}{\partial y} - V_2 \frac{\partial F_2}{\partial y}, \quad Y_3 = \frac{\partial F_4}{\partial x} + V_1 \frac{\partial F_3}{\partial x} + V_2 \frac{\partial F_2}{\partial x},$$

where  $F_4 = F_4(x, y)$  is an arbitrary homogenous polynomial of degree 4. By continuing this process we get

$$\{F_j, V_1\} + \dots + \{F_2, V_{j-1}\} = \frac{\partial X_j}{\partial x} + \frac{\partial Y_j}{\partial y},$$

by considering that  $\{F_k, V_n\} = \frac{\partial}{\partial y} \left( V_n \frac{\partial F_j}{\partial x} \right) - \frac{\partial}{\partial x} \left( V_n \frac{\partial F_j}{\partial y} \right)$  we deduce the relation

$$\frac{\partial}{\partial x} \left( X_j + V_1 \frac{\partial F_j}{\partial y} + \dots + \dots + V_{j-1} \frac{\partial F_2}{\partial y} \right) + \frac{\partial}{\partial y} \left( X_j - V_1 \frac{\partial F_j}{\partial x} - \dots - V_{j-1} \frac{\partial F_2}{\partial x} \right) = 0.$$

By using the notation

$$\tilde{\mathcal{X}}_0 = \sum_{j=1}^k \tilde{\mathcal{X}}_j = \sum_{j=1}^k (\{F_{j+1}, \cdot\} + V_1 \{F_j, \cdot\} + \dots + V_{j-1} \{F_2, \cdot\}),$$

where  $k \leq \infty$ ,  $V_j = V_j(x, y)$  and  $F_j = F_j(x, y)$  are homogenous polynomial in the variables  $x$  and  $y$  of degree  $j$ , for  $j = 1, 2, \dots, k$ , we get that

$$\begin{aligned} \tilde{\mathcal{X}}_1(x) &= \{F_2, x\} = X_1, \\ \tilde{\mathcal{X}}_1(y) &= \{F_2, y\} = Y_1, \\ \tilde{\mathcal{X}}_2(x) &= \{F_3, x\} + V_1 \{F_2, x\} = X_2, \\ \tilde{\mathcal{X}}_2(y) &= \{F_3, y\} + V_1 \{F_2, y\} = Y_2, \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ \tilde{\mathcal{X}}_m(x) &= \{F_{m+1}, x\} + V_1 \{F_m, x\} + \dots + V_{m-1} \{F_2, x\} = X_m, \\ \tilde{\mathcal{X}}_m(y) &= \{F_{m+1}, y\} + V_1 \{F_m, y\} + \dots + V_{m-1} \{F_2, y\} = Y_m, \end{aligned} \tag{40}$$

$$\begin{aligned} \tilde{\mathcal{X}}_{m+k-1}(x) &= \{F_{m+k}, x\} + V_1 \{F_{m-1+k}, x\} + \dots + V_{m+k-2} \{F_2, x\} = X_{m+k-1}, \\ \tilde{\mathcal{X}}_{m+k-1}(y) &= \{F_{m+k}, y\} + V_1 \{F_{m-1+k}, y\} + \dots + V_{m+k-2} \{F_2, y\} = Y_{m+k-1}, \end{aligned} \tag{41}$$

where  $k > 1$  and  $F_k = F_k(x, y)$  is an arbitrary homogenous polynomial of degree  $j$ , for  $j \geq 3$ , such that the series  $F = \sum_{j=2}^{\infty} F_j$ , converges at neighborhood of the origin. By considering that we are interesting in studying the linear type center then  $X_1 = -y$  and  $X_1 = x$  then we have that  $F_2 = (x^2 + y^2)/2$ . Therefore

$$F = (x^2 + y^2)/2 + F_3 + F_4 + \dots,$$

is a Poincaré-Liapunov local first integral this prove the “ if ” part of the theorem.

Moreover, by summing we get

$$\begin{aligned}
\dot{x} &= -y + \sum_{j=2}^{\infty} X_j = \sum_{j=2}^{\infty} (\{F_{j+1}, x\} + V_1\{F_j, x\} + \dots + V_{j-1}\{F_2, x\}) \\
&= \left(1 + \sum_{j=2}^{\infty} V_j\right) \{F, x\} \\
\dot{y} &= x + \sum_{j=2}^{\infty} X_j = \sum_{j=2}^{\infty} (\{F_{j+1}, y\} + V_1\{F_j, y\} + \dots + V_{j-1}\{F_2, y\}) \\
&= \left(1 + \sum_{j=2}^{\infty} V_j\right) \{F, y\},
\end{aligned}$$

Thus the proof of the theorem follows.  $\square$

**Remark 28.** From the “only if” part follows that the arbitrariness which we determine the vector fields with the given Poincaré-Liapunov local first integral is related with the Reeb’s inverse integrating factor  $V = 1 + \sum_{j=2}^{\infty} g_j$  and from the ”if” part follows that the arbitrariness which we determine the vector fields with the given Reeb’s inverse integrating factor is related with the Poincaré-Liapunov local first integral  $F = (x^2 + y^2)/2 + F_3 + F_4 + \dots$

*Proof of Theorem 13.* Now we assume that the vector field  $\mathcal{X}$  is polynomial of degree  $m$ . First we prove the ”only if” part. From (37) it follows that if  $X_n = Y_n = 0$  for  $n > m + 1$ , then

$$\begin{aligned}
(42) \quad \dot{x} &= -y + \sum_{j=2}^{\infty} \mathcal{X}_j(x) = -y + \sum_{j=2}^m \mathcal{X}_j(x) \\
&= -y + \{H_{m+1}, x\} + (1 + g_1)\{H_m, x\} + \dots + (1 + g_1 + \dots + g_{m-1})\{H_2, x\}, \\
\dot{y} &= x + \sum_{j=2}^{\infty} \mathcal{X}_j(y) = x + \sum_{j=2}^m \mathcal{X}_j(y) \\
&= x + \{H_{m+1}, y\} + (1 + g_1)\{H_m, y\} + \dots + (1 + g_1 + \dots + g_{m-1})\{H_2, y\}.
\end{aligned}$$

Clearly, if  $X_n = Y_n = 0$  for  $n \geq m + 1$ , then

$$\begin{aligned}
(43) \quad \mathcal{X}_{m+1}(x) &= \{H_{m+2}, x\} + g_1\{H_{m+1}, x\} + \dots + g_{m-1}\{H_3, x\} + g_m\{H_2, x\} = 0, \\
\mathcal{X}_{m+1}(y) &= \{H_{m+2}, y\} + g_1\{H_{m+1}, y\} + \dots + g_{m-1}\{H_3, y\} + g_m\{H_2, y\} = 0, \\
\mathcal{X}_{m+k}(x) &= \{H_{m+k+1}, x\} + g_1\{H_{m+k}, x\} + \dots + g_{m+k-1}\{H_2, x\} = 0, \\
\mathcal{X}_{m+k}(y) &= \{H_{m+k+1}, y\} + g_1\{H_{m+k}, y\} + \dots + g_{m+k-1}\{H_2, y\} = 0,
\end{aligned}$$

for  $k > 2$ . This system of partial differential equations of first order is compatible if and only if the following relations hold

$$(44) \quad \begin{aligned} \{H_{m+1}, g_1\} + \{H_m, g_2\} + \dots + \{H_3, g_{m-1}\} + \{H_2, g_m\} &= 0, \\ \{H_{m+k}, g_1\} + \{H_{m+k-1}, g_2\} + \dots + \{H_2, g_{m+k-1}\} &= 0, \end{aligned}$$

for  $k \geq 2$ . Hence, in view of Proposition 8 we get that

$$\int_0^{2\pi} \left( \{H_{m+1}, g_1\} + \{H_m, g_2\} + \dots + \{H_3, g_{m-1}\} \right) \Big|_{x=\cos t, y=\sin t} dt = 0.$$

We shall study partial differential equations (43) under the conditions (44).

For  $k = 2$  from (43) we get

$$(45) \quad dH_{m+2} = -g_1 dH_{m+1} - g_2 dH_m + \dots - g_{m-1} dH_3 - g_m dH_2,$$

where  $g_m = g_m(x, y)$  is an arbitrary homogenous polynomial of degree  $m$  which satisfies the first order partial differential equations (see (44) for  $n = m + 1$ .) Hence the two first partial differential system (43) is compatible, consequently integrating the 1-form (45) we obtain

$$H_{m+2} = - \int_{\gamma} (g_1 dH_{m+1} + g_2 dH_m + \dots + g_{m-1} dH_3 + g_m dH_2).$$

On the other hand from (43) and using that  $H_j$  are homogenous polynomial of degree  $j$ , we get that

$$H_{m+2} = - \frac{1}{m+2} ((m+1)g_1 H_{m+1} + \dots + 3g_{m-1} H_3 + 2g_m H_2).$$

For  $k = 3$  system (43) becomes

$$(46) \quad \begin{aligned} \{H_{m+3}, x\} + g_1 \{H_{m+2}, x\} + \dots + g_m \{H_3, x\} + g_{m+1} \{H_2, x\} &= 0, \\ \{H_{m+3}, y\} + g_1 \{H_{m+2}, y\} + \dots + g_m \{H_3, y\} + g_{m+1} \{H_2, y\} &= 0, \end{aligned}$$

which in view of (45) system (46) can be written as

$$(47) \quad \begin{aligned} \{H_{m+3}, x\} + (g_1^2 - g_2) \{H_{m+1}, x\} + (g_1 g_2 - g_3) \{H_m, x\} \\ + \dots + (g_1 g_{m-1} - g_m) \{H_3, x\} + (g_1 g_m + g_{m+1}) \{H_2, x\} &= 0, \\ \{H_{m+3}, y\} + (g_1^2 - g_2) \{H_{m+1}, y\} + (g_1 g_2 - g_3) \{H_m, y\} \\ + \dots + (g_1 g_{m-1} - g_m) \{H_3, y\} + (g_1 g_m + g_{m+1}) \{H_2, y\} &= 0, \end{aligned}$$

where  $g_{m+1} = g_{m+1}(x, y)$  is an arbitrary homogenous polynomial of degree  $m + 1$  which satisfies the first order partial differential equation

$$(48) \quad \begin{aligned} \{(g_1^2 - g_2), H_{m+1}\} + \{(g_1 g_2 - g_3), H_m\} \\ + \dots + \{(g_1 g_{m-1} - g_m), H_3\} + \{(g_1 g_m - g_{m+1}), H_2\} &= 0. \end{aligned}$$

Hence, in view of Proposition 8 we get that

$$\begin{aligned} - \int_0^{2\pi} \left( \{(g_1^2 - g_2), H_{m+1}\} + \{(g_1 g_2 - g_3), H_m\} \right. \\ \left. + \dots + \{(g_1 g_{m-1} - g_m), H_3\} \right) \Big|_{x=\cos t, y=\sin t} dt = 0. \end{aligned}$$

On the other hand, from (47) and in view of the fact that  $H_j$  are homogenous polynomial of degree  $j$  we get that

$$\begin{aligned} H_{m+3} = & -\frac{1}{m+3} \left( (m+1)(g_1^2 - g_2) H_{m+1} - m(g_1 g_2 - g_3) H_m \right. \\ & \left. - \dots + 3(g_1 g_{m-1} - g_m) H_3 - 2(g_1 g_m - g_{m+1}) H_2 \right). \end{aligned}$$

Under the condition (48) system (47) is compatible, consequently after the integration de 1-form

$$\begin{aligned} dH_{m+3} = & -(g_1^2 - g_2) dH_{m+1} - (g_1 g_2 - g_3) dH_m \\ & - \dots + (g_1 g_{m-1} - g_m) dH_3 - (g_1 g_m - g_{m+1}) dH_2, \end{aligned}$$

we get that

$$\begin{aligned} H_{m+3} = & -\int_{\gamma} (g_1^2 - g_2) dH_{m+1} - (g_1 g_2 - g_3) dH_m \\ & - \dots + (g_1 g_{m-1} - g_m) dH_3 - (g_1 g_m - g_{m+1}) dH_2 \\ = & -\frac{1}{m+3} \left( (m+1)(g_1^2 - g_2) H_{m+1} - m(g_1 g_2 - g_3) H_m \right. \\ & \left. - \dots + 3(g_1 g_{m-1} - g_m) H_3 - 2(g_1 g_m - g_{m+1}) H_2 \right) \end{aligned}$$

For  $k = 4$  system (43) becomes

$$\begin{aligned} \{H_{m+4}, x\} + g_1 \{H_{m+3}, x\} + \dots + g_m \{H_4, x\} + g_{m+1} \{H_3, x\} + g_{m+2} \{H_2, x\} &= 0, \\ \{H_{m+4}, y\} + g_1 \{H_{m+3}, y\} + \dots + g_m \{H_4, y\} + g_{m+1} \{H_3, y\} + g_{m+2} \{H_2, y\} &= 0, \end{aligned}$$

which in view of (45) system (46) can be written as

$$\begin{aligned} (49) \quad & \{H_{m+4}, x\} + (-g_1^3 + 2g_1 g_2 - g_3) \{H_{m+1}, x\} \\ & + (-g_1^2 g_2 + g_3 g_1 + g_2^2 - g_4) \{H_m, x\} + \dots + (g_1 g_{m+1} - g_1^2 g_{m-1} - g_{m+2}) \{H_2, x\} = 0, \\ & \{H_{m+4}, y\} + (-g_1^3 + 2g_1 g_2 - g_3) \{H_{m+1}, y\} \\ & + (-g_1^2 g_2 + g_3 g_1 + g_2^2 - g_4) \{H_m, y\} + \dots + (g_1 g_{m+1} - g_1^2 g_{m-1} - g_{m+2}) \{H_2, y\} = 0, \end{aligned}$$

where  $g_{m+2} = g_{m+2}(x, y)$  is an arbitrary homogenous polynomial of degree  $m+2$  which satisfies the first order partial differential equation

$$(50) \quad \begin{aligned} 0 = & \{-g_1^3 + 2g_1 g_2 - g_3, H_{m+1}\} + \{-g_1^2 g_2 + g_3 g_1 + g_2^2 - g_4, H_m\} \\ & + \dots + \{(g_2 - g_1^2)g_m + g_1 g_{m+1} - g_{m+2}, H_2\}. \end{aligned}$$

$$\begin{aligned} & \int_0^{2\pi} \left( \{(-g_1^3 + 2g_1 g_2 - g_3), H_{m+1}\} \right. \\ & \left. + \dots + \{((g_2 - g_1^2)g_{m-1} + g_1 g_m - g_{m+1}), H_3\} \right) \Big|_{x=\cos t, y=\sin t} dt = 0. \end{aligned}$$

Under this condition system (49) is compatible, consequently after the integration of the 1-form

$$dH_{m+4} = -(-g_1^3 + 2g_1 g_2 - g_3) dH_{m+1} - \dots - ((g_2 - g_1^2)g_m + g_1 g_{m+1} - g_{m+2}) dH_2$$



and using the property of homogenous polynomial we get that

$$\begin{aligned} H_{m+4} &= \int_{\gamma} \sum_{j=1}^m (g_j(g_2 - g_1^2) + g_1 g_{j+1} - g_{j+2}) dH_{m+2-j}. \\ &= \frac{1}{m+4} \sum_{j=1}^m (m+2-j) ((g_j(g_2 - g_1^2) + g_1 g_{j+1} - g_{j+2}) H_{m+2-j}). \end{aligned}$$

By continuing this process we deduce that

$$dH_{m+k} = -\beta_{k-1} dH_{m+1} - \beta_k dH_m - \dots - (\beta_{m+k-2} + g_{m+k-2}) dH_2 \quad \text{for } k \geq 5,$$

where  $\beta_j = \beta_j(x, y)$  are homogenous polynomial of degree  $j$ , and  $g_{m+k-2}$  is an arbitrary homogenous polynomial of degree  $m+k-2$  which we choose as a solution of the first order partial differential equation

$$(51) \quad \{\beta_{k-1}, H_{m+1}\} + \dots + \{\beta_{m+k-2} + g_{m+k-2}, H_2\} = 0 \quad \text{for } k \geq 5,$$

Thus

$$\begin{aligned} H_{m+k} &= \int_{\gamma} (-\beta_{k-1} dH_{m+1} - \beta_k dH_m - \dots - (\beta_{m+k-2} + g_{m+k-2}) dH_2) \\ &= \frac{1}{m+k} ((m+1)\alpha_k H_{m+1} + m\alpha_{k+1} H_m + \dots + 2\alpha_{m+k-2} H_2) \end{aligned}$$

for  $k \geq 5$ , where  $\alpha_k$  is a convenient homogenous polynomial of degree  $k$ .

From these results it follows that the homogenous polynomials  $H_{j+1}$  and  $g_{j-1}$  for  $j > m$  we determine by the line integral and as a solution of the linear partial differential equation respectively.

By summing we finally obtain

$$\begin{aligned} (52) \quad H &= \sum_{j=2}^{\infty} H_j = (x^2 + y^2)/2 + \sum_{j=3}^{\infty} H_j \\ &= \int_{\gamma} \left(1 - g_1 - g_2 - \dots - g_{m-1} - g_m \dots + g_1^2 + \dots + g_{m-1}^2 + g_m^2 + \dots\right) dH_{m+1} \\ &+ \int_{\gamma} \left(1 - g_2 - g_3 - \dots - g_{m-1} - g_m \dots - g_1 g_2 - g_1 g_3 + \dots\right) dH_m \\ &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ &+ \int_{\gamma} \left(1 - g_m - g_{m+1} - \dots - 2g_1 g_2 - 2g_1 g_3 - \dots - 2g_1 g_{m-1} - g_1^2 - g_2^2 \dots\right) dH_2 \\ &= \tau_{m+1} H_{m+1} + \tau_m H_m + \dots + \tau_2 H_2, \end{aligned}$$

where  $\tau_j = \tau_j(x, y)$  is a convenient analytical function, for  $j = 2, \dots, m+1$ .



System of partial differential equations of first order (53) is compatible if and only if

$$(54) \quad \{V_1, F_k\} + \{V_2, F_{k-1}\} + \dots + \{V_{k-1}, F_2\} = 0,$$

where  $k \geq m + 1$ . The proof of statement (ii) can be obtained analogously to the proof of statement (i), if we take  $g_j = V_j$  and  $H_{j+1} = F_{j+1}$  for  $j = 1, \dots, m$ .

Finally we observe that from (15) it follows that

$$\begin{aligned} \frac{\partial F}{\partial y} &= \Omega(\{H_{m+1}, x\} + (1 + g_1)\{H_m, x\} + \dots + (1 + g_1 + \dots + g_{m-1})\{H_2, x\}), \\ \frac{\partial F}{\partial x} &= \Omega(\{H_{m+1}, y\} + (1 + g_1)\{H_m, y\} + \dots + (1 + g_1 + \dots + g_{m-1})\{H_2, y\}), \end{aligned}$$

From the condition  $\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}$ , we get the condition (16). In short the theorem is proved.  $\square$

**Example 29.** We shall determine the quadratic system having the Reeb integrating factor

$$\begin{aligned} V &= (1a + Ay)^{2b/A-1} = 1 + (2b - A)y + (A - 2b)(A - b)y^2 \\ &\quad + \frac{1}{3}(A - 2b)(A - b)(3A - 2b)y^3 + \dots \\ &= 1 + V_1 + V_2 + V_3 + \dots, \end{aligned}$$

where  $A$  and  $b$  are nonzero constants. The quadratic polynomial differential system (39) in this case becomes

$$X_2 = -\frac{\partial F_3}{\partial y} - (2b - A)y^2, \quad Y_2 = \frac{\partial F_3}{\partial x} + (2b - A)xy,$$

where  $F_3$  is a homogenous polynomial of degree 3, which satisfies the conditions (54) for  $k = 3$ . Hence we obtain that  $F_3 = bx^2y + \kappa y^3$ , where  $\kappa$  is a constant. Therefore

$$X_2 = -bx^2 - (A - 2b + 3\kappa)y^2, \quad Y_2 = 2bxy + (A - 2b)xy = Axy.$$

## 5. THE PROOFS OF SUBSECTION 3.2

*Proof of Theorem 15. Necessity* We suppose that system (4) has a weak center at the origin. Consequently there exists an analytic local first integral  $H = H_2(1 + \sum_{j=1}^{\infty} \Upsilon_j) := H_2\Phi$ . Then from Theorem 12 it follows the necessary and sufficient conditions on the existence of a linear type center for an analytic differential system

differential system. Thus (13) becomes

(55)

$$\begin{aligned}
\dot{x} &= V\{H, x\} = -V\left(y\Phi + H_2\frac{\partial\Phi}{\partial y}\right) = -Vy\left(\Phi + \frac{y}{2}\frac{\partial\Phi}{\partial y} + \frac{x}{2}\frac{\partial\Phi}{\partial x}\right) + V\frac{x}{2}\{\Phi, H_2\} \\
&= \left(1 + \sum_{j=1}^{\infty} g_j\right) \left(-y\left(1 + \sum_{j=1}^{\infty} \frac{j+2}{2}\Upsilon_j\right) + \frac{x}{2}\sum_{j=1}^{\infty} \frac{j+2}{2}\{\Upsilon_j, H_2\}\right) \\
&= -y\left(1 + \sum_{j=2}^{\infty} \left(\frac{j+1}{2}\Upsilon_{j-1} + \frac{j}{2}g_1\Upsilon_{j-2} + \dots + \frac{3}{2}g_{j-2}\Upsilon_1 + g_{j-1}\right)\right) \\
&\quad + \frac{x}{2}\sum_{j=2}^{\infty} \left(\{\Upsilon_{j-1}, H_2\} + g_1\{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2}\{\Upsilon_1, H_2\}\right) \\
&= -y\left(1 + \sum_{j=2}^{\infty} \Lambda_j\right) + \frac{x}{2}\sum_{j=2}^{\infty} \Omega_j \\
\dot{y} &= V\{H, y\} = V\left(x\Phi + H_2\frac{\partial\Phi}{\partial x}\right) = Vx\left(\Phi + \frac{y}{2}\frac{\partial\Phi}{\partial y} + \frac{x}{2}\frac{\partial\Phi}{\partial x}\right) + V\frac{y}{2}\{\Phi, H_2\} \\
&= \left(1 + \sum_{j=1}^{\infty} g_j\right) \left(x\left(1 + \sum_{j=1}^{\infty} \frac{j+2}{2}\Upsilon_j\right) + \frac{y}{2}\sum_{j=1}^{\infty} \frac{j+2}{2}\{\Upsilon_j, H_2\}\right) \\
&= x\left(1 + \sum_{j=2}^{\infty} \left(\frac{j+1}{2}\Upsilon_{j-1} + \frac{j}{2}g_1\Upsilon_{j-2} + \dots + \frac{3}{2}g_{j-2}\Upsilon_1 + g_{j-1}\right)\right) \\
&\quad + \frac{y}{2}\sum_{j=2}^{\infty} \left(\{\Upsilon_{j-1}, H_2\} + g_1\{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2}\{\Upsilon_1, H_2\}\right) \\
&= x\left(1 + \sum_{j=2}^{\infty} \Lambda_j\right) + \frac{y}{2}\sum_{j=2}^{\infty} \Omega_j.
\end{aligned}$$

*Sufficiency* Now we suppose that (19) holds and show that then the origin is a weak center. Indeed, from (55) we obtain that  $\dot{H}_2 = 2VH_2\{\Phi, H_2\}$ ,  $\dot{\Phi} = -2V\Phi\{\Phi, H_2\}$ , thus  $\frac{d\Phi}{dH_2} = -\frac{\Phi}{H_2}$  then  $H = H_2\Phi$  is a first integral, consequently the origin is a weak center.

The second statement we prove as follows. Under the assumption

$$\begin{aligned}
&-y\sum_{j=m}^{\infty} \left(\frac{j+1}{2}\Upsilon_{j-1} + \frac{j}{2}g_1\Upsilon_{j-2} + \dots + \frac{3}{2}g_{j-2}\Upsilon_1 + g_{j-1}\right) \\
&+ \frac{x}{2}\sum_{j=m}^{\infty} \left(\{\Upsilon_{j-1}, H_2\} + g_1\{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2}\{\Upsilon_1, H_2\}\right) = 0 \\
&x\sum_{j=m}^{\infty} \left(\frac{j+1}{2}\Upsilon_{j-1} + \frac{j}{2}g_1\Upsilon_{j-2} + \dots + \frac{3}{2}g_{j-2}\Upsilon_1 + g_{j-1}\right)
\end{aligned}$$

$$+\frac{y}{2} \sum_{j=m}^{\infty} \left( \{\Upsilon_{j-1}, H_2\} + g_1 \{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2} \{\Upsilon_1, H_2\} \right) = 0$$

which is equivalent to the equations

$$\begin{aligned} \frac{j+1}{2} \Upsilon_{j-1} + \frac{j}{2} g_1 \Upsilon_{j-2} + \dots + \frac{3}{2} g_{j-2} \Upsilon_1 + g_{j-1} &= 0 \\ \{\Upsilon_{j-1}, H_2\} + g_1 \{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2} \{\Upsilon_1, H_2\} &= 0 \end{aligned}$$

for  $j > m + 1$ , from (55) we get the following polynomial differential equations of degree  $m$ .

$$\begin{aligned} \dot{x} &= -y \left( 1 + \sum_{j=2}^m \left( \frac{j+1}{2} \Upsilon_{j-1} + \frac{j}{2} g_1 \Upsilon_{j-2} + \dots + \frac{3}{2} g_{j-2} \Upsilon_1 + g_{j-1} \right) \right) \\ &\quad + \frac{x}{2} \sum_{j=2}^m \left( \{\Upsilon_{j-1}, H_2\} + g_1 \{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2} \{\Upsilon_1, H_2\} \right), \\ \dot{y} &= x \left( 1 + \sum_{j=2}^m \left( \frac{j+1}{2} \Upsilon_{j-1} + \frac{j}{2} g_1 \Upsilon_{j-2} + \dots + \frac{3}{2} g_{j-2} \Upsilon_1 + g_{j-1} \right) \right) \\ &\quad + \frac{y}{2} \sum_{j=2}^m \left( \{\Upsilon_{j-1}, H_2\} + g_1 \{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2} \{\Upsilon_1, H_2\} \right) \end{aligned}$$

Here  $\Upsilon_j$  is a convenient homogenous polynomial of degree  $j$ , such that  $H_2 = (x^2 + y^2)/2$ ,  $H_{j+2} = H_2 \Upsilon_j$ , for  $j = 1 \dots m + 1$  and  $g_j$  is an arbitrary homogenous polynomial of degree  $j$  satisfying (16). Consequently in view of (15) we obtain the local first integral (20). Thus the theorem is proved  $\square$

**Example 30.** *The following cubic polynomial differential system has a center at the origin (see [35])*

$$\begin{aligned} \dot{x} &= -y + \frac{1}{2} (x^2 - xy - 2y^2 - xy^2 - y^3) = -y \left( 1 + y + \frac{y^2}{2} \right) + \frac{x}{2} (x - y - y^2) \\ \dot{y} &= x + \frac{1}{2} (3xy - y^2 + xy^2 - y^3) = x \left( 1 + y + \frac{y^2}{2} \right) + \frac{y}{2} (x - y - y^2), \end{aligned}$$

Consequently this system can be rewritten as (18) with the functions  $\varphi$  and  $\Lambda$  determined as follows  $1 + \Lambda = \frac{1}{2} (1 + (y + 1)^2)$ ,  $\varphi = x - y - y^2$ . and hence the center is a weak center.

In order to illustrate Theorem 13 we study the following quadratic systems.

**Example 31.** *For the quadratic differential system*

$$\dot{x} = -y - bx^2 - dy^2 \quad \dot{y} = x + Axy,$$

the functions  $H_3$  and  $g_1$  are  $H_3 = 2byH_2 + \frac{d-b-A}{3}y^3$  and  $g_1 = (A-2b)y$ . It is easy to show that the solution of equation (16) for  $m = 2$ , i.e. the equation

$H_3, \Omega\} + \{H_2, (1 + g_1)\Omega\} = 0$ , is  $\Omega = (1 + Ay)^{2b/A-1}$ . Consequently from (15) the quadratic system has the following first integral

$$\begin{aligned} h(x, y) &= \int (1 + Ay)^{2b/A-1} (dH_3 + (1 + g_1)dH_2) \\ &= (1 + Ay)^{2b/A} (2(b(A + b)(A + 2b)H_2 + (d - A - b)(1 - 2by + by^2(A + 2b))), \end{aligned}$$

and from it we obtain the Poincaré-Liapunov first integral  $H = \frac{A + b - d + h(x, y)}{2b(A + b)(A + 2b)} = H_2 + h.o.t..$

In particular if  $d - A - b = 0$ , then the quadratic polynomial differential system has a weak center at the origin with  $H_3 = 2byH_2$ , and Poincaré-Liapunov first integral  $H = (1 + Ay)^{2b/A-1}H_2$ .

*Proof of Corollary 17.* From the equation  $\Lambda = 0$  and in view of of (22) we get

$$\sum_{j=2}^{\infty} \left( \frac{j+1}{2} \Upsilon_{j-1} + \frac{j}{2} g_1 \Upsilon_{j-2} + \dots + \frac{3}{2} g_{j-2} \Upsilon_1 + g_{j-1} \right) = 0,$$

hence by considering that  $H_j = H_2 \Upsilon_{j-2}$  we obtain that

$$H_{j+1} = -\frac{1}{j+1} (j g_1 H_j + \dots + 3 g_{j-2} H_3 + 2 g_{j-1} H_2),$$

for  $j \geq 2$ . On the other hand, in view of the previous relation it follows from (22) that

$$\begin{aligned} 2\varphi &= \sum_{j=2}^{\infty} \left( \{\Upsilon_{j-1}, H_2\} + g_1 \{\Upsilon_{j-2}, H_2\} + \dots + g_{j-2} \{\Upsilon_1, H_2\} \right) \\ &= \frac{1}{H_2} \sum_{j=2}^{\infty} \left( \{H_{j+1}, H_2\} + g_1 \{H_j, H_2\} + \dots + g_{j-2} \{H_3, H_2\} \right) \\ &= \frac{1}{H_2} \sum_{j=2}^{\infty} \left( \frac{-1}{j+1} \{j g_1 H_j + \dots + 2 g_{j-1} H_2, H_2\} + g_1 \{H_j, H_2\} + \dots + g_{j-2} \{H_3, H_2\} \right) \\ &= -\frac{1}{H_2} \sum_{j=2}^{\infty} \left( \frac{1}{j+1} (\{j g_1 H_j + \dots + 2 g_{j-1} H_2, H_2\} - g_1 \{H_j, H_2\} - \dots - g_{j-2} \{H_3, H_2\}) \right) \\ &= -\frac{1}{H_2} \sum_{j=2}^{\infty} \left( \frac{1}{j+1} (\{j g_1 H_j, H_2\} - (j+1) g_1 \{H_j, H_2\} + \dots) \right) \\ &= -\frac{1}{H_2} \sum_{j=2}^{\infty} \left( \frac{1}{j+1} (j H_j \{g_1, H_2\} - g_1 \{H_j, H_2\} + \dots) \right) \\ &= -\frac{1}{H_2} \sum_{j=2}^{\infty} \left( \frac{1}{j+1} \left( \left( x \frac{\partial H_j}{\partial x} + y \frac{\partial H_j}{\partial y} \right) \{g_1, H_2\} - \left( x \frac{\partial g_1}{\partial x} + y \frac{\partial g_1}{\partial y} \right) (\{H_j, H_2\} + \dots) \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{H_2} \sum_{j=2}^{\infty} \frac{1}{j+1} (x^2 + y^2) \left( \{H_j, g_1\} + \dots + \{H_2, g_{j-1}\} \right) \\
&= 2 \sum_{j=2}^{\infty} \frac{1}{j+1} \left( \{H_j, g_1\} + \dots + \{H_2, g_{j-1}\} \right).
\end{aligned}$$

Hence, if we assume that  $\{H_j, g_1\} + \dots + \{H_2, g_{j-1}\} = 0$  for  $j > m + 1$  then we obtain the conditions under which the polynomial differential system has a uniform isochronous center at the origin. Thus the proof of the corollary follows.  $\square$

*Proof of Corollary 16.* First we observe that differential equations (18) in polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$  becomes

$$\dot{r} = r\varphi(r \cos \theta, r \sin \theta), \quad \dot{\theta} = 1 + \Lambda(r \cos \theta, r \sin \theta),$$

hence in view of that the center is weak center, then the polar coordinates must be such that  $H(r \cos \theta, r \sin \theta) = C = \text{constant}$ . Hence we get that the weak center is an isochronous center if and only if (21) holds, thus the corollary is proved.  $\square$

*Proof of Proposition 22.* Since at the origin of system (18) there is a weak center, we have an analytic first integral  $H = H_2 f$  in a neighborhood of the origin. So clearly  $H_2 = 0$  and  $f = 0$  are invariant curves of system (18). It is easy to check that  $\frac{dH_2}{dt} = 2H_2\varphi$ . From the first integral  $H = H_2 f$  we get that

$$\frac{dH_2}{dt} f + H_2 \frac{df}{dt} = 2H_2\varphi f + H_2 \frac{df}{dt} = 0.$$

Thus  $\frac{df}{dt} = -2\varphi f$ , and the proposition is proved.  $\square$

*Proof of Corollary 18.* From the Cauchy–Riemann conditions it is easy to obtain condition

$$\frac{\partial \Lambda}{\partial x} + \frac{\partial \varphi}{\partial y} = 0, \quad \frac{\partial \Lambda}{\partial y} - \frac{\partial \varphi}{\partial x} = 0 \iff \frac{\partial(\varphi + i\Lambda)}{\partial \bar{z}} = 0,$$

i.e. the functions  $\Lambda$  and  $\varphi$  are harmonic functions. Moreover differential system (18) in complex coordinates  $z = x + iy$  and  $\bar{z} = x - iy$  becomes

$$(56) \quad \dot{z} = iz + z\Phi(z) := f(z),$$

where  $f = f(z)$  is a holomorphic function,  $\Phi(z) = \varphi + i\Lambda$ . Clearly if we have a differential system  $\dot{z} = z\Psi(z)$  where  $\Psi = u(x, y) + iv(x, y)$  then this differential system can be rewritten as follows

$$\dot{x} = -yv + xu \quad \dot{y} = xv + yu,$$

i.e. can be rewritten as (18) with  $1 + \Lambda = v$  and  $\varphi = u$ .

From equations (56) we get

$$\frac{dz}{f(z)} - \overline{\left( \frac{dz}{f(z)} \right)} = 0,$$

hence after the integration the existence of the first integral

$$(57) \quad \int_{\gamma} \left( \frac{dz}{f(z)} - \overline{\left( \frac{dz}{f(z)} \right)} \right) = 2i \left( \int_{\gamma} \text{Im} \frac{dz}{f(z)} \right) = \text{Const..}$$

follows.

Now we prove that a holomorphic center for a polynomial differential system is Darboux integrable. Indeed from (57) it follows that  $F = 2 \int \operatorname{Im} \frac{dz}{f(z)}$  is a first integral.

We shall study the case when

$$f(z) = z(i + \Phi(z)) = z \prod_{s=1}^{m-1} (z - z_s)^{k_s} \quad \text{for } k_s \in \mathbb{N} : k_1 + k_2 + \dots + k_{m-1} = m-1,$$

where  $z_s$  are complex numbers for  $s = 1, \dots, m-1$ , such that

$$(58) \quad \operatorname{Re} f'(0) := 0, \quad \operatorname{Im} f'(0) = i.$$

Under this condition the origin is a holomorphic isochronous center.

We develop  $1/f(z)$  and  $1/f(\bar{z})$  as follows

$$\begin{aligned} \frac{1}{f(z)} &= \frac{A_0^{(1)}}{z} + \sum_{s=1}^{m-1} \frac{A_s^{(1)}}{z - z_s} + \sum_{s=1}^{m-1} \frac{A_s^{(2)}}{(z - z_s)^2} + \dots + \sum_{s=1}^{m-1} \frac{A_s^{(k_s)}}{(z - z_s)^{k_s}}, \\ \frac{1}{f(\bar{z})} &= \overline{\left( \frac{1}{f(z)} \right)}, \end{aligned}$$

where  $A_s^{(k_s)} = \frac{k_s!}{f^{(k_s)}(z_s)} = \alpha_s^{(k_s)} + i\beta_s^{(k_s)}$ ,  $\alpha_0^{(1)} = 0$ ,  $\beta_0^{(1)} = 1$ , with  $\alpha_s^{(k_s)}, \beta_s^{(k_s)} \in \mathbb{R}$  and  $f^{(n)}(z) = \frac{d^n f(z)}{dz^n}$ . We get after the integration

$$\begin{aligned} & \int_{\gamma} \left( \frac{A_0^{(1)}}{z} + \sum_{s=1}^{m-1} \frac{A_s^{(1)}}{z - z_s} + \sum_{s=1}^{m-1} \frac{A_s^{(2)}}{(z - z_s)^2} + \dots + \sum_{s=1}^{m-1} \frac{A_s^{(k_s)}}{(z - z_s)^{k_s}} \right) dz \\ & - \int_{\gamma} \left( \frac{\bar{A}_0^{(1)}}{\bar{z}} + \sum_{s=1}^{m-1} \frac{\bar{A}_s^{(1)}}{\bar{z} - \bar{z}_s} + \sum_{s=1}^{m-1} \frac{\bar{A}_s^{(2)}}{(\bar{z} - \bar{z}_s)^2} + \dots + \sum_{s=1}^{m-1} \frac{\bar{A}_s^{(k_s)}}{(\bar{z} - \bar{z}_s)^{k_s}} \right) d\bar{z} \\ & = \int_{\gamma} \left( \frac{dz}{f(z)} - \overline{\left( \frac{dz}{f(z)} \right)} \right). \end{aligned}$$

After some computations we obtain the following expression for the first integral

$$\begin{aligned} \tilde{F} &= \log \left( \frac{\prod_{s=0}^{m-1} (z - z_s)^{A_s^{(1)}}}{\prod_{s=0}^{m-1} (\bar{z} - \bar{z}_s)^{\bar{A}_s^{(1)}}} \right) \\ & - \sum_{s=1}^{m-1} \frac{A_s^{(2)}}{(z - z_s)} + \dots + \sum_{s=1}^{m-1} \frac{1}{(1 - k_s)} \frac{A_s^{(k_s)}}{(z - z_s)^{-1+k_s}} \\ & - \sum_{s=1}^{m-1} \frac{\bar{A}_s^{(2)}}{(\bar{z} - \bar{z}_s)} + \dots + \sum_{s=1}^{m-1} \frac{1}{(1 - k_s)} \frac{\bar{A}_s^{(k_s)}}{(\bar{z} - \bar{z}_s)^{-1+k_s}}. \end{aligned}$$



where  $z_0 = 0$ . In view of the relation

$$z - z_s = \sqrt{(x - x_s)^2 + (y - y_s)^2} e^{i \arctan \frac{y - y_s}{x - x_s}}.$$

Hence

$$\begin{aligned} \tilde{F} &= \log \left( \prod_{s=0}^{m-1} (|z - z_s|)^{A_s^{(1)} - \bar{A}_s^{(1)}} e^{i(A_s^{(1)} + \bar{A}_s^{(1)}) \arctan \frac{y - y_s}{x - x_s}} \right) + \Psi \\ \Psi &:= - \sum_{s=1}^{m-1} \frac{A_s^{(2)}}{(z - z_s)} + \dots + \sum_{s=1}^{m-1} \frac{1}{(1 - k_s)} \frac{A_s^{(k_s)}}{(z - z_s)^{-1+k_s}} \\ &\quad - \sum_{s=1}^{m-1} \frac{\bar{A}_s^{(2)}}{(\bar{z} - \bar{z}_s)} - \dots - \sum_{s=1}^{m-1} \frac{1}{(1 - k_s)} \frac{\bar{A}_s^{(k_s)}}{(\bar{z} - \bar{z}_s)^{-1+k_s}}. \end{aligned}$$

and by considering condition (58) after tedious computations we obtain that the first integral is a Darboux first integral:

$$\begin{aligned} F &= e^{-i\tilde{F}} = e^{-i\Psi} \prod_{s=0}^{m-1} ((x - x_s)^2 + (y - y_s)^2)^{\beta_s^{(1)}} e^{\sum_{s=0}^{m-1} \alpha_s^{(1)} \arctan \frac{y - y_s}{x - x_s}} \\ &= (x^2 + y^2) \left( e^{-i\Psi} \prod_{s=1}^{m-1} ((x - x_s)^2 + (y - y_s)^2)^{\beta_s^{(1)}} e^{\sum_{s=1}^{m-1} \alpha_s^{(1)} \arctan \frac{y - y_s}{x - x_s}} \right). \end{aligned}$$

So the first integral  $F$  has a Taylor expansion in the neighborhood of the origin has the form  $F = (x^2 + y^2)(1 + h.o.t.)$ . Thus the holomorphic isochronous center is a weak center. In short the proposition is proved.  $\square$

We observe that the problem on the existence the first integral for the complex differential system was study in particular in [27]

*Proof of Proposition 23.* From (24) it follows that

$$x(-y + X - \lambda x\varphi) + y(x + Y - y\varphi) = 0,$$

where  $\lambda = 2/\mu$  and  $\varphi = \frac{\partial(-y + X)}{\partial x} + \frac{\partial(x + Y)}{\partial y}$ . Thus

$$-y + X = -\nu y + x\varphi, \quad x + Y = x\nu + y\varphi,$$

where  $\nu = \nu(x, y)$  is an arbitrary function. Denoting  $\nu = 1 + \Lambda$  we get that differential equations (4) coincide with (18). On the other hand in view of the relations

$$(-y + X) \frac{\partial H_2}{\partial x} + (x + Y) \frac{\partial H_2}{\partial y} = \lambda H_2 \varphi = \lambda H_2 \left( \frac{\partial(-y + X)}{\partial x} + \frac{\partial(x + Y)}{\partial y} \right),$$

which is equivalent to

$$(59) \quad \frac{\partial}{\partial x} \left( \frac{-y + X}{(x^2 + y^2)^{\mu/2}} \right) + \frac{\partial}{\partial y} \left( \frac{x + Y}{(x^2 + y^2)^{\mu/2}} \right) = 0.$$

i.e.  $H_2^\lambda$  is inverse integrating factor. Thus differential system (4) can be written as (25) with  $F$  given by the formula (26). In short corollary is proved.  $\square$

*Proof of Proposition 24.* Suppose that  $P$  and  $Q$  can be written as in (28) where  $H$  and  $G$  are polynomials, and we shall see that such polynomials exist when (27) holds. Then

$$\frac{\partial \tilde{H}}{\partial y} = -yG - P, \quad \frac{\partial \tilde{H}}{\partial x} = -xG + Q.$$

Hence by considering that  $\frac{\partial^2 \tilde{H}}{\partial x \partial y} = \frac{\partial^2 \tilde{H}}{\partial y \partial x}$ , we get that

$$x \frac{\partial G}{\partial y} - y \frac{\partial G}{\partial x} = \{H_2, G\} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

By considering that (27) holds, then in view of Corollary 8 we deduce that there

exists a polynomial  $G = \sum_{j=1}^{m-1} G_j$  such that

$$(60) \quad x \frac{\partial G_j}{\partial y} - y \frac{\partial G_j}{\partial x} = \{H_2, G_j\} = \frac{\partial P_j}{\partial x} + \frac{\partial Q_j}{\partial y}.$$

We can determine the function  $H$  as follows

$$\tilde{H} = \int_{x_0}^x \left( -x \sum_{j=1}^{m-1} G_j + Q \right) dx - \int_{y_0}^y \left( y \sum_{j=1}^{m-1} G_j + P \right) dy \Big|_{x=x_0},$$

where  $G_j = G_j(x, y)$  is the solution of equation (60). In short the proposition is proved.  $\square$

We observe that from (28) it follows that

$$xP + yQ = \{\tilde{H}, H_2\}, \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \{H_2, G\},$$

thus in view of Proposition 6 we obtain that

$$\int_0^{2\pi} (xP(x, y) + yQ(x, y)) \Big|_{x=\cos t, y=\sin t} dt = 0,$$

$$\int_0^{2\pi} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Big|_{x=\cos t, y=\sin t} dt = 0.$$

We consider an analytic differential system (7) under the assumptions

$$(61) \quad (x^2 + y^2) \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) = \mu(xX + yY),$$

$$\int_0^{2\pi} \left( \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \Big|_{x=\cos t, y=\sin t} dt = 0,$$

where  $\mu \in \mathbb{R} \setminus \{0\}$ .

The previous result can be extended for the analytic vector field. Thus we have the following proposition.

**Proposition 32.** *Let (2) be an analytic differential system which satisfies the relation*

$$\int_0^{2\pi} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Big|_{x=\cos t, y=\sin t} dt = 0.$$

*Then there exist analytic functions  $\tilde{H} = \sum_{j=2}^{\infty} H_j$  and  $G = \sum_{j=1}^{\infty} G_j$  such that*

$$\dot{x} = P = -\frac{\partial \tilde{H}}{\partial y} - yG = \{\tilde{H}, x\} + G\{H_2, x\}, \quad \dot{y} = Q = \frac{\partial \tilde{H}}{\partial x} + xG = \{\tilde{H}, y\} + G\{H_2, y\}.$$

*Proof.* It is analogous to the proof of Proposition 24. □

Finally We have the following remarks related with differential system (18).

**Remark 33.** (a) *It is easy to observe that the singular points of differential system (18) lies on the intersection of the curves*

$$(x^2 + y^2) \varphi(x, y) = 0 \quad (x^2 + y^2) (1 + \Lambda(x, y)) = 0.$$

*In particular if  $\Lambda = 0$  then the only singular point is the origin. If the vector field is polynomial of degree  $m$ , then by Bezout Theorem the maximum number of singular points of system (18) is  $(m - 1)^2 + 1$ .*

(b) *If in (18) we assume that  $\Lambda = \omega(x^2 + y^2) - 1$ , and  $\varphi = \lambda(x^2 + y^2)$ , then system (18) becomes*

$$\dot{x} = -y\omega(x^2 + y^2) + x\lambda(x^2 + y^2), \quad \dot{y} = x\omega(x^2 + y^2) + y\lambda(x^2 + y^2),$$

*which is called the lambda-omega system (see for instance [15]).*

(c) *It is well known the following result.*

*Let  $\mathcal{X}$  be an analytic vector field associated to differential system (4). Then  $\mathcal{X}$  has either the focus or a center at the origin, and under a formal change of coordinates differential system associated to  $\mathcal{X}$  can be reduced to the Birkhoff normal form*

$$\begin{aligned} \dot{x} &= -y(1 + S_2(x^2 + y^2)) + xS_1(x^2 + y^2), \\ \dot{y} &= (1 + S_2(x^2 + y^2)) + yS_1(x^2 + y^2), \end{aligned}$$

*where  $S_j = S_j(x^2 + y^2)$  for  $j = 1, 2$  are formal series in the variable  $x^2 + y^2$  (see for instance [3]). Clearly these differential equations are particular case of (18).*

## 6. THE PROOFS OF SUBSECTION 3.3

*Proof of Theorem 25.* We shall study only the case when the differential system is a polynomial differential systems of degree  $m$ .

It is possible to show that condition (24) is equivalent to (59). Hence from the first of condition of (61) and in view of Proposition 23 we get that a polynomial differential system (4) can be written as (25) with  $F$  given in the formula (26).

On the other hand in view of Proposition 24 and the second of conditions (61) we get that there exist polynomials  $\tilde{H} = \tilde{H}(x, y)$  and  $G = G(x, y)$  of degree  $m + 1$  and  $m - 1$  respectively, such that the following relations hold

$$(62) \quad \begin{aligned} \dot{x} &= -y + X = \{\tilde{H}, x\} + G\{H_2, x\}, \\ \dot{y} &= x + Y = \{\tilde{H}, y\} + G\{H_2, y\}, \end{aligned}$$

Hence

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = \{H_2, G\}, \quad xX + yY = \{\tilde{H}, H_2\},$$

consequently condition (24) becomes  $2H_2\{H_2, G\} = \mu\{\tilde{H}, H_2\}$ . Thus

$$\lambda H_2\{H_2, G\} = \{\tilde{H}, H_2\} \iff \{H_2, \tilde{H} + \lambda H_2 G\} = 0,$$

where  $\lambda = 2/\mu$ . Hence

$$(63) \quad \tilde{H} = -\lambda H_2 G + \lambda p(H_2) := H_2 \Upsilon$$

here  $p(H_2)$  is a polynomial of degree  $[(m - 1)/2]$ , where  $[(m - 1)/2]$  is the integer part of  $(m - 1)/2$  such that

$$p(H_2) = a_1 H_2 + a_2 H_2^2 + \dots + \alpha_m H_2^{[(m-1)/2]} := H_2 q(H_2) \quad \text{and} \quad G = -1/\lambda \Upsilon + q(H_2).$$

Thus by putting (63) into differential system (62) we get

$$\begin{aligned} \dot{x} &= -y - \frac{\partial \tilde{H}}{\partial y} - yG = -H_2 \frac{\partial \Upsilon}{\partial y} - y(1 + \Upsilon + G) \\ &= -H_2 \frac{\partial \Upsilon}{\partial y} - y\left(1 + \frac{\lambda - 1}{\lambda} \Upsilon + q(H_2)\right), \\ &= -y \left(1 + q(H_2) + (1 - 1/\lambda)\Upsilon + 1/2 \left(x \frac{\partial \Upsilon}{\partial x} + y \frac{\partial \Upsilon}{\partial y}\right)\right) + \frac{x}{2} \{\Upsilon, H_2\}, \\ \dot{y} &= x + \frac{\partial \tilde{H}}{\partial x} + xG = H_2 \frac{\partial \Upsilon}{\partial x} + x(1 + \Upsilon + G) \\ &= H_2 \frac{\partial \Upsilon}{\partial x} + x\left(1 + \frac{\lambda - 1}{\lambda} \Upsilon + q(H_2)\right) \\ &= x \left(1 + q(H_2) + (1 - 1/\lambda)\Upsilon + 1/2 \left(x \frac{\partial \Upsilon}{\partial x} + y \frac{\partial \Upsilon}{\partial y}\right)\right) + \frac{y}{2} \{\Upsilon, H_2\}. \end{aligned}$$

Consequently

$$\begin{aligned} \dot{H}_2 &= H_2 \{\Upsilon, H_2\}, \\ \dot{\Upsilon} &= - \left(1 + \frac{\lambda - 1}{\lambda} \Upsilon + q(H_2)\right) \{\Upsilon, H_2\}, \end{aligned}$$

hence

$$\frac{d\Upsilon}{dH_2} = \frac{1 - \lambda}{\lambda H_2} \Upsilon - \frac{1 + q(H_2)}{H_2}.$$

After the integration this first order linear ordinary differential equations we have the following solution

$$\Upsilon = H_2^{1/\lambda - 1} \left( C - \int \frac{1 + q(H_2)}{H_2^{1/\lambda}} dH_2 \right),$$

where  $C$  is an arbitrary constant. Consequently we have the following particular cases.

(i) If  $\lambda \neq 1$  and  $\prod_{n=2}^{[(m-1)/2]} (n - 1/\lambda) \neq 0$ , then

$$\Upsilon = H_2^{1/\lambda-1} C - \frac{1 + \alpha_1}{1 - 1/\lambda} - \frac{\alpha_2 H_2}{2 - 1/\lambda} - \dots - \frac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2] - 1/\lambda}.$$

(ii) If  $\lambda = 1$ , then

$$\Upsilon = C - (1 + \alpha_1) \log H_2 - \alpha_2 H_2 - \frac{\alpha_3 H_2^2}{2} - \dots - \frac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2] - 1}.$$

(iii) If  $1 < \lambda = 1/k \leq 1/[(m-1)/2]$  and  $\prod_{n=2, n \neq k}^{[(m-1)/2]} (n - k) \neq 0$ , then

$$\begin{aligned} \Upsilon = & -H_2^{k-1} \left( -C + \frac{1 + \alpha_1}{1 - k} H_2^{1-k} - \frac{\alpha_2}{2 - k} H_2^{2-k} - \dots \right. \\ & \left. - \alpha_k \log H_2 - \alpha_{k+1} H_2 - \frac{\alpha_{k+2}}{2} H_2^2 - \dots - \frac{\alpha_m}{[(m-1)/2] - k} H_2^{[(m-1)/2]-k} \right). \end{aligned}$$

(iv) If  $1/[(m-1)/2] \leq \lambda = 1/k < 1$ ,  $\alpha_k = 0$  and  $\prod_{n=2, n \neq k}^{[(m-1)/2]} (n - k) \neq 0$ , then

$$\Upsilon = H_2^{k-1} C - \frac{1 + \alpha_0}{1 - k} - \sum_{j=2, j \neq k}^{[(m-1)/2]} \frac{\alpha_j}{j - k} H_2^{j-1}.$$

Excluding the constant  $C$  in the obtained solutions we deduce the first integrals  $F$  given in formula (30), (31), (32), (33).  $\square$

**Remarks 34.** (a) *If in the equation (29) the relation  $1 + \alpha_1 + \frac{\lambda - 1}{\lambda} \Upsilon(0, 0) = 0$ , holds then the origin is not linear type. Indeed, under this condition the given equation becomes*

$$\begin{aligned} \dot{x} = & -y \left( \sum_{j=1}^{m-1} \left( (1 - 1/\lambda) + \frac{j}{2} \right) \Upsilon_j + \sum_{j=2}^{[(m-1)/2]} \alpha_j H_2^{j-1} \right) + \frac{x}{2} \{ \Upsilon, H_2 \}, \\ \dot{y} = & x \left( \sum_{j=1}^{m-1} \left( (1 - 1/\lambda) + \frac{j}{2} \right) \Upsilon_j + \sum_{j=2}^{[(m-1)/2]} \alpha_j H_2^{j-1} \right) + \frac{y}{2} \{ \Upsilon, H_2 \}. \end{aligned}$$

where  $\Upsilon = \sum_{j=1}^{m-1} \Upsilon_j$ , and  $\Upsilon_j$  is a homogenous polynomial of degree  $j$  for  $j = 1, \dots, m-1$ .

(b) *The first condition (61) is necessary as it follows from the next example. Consider the analytic differential system*

$$(64) \quad \begin{aligned} \dot{x} = & -y \left( 1 + \frac{\beta}{\beta + 1} G \right) + \frac{H_2}{\beta + 1} \frac{\partial G}{\partial y} + \alpha x (x^2 + y^2)^\beta = -y + X, \\ \dot{y} = & x \left( 1 + \frac{\beta}{\beta + 1} G \right) - \frac{H_2}{\beta + 1} \frac{\partial G}{\partial x} + \alpha y (x^2 + y^2)^\beta = x + Y, \end{aligned}$$

where  $\alpha$  (Liapunov constant) and  $\beta \geq 0$  are real numbers,  $G = G(x, y)$  is an analytic function in  $\mathbb{R}^2$ . It is easy to prove that (64) satisfies the first of condition (61) for arbitrary  $\alpha \neq 0$  with  $\mu = 2(\beta + 1)$  and  $\lambda = 1/(\beta + 1)$ , but the origin of this system is a focus. Moreover differential system (64) can be rewritten as

$$\dot{x} = (x^2 + y^2)^{\beta+1}\{F, x\}, \quad \dot{y} = (x^2 + y^2)^{\beta+1}\{F, y\},$$

where  $V = (x^2 + y^2)^{\beta+1}$  is an inverse integrating factor and a first integral is

$$F = \begin{cases} 1 + \frac{\beta}{\beta+1}G \\ \frac{2\beta(x^2 + y^2)^\beta}{2\beta(x^2 + y^2)^\beta} + \alpha \arctan(y/x) & \text{if } \beta \neq 0, \\ H_2 e^{-2G - 2\alpha \arctan(y/x)} & \text{if } \beta = 0, \end{cases}$$

defined in  $\mathbb{R}^2 \setminus (0, 0)$ . It is easy to prove that in this case

$$\frac{\partial(-y + X)}{\partial x} + \frac{\partial(x + Y)}{\partial y} = \{H_2, G\} + 2\alpha(\beta + 1)(x^2 + y^2)^\beta,$$

consequently

$$\int_0^{2\pi} \left( \frac{\partial(-y + X)}{\partial x} + \frac{\partial(x + Y)}{\partial y} \right) \Big|_{x=\cos t, y=\sin t} dt = 2\pi\alpha(\beta + 1) \neq 0.$$

- (c) The second of condition (61) for analytic differential system is necessary as it follows from the next example. If it holds then in view of Proposition 24 we get that analytic differential system (7) can be written as (28). In this case we have

$$\begin{aligned} & \int_0^{2\pi} \left( \frac{\partial(-y + X)}{\partial x} + \frac{\partial(x + Y)}{\partial y} \right) \Big|_{x=\cos t, y=\sin t} dt \\ &= \int_0^{2\pi} \{H_2, G\} \Big|_{x=\cos t, y=\sin t} dt = 0, \end{aligned}$$

and that differential system (28) satisfies the second conditions of (61) for arbitrary  $G$  and  $\tilde{H}$ . Clearly there exist analytic function  $G$  and  $\tilde{H}$  for which the origin is a focus and the first conditions of (61) does not hold.

## 7. THE PROOFS OF SUBSECTION 3.4

*Proof of Proposition 26.* From (14) and (34) it follows that

$$(65) \quad \begin{aligned} \frac{\partial H_{m+1}}{\partial y} + g_1 \frac{\partial H_m}{\partial y} + \dots + g_{m-1}y &= -xR_{m-1}, \\ \frac{\partial H_{m+1}}{\partial x} + g_1 \frac{\partial H_m}{\partial x} + \dots + g_{m-1}x &= yR_{m-1}. \end{aligned}$$

Thus

$$y \frac{\partial H_{m+1}}{\partial y} + x \frac{\partial H_{m+1}}{\partial x} + g_1 \left( y \frac{\partial H_m}{\partial y} + x \frac{\partial H_m}{\partial x} \right) + \dots + g_{m-1} \left( y \frac{\partial H_2}{\partial y} + x \frac{\partial H_2}{\partial x} \right) = 0.$$

Hence by considering that  $H_j$  is homogenous polynomial of degree  $j$  we get that

$$H_{m+1} = -\frac{1}{m+1} (mg_1 H_m + (m-1)g_2 H_{m-1} + \dots + 2g_{m-1} H_2).$$

Substituting this polynomial into (65) we obtain

$$\begin{aligned}
& \frac{1}{m+1} \left( m \frac{\partial H_m}{\partial y} g_1 + m H_m \frac{\partial g_1}{\partial y} - (m+1) g_1 \frac{\partial H_m}{\partial y} \right. \\
& + \dots + 2 \frac{\partial H_2}{\partial y} g_{m-1} + 2 H_2 \frac{\partial g_{m-1}}{\partial y} - (m+1) g_{m-1} \frac{\partial H_2}{\partial y} \left. \right) \\
& = \frac{1}{m+1} \left( \left( x \frac{\partial H_m}{\partial x} + y \frac{\partial H_m}{\partial y} \right) \frac{\partial g_1}{\partial y} - \left( x \frac{\partial g_1}{\partial x} + y \frac{\partial g_1}{\partial y} \right) \frac{\partial H_m}{\partial y} \right) \\
& + \dots + \frac{1}{m+1} \left( \left( x \frac{\partial H_2}{\partial x} + y \frac{\partial H_2}{\partial y} \right) \frac{\partial g_{m-1}}{\partial y} - \left( x \frac{\partial g_{m-1}}{\partial x} + y \frac{\partial g_{m-1}}{\partial y} \right) \frac{\partial H_2}{\partial y} \right) \\
& = \frac{x}{m+1} \sum_{j=1}^{m-1} \{H_{m+1-j}, g_j\} = x R_{m-1}.
\end{aligned}$$

Thus  $R_{m-1} = \frac{1}{m+1} \sum_{j=1}^{m-1} \{H_{m+1-j}, g_j\}$ . So we proved the equation of  $\dot{x}$  in (35).

The proof for  $\dot{y}$  is similar.  $\square$

*Proof of Proposition 26.* Follows from the equalities

$$\begin{aligned}
\dot{x} &= -y(1 + \Lambda) + x\varphi = -y \left( 1 + \sum_{j=1}^{m-2} \Lambda_j \right) + x \sum_{j=1}^{m-2} \Lambda_j - y\Lambda_{m-1} + x\varphi_{m-1} \\
\dot{y} &= x(1 + \Lambda) + y\varphi = x \left( 1 + \sum_{j=1}^{m-2} \Lambda_j \right) + y \sum_{j=1}^{m-2} \Lambda_j + x\Lambda_{m-1} + y\varphi_{m-1}
\end{aligned}$$

Thus the following relations must be hold

$$-y\Lambda_{m-1} + x\varphi_{m-1} = xR_{m-1}, \quad x\Lambda_{m-1} + y\varphi_{m-1} = yR_{m-1},$$

hence  $\Lambda_{m-1} = 0$  and  $\varphi_{m-1} = R_{m-1}$ . In short the proposition is proved.  $\square$

## 8. APPLICATION OF THEOREM 12 AND 13

In this section we study how to determine the Poincaré-Liapunov first integral and the Reeb inverse integrating factor for a given analytic or polynomial differential system. This problem is solved by applying Theorems 12 and 13.

Given an analytic vector field  $\mathcal{X}$  with a linear type center at the origin of coordinates, we shall use the expression of (13) to determine its first integral  $H$  and its Reeb inverse integrating factor. Thus, from (13) equating the terms of the same degree we get

$$\begin{aligned}
\{H_{j+1}, x\} + g_1 \{H_j, x\} + \dots + g_{j-1} \{H_2, x\} &= X_j, \\
\{H_{j+1}, y\} + g_1 \{H_j, y\} + \dots + g_{j-1} \{H_2, y\} &= Y_j,
\end{aligned}$$

for  $j \geq 2$ . Hence

$$\begin{aligned}
\frac{\partial H_3}{\partial y} &= -X_2 - yg_1, \\
\frac{\partial H_3}{\partial x} &= Y_2 - xg_1, \\
\frac{\partial H_4}{\partial y} &= -X_3 - g_1 \frac{\partial H_3}{\partial y} - yg_2, \\
\frac{\partial H_4}{\partial x} &= Y_3 - g_1 \frac{\partial H_3}{\partial x} - xg_2, \\
(66) \quad \frac{\partial H_5}{\partial y} &= -X_4 - g_1 \frac{\partial H_4}{\partial y} - g_2 \frac{\partial H_3}{\partial y} - yg_3, \\
\frac{\partial H_5}{\partial x} &= Y_4 - g_1 \frac{\partial H_4}{\partial x} - g_2 \frac{\partial H_3}{\partial x} - xg_3, \\
\frac{\partial H_6}{\partial y} &= -X_5 - g_1 \frac{\partial H_5}{\partial y} - g_2 \frac{\partial H_4}{\partial y} - g_3 \frac{\partial H_3}{\partial y} - yg_4, \\
\frac{\partial H_6}{\partial x} &= Y_5 - g_1 \frac{\partial H_5}{\partial x} - g_2 \frac{\partial H_4}{\partial x} - g_3 \frac{\partial H_3}{\partial x} - xg_4, \\
\vdots & \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots
\end{aligned}$$

From the first two equation of (66) it follows that  $g_1$  must satisfy the first order partial differential equation

$$(67) \quad \{H_2, g_1\} = \frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial x},$$

which by Corollary 8 has a unique solution  $g_1$ . Substituting  $g_1$  into the first two equations of (66) and using the Euler's Theorem for homogenous polynomial we obtain  $H_3 = \frac{1}{3}(xY_2 - yX_2 - 2g_1H_2)$ .

We determine  $g_2$  as the solution of the first order partial differential equation

$$\{H_2, g_2\} = \frac{\partial X_3}{\partial x} + \frac{\partial Y_3}{\partial x} - \{H_3, g_1\},$$

where  $g_1$  is the solution of (67). Then by Corollary 8 we get that under the condition

$$\int_0^{2\pi} \left( \frac{\partial X_3}{\partial x} + \frac{\partial Y_3}{\partial x} - \{H_3, g_1\} \right) \Big|_{x=\cos t, y=\sin t} dt = 0,$$

there exists  $g_2 = g_2(x, y)$  of the form  $\tilde{g}_2(x, y) + cH_2$  where  $c$  is an arbitrary constant. Hence from the third and fourth equation of (66) we get

$$H_4 = \frac{1}{4}(xY_3 - yX_3 - 3g_1H_3 - 2g_2H_2).$$

We determine  $g_3$  as the solution of the first order partial differential equation

$$\{H_2, g_3\} = \frac{\partial X_4}{\partial x} + \frac{\partial Y_4}{\partial x} - \{H_4, g_1\} - \{H_3, g_2\},$$

where  $g_1, g_2, H_3$  and  $H_4$  are solutions of the previous differential equations. Then by Corollary 8 we get that there exists a unique solution  $g_3$ . Hence from the fifth



and sixth equation of (66) we get

$$H_5 = \frac{1}{5} (xY_4 - yX_4 - 4g_1H_4 - 3g_2H_3 - 2g_3H_2).$$

By continuing this process we obtain the expression  $H$  and of the inverse integrating factor  $\left(1 + \sum_{j=1}^{\infty} g_j\right)^{-1}$ .

In order to illustrate this previous algorithm for computing the homogenous polynomials  $g_j$ 's and  $H_j$ 's we have the following proposition.

**Proposition 35.** *The polynomial differential system of degree four*

$$(68) \quad \begin{aligned} \dot{x} &= -y(1 + n_1x + n_2y) \\ &\quad + x(a_1x + a_2y + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2), \\ \dot{y} &= x(1 + n_1x + n_2y) \\ &\quad + y(a_1x + a_2y + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2), \end{aligned}$$

has a weak center at the origin if and only if one of the two set of condition holds

$$(69) \quad \begin{aligned} a_1^2 + a_2^2 &\neq 0, \\ a_1n_1 + a_2n_2 &= 0, \\ 2a_2^2a_6 + a_1(a_1^2 - 3a_2^2)a_7 + a_2(a_2^2 - a_1^2)a_9 &= 0, \\ 2a_2^2a_1a_8 + a_2(a_2^2 - 3a_1^2)a_9 + 3a_1(a_1^2 - a_2^2)a_7 &= 0. \end{aligned}$$

or

$$(70) \quad \begin{aligned} n_1^2 + n_2^2 &\neq 0, \\ a_1n_1 + a_2n_2 &= 0, \\ 2n_1^3n_6 + a_1(3n_1^2 - n_2^2)a_7 + n_1(n_1^2 - n_2^2)a_9 &= 0, \\ 2n_2^2n_1a_8 + 3n_2(n_2^2 - n_1^2)a_7 + n_1(3n_2^2 - n_1^2)a_9 &= 0. \end{aligned}$$

Moreover system (68) under conditions (69) or (70) is invariant with respect to a straight line.

*Proof. Necessity.* We suppose that the origin is a center of (68) and we shall prove that (69) or (70) holds. First we prove the necessity of the condition  $a_1n_1 + a_2n_2 = 0$ . Indeed, from Theorem 13 it follows that the differential system (68) can be written as

$$(71) \quad \begin{aligned} &\{H_5, x\} + (1 + g_1)\{H_4, x\} + (1 + g_1 + g_2)\{H_3, x\} \\ &\quad + (1 + g_1 + g_2 + g_3)\{H_2, x\} \\ &= -y(1 + n_1x + n_2y) + x(a_1x + a_2y + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2), \\ &\{H_5, y\} + (1 + g_1)\{H_4, y\} + (1 + g_1 + g_2)\{H_3, y\} \\ &\quad + (1 + g_1 + g_2 + g_3)\{H_2, y\}, \\ &= x(1 + n_1x + n_2y) + y(a_1x + a_2y + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2) \end{aligned}$$

We prove that this partial differential system has solution if and only if  $a_1n_1 + a_2n_2 = 0$ . Indeed, from (71) equating the terms of the same degree we obtain

$$\begin{aligned} \{H_3, x\} + g_1\{H_2, x\} &= -y(n_1x + n_2y) + x(a_1x + a_2y), \\ \{H_3, y\} + g_1\{H_2, y\} &= x(n_1x + n_2y) + y(a_1x + a_2y). \end{aligned}$$

By determining the homogenous polynomial  $g_1$  as the unique solution of the equation  $\{H_2, g_1\} = (n_2 + 3a_1)x + (3a_2 - n_1)y$ , we get that  $g_1 = (n_1 - 3a_2)x + (n_2 + 3a_1)y$ . By the homogeneity we finally obtain that  $H_3 = 2H_2(a_2x - a_1y)$ . By inserting these polynomials into the system

$$(72) \quad \begin{aligned} \{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2, x\} &= 0, \\ \{H_4, y\} + g_1\{H_3, y\} + g_2\{H_2, y\} &= 0, \end{aligned}$$

Hence we get  $g_2$  from the equation  $\{H_2, g_2\} + \{H_3, g_1\} = 0$ .

By Corollary 8 we deduce that this equation has solution if

$$(73) \quad \int_0^{2\pi} \{H_3, g_1\}|_{x=\cos t, y=\sin t} dt = 4\pi(a_1n_1 + a_2n_2) = 0.$$

We study the case when  $a_2a_1 \neq 0$ . Then we consider system (68) with  $n_1 = -\frac{a_2n_2}{a_1}$ . In view of (73) and by the homogeneity of  $H_4$  from (72) we get that

$$\begin{aligned} H_4 &= \frac{3a_1 + n_2}{a_1} H_2 y ((a_1^2 - a_2^2)y - 2a_1a_2x) + c_1 H_2^2, \\ g_2 &= \frac{3a_1 + n_2}{a_1} (a_1y - a_2x)^2 + c_2 H_2, \end{aligned}$$

where  $c_1$  and  $c_2$  are arbitrary constants. By inserting  $g_1, g_2, H_3$  and  $H_4$  into the equation

$$(74) \quad \begin{aligned} \{H_5, x\} + g_1\{H_4, x\} + g_2\{H_3, x\} + g_3\{H_2, x\} \\ &= x(a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2), \\ \{H_5, y\} + g_1\{H_4, y\} + g_2\{H_3, y\} + g_3\{H_2, y\} \\ &= y(a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2) \end{aligned}$$

and by Corollary 8 we get that the equation

$$\{H_4, g_1\} + \{H_3, g_2\} + \{H_2, g_3\} = 5(a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2)$$

has a unique solution  $g_3$ . From (74) we can obtain the homogenous polynomial  $H_5$ . Consequently we get that equations (71) have solutions if and only if the relation  $a_1n_1 + a_2n_2 = 0$  holds. The case when  $a_1a_2 = 0$  it is easy to study. Thus the necessity of this condition is proved.

Now we study the remain equations of (66), i.e.

$$\begin{aligned} \{H_{j+1}, x\} + g_1\{H_j, x\} + \dots + g_{j-1}\{H_2, x\} &= 0 \\ \{H_{j+1}, y\} + g_1\{H_j, y\} + \dots + g_{j-1}\{H_2, y\} &= 0, \end{aligned}$$

for  $j \geq 6$ . For  $j = 6$  we get

$$(75) \quad \begin{aligned} \frac{\partial H_6}{\partial y} &= -g_1 \frac{\partial H_5}{\partial y} - g_2 \frac{\partial H_4}{\partial y} - g_3 \frac{\partial H_3}{\partial y} - yg_4, \\ \frac{\partial H_6}{\partial x} &= -g_1 \frac{\partial H_5}{\partial x} - g_2 \frac{\partial H_4}{\partial x} - g_3 \frac{\partial H_3}{\partial x} - xg_4, \end{aligned}$$

where  $H_3, H_4, H_5, g_1, g_2$  and  $g_3$  are homogenous polynomials obtained in the previous equations. The homogenous polynomial  $H_5$  which satisfies (78) exists if and only if the homogenous polynomial  $g_4$  is a solution of the equation

$$(76) \quad \{H_5, g_1\} + \{H_4, g_2\} + \{H_3, g_3\} + \{H_2, g_4\} = 0,$$

and by Corollary 8 we need that

$$(77) \quad \begin{aligned} 0 &= \int_0^{2\pi} (\{H_3, g_3\} + \{H_4, g_2\} + \{H_5, g_1\})|_{x=\cos t, y=\sin t} dt \\ &= -\frac{3\pi(n_1 + 2a_2)}{a_2} (a_1\lambda_2 - a_2\lambda_1), \end{aligned}$$

where  $\lambda_1$  and  $\lambda_2$  are defined as

$$\begin{aligned} \lambda_1 &:= \frac{1}{2a_2^2} (2a_2^2a_6 + a_1(a_1^2 - 3a_2^2)a_7 + a_2(a_2^2 - a_1^2)a_9), \\ \lambda_2 &:= \frac{1}{2a_2^2a_1} (2a_2^2a_1a_8 + a_2(a_2^2 - 3a_1^2)a_9 + 3a_1(a_1^2 - a_2^2)a_7). \end{aligned}$$

Under the condition (78) differential equation (76) has a solution  $g_4$  which by Corollary 8 can be obtained with arbitrary term of the type  $c(x^2 + y^2)^2$ .

By using the homogeneity of  $H_6$  we get

$$H_6 = -\frac{5}{6}g_1H_5 - \frac{4}{6}g_2H_4 - \frac{3}{6}g_3H_3 - \frac{2}{6}g_4H_2.$$

Since the integral of homogenous polynomial of degree 5

$$\int_0^{2\pi} (\{H_6, g_1\} + \{H_5, g_2\} + \{H_4, g_3\} + \{H_3, g_4\})|_{x=\cos t, y=\sin t} dt \equiv 0,$$

then by Corollary 8 we obtain that there is a unique homogenous polynomial  $g_5$  satisfying

$$\{H_6, g_1\} + \{H_5, g_2\} + \{H_4, g_3\} + \{H_3, g_4\} + \{H_2, g_5\} = 0.$$

Partial differential equations of first order

$$(78) \quad \begin{aligned} \frac{\partial H_7}{\partial y} &= -g_1 \frac{\partial H_6}{\partial y} - g_2 \frac{\partial H_5}{\partial y} - g_3 \frac{\partial H_4}{\partial y} - g_4 \frac{\partial H_3}{\partial y} - yg_5, \\ \frac{\partial H_7}{\partial x} &= -g_1 \frac{\partial H_6}{\partial x} - g_2 \frac{\partial H_5}{\partial x} - g_3 \frac{\partial H_4}{\partial x} - g_4 \frac{\partial H_3}{\partial x} - xg_5, \end{aligned}$$

have a solution if and only if

$$\begin{aligned} &\int_0^{2\pi} (\{H_7, g_1\} + \{H_6, g_2\} + \{H_4, g_3\} + \{H_3, g_4\})|_{x=\cos t, y=\sin t} dt \\ &= \frac{\pi(3a_2 - n_1)}{252a_2^3} (\mu_1\lambda_1 + \mu_2\lambda_2) = 0 \end{aligned}$$

where  $H_j$  and  $g_{j-1}$  for  $j = 2, 3, 4, 5, 6$  are homogenous polynomial of degree  $j$ , and  $\mu_1$  and  $\mu_2$  are

$$\begin{aligned} & 2781a_1^2a_2^3 + 2673a_1^2a_2^2n_1 + 690a_1^2a_2n_1^2 + 819a_2^5 + 927a_2^4n_1 + 230a_2^2n_1^2 - 216ca_2^3, \\ & 5301a_1^3a_2^2 + 1305a_1^3a_2n_1 - 414a_1^3n_1^2 - 2457a_1a_2^4 - 2781a_1a_2^3n_1 - 690a_1a_2^2n_1^2 + 648a_1a_2^2c, \end{aligned}$$

respectively, where  $c$  is a constant. Under the previous condition the homogenous polynomial  $H_7$  can be calculated and we obtain

$$H_7 = -\frac{6}{7}g_1H_5 - \frac{5}{7}g_2H_5 - \frac{4}{7}g_3H_4 - \frac{3}{7}g_4H_3 - \frac{2}{7}g_5H_2.$$

By solving the system

$$\begin{aligned} \frac{\pi(2a_2 + n_1)}{2a_2} (a_2\lambda_1 - 3a_1\lambda_2) &= 0, \\ \frac{\pi(3a_2 - n_1)}{252a_2^3} (\mu_1\lambda_1 + \mu_2\lambda_2) &= 0, \end{aligned}$$

with respect to  $\lambda_1$  and  $\lambda_2$  and by considering that the determinant of the matrix of coefficients is

$$(2a_2 + n_1)(3a_2 - n_1)(379a_2^2 + 259a_2n_1 + 46n_1^2),$$

and assuming that  $(2a_2 + n_1)(3a_2 - n_1) \neq 0$  we deduce that  $\lambda_1 = \lambda_2 = 0$ . It is possible to study the case when  $(2a_2 + n_1)(3a_2 - n_1) = 0$  and the case when  $a_1a_2 = 0$ . Thus we obtain the necessity of condition (69). In analogous way we can study the case  $n_2n_1 \neq 0$ .

*Sufficiency.* We need the following results. Let

$$(79) \quad x = \kappa_1X + \kappa_2Y, \quad y = -\kappa_2X + \kappa_1Y,$$

be a non-degenerated linear transformation, i.e.  $\kappa_1^2 + \kappa_2^2 \neq 0$ . Then differential system (18) becomes

$$(80) \quad \begin{aligned} \dot{X} &= -Y \left( 1 + \tilde{\Lambda}(X, Y) \right) + X\tilde{\Omega}(X, Y), \\ \dot{Y} &= X \left( 1 + \tilde{\Lambda}(X, Y) \right) + Y\tilde{\Omega}(X, Y), \end{aligned}$$

where  $\tilde{\Lambda}(X, Y) = \Lambda(\kappa_1X + \kappa_2Y, -\kappa_2X + \kappa_1Y)$  and  $\tilde{\varphi}(X, Y) = \varphi(\kappa_1X + \kappa_2Y, -\kappa_2X + \kappa_1Y)$ .

The proof of the next claim is easy. The differential system (80) is invariant under the transformation  $(X, Y, t) \rightarrow (-X, Y, -t)$  if and only if it can be written as

$$(81) \quad \begin{aligned} \dot{X} &= -Y \left( 1 + \Theta_1(X^2, Y) \right) + X^2\Theta_2(X^2, Y), \\ \dot{Y} &= X \left( 1 + \Theta_1(X^2, Y) \right) + XY\Theta_2(X^2, Y), \end{aligned}$$

and the differential system (80) is invariant under the transformation  $(X, Y, t) \rightarrow (X, -Y, -t)$  if and only if it can be written as

$$(82) \quad \begin{aligned} \dot{X} &= -Y \left( 1 + \Theta_1(X, Y^2) \right) + XY\Theta_2(X, Y^2), \\ \dot{Y} &= X \left( 1 + \Theta_1(X, Y^2) \right) + Y^2\Theta_2(X, Y^2), \end{aligned}$$

Doing the change of variables (79) to system (68) we obtain system (80) for  $m = 4$ . From the claim system (68) written in the form (81) is invariant under the transformation  $(X, Y, t) \rightarrow (-X, Y, -t)$  if and only if

$$(83) \quad \begin{aligned} k_1 n_1 + k_2 n_2 &= 0, \\ k_1 a_2 - k_2 a_1 &= 0, \\ 2\kappa_2^2 a_6 + \kappa_1(\kappa_1^2 - 3\kappa_2^2)a_7 + \kappa_2(\kappa_2^2 - \kappa_1^2)a_9 &= 0, \\ 2\kappa_2^2 \kappa_1 a_8 + \kappa_2(\kappa_2^2 - 3\kappa_1^2)a_9 + 3\kappa_1(\kappa_1^2 - \kappa_2^2)a_7 &= 0, \end{aligned}$$

and it is invariant under the transformation  $(X, Y, t) \rightarrow (X, -Y, -t)$  if and only if

$$(84) \quad \begin{aligned} k_1 n_2 - k_2 n_1 + k_1 a_1 + k_2 a_2 &= 0, \\ 2\kappa_1^3 a_6 + \kappa_1(3\kappa_1^2 - \kappa_2^2)a_7 + \kappa_1(\kappa_1^2 - \kappa_2^2)a_9 &= 0, \\ 2\kappa_2^2 \kappa_1 a_8 + 3\kappa_2(\kappa_2^2 - \kappa_1^2)a_7 + \kappa_1(3\kappa_2^2 - \kappa_1^2)a_9 &= 0. \end{aligned}$$

We suppose that (69) or (70) hold and we claim that then the origin is a center of system (68). Now we prove this claim. First we study the case when  $a_1^2 + a_2^2 \neq 0$ . Then after the change  $x = a_1 X - a_2 Y$ ,  $y = a_2 X + a_1 Y$ , we get that this system coincides with system (81) for  $m = 4$  and with  $\kappa_1 = a_1$  and  $\kappa_2 = a_2$ , and consequently system (68) is invariant under the change  $(X, Y, t) \rightarrow (-X, Y, -t)$  i.e. it is reversible. Thus by Poincaré Theorem 9 we get that the origin is a center. We suppose that  $n_1^2 + n_2^2 \neq 0$ . Then after the change  $x = n_1 X - n_2 Y$ ,  $y = n_2 X + n_1 Y$ , we get that this system coincides with system (82) for  $m = 4$  with  $\kappa_1 = n_1$  and  $\kappa_2 = n_2$ , and consequently system (68) is invariant under the change  $(X, Y, t) \rightarrow (X, -Y, -t)$  i.e. it is reversible. Thus in view of the Poincaré Theorem we get that the origin is a center. By considering that  $\kappa_1^2 + \kappa_2^2 \neq 0$ . Then from the two first conditions of (83) and (84) it follows that  $a_1 n_1 + a_2 n_2 = 0$ . In short the proposition is proved.  $\square$

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