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On the limit cycles of planar discontinuous piecewise linear differential systems with a unique equilibrium

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Abstract This paper deals with planar discontinuous piecewise linear differential systems with two zones separated by a vertical straight line $x = k$. We assume that the left linear differential systems $(x < k)$ and the right linear differential systems $(x > k)$ sharing the same equilibrium, which is located at the origin $O(0, 0)$ without loss of generality.

Our results show that if $k = 0$, that is when the unique equilibrium $O(0, 0)$ is located in the separating line, then the planar discontinuous piecewise linear differential systems have no limit cycles. While for the case $k \neq 0$, we bound the number of limit cycles of these planar discontinuous piecewise linear differential systems, see Table 2.

Keywords Limit cycle · piecewise discontinuous linear systems

1 Introduction and statement of the main results

Planar piecewise linear differential systems are the natural extension of linear differential systems in order to investigate nonlinear dynamic. It is obvious that this class of piecewise linear differential systems with two zones separated by a straight line is the simplest class of these piecewise differential systems. Without loss of generality we can assume that the separating straight line is

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$x = k$, then we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{1,1}^- & a_{1,2}^- \\ a_{2,1}^- & a_{2,2}^- \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix} & \text{if } x < k, \\ \begin{pmatrix} a_{1,1}^+ & a_{1,2}^+ \\ a_{2,1}^+ & a_{2,2}^+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix} & \text{if } x > k, \end{cases} \tag{1}$$

where the dot denote the derivative with respect to t. We call systems (1) with $x < k$ (resp. $x > k$) the left (resp. right) linear differential systems.

In 1990 Lum and Chua [20] conjectured that a continuous piecewise linear differential systems (1) has at most one limit cycle. This conjecture has been solved by Freire et al [4] in 1998, for a shorter proof see [15]. While for the discontinuous piecewise linear differential systems (1) the situation becomes more complicate because such systems have twelve parameters in general.

We denote

$$\begin{aligned} \Sigma_C &= \{(k, y) | (a_{1,1}^- k + a_{1,2}^- y + b_1^-)(a_{1,1}^+ k + a_{1,2}^+ y + b_1^+) > 0\}, \\ \Sigma_S &= \{(k, y) | (a_{1,1}^- k + a_{1,2}^- y + b_1^-)(a_{1,1}^+ k + a_{1,2}^+ y + b_1^+) \leqslant 0\}. \end{aligned} \tag{2}$$

The sets Σ_C and Σ_S are called the *crossing set* and *sliding set* of systems (1), respectively. If an isolated periodic orbit of systems (1) have sliding points, then it will be called a *sliding limit cycle*. Otherwise it will be called a *crossing limit cycle*.

In order to simplify the analysis of the crossing limit cycles of discontinuous piecewise linear differential systems (1), Freire, Ponce and Torres in [6] reduced the study of systems (1) to study the following Liénard piecewise linear differential systems with seven parameters.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} T^- & -1 \\ D^- & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a^- \end{pmatrix} & \text{if } x < 0, \\ \begin{pmatrix} T^+ & -1 \\ D^+ & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -b \\ a^+ \end{pmatrix} & \text{if } x > 0, \end{cases} \tag{3}$$

where $T^{\pm}$ and $D^{\pm}$ denote the traces and determinants of the right and left linear differential systems of (1), respectively.

Later on Freire, Ponce and Torres in [7] obtain a more simple canonical form with just five parameters as follows

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2 - \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a_L \end{pmatrix} & \text{if } x < 0, \\ \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2 - \beta^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -b \\ a_R \end{pmatrix} & \text{if } x > 0, \end{cases} \tag{4}$$

where the parameters $\alpha = \sqrt{\text{sign}(\Delta^-)} \in \{i, 0, 1\}$, $\beta = \sqrt{\text{sign}(\Delta^+)} \in \{i, 0, 1\}$ with $\Delta^{\pm} = (T^{\pm})^2 - 4D^{\pm}$.

Applying the canonical forms (3) or (4), the number of crossing limit cycles for planar discontinuous piecewise linear differential systems (1) have been

Table 1 Lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems (3) known up to now. F, S and N denote a linear differential systems having a focus or a center, a saddle and a node, respectively. In the column there is the linear differential systems on $x > k$, and on the row the linear differential systems in $x < k$.

	F	S	N
F	3	3	3
S	3	2	2
N	3	2	2

studied in several papers, see for instance [1,7,9,10,12,18]. In summary, the maximum known number of limit cycles of planar discontinuous piecewise linear differential systems (3) up to now are given in Table 1. In this table F, S and N denotes linear differential systems with a focus or center, a saddle and a node, respectively. And for instance, when in the table we intersect the column S with the row N, and we obtain the number 2, this means that systems (3) having in $x < 0$ a linear saddle and in $x > 0$ a node the maximum number of limit cycles that we know for such systems is 2.

From Table 1 a natural open question arise: *Is* 3 *the upper bound for the maximum number of limit cycles that a discontinuous piecewise linear differential systems* (3) *with a straight line of separation can have*?

There are several papers [2,3,6,13,14,19] which investigate the upper bounds of limit cycles of systems (3) under some special conditions. Euzébio and Llibre [2] proved that if one of the linear differential systems of (3) has its equilibrium point on the line of discontinuity, that is $a^-a^+ = 0$, then systems (3) have at most four limit cycles. Later on, Llibre, Novaes and Teixeira [13, 14] reduced these upper bounds of [2] to two limit cycles and proved that this upper bound is reached. Giannakopoulos and Pliete [3] showed that systems (3) with $\mathcal{Z}_2$ symmetry have at most two crossing limit cycles. In [14], and later on in [19] it is proved that if one of the two linear differential systems (3) is a center, the maximum number of crossing limit cycles is two, and that this upper bound is reached. Freire, Ponce and Torres [6] investigated systems (3) with a maximal crossing set (that is, $b = 0$), and with a focus-focus dynamics, they proved that if either $a^+ \leqslant 0 \leqslant a^-$ or $a^-a^+ > 0$, then systems (3) have at most one limit cycle. Recently Ponce, Ros and Vela [21] showed that systems (3) of the focus-saddle type with $b = 0$ have at most one limit cycle.

In the present paper we consider the number of crossing limit cycles for discontinuous piecewise linear differential systems (1) sharing a unique nondegenerate equilibrium. Without loss of generality we assume that the unique equilibrium is located at the origin $O(0,0)$ and $k \geqslant 0$. In order to suppose that the orientation of limit cycles are counter clockwise, we impose that $a^-_{1,2} = -1$

and $a_{1,2}^{+}<0$. Thus systems (1) become

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}a_{1,1}^{-} & -1\\ a_{2,1}^{-} & a_{2,2}^{-}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} & \text{if } x<k,\\ \begin{pmatrix}a_{1,1}^{+} & a_{1,2}^{+}\\ a_{2,1}^{+} & a_{2,2}^{+}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} & \text{if } x>k,\end{cases} \tag{5}$$

with $D^{\pm}\neq 0$.

According to the orientation of the flows of systems (5), it is easy to check that if the left linear differential systems of (5) is one type of nodes, including diagonal node with distinct eigenvalues (N), non-diagonal node (N$'$) or diagonal node with equal eigenvalues (N*), or if the right linear differential systems of (5) is a saddle (S) or a diagonal node with equal eigenvalues (N*), then the crossing limit cycles can not exist. So we just need to study the cases that the left linear differential systems of (5) is a focus (F), a center (C) and a saddle (S), and the right linear differential systems of (5) is a focus (F), a center (C), a diagonal node (N) or a non-diagonal node (N$'$), see Table 2.

Our first result is concerned with the maximum number of limit cycles of systems (5) whose unique equilibrium $O(0,0)$ located in the separating line, that is $k=0$.

Theorem 1 *Planar discontinuous piecewise linear differential systems* (5) *with $k=0$ have no limit cycles.*

It is worth to remark that from [14] we can conclude that systems (3) have at most one limit cycle when the equilibrium of both the left and the right linear differential systems are located in the separating line $x=0$. Thus we reduce the upper bounds of limit cycles of systems (3) when both the left and the right linear differential systems share a unique equilibrium which is located at the origin $O(0,0)$. The proof of Theorem 1 will be given in section 2.

We denote by $\mathcal{N}(L,R)$ the maximum number of crossing limit cycles that systems (5) with L and R dynamics in $x<k$ and $x>k$ respectively. Note that a system (5) is not symmetry with respect to separating line $x=k>0$, because the unique equilibrium is located in the left zone $x<k$, thus in general $\mathcal{N}(L,R)\neq\mathcal{N}(R,L)$, see Table 2.

Our second main result is study the number of limit cycles of systems (5) with $k>0$.

Theorem 2 *The following statements hold for a planar discontinuous piecewise linear differential systems* (5) *with $k>0$.*

(i) $\mathcal{N}(\mathrm{F},\mathrm{F})\geqslant 3$, $\mathcal{N}(\mathrm{F},\mathrm{C})\geqslant 2$, $\mathcal{N}(\mathrm{F},\mathrm{N})\geqslant 1$, $\mathcal{N}(\mathrm{F},\mathrm{N}')\geqslant 1$*;*
(ii) $\mathcal{N}(\mathrm{C},\mathrm{F})\leqslant 1$, $\mathcal{N}(\mathrm{C},\mathrm{C})\leqslant 0$, $\mathcal{N}(\mathrm{C},\mathrm{N})\leqslant 1$, $\mathcal{N}(\mathrm{C},\mathrm{N}')\leqslant 1$*;*
(iii) $\mathcal{N}(\mathrm{S},\mathrm{F})\geqslant 1$, $\mathcal{N}(\mathrm{S},\mathrm{C})\geqslant 1$, $\mathcal{N}(\mathrm{S},\mathrm{N})\geqslant 1$, $\mathcal{N}(\mathrm{S},\mathrm{N}')\geqslant 1$.

These results are summarized in Table 2.

Table 2 The lower bounds for the maximum number of limit cycles of discontinuous piecewise linear differential systems (5) with $k > 0$. See Theorem 2.

	F	C	N	N′
F	3	2	1	1
C	1	0	1	1
S	1	1	1	1

The proof of Theorem 2 will be given in section 3.

From statement (ii) of Theorem 2, we know that systems (5) with $k > 0$ have at most one crossing limit cycle when the left linear differential systems of (5) is a center. While for the other cases we only give examples to show the lower bounds of limit cycles of systems (5) with $k > 0$.

Huan and Yang [8] investigated the number of limit cycles of systems (5) with $k = 1$ and of focus-focus type. They provided strong numerical evidence that those systems can have three limit cycles. Later on Llibre and Ponce in [16] gave an analytic proof that the following discontinuous piecewise linear differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 4/3 & -20/3 \\ 377/750 & -26/15 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x < 1, \\ \begin{pmatrix} 19/50 & -1 \\ 1 & 19/50 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x > 1, \end{cases} \tag{6}$$

of focus-focus type sharing the same equilibrium point has three limit cycles.

The organization of the rest paper is as follows. In section 2 we prove Theorem 1. In section 3 we divide the proof of Theorem 2 into three parts according if the left linear differential systems of (5) is a focus, a center or a saddle.

2 Proof of Theorem 1

In this section we consider the number of crossing limit cycles for systems (5) with $k = 0$. Thus we consider the planar piecewise discontinuous linear differential systems (5). It is obvious that the unique equilibrium is located in the separating line $x = 0$. In order to have the possibility of existence of limit cycles we need only to study the cases for focus or center, that is, $D^{\pm} > 0$ and $(T^{\pm})^2 - 4D^{\pm} < 0$.

We give a well known result on the necessary condition for the existence of crossing limit cycles of systems (5).

Proposition 1 *If a planar discontinuous piecewise linear differential system (5) has a crossing limit cycle $\Gamma = \Gamma^+ \cup \Gamma^-$ that intersects the separating line $x = 0$ at the two points $(0, y_0)$ and $(0, y_1)$, then*

$$T^- \sigma^- + T^+ \sigma^+ = 0, \tag{7}$$

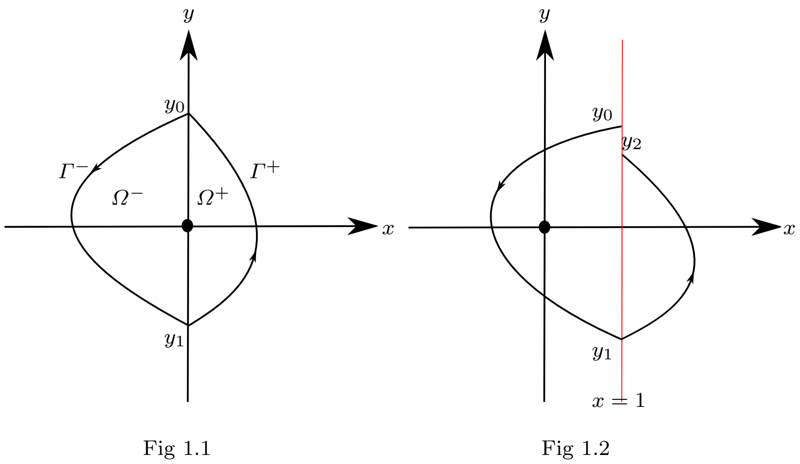

Fig. 1 Fig 1.1. A limit cycle of systems (5). Fig 1.2. Poincaré map of systems (5).

where $\sigma^{\pm}$ are denote the areas of $\Omega^{\pm}$ respectively, see Figure 1.1

The proof of Proposition 1 is deduced from the Green's formula, see for instance [5]. According to Proposition 1 a necessary condition for the existence of a crossing limit cycle for a system (5) is either $T^+T^- < 0$ or $T^+ = T^- = 0$.

Proof of Theorem 1. Let $X = x, Y = a^-_{2,2}x + y$ when $x \leq 0$ and $X = \dfrac{-1}{a^+_{1,2}}x, Y = -\dfrac{a^+_{2,2}}{a^+_{1,2}}x + y$ when $x \geq 0$. Note that both changes of variables coincide on $x = 0$, and that the straight line $x = 0$ remains invariant by this change of variables. Then in the new variables systems (5) becomes

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{cases} \begin{pmatrix} T^- & -1 \\ D^- & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} & \text{if } X < 0, \\ \begin{pmatrix} T^+ & -1 \\ D^+ & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} & \text{if } X > 0. \end{cases} \tag{8}$$

First we consider the focus-center type. It is obvious that $T^+ = 0$ and $T^- \neq 0$. By Proposition 1 we obtain that systems (5) have no limit cycles for the focus-center type. The proof of the center-focus is similar to the one of the focus-center.

Second we study the focus-focus type. Applying the change of variables given in Proposition 4.1 of [5], we can write systems (8) into the canonical form (4) with $a_L = a_R = b = 0$. According with Theorem 4.3 of [5] we know that systems (8) and consequently systems (5) have no limit cycles.

Finally we consider the center-center type, that is $T^{\pm} = 0$, thus we cannot use Proposition 1, but from Theorem 3 of [17] or Theorem 1 of [19] it follows that such systems have no limit cycles. This completes the proof of Theorem 1.

3 Proof of Theorem 2

For systems (5) with $k > 0$ we take $k = 1$ without loss of generality.

In order to investigate the crossing limit cycles of systems (5), we construct the left Poincaré map and the right Poincaré map of systems (5) as follows. Assume that the orbits starting at the point $(1, y_0)$ with $y_0 > 0$ go into the left zone $x < 1$ under the flow of the left linear differential systems. If these orbits can reach $x = 1$ again at some point $(1, y_1)$ with $y_1 < 0$ after some time $t_- > 0$, then we can define a left Poincaré map

$$y_1 = P_L(y_0), \quad y_0 > 0. \tag{9}$$

Similarly the orbits of systems (5) starting at the point $(1, y_1)$ with $y_1 < 0$ will go into the right zone $x > 1$ under the flow of the right linear differential systems. If the orbits can go back to $x = 1$ again after some time $t_+ > 0$ and intersect the line $x = 1$ at $(1, y_2)$ with $y_2 > 0$, then we can define a right Poincaré map

$$y_2 = P_R(y_1), \quad y_1 < 0. \tag{10}$$

Composite the left Poincaré map P_L and the right Poincaré map P_R, we obtain the full Poincaré map

$$y_2 = P(y_0) = P_R \circ P_L(y_0), \quad y_0 > 0. \tag{11}$$

It is obvious that the zeros of

$$F(y_0) \triangleq P(y_0) - y_0 = P_R \circ P_L(y_0) - y_0, \quad y_0 > 0, \tag{12}$$

correspondence to the limit cycles of the discontinuous piecewise linear differential systems (5), see Figure 1.2.

Since systems (5) have a vertical separating line $x = 1$ we need to use linear change of variables which preserve the vertical line $x = 1$, for obtaining the canonical forms of a piecewise linear differential systems (5).

Proposition 2 *Consider the following planar linear differential systems*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{13}$$

After a vertical line-preserving linear change of variables and a time-rescaling, then systems (13) *becomes one of the following three linear differential systems:*

(i) either a focus (resp. a center) of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A & -1 \\ 1 & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{14}$$

with $A \neq 0$ *(resp. A=0);*

(ii) *or a saddle (resp. a diagonal node) of the form*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A & -1 \\ -1 & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{15}$$

with $|A| < 1$ *(resp.* $|A| > 1$*);*

(iii) *or a non-diagonal node of the form*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{16}$$

For a proof of Proposition 2 see Proposition 4.3.1 of [14].

We divide the proof of Theorem 2 into three cases according if the left linear differential systems of (5) is a center, a saddle or a focus.

3.1 The left linear differential systems of (5) is of center type

In this subsection we assume that the left linear differential systems of (5) is of center type. Using the canonical form of (14), we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x < 1. \\ \begin{pmatrix} a^+_{1,1} & a^+_{1,2} \\ a^+_{2,1} & a^+_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x > 1, \end{cases} \tag{17}$$

with $a^+_{1,2} < 0$.

When we analyze the crossing limit cycles of discontinuous piecewise linear systems (5), we cannot choose the canonical forms given in Proposition 2 in both the left zone and the right zone. Thus we need to assume that the right zone with a general canonical form as follows.

Proposition 3 *The origin of a planar linear differential systems* (13) *in the right zone is*

(i) *a general focus (resp. a general center) when*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ \dfrac{-(a-c)^2 - d^2}{b} & 2c - a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \textit{if } x > 1, \tag{18}$$

with $b < 0$ *and* $c \neq 0$ *(resp.* $b < 0$ *and* $c = 0$*);*

(ii) *a general diagonal node (resp. a general non-diagonal node) when*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ \dfrac{-(a-c)^2 + d^2}{b} & 2c - a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \textit{if } x > 1, \tag{19}$$

with $c^2 > d^2 > 0$ *and* $b < 0$ *(resp.* $d = 0$ *and* $b < 0$*).*

Proof The eigenvalues of systems (13) with respect to the origin $O(0,0)$ are

$$\lambda_\pm = \frac{a_{1,1}+a_{2,2} \pm \sqrt{(a_{1,1}-a_{2,2})^2+4a_{1,2}a_{2,1}}}{2}. \tag{20}$$

(i) If we impose $a_{1,1}+a_{2,2}=2c$ and $(a_{1,1}-a_{2,2})^2+4a_{1,2}a_{2,1}=-4d^2$, then the origin is a general focus (resp. center) with eigenvalues $c\pm id$ when $c\neq 0$ (resp. $c=0$). From the above two equations we have $a_{2,1}=\dfrac{-(a_{1,1}-c)^2-d^2}{a_{1,2}}$ and $a_{2,2}=2c-a_{1,1}$. Replace $a_{1,1}=a, a_{1,2}=b$ into (13) then we obtain systems (18).

(ii) We assume that $a_{1,1}+a_{2,2}=2c$ and $(a_{1,1}-a_{2,2})^2+4a_{1,2}a_{2,1}=4d^2$, then the origin is a general diagonal node (resp. non-diagonal node) with eigenvalues $c\pm d$ when $d\neq 0$ (resp. $d=0$). It is easy to know that $a_{2,1}=\dfrac{-(a_{1,1}-c)^2+d^2}{a_{1,2}}$ and $a_{2,2}=2c-a_{1,1}$. Replace $a_{1,1}=a, a_{1,2}=b$ into (13) then we obtain systems (19).

With the help of Proposition 2 and Proposition 3, we can prove the statement (ii) of Theorem 2.

Proof of statement (ii) *of Theorem* 2. From the left linear differential systems of (17) we obtain the left Poincaré map

$$P_L(y_0) = -y_0, \quad y_0>0. \tag{21}$$

In the following we need to study the right Poincaré map, we distinguish four cases.

Case (i): First we consider the center-focus case of systems (17) satisfying $(18)|_{c\neq 0}$.

From elementary calculations we obtain the solution of systems $(18)|_{c\neq 0}$ starting at the point $(1, y_1)$ when $t_+=0$:

$$\begin{pmatrix} x(t_+) \\ y(t_+) \end{pmatrix} = \frac{e^{ct_+}}{d}\begin{pmatrix} d\cos(dt_+)+(a-c+by_1)\sin(dt_+) \\ dy_1\cos(dt_+)-\dfrac{d^2+(a-c)(a-c+by_1)}{b}\sin(dt_+) \end{pmatrix}. \tag{22}$$

If $x(t_+)=1$ then from (22) we get the parametric representation of the right Poincaré map

$$y_2(t_+) = \frac{-a+c+d\cot(dt_+)-de^{ct_+}\csc(dt_+)}{b}. \tag{23}$$

Note that

$$y_1(t_+) = -a+c-d\cot(dt_+)+de^{-ct_+}\csc(dt_+).$$

Recall that $y_1=-y_0$ from (21), the zeros of $F(y_0)$ is equivalent to the zeros of

$$\begin{aligned} G(t_+) &\triangleq y_2(t_+)-y_0 = y_2(t_+)+y_1(t_+) \\ &= \frac{-2}{b\sin(dt_+)}((a-c)g_0(t_+)+dg_1(t_+)), \end{aligned} \tag{24}$$

where $t_+ \in (0, \pi/d)$ in order that the function $G(t_+)$ be well defined, and

$$g_0(t_+) = \sin(dt_+), \quad g_1(t_+) = \sinh(ct_+). \tag{25}$$

In the following we will show that

$$(a - c)g_0(t_+) + dg_1(t_+) = 0, \tag{26}$$

with $t_+ \in (0, \pi/d)$ has at most one zero.

It is obvious that the coefficients $a - c$ and d can be chosen arbitrarily. The Wronskian of the functions $g_0(t_+)$ and $g_1(t_+)$ is

$$W(t_+) \triangleq \begin{vmatrix} g_0(t_+) & g_1(t_+) \\ g_0^{'}(t_+) & g_1^{'}(t_+) \end{vmatrix} = c\cosh(ct_+)\sin(dt_+) - d\cos(dt_+)\sinh(ct_+). \tag{27}$$

Since $W(0) = 0$ and $W^{'}(t_+) = (c^2 + d^2)\sin(dt_+)\sinh(ct_+) \neq 0$ if $t_+ \in (0, \pi/d)$, the Wronskian $W(t_+) \neq 0$. Thus $G(t_+)$ has at most one zero when $t_+ \in (0, \pi/d)$, and then, using Theorem 3 of the appendix, we can conclude that a system (17) satisfying $(18)|_{c\neq 0}$ has at most one crossing limit cycle.

Case (ii): Second we consider the center-center case of systems (17) satisfying $(18)|_{c=0}$. Taking into account that $c = 0$ in (18) we have $P_R(y_1) = -2a/b - y_1$. Then we can obtain $P(y_0) = -2a/b + y_0$ because $y_1 = -y_0$. Thus $F(y_0) = -2a/b$ and then a system $(17){+}(18)|_{c=0}$ has no crossing limit cycles. In fact, this result also follows from Theorem 3 of [17] or Theorem 1 of [19].

Case (iii): Now we consider the center-diagonal node case of systems $(17){+}(19)|_{d\neq 0}$.

Similar to the case (i) we get the parametric representation of the right Poincaré map of systems $(19)|_{d\neq 0}$ as

$$\begin{aligned} y_1(t_+) &= \frac{-a + c - d\coth(dt_+) + de^{-ct_+}\operatorname{csch}(dt_+)}{b}, \\ y_2(t_+) &= \frac{-a + c + d\coth(dt_+) - de^{ct_+}\operatorname{csch}(dt_+)}{b}, \end{aligned} \tag{28}$$

and then we obtain

$$G(t_+) \triangleq y_1(t_+) + y_2(t_+) = -\frac{2}{b}(a - c + d\operatorname{csch}(dt_+)\sinh(ct_+)), \quad t_+ > 0. \tag{29}$$

In the following we will prove that $G(t_+)$ has at most one zero in $t_+ > 0$. It is obvious that

$$G^{'}(t_+) = \frac{2d}{b\sinh(ct_+)\sinh(dt_+)}\left(d\coth(dt_+) - c\coth(ct_+)\right). \tag{30}$$

Since $c^2 > d^2$ we need to consider two cases $c > d > 0$ and $c < d < 0$. If $c > d > 0$, then $c\coth(ct_+) > d\coth(dt_+)$ because $\coth(t_+)$ is an increasing function with respect to t_+. If $c < d < 0$, then we have $-c > -d > 0$. From $-c\coth(-ct_+) > -d\coth(-dt_+)$, we can deduce that $c\coth(ct_+) > d\coth(dt_+)$ because $\coth(t)$ is an odd function with respect to t. Thus we obtain that $G^{'}(t_+) \neq 0$, and then we can conclude that a system (17) satisfying $(19)|_{d\neq 0}$ has at most one crossing limit cycle.

Case (iv): Finally we consider the center-nondiagonal node case of systems (17)+(18)$|_{d=0}$. From direct computations we obtain the parametric representation of the right Poincaré map of systems (19)$|_{d=0}$ as

$$y_1(t_+) = \frac{(c-a)t_+ + e^{-ct_+} - 1}{bt_+}, \quad y_2(t_+) = \frac{(c-a)t_+ + 1 - e^{ct_+}}{bt_+}, \quad t_+ > 0. \tag{31}$$

Take $G(t_+) \triangleq y_1(t_+) + y_2(t_+)$ derivative with respect to t_+ we have

$$G^{'}(t_+) = 2\frac{\sinh(ct_+) - ct_+ \cosh(ct_+)}{bt_+^2}, \quad t_+ > 0. \tag{32}$$

It is obvious that $G_1(t_+) = \sinh(ct_+) - ct_+ \cosh(ct_+) \neq 0$ because $G_1^{'}(t_+) = -c^2 t_+ \sinh(ct_+) \neq 0$ and $G_1(0) = 0$. From (32) we have $G^{'}(t_+) \neq 0$. From the above analysis, we can conclude that a system (17) satisfying (19)$|_{d=0}$ has at most one crossing limit cycle.

At the end of this subsection we give three discontinuous piecewise linear differential systems (5) having one crossing limit cycle of the type center-focus, center-node and center non-diagonal node.

Example 1 Consider a system (17) with $a_{1,1}^+ = 1, a_{1,2}^+ = -1, a_{2,1}^+ = 5, a_{2,2}^+ = -3$, then this system is of center-focus type. It has exactly one limit cycle, which intersects the switching line $x = 1$ at the two points

$$y_0 \approx 2.104005385670952, \quad y_1 = -y_0,$$

with the times

$$t_- \approx 4.028952638215224, \quad t_+ \approx 1.4354222030070583,$$

respectively, see Figure 2.1.

Example 2 Consider a system (17) with $a_{1,1}^+ = a_{1,2}^+ = -1, a_{2,1}^+ = 15, a_{2,2}^+ = 7$, then this system is of center-node (diagonal) type. It has exactly one limit cycle, which intersects the switching line $x = 1$ at the two points

$$y_0 \approx 2.2360679774997894, \quad y_1 = -y_0,$$

with the times

$$t_- \approx 3.9826613241577236, \quad t_+ \approx 0.4812118250596032,$$

respectively, see Figure 2.2.

Example 3 Consider a system (17) with $a_{1,1}^+ = a_{1,2}^+ = -1, a_{2,1}^+ = 4, a_{2,2}^+ = 3$, then this system is of center-node (non-diagonal) type. It has exactly one limit cycle, which intersects the switching line $x = 1$ at the two points

$$y_0 \approx 1.5927767117205982, \quad y_1 = -y_0,$$

with the times

$$t_- \approx 4.262862522751957, \quad t_+ \approx 2.1773189849653076,$$

respectively, see Figure 2.3.

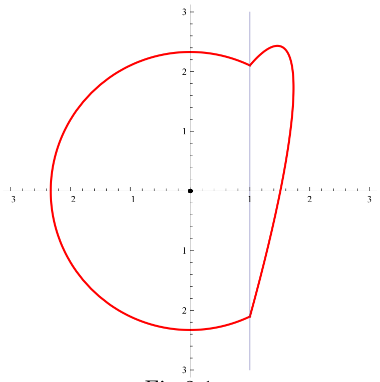

Fig 2.1

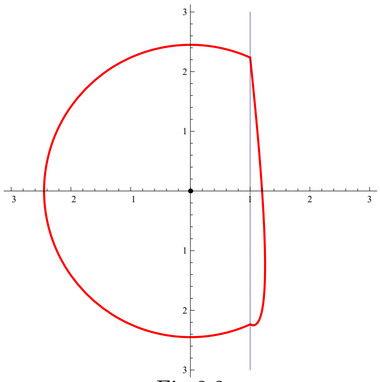

Fig 2.2

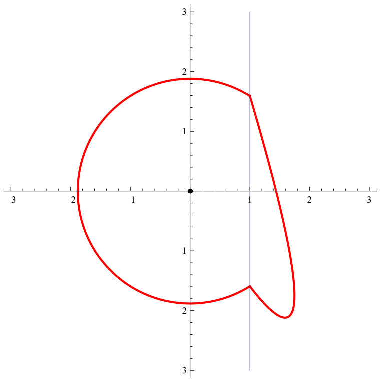

Fig 2.3

Fig. 2 The unique limit cycle of some systems (17). Fig 2.1. Center-Focus type. Fig 2.2. Center-Node (diagonal) type. Fig 2.3. Center-Node (non-diagonal) type.

3.2 The left linear differential systems of (5) is of saddle type

In this section we study the number of limit cycles of systems (5) with a saddle dynamics in the left zone.

Using the canonical form of (15) we need to consider the following differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} A & -1 \\ -1 & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x < 1, \\ \begin{pmatrix} a^+_{1,1} & a^+_{1,2} \\ a^+_{2,1} & a^+_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x > 1, \end{cases} \tag{33}$$

where $|A| < 1$ and $a^+_{1,2} < 0$.

Example 4 Consider a system (33) with $A = \dfrac{2}{5}$ and $a^+_{1,1} = 0, a^+_{1,2} = -1, a^+_{2,1} = \dfrac{5}{4}, a^+_{2,2} = 1$, then this system is of saddle-focus type. It has exactly one limit cycle, which intersects the switching line $x = 1$ at the two points

$$y_0 \approx 0.9718001266500934, \quad y_1 \approx -0.6557898066657364,$$

with the times

$$t_- \approx 2.9090827941080404, \quad t_+ \approx 1.128550128416395,$$

respectively, see Figure 3.1.

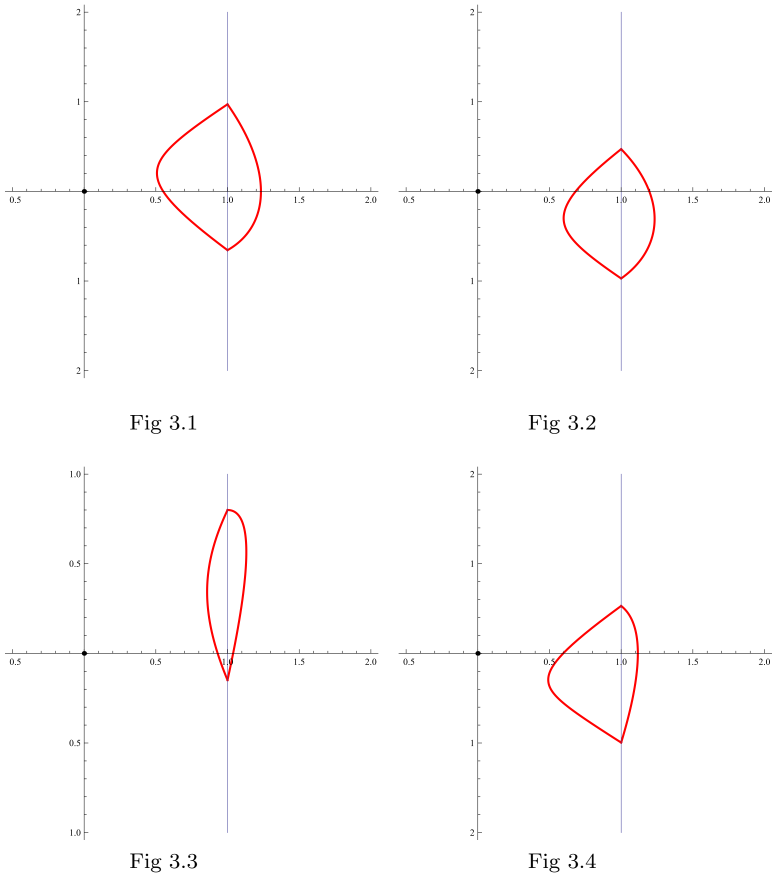

Fig. 3 Limit cycles of some systems (33). Fig 3.1. Saddle-Focus type. Fig 3.2. Saddle-Center type. Fig 3.3. Saddle-Node (diagonal) type. Fig 3.4. Saddle-Node (non-diagonal) type.

Example 5 Consider a system (33) with $A = -\dfrac{1}{2}$ and $a^+_{1,1} = -\dfrac{1}{4}, a^+_{1,2} = -1, a^+_{2,1} = \dfrac{17}{16}, a^+_{2,2} = \dfrac{1}{4}$, then this system is of saddle-center type. It has exactly one limit cycle, which intersects the switching line $x = 1$ at the two points

$$y_0 \approx 0.4716878364870325, \quad y_1 \approx -0.9716878364870325,$$

with the times

$$t_- \approx 2.633915793849635, \quad t_+ \approx 1.250267504826089,$$

respectively, see Figure 3.2.

Example 6 Consider a system (33) with $A = \frac{2}{5}$ and $a^+_{1,1} = \frac{1}{2}, a^+_{1,2} = -1, a^+_{2,1} = 2, a^+_{2,2} = -\frac{5}{2}$, then this system is of saddle-node type. It has exactly one limit cycle, which intersects the switching line $x = 1$ at the two points

$$y_0 \approx 0.7998728207036497, \quad y_1 \approx -0.14894096273908286,$$

with the times

$$t_- \approx 1.2483163101194772, \quad t_+ \approx 1.1650952227781781,$$

respectively, see Figure 3.3.

Example 7 Consider a system (33) with $A = -\frac{3}{5}$ and $a^+_{1,1} = a^+_{1,2} = -1, a^+_{2,1} = \frac{9}{4}, a^+_{2,2} = -3$, then this system is of saddle-node (non-diagonal) type. It has exactly one limit cycle, which intersects the switching line $x = 1$ at the two points

$$y_0 \approx 0.5292900080573865, \quad y_1 \approx -0.9952630050743477,$$

with the times

$$t_- \approx 3.6107224418352737, \quad t_+ \approx 0.6294144408512573,$$

respectively, see Figure 3.4.

3.3 The left linear differential systems of (5) of focus type

In this subsection we consider the limit cycles of systems (5) with a focus dynamics in the left zone.

We assume that the left linear differential systems of (5) is a general focus, then systems (5) become

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} a & b \\ \dfrac{-(a-c)^2 - d^2}{b} & 2c - a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x < 1, \\ \begin{pmatrix} a^+_{1,1} & a^+_{1,2} \\ a^+_{2,1} & a^+_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x > 1, \end{cases} \tag{34}$$

with $b < 0$ and $a^+_{1,2} < 0$.

Example 8 The discontinuous piecewise linear differential system (6) is of focus-focus type, it is the systems studied in [16]. It has exactly three limit cycles $\Gamma_i, i = 1, 2, 3$. The smallest one Γ_1 intersects the switching line $x = 1$ at the two points

$$y_0 \approx 0.6188541651825252, \quad y_1 \approx 0.1638262276270832,$$

with the times

$$t_- \approx 6.220096927394248, \quad t_+ \approx 3.9178388598443292,$$

respectively. The middle limit cycle Γ_2 intersects the switching line $x = 1$ at the two points

$$\tilde{y}_0 \approx 0.9657998558466794, \quad \tilde{y}_1 \approx -0.08891408094287911,$$

with the times

$$t_- \approx 5.043121491547688, \quad t_+ \approx 8.566832229211103,$$

respectively. The biggest limit cycle Γ_3 intersects the switching line $x = 1$ at the two points

$$\bar{y}_0 \approx 1.6811945105189345, \quad \bar{y}_1 \approx -0.48606164912659217,$$

with the times

$$t_- \approx 4.601444430614628, \quad t_+ \approx 14.866328036550136,$$

respectively, see Figure 4.1.

Example 9 Consider a system (34) with $a = 1$, $b = -1$, $c = \dfrac{1}{10}$, $d = 8$, $a^+_{1,1} = a^+_{2,2} = 0$, $a^+_{1,2} = -1$ and $a^+_{2,1} = 1$, then this system is of focus-center type. It has exactly two limit cycles Γ_1 and Γ_2. The smallest one Γ_1 intersects the switching line $x = 1$ at the two points

$$y_0 \approx 2.8326371864989257, \quad y_1 = -y_0,$$

with the times

$$t_- \approx 0.7011989432509553, \quad t_+ \approx 2.462853167680777,$$

respectively. The largest limit cycle Γ_2 intersects the switching line $x = 1$ at the two points

$$\tilde{y}_0 \approx 38.96308262220769, \quad \tilde{y}_1 = -\tilde{y}_0,$$

with the times

$$t_- \approx 0.4433513431734112, \quad t_+ \approx 3.0902732789460954,$$

respectively, see Figure 4.2.

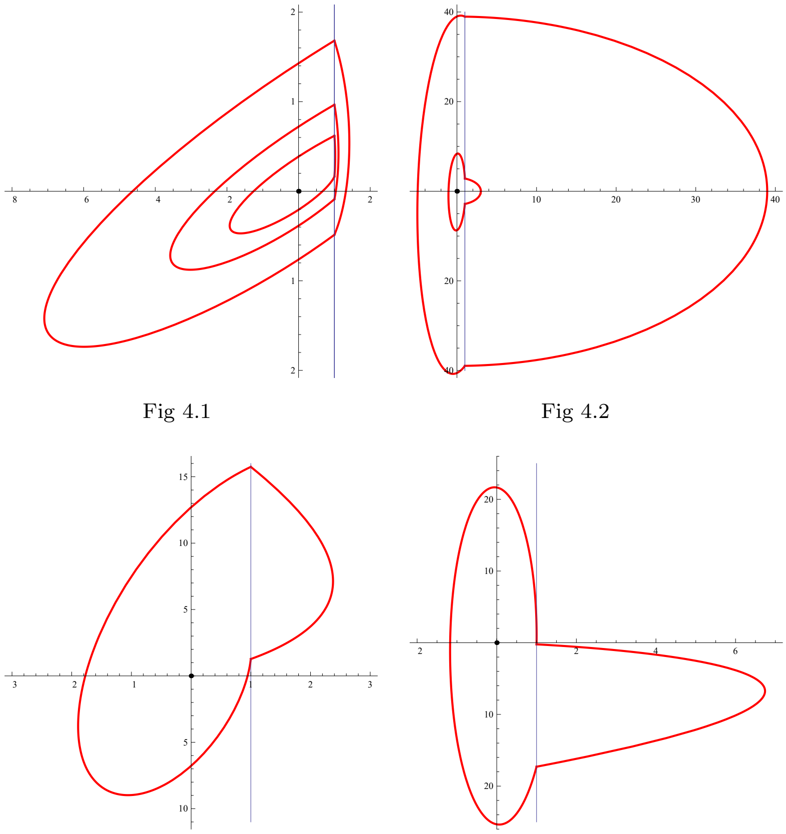

Fig 4.1 Fig 4.2

Fig 4.3 Fig 4.4

Fig. 4 Limit cycles of some systems (33). Fig 4.1. Focus-Focus type. Fig 4.2. Focus-Center type. Fig 4.3. Focus-Node (diagonal) type. Fig 4.4. Focus-Node (non-diagonal) type.

Example 10 Consider a system (34) with $a = 2$, $b = c = -1$, $d = 5$, $a^+_{1,1} = a^+_{2,2} = 3$ and $a^+_{1,2} = a^+_{2,1} = -1$, then this system is of focus-node type. It ha exactly one limit cycle, which intersects the switching line $x = 1$ at the two points

$$y_0 \approx 15.74300241360837, \quad y_1 \approx 1.2713151560165141,$$

with the times

$$t_- \approx 0.9506860894050889, \quad t_+ \approx 0.9988106885099418,$$

respectively, see Figure 4.3.

Example 11 Consider a system (34) with $a = c = 1$, $b = -1$, $d = 20$, $a^+_{1,1} = a^+_{2,2} = -1$, $a^+_{1,2} = -1$ and $a^+_{2,1} = 0$, then this system is of focus-node (no-diagonal) type. It has exactly one limit cycle, which intersects the switching line $x = 1$ at the two points

$$y_0 \approx -0.227958611935782, \quad y_1 \approx -17.293759360246668,$$

with the times

$$t_- \approx 0.27908115601573263, \quad t_+ \approx 4.328936898906302,$$

respectively, see Figure 4.4.

These examples complete the result presented in Table 2, and consequently prove Theorem 2.

Appendix: Extended Complete Chebyshev systems

The set ot functions $(f_0, \ldots, f_n)$ defined on the interval I form an *Extended Chebyshev systems* on I if and only if any nontrivial linear combination of these functions has at most n zeros counting their multiplicities and this number is reached.

The functions $(f_0, \ldots, f_n)$ form an *Extended Complete Chebyshev systems* on I if and only if for any $k \in \{0, 1, \ldots, n\}$, $(f_0, \ldots, f_k)$ form an Extended Chebyshev systems.

Theorem 3 *Let $f_0, \ldots, f_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. Then the set of functions $(f_0, \ldots, f_n)$ form an Extended Complete Chebyshev systems on I if and only if for each $k \in \{0, 1, \ldots, n\}$ and all $y \in I$ the Wronskian*

$$W(f_0, \ldots, f_k)(y) = \begin{vmatrix} f_0(y) & f_1(y) & \cdots & f_k(y) \\ f_0'(y) & f_1'(y) & \cdots & f_k'(y) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(y) & f_1^{(k)}(y) & \cdots & f_k^{(k)}(y) \end{vmatrix} \neq 0.$$

See the book [11] for a proof of the previous theorem.

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