

# UNIVALENT WANDERING DOMAINS IN THE EREMENKO-LYUBICH CLASS

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**ABSTRACT.** We use the folding theorem of [Bis15] to construct an entire function  $f$  in class  $\mathcal{B}$  and a wandering domain  $U$  of  $f$  such that  $f$  restricted to  $f^n(U)$  is univalent, for all  $n \geq 0$ . The components of the wandering orbit are bounded and surrounded by the postcritical set.

## 1. INTRODUCTION

We consider the dynamical system formed by the iterates of an entire map  $f : \mathbb{C} \rightarrow \mathbb{C}$ . We will consider only *transcendental*  $f$ , namely those maps  $f$  with an essential singularity at  $\infty$ . Such dynamical systems appear naturally as complexifications of one-dimensional real-analytic systems (interval maps or circle maps for instance) or as restrictions of analytic maps of  $\mathbb{R}^{2n}$  to certain invariant one complex-dimensional manifolds.

The dynamics of  $f$  splits the complex plane into two complementary and totally invariant sets: The *Fatou set* (or *stable set*), where the iterates form a normal family, and its closed complement, the *Julia set*,  $J(f)$ , often a fractal formed by chaotic orbits. The Fatou set is open and is generally composed of infinitely many connected components, known as *Fatou components*, which map among each other under the function  $f$ .

It was already Fatou [Fat20] who gave a complete classification of periodic Fatou components in terms of the possible limit functions of the sequence of iterates. His classification theorem states that an invariant Fatou component is either an *immediate basin of attraction* of an attracting or parabolic fixed point; or a *Siegel disk*, i.e. a topological disk on which  $f$  is conformally conjugate to a rigid irrational rotation; or a *Baker domain* if the iterates converge uniformly to infinity. This classification extends to periodic Fatou components, since a component of period  $p > 1$  is invariant under  $f^p$ .

It is well-known that each of the periodic cycles of Fatou components is in some sense associated to the orbit of a *singular value*, that is, a point around which not all branches of  $f^{-1}$  are well defined. Singular values may be *critical values* (images of zeroes of  $f'$ ) or also *asymptotic values* which, informally speaking, are points that have at least one preimage “at infinity”, such as 0 for  $z \rightarrow e^z$ .

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The set of singular values,  $S(f)$ , plays a crucial role in holomorphic dynamics precisely because of its relation with the periodic Fatou components of  $f$ . We mention two examples. First, basins of attraction must contain a singular value (and hence its forward orbit). Second, it is known that the *postsingular set*,  $P(f)$ , i.e. the singular values of  $f$  together with their forward orbits, must accumulate on the boundary of a Siegel disk [Fat20, Mil06]. The relation with Baker domains is weaker and not so easy to state, and we refer the reader to [Ber95]. This connection allows one to glean information about possible stable orbit behaviours (namely periodic Fatou components) of a given map by understanding the dynamics of its set of singular values. For this reason, many classes of entire functions have been singled out in terms of properties of their singular set. Important examples are the *Speiser class*  $\mathcal{S}$  of maps with a finite number of singular values (whose members are also called of *finite type*) or the *Eremenko-Lyubich class*

$$\mathcal{B} = \{f : \mathbb{C} \rightarrow \mathbb{C} \text{ entire} \mid S(f) \text{ is bounded}\}.$$

In the presence of an essential singularity at infinity there may exist Fatou components which are neither periodic nor eventually periodic. These are called *wandering domains* and are the subject of this paper. More precisely, a Fatou component  $U$  is a wandering domain if  $f^k(U) \cap f^j(U) = \emptyset$  for all  $k, j \in \mathbb{N}$ ,  $k \neq j$ .

Perhaps because wandering domains do not exist for rational maps [Sul85], nor for maps in the Speiser class  $\mathcal{S}$  [EL92, GK86], these rare Fatou components have not been subject of attention until quite recently, when maps with infinitely many singular values (like Newton's method applied to entire functions) have started to emerge as interesting objects. Nevertheless, many recent breakthrough results about wandering domains have appeared in the last several years. After the classical result [EL92] which states that maps in class  $\mathcal{B}$  cannot have wandering domains whose orbits converge to infinity uniformly (called *escaping wandering domains*), there was reasonable doubt of whether functions in class  $\mathcal{B}$  could have wandering domains at all. This question was answered affirmatively by C. Bishop in [Bis15] who constructed a function in class  $\mathcal{B}$  with an *oscillating* wandering domain, that is, a wandering domain whose orbits accumulate both at infinity and on a compact set (the collection of points that oscillate as such under  $f$  is termed the *Bungee set*  $BU(f)$  - see [OS16] for details and properties of this set). This construction depends on a technique, also developed in [Bis15], termed *quasiconformal folding*. An oscillating wandering domain was constructed previously in [EL87] using approximation theory techniques, however it is not known whether this example is in class  $\mathcal{B}$ . Indeed, these techniques give insufficient control over the singular set for this purpose. The question of existence of *dynamically bounded* wandering domains, i.e., wandering domains whose orbits do not accumulate at infinity, is unknown (this problem was first posed in [EL87]).

It is a wide open problem to find the sharp relationship between wandering domains and the postsingular set  $P(f)$ , or even with the singular set  $S(f)$ . Results up to now show that some relation exists: if the domain is oscillating (i.e. lies inside  $BU(f)$ ), any finite limit

function must be a constant in  $J(f) \cap \overline{P(f)}$  [Bak02] (see also [BHK<sup>+</sup>93]) and, in any case, there must be postsingular points inside or nearby the wandering components (see [BFJK] and [MBRG13] for the precise statements).

A very related and natural question is whether a wandering domain could exist such that the function were univalent on each of the orbit components. Outside the class  $\mathcal{B}$  the answer is, not surprisingly, affirmative, as shown in [EL87] and [FH09, Example 1]. The example of [EL87] is obtained using approximation theory, whereas the escaping wandering domain of [FH09, Example 1] is a logarithmic lift of an appropriately chosen invariant Siegel disk. But it is inside class  $\mathcal{B}$  where this question makes most sense to be asked. As we mention above, wandering domains in class  $\mathcal{B}$  can only be oscillating, that is, they must return infinitely often to a compact set, and hence a large amount of contraction is necessary. Let us keep in mind that Bishop's example contains a critical point of very high order inside infinitely many of its components, which allows for this large contraction. Our main result in this paper shows that, nevertheless, univalent wandering domains are also possible inside this class of maps.

**Main Theorem.** *There exists an entire transcendental function  $f \in \mathcal{B}$  and a wandering Fatou component  $U$  of  $f$  such that  $f|_{f^n(U)}$  is univalent for all  $n \geq 0$ .*

Our example uses the Folding Theorem in [Bis15] (see Section 2) and is in fact a careful modification of Bishop's original construction. Very roughly speaking Bishop's function behaves like  $(z - z_n)^{d_n}$  for some  $d_n \rightarrow \infty$ , on some subsequence of wandering components. We replace these maps by  $(z - z_n)^{d_n} + \delta_n \cdot (z - z_n)$  on subsets of the same components, which are univalent near the origin, and show that the critical values can be kept outside (but very close to) the actual wandering components. This is achieved by trapping the boundary of the wandering components inside annuli of decreasing moduli which separate the domains from the critical values.

This paper is organized as follows. In section 2 we recall Bishop's construction of a wandering domain in class  $\mathcal{B}$ , together with other preliminary results we use later on in the proof. Section 3 describes the new map on  $D$ -components that we shall work with in the construction. The general strategy is then explained in Section 4, followed in Sections 5 and 6 by the estimates that show that a wandering domain exists. Finally Section 7 is dedicated to showing that the critical values do not belong to the new wandering domain orbit.

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## 2. PRELIMINARIES

The main result of this paper is based on Bishop's construction of transcendental entire maps via quasiconformal folding [Bis15, Theorem 1.7]. Roughly speaking, given an infinite bipartite tree  $T$ , Bishop's Theorem provides an entire function in  $\mathcal{B}$  (bounded singular set) with prescribed behaviour (up to pre-composition with a quasiconformal map close to the identity) off a small neighborhood of  $T$ . As opposed to what occurs with other existence theorems (for example those in approximation theory - see for instance [Gai87]), one has fine control of the singular set of the final map, which makes this tool effective when constructing examples in restricted classes of functions such as class  $\mathcal{B}$ .

Our goal in this section is to provide the reader with the essential background to state (a simplified version of) Bishop's Theorem (Subsection 2.1), and its application to produce a transcendental entire function in  $\mathcal{B}$  having an oscillating wandering domain (Subsection 2.2). For a deeper discussion we refer to the source [Bis15] or to [FGJ15, Laz17] where some details are more explicit. Additionally, at the end of this section we recall some additional tools that will be used throughout the paper.

**2.1. On Bishop's quasiconformal folding construction.** Let  $T$  be an unbounded connected bipartite graph with vertex labels in  $\{-1, +1\}$ . Then the connected components of  $\mathbb{C} \setminus T$  are simply connected domains in  $\mathbb{C}$ . We denote by *R-components* (respectively *D-components*) the unbounded (respectively bounded) components of  $\mathbb{C} \setminus T$ . We will assume that  $T$  has *uniformly bounded geometry*, i.e. that edges are (uniformly)  $\mathcal{C}^2$  and that the diameters of edges satisfy certain uniform bounds (see Theorem 1.1 of [Bis15]). We define a neighbourhood of the graph given by

$$T(r) := \bigcup_{e \text{ edge of } T} \left\{ z \in \mathbb{C} \mid \text{dist}(z, e) < r \text{diam}(e) \right\},$$

where  $\text{dist}$  and  $\text{diam}$  denote the Euclidean distance and diameter respectively.

We denote by  $\mathbb{H}_r = \{z = x + iy \in \mathbb{C} \mid x > 0\}$  the right half plane and by  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$  the unit disk. For each connected component  $\Omega_j$  of  $\mathbb{C} \setminus T$ , let  $\tau_j : \Omega_j \rightarrow \Delta_j$  be the Riemann map where  $\Delta_j = \mathbb{H}_r$  or  $\Delta_j = \mathbb{D}$  depending on whether  $\Omega_j$  is an unbounded or bounded component. We call  $\Delta = \Delta_j$  the *standard domain* for  $\Omega_j$ . We shall denote by  $\tau$  the global map defined on  $\cup_j \Omega_j$  such that  $\tau|_{\Omega_j} = \tau_j$ . Each edge  $e$  of  $T$  is the common boundary of at most two complementary domains but corresponds via  $\tau$  to exactly two intervals on  $\partial\mathbb{H}_r$ , or one interval on  $\partial\mathbb{H}_r$  and one in  $\partial\mathbb{D}$  (see condition (i) in Theorem 2.1). The  $\tau$ -size of an edge  $e$  is defined to be the minimum among the lengths of the two images of  $e$  under  $\tau$ .

Moreover we also define a map  $\sigma$  from the standard domains  $\Delta_j$  to  $\mathbb{C}$  depending on whether  $\Delta_j$  equals  $\mathbb{H}_r$  or  $\mathbb{D}$ . More precisely we define  $\sigma(z) := \cosh(z)$  if  $\Delta_j = \mathbb{H}_r$ . Otherwise, if  $\Delta_j = \mathbb{D}$ , then  $\sigma(z) := z^d$ ,  $d \geq 2$  possibly followed by a quasiconformal map  $\rho : \mathbb{D} \mapsto \mathbb{D}$  which sends 0 to some  $w \in \text{int}(\mathbb{D})$  and is the identity on  $\partial\mathbb{D}$ .

Now we are ready to state the main result in [Bis15], in a simplified version which is sufficient for our purposes.

**Theorem 2.1.** *Let  $T$  be an unbounded connected graph and let  $\tau$  be a conformal map defined on each complementary domain  $\mathbb{C} \setminus T$  as above. Assume that:*

- (i) *No two  $D$ -components of  $\mathbb{C} \setminus T$  share a common edge.*
- (ii)  *$T$  is bipartite with uniformly bounded geometry.*
- (iii) *The map  $\tau$  on a  $D$ -component with  $2n$  edges maps the vertices to the  $2n^{\text{th}}$  roots of unity.*
- (iv) *On  $R$ -components the  $\tau$ -sizes of all edges are uniformly bounded from below.*

*Then there is an  $r_0 > 0$ , a transcendental entire  $f$ , and a  $K$ -quasiconformal map  $\phi$  of the plane, with  $K$  depending only on the uniformly bounded geometry constants, so that  $f = \sigma \circ \tau \circ \phi$  off  $T(r_0)$ . Moreover the only critical values of  $f$  are  $\pm 1$  - corresponding to the vertices of  $T$ , and those critical values assigned by the  $D$ -components.*

As mentioned, the proof of the theorem is based on some quasi-conformal deformations of the maps  $\tau$  and  $\sigma$  inside  $T_{r_0}$  so that the modified global map  $g = \sigma \circ \tau$  is quasiregular. In this situation the Measurable Riemann Mapping Theorem provides the quasiconformal map  $\phi$  in the statement. We sketch here some details needed for our construction but we refer again to [Bis15] for a more detailed approach.

If  $\Delta = \mathbb{H}_r$  we first divide  $\partial\mathbb{H}_r$  into intervals  $I$  of length  $2\pi$  and vertices in  $2\pi i\mathbb{Z}$ . These intervals will correspond, after Bishop's folding construction, to the images of the edges of  $T' \supset T$  (where  $T'$  is  $T$  with some *decorations* added, all of which are contained in a sufficiently small neighborhood  $T(r_0)$  of  $T$ ) by a suitable quasiconformal deformation  $\hat{\eta}$  of  $\tau$ . Secondly we define  $\sigma$  on  $\partial\mathbb{H}_r$ . There are two cases to consider: either  $I$  is identified with a common arc of two  $R$ -components in which case we define  $\sigma(iy) := \cosh(iy)$  for every  $iy \in I$ , or  $I$  is identified with a common arc of one  $R$ -component and one  $D$ -component, in which case we define  $\sigma(iy) := \exp(iy)$  for every  $iy \in I$ . Finally we extend  $\sigma$  to  $\mathbb{H}_r$  as a quasiconformal map which is  $\cosh(x + iy)$  for  $x > 2\pi$ .

**2.2. The prototype map.** In [Bis15, Section 17], the author gives an application of Theorem 2.1 in order to construct a family of entire functions in class  $\mathcal{B}$  depending on infinitely many parameters. One defines an unbounded connected graph  $T$  and, on some relevant complementary domains, one defines the maps  $\sigma \circ \tau$  depending on the parameters. By choosing the parameters appropriately, one is able to ensure that the resulting function has oscillating wandering domains. Since our Theorem A is a modification of this example, we briefly describe it here. Again we refer to [Bis15] or [FGJ15, Laz17] for a detailed discussion.

Consider the open half strip

$$S^+ := \left\{ x + iy \in \mathbb{C} \mid x > 0 \text{ and } |y| < \frac{\pi}{2} \right\}.$$

Following the previous notation we denote by  $(\sigma \circ \tau)|_{S^+}$  the composition  $z \mapsto \cosh(\lambda \sinh(z))$ . We remark that this map extends continuously to the boundary, sending  $\partial S^+$  onto the real segment  $[-1, +1]$ . On the upper horizontal boundary of  $S^+$ , we select points  $(a_n \pm i\pi/2)_{n \geq 1}$  which are sent to  $\{-1, +1\}$  by  $(\sigma \circ \tau)|_{S^+}$ , such that  $a_n$  is close to  $n\pi$  for every  $n \geq 1$  (see [FGJ15] for more details).

The following open disks will belong to the graph  $T$ :

$$\forall n \geq 1, \quad D_n := \{z \in \mathbb{C} \mid |z - z_n| < 1\} \quad \text{where} \quad z_n := a_n + i\pi.$$

We complete the construction of  $T$  by firstly adding segments connecting vertically the points  $(a_i, \pi/2)$  and  $(a_i, \pi - 1)$ , and the point  $(a_i, \pi + 1)$  with infinity and secondly copying the structure through the symmetries  $z \rightarrow \pm \bar{z}$ .

Again, following above notation, we will denote by  $(\sigma \circ \tau)|_{D_n}$  the composition  $z \mapsto \rho_n(z - z_n)^{d_n}$  for every  $n \geq 1$  where, for every  $n$ ,  $d_n$  is a parameter to be fixed and  $\rho_n$  is a quasiconformal map sending 0 to  $w_n$  (near  $1/2$ ) and such that  $\rho_n|_{\partial \mathbb{D}} = \text{Id}$ . Figure 1 summarizes the construction.

For suitable choices of the parameters  $\{w_n, d_n, \lambda\}$ , Theorem 2.1 gives an example of a transcendental map in  $\mathcal{B}$  with an oscillating (non-univalent) wandering domain. The contribution of the present work is to show that by modifying the quasiregular maps on  $D$ -components (see Section 3), one is able to construct a function in class  $\mathcal{B}$  which is then proven to have a *univalent* wandering domain.

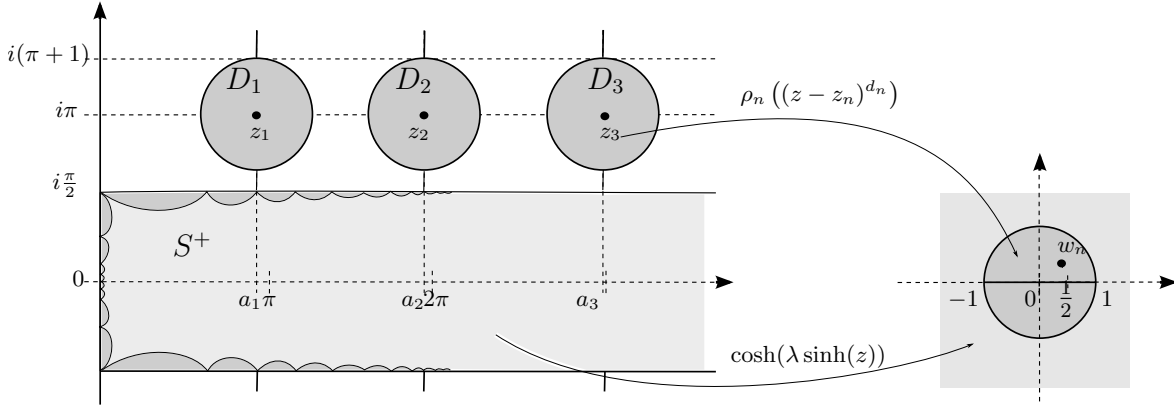


FIGURE 1. The domains  $S^+$  and  $(D_n)_{n \geq 1}$  are depicted on the left. The dark gray areas represent for the preimages of the unit disk  $\mathbb{D}$  under the map  $\sigma \circ \tau$ .

**2.3. Other tools.** The following statement is Koebe's one quarter Theorem and a part of his distortion theorem.

**Theorem 2.2** ([Pom75a, Section 1.3]). *Let  $F$  be a univalent function on the disk  $D(a, r)$  for some  $a \in \mathbb{C}$  and  $r > 0$ . Then*

- (a)  $f(D(a, r)) \supset D(F(a), \frac{1}{4}|F'(a)|r)$ .
- (b) For all  $z \in D(a, r)$ .

$$\frac{r^2|z - a||F'(a)|}{(r + |z - a|)^2} \leq |F(z) - F(a)| \leq \frac{r^2|z - a||F'(a)|}{(r - |z - a|)^2}.$$

As explained above the key idea behind Theorem 2.1 is to obtain the desired entire function  $f$  as the composition of a quasiregular map  $\sigma \circ \eta$ , and a quasiconformal map  $\phi$  given by the Measurable Riemann Mapping Theorem, that is  $f := (\sigma \circ \eta) \circ \phi$ . In particular,  $f$  and  $\sigma \circ \eta$  are not conjugate to each other. As it turns out, we shall have an explicit expression for  $\sigma \circ \eta$ , at least in the domains where the relevant dynamics occur. In order then to control the dynamics of  $f$  one needs control on the correction map  $\phi$ . The result that follows will be used to show that, in the cases we are interested in, we may assume that  $\phi$  is arbitrarily close to the identity uniformly on  $\mathbb{C}$ . Roughly speaking the next result states that if  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a  $K$ -quasiconformal map with dilatation  $\mu$  supported on a planar set whose area is exponentially small near infinity (see Definition 2.4 in [FGJ15] for a precise definition of  $(\varepsilon, h)$ -thin *small* set) then we may expect  $\Phi$  to be close to a conformal map in  $\mathbb{C}$ , and so, close to the identity after normalization.

**Theorem 2.3** (Bishop, personal communication). *Suppose  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is  $K$ -quasiconformal and is normalized to fix 0 and 1. Let  $E$  be the support of the dilatation of  $\Phi$  (possibly unbounded) and assume that  $E$  is  $(\varepsilon, h)$ -thin. Then*

$$(2.1) \quad \forall z, w \in \mathbb{R}^2, (1 - Ce^\beta)|z - w| - Ce^\beta \leq |\Phi(z) - \Phi(w)| \leq (1 + Ce^\beta)|z - w| + e^\beta$$

where  $C$  and  $\beta$  only depend on  $K$  and  $h$ .

For instance, as a consequence of the above result, and under suitable conditions on the behaviour and smoothness of  $\Phi$  in a neighborhood of the real line, one can deduce bounds on the derivative of the map. Again we refer to [FGJ15] for precise statements and discussion.

### 3. THE MAP ON $D$ -COMPONENTS

In this section we describe a quasiregular map  $g$  in each of the  $D$ -components  $D_n$  of our graph that will be used in accordance with Theorem 2.1. Recall  $D_n$  is a disc of radius 1 centered at a point  $z_n$ . We first translate  $D_n$  to the unit disc  $\mathbb{D}$  by the map  $z \rightarrow z - z_n$ . Next we post-compose with a quasiregular map  $\psi$  which will be defined as  $z \rightarrow z^m + \delta z$  in a subdisc of  $\mathbb{D}$ , interpolated with  $z \rightarrow z^m$  on  $\partial\mathbb{D}$ .

More precisely, consider for now the map  $z \rightarrow z^m + \delta z$  defined on  $\mathbb{D}$ . The  $m - 1$  critical points  $(c_k)$  and  $m - 1$  critical values  $(v_k)$  of this map are given by

$$c_k = \left(\frac{-\delta}{m}\right)^{\left(\frac{1}{m-1}\right)}, \quad v_k = \delta \left(\frac{-\delta}{m}\right)^{\left(\frac{1}{m-1}\right)} \left(\frac{m-1}{m}\right).$$

If we fix  $\delta$  and let  $m \rightarrow \infty$ , the critical points will tend to  $\partial\mathbb{D}$ , while the critical values tend, in absolute value, to  $\delta$ .

Fix  $\delta \ll 1/2$ . In order to use Theorem 2.1 we would like to consider a map  $\psi$  which is the above described map  $z \rightarrow z^m + \delta z$  on a subdisc of  $\mathbb{D}$ , interpolated with  $z \rightarrow z^m$  on  $\partial\mathbb{D}$ . More precisely, consider a Euclidean circle  $\gamma$  centered at the origin of radius  $\frac{3}{2}\delta$ , and note that for sufficiently large  $m$ , the critical values  $v_k$  belong to  $\text{int}(\gamma)$ , where  $\text{int}(\gamma)$  denotes the bounded component of  $\mathbb{C} \setminus \gamma$ . Taking the pullback of  $\gamma$  under  $z \rightarrow z^m + \delta z$  we get a simple closed curve in  $\mathbb{D}$ , that we denote  $\gamma^{-1}$ , so that the critical points  $c_k$  belong to  $\text{int}(\gamma^{-1})$ . Indeed, notice that  $\text{int}(\gamma^{-1})$  is mapped to  $\text{int}(\gamma)$ , while the unit circle is mapped outside the disk of radius  $1 - \delta \gg \delta$ , so  $\gamma^{-1} \subset \mathbb{D}$ . See Figure 2. We define the quasiregular map  $\psi$  on  $\mathbb{D}$  by

$$(3.1) \quad \psi = \begin{cases} z^m + \delta z, & \text{if } z \in \text{int}(\gamma^{-1}) \\ z^m & \text{if } z \in \partial\mathbb{D}, \end{cases}$$

and by linearly interpolating between the two definitions on the annulus  $\mathbb{D} \setminus \text{int}(\gamma^{-1})$ .

It will be important that we establish that the dilatation of the map  $\psi$  is uniformly bounded independently of  $m$  and  $\delta$ . Consider Figure 2. By symmetry, it is enough to bound the dilatation needed to interpolate between  $z \rightarrow z^m$  on  $\partial\mathbb{D}$  and  $z \rightarrow z^m + \delta z$  on  $\gamma^{-1}$  in the shaded grey sector, denoted by  $Q$ , where the black vertex and white vertex are two preimages of  $+1, -1$  under  $z \rightarrow z^m$ , respectively. We label  $w$  (the white vertex) the preimage of  $-1$ . Observe that the required dilatation is the ratio of the moduli of the two quadrilaterals shown at the bottom of Figure 2, since the map  $z^m$  is conformal in the region under consideration and hence preserves moduli. We now argue that this quantity is uniformly bounded as  $\delta \rightarrow 0$  and  $m \rightarrow \infty$ .

Indeed consider  $\psi(Q)$  as pictured, where the  $\frac{3}{2}\delta$  term is clear since we know  $\gamma^{-1}$  is sent by  $\psi$  to the circle centered at the origin of radius  $\frac{3}{2}\delta$ , and  $z^m + \delta z$  does not change the argument of a complex number  $z \in \mathbb{R}$ . On the other hand,  $\arg(w^m) = \pi$ , and so  $\arg(w^m + \delta w) = \pi + o(1)$  as  $m \rightarrow \infty$ . From this we get the  $\frac{3}{2}\delta e^{i\theta}$  term, with  $\theta = \pi + o(1)$ . We now explain the terms in the figure for the image of  $Q$  under  $z^m$ . Notice that for  $x > 0$ , if  $x^m + \delta x = \frac{3}{2}\delta$  and  $x = 1 + o(1)$ , then  $x^m = \frac{1}{2}\delta + o(1)$  as  $m \rightarrow \infty$ . And if  $\arg(x) = \arg(w)$  with  $x^m + \delta x = \frac{3}{2}\delta e^{i(\pi+o(1))} = -\frac{3}{2}\delta e^{io(1)}$ , then  $x^m = -\frac{5}{2}\delta + o(1)$  as  $m \rightarrow \infty$ .

Observe that  $\log(\psi(Q))$  tends to a Euclidean rectangle while  $\log(h(Q))$ , with  $h(z) = z^m$ , tends to a quadrilateral as  $m \rightarrow \infty$ . In particular we have

$$(3.2) \quad \begin{aligned} \text{mod}(\psi(Q)) &= \frac{1}{\pi} (\log(\delta) + \log(3/2) + o(1)), \text{ and} \\ \frac{1}{\pi} (\log(\delta) + \log(5/2) + o(1)) &\leq \text{mod}(Q) \leq \frac{1}{\pi} (\log(\delta) + \log(1/2) + o(1)), \end{aligned}$$

which implies that the quotient is uniformly bounded with  $m \rightarrow \infty$  and  $\delta \rightarrow 0$ .



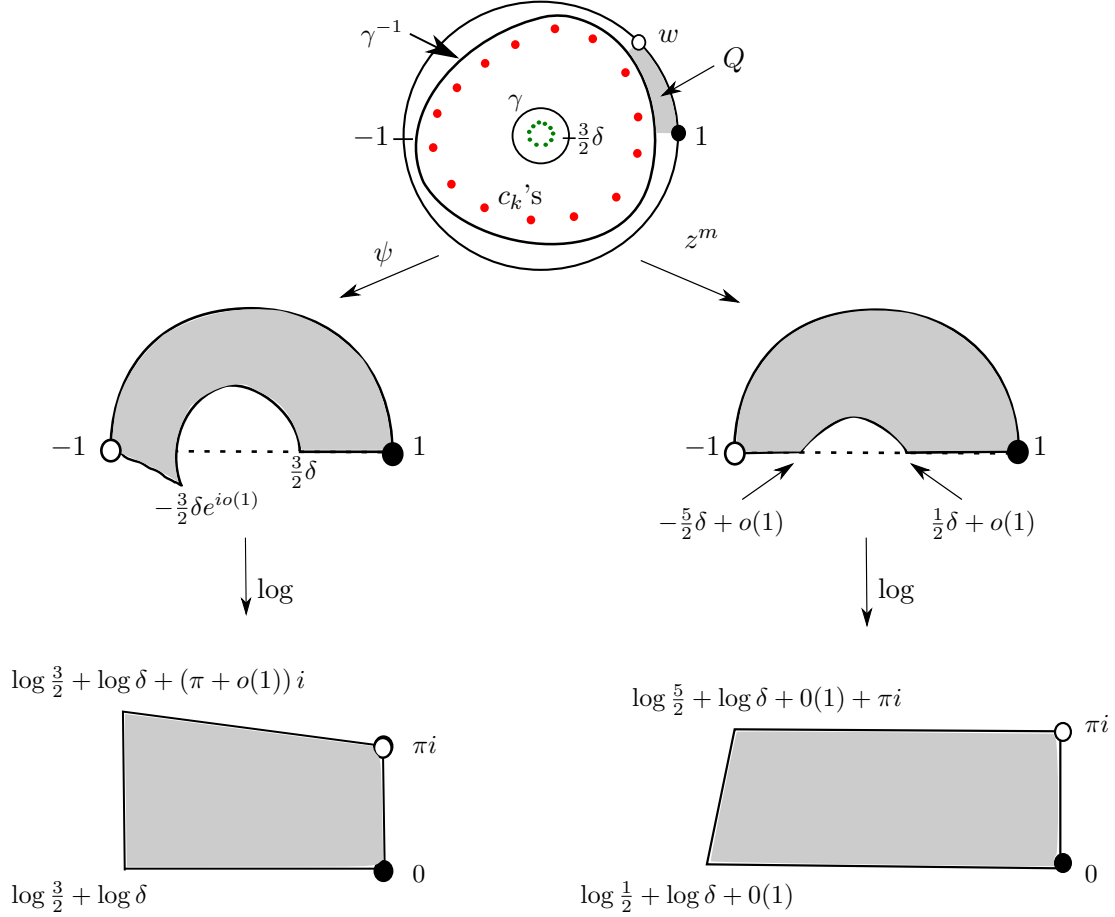


FIGURE 2. This figure illustrates the argument showing that the dilatation of  $\psi$  is uniformly bounded independent of  $m$  and  $\delta < 1/2$ .

Furthermore it will be important to note that the dilatation of the map  $\psi$  is supported on a region whose area tends to zero as  $m \rightarrow \infty$ . This, together with the fact that the dilatation of  $\psi$  is bounded independent of  $m, \delta$ , will be vital to our construction.

We have described the map  $\psi$  on  $\mathbb{D}$ . We recall we are defining a quasiregular map  $g$  on  $D_n$ , and that so far we have  $g(z) = \psi(z - z_n)$ . We will post-compose with one more quasiconformal map  $\rho$  that is nearly identical to the one in [Bis15]. Namely  $\rho$  is a quasiconformal map of  $\mathbb{D}$  which equals  $z \rightarrow z + w$  in  $\frac{1}{4}\mathbb{D}$ , and is a linear interpolation between this definition on  $\frac{1}{4}\mathbb{D}$  and the identity on  $\partial\mathbb{D}$ . The value  $w$  is a parameter in a small neighborhood of  $1/2$  that we will adjust later in the course of our construction. Thus our quasiregular map in  $D$ -components is  $g(z) = \rho \circ \psi(z - z_n)$ . There are  $m - 1$  critical points, and  $m - 1$  critical values that are located on a symmetric Euclidean circle of modulus approximately  $\delta$  which

is centered around  $w$ . Lastly, notice that the bounded dilatation of  $g(z) = \rho \circ \psi(z - z_n)$  in  $D_n$  is still concentrated in the annular region attached to  $\partial D_n$  whose area tends to 0 as  $m \rightarrow \infty$ .

#### 4. A GENERAL STRATEGY FOR THE CONSTRUCTION

Our construction will closely resemble the prototype explained in Section 2.2, with some modifications that we now describe.

We have a collection of parameters  $\lambda, (w_n), (\delta_n), (d_n)$  whose definitions we now recall. In the strip  $S^+$  we have the holomorphic map  $g(z) = \cosh(\lambda \sinh(z))$ . In the disc components  $D_n$  centered at  $z_n$ , we have  $g(z) = \rho_n \circ \psi_n(z - z_n)$ , as described in the previous section. Namely,  $\psi_n(z) = z^{d_n}$  on  $\partial \mathbb{D}$  and  $\psi_n(z) = z^{d_n} + \delta_n z$  on a subdisc of  $\mathbb{D}$ . We colloquially refer to  $(\delta_n)$  as the dilation parameters, since  $\psi_n$  effectively acts as a dilation by a factor of  $\delta_n$  on a subdisc of  $\mathbb{D}$ . Finally  $\rho_n(0) = w_n$ , with  $w_n$  contained in a small neighborhood of  $1/2$ . Note that this means the critical values of  $\rho_n \circ \psi_n(z - z_n)$  are centered around  $w_n$ .

Theorem 2.1 yields a quasiregular extension of  $g$  to all of  $\mathbb{C}$  without modifying the definition of  $g$  in the disc components  $D_n$  or the strip  $S^+$ . The essential part of this theorem is that the quasiregular constant is independent of the choices of the parameters  $\lambda, (w_n), (\delta_n), (d_n)$  - indeed varying the parameters does not affect the bounded geometry constants as verified explicitly in [FGJ15]. The entire function we consider will be  $f = g \circ \phi$ , where  $\phi$  is a quasiconformal map gotten from invoking the Measurable Riemann Mapping Theorem, so that  $g \circ \phi$  is indeed holomorphic. Furthermore, we normalize  $\phi$  to be the identity near infinity (see [Bis15]).

Our strategy to construct a wandering domain will be similar to the original argument given in [Bis15]. There the desired map  $f = g \circ \phi$  was obtained without modifying  $g$  on  $S^+$  (that is, fixing  $\lambda$  conveniently), and then modifying  $g$  on the discs  $D_n$  for increasing  $n$ . At each step  $k$  of the construction the integrating map  $\phi_k$  will change the definition of  $f$  itself, but the key point of the argument is to show that, on the one hand this process converges to a holomorphic map  $f$ , and on the other hand that the modifications do not destroy the desired dynamics.

By ensuring the constants  $\lambda, (d_n)$  are sufficiently large, it may be verified as in [Bis15] or [FGJ15] that the orbit of  $\frac{1}{2}$  under  $f$  escapes to infinity, and that  $\phi$  is  $\varepsilon$ -close to the identity uniformly on  $\mathbb{C}$  for some small  $\varepsilon$ . Henceforth we will not change the parameter  $\lambda$ . We select a subsequence of discs  $D_{p_n}$  by defining  $p_n$  to be the integer so that  $D_{p_n}$  is the disc closest to  $f^n(\frac{1}{2})$ . We let  $\tilde{D}_n := D_{p_n}$ . Henceforth we use the notations  $(\tilde{d}_n), (\tilde{w}_n), (\tilde{\delta}_n)$  to denote the parameters corresponding to the subsequence of discs  $(\tilde{D}_n)$ .

Define an annulus  $A_n := A_{n, \mu_n}$  to be the annulus containing  $\partial D_n$  bounded by two concentric circles of a distance  $\mu_n$  away from  $\partial D_n$  (see Figure 3). Since  $\phi$  is normalized to be the identity near infinity, we may now fix  $(\mu_n) \rightarrow 0$ , so that the bounded component of  $\mathbb{C} \setminus A_n$  is mapped by  $\phi$  inside of the bounded component of  $\mathbb{C} \setminus A_{n, \mu_n/2}$ , and so that the unbounded

component of  $\mathbb{C} \setminus A_n$  is mapped by  $\phi$  into the unbounded component of  $\mathbb{C} \setminus A_{n,\mu_n/2}$ , for all  $n > 0$ . We denote the bounded component of  $\mathbb{C} \setminus A_n$  as  $D_{n,\mu_n}$ . We further use the notation  $\tilde{\mu}_n = \mu_{p_n}$  and  $\tilde{A}_n$  to denote the annulus  $A_{p_n} = A_{p_n,\tilde{\mu}_n}$  surrounding  $\partial\tilde{D}_n$ . Likewise,  $\tilde{D}_{n,\tilde{\mu}_n} = D_{p_n,\mu_{p_n}}$ .

The general strategy for the first step in the construction of  $f$ , shown in Figure 3, is to assign the critical values (arising from critical points in  $\tilde{D}_1$ ) to land outside (but close to) the second preimage under  $f$  of the outer boundary of  $\tilde{A}_2$ . With this, an argument will show that these critical values do not lie in the same Fatou component as the wandering domain. On the other hand, as long as  $\tilde{\mu}_1 > \tilde{\mu}_2$ , we will be able to ensure that  $f(\tilde{D}_{1,\tilde{\mu}_1}) \subset f^{-2}(\tilde{D}_{2,\tilde{\mu}_2})$ . Note that the distance of the critical values (arising from critical points in  $\tilde{D}_1$ ) from the point  $w_1$  is determined by  $\delta_1$ , and  $w_1$  determines where the center of the disc  $\tilde{D}_1$  is sent. In the subsequent sections we will perform this procedure iteratively over  $n$  to yield an oscillating bounded wandering domain containing the orbit of  $\tilde{D}_{1,\tilde{\mu}_1}$ . Then we will prove that indeed there are no critical values in the orbit of our wandering domain. This will imply that the function  $f$  acts univalently on the whole orbit.

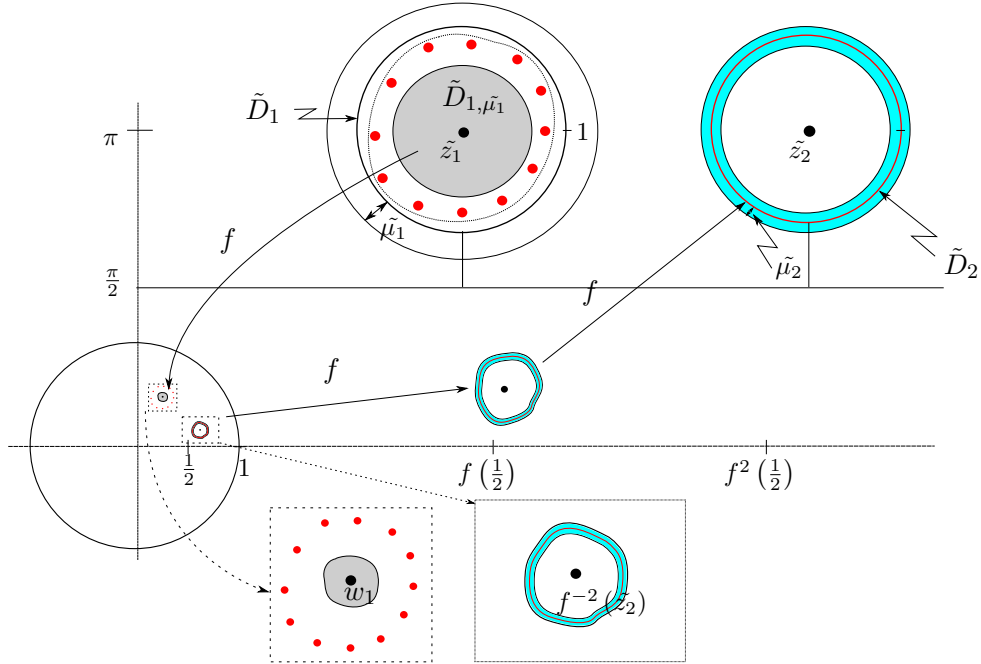


FIGURE 3. This picture illustrates the desired dynamics of  $f$  before adjusting  $w_1$  to be inside  $f^{-2}(\tilde{D}_{2,\mu_2})$ . After this adjustment, we will have  $f(\tilde{D}_{1,\mu_1}) \subset f^{-2}(\tilde{D}_{2,\mu_2})$ . As noted later in this section, changing  $w_1$  results in an extra dilatation for the quasiregular map  $g$ . We need to ensure that even after making the corresponding correction, we still have  $f(\tilde{D}_{1,\mu_1}) \subset f^{-2}(\tilde{D}_{2,\mu_2})$ .

## 5. DETERMINING THE EXPONENT AND DILATION PARAMETERS

As noted earlier, we have fixed the parameter  $\lambda$  sufficiently large so that  $f^n(1/2)$  escapes to infinity independently of further adjustments to  $(w_n), (\delta_n), (d_n)$ . Observe that as each of the exponents  $d_1, d_2, \dots \rightarrow \infty$ , the corresponding correction maps  $\phi_{(d_n)}$  converge<sup>1</sup> uniformly on compact subsets to some fixed quasiconformal map  $\phi_0$ . Moreover in the region  $S^+$  we know that our function is  $f(z) = \cosh(\lambda \sinh(\phi_{(d_n)}(z)))$ . We can take a univalent preimage of  $\tilde{D}_1$  under  $f(z) = \cosh(\lambda \sinh(\phi_{(d_n)}(z)))$  in  $S^+$ . Since  $\phi_{(d_n)}$  converges uniformly on compact subsets to some fixed quasiconformal map  $\phi_0$ , we know that  $f^{-1}(\tilde{D}_1)$  converges to some fixed region, namely  $(\cosh \circ \lambda \sinh \circ \phi_0)^{-1}(\tilde{D}_1)$ . Indeed by a similar argument, as each  $d_n \rightarrow \infty$  over all  $n$ ,  $f^{-n}(\tilde{D}_n)$  converges to some fixed region near  $1/2$ , namely  $((\cosh \circ \lambda \sinh \circ \phi_0)^{-1})^{\circ n}(\tilde{D}_n)$ .

We now iteratively define the sequence of exponents  $(d_n)$  and the dilation parameters  $(\delta_n)$ . We will determine how to fix  $\tilde{d}_n$  only at step  $n$ , and we will not change the exponents  $d_1, d_2, \dots, \tilde{d}_{n-1}$  at step  $n$ , once we have determined these exponents at previous steps. We start with step 1. Consider increasing each  $d_n \rightarrow \infty$ . For reasons noted earlier, we have  $f^{-2}(\tilde{A}_2)$  converging to some fixed topological annulus  $(\cosh \circ \lambda \sinh \circ \phi_0)^{-2}(\tilde{A}_2)$ . Consider, as pictured in Figure 4, the two concentric circles  $C_2, C'_2$  minimizing the ratio

$$\frac{\text{radius}(C'_2)}{\text{radius}(C_2)}$$

so that

$$\text{int}(C'_2) \supset (\cosh \circ \lambda \sinh \circ \phi_0)^{-2}(\tilde{A}_2),$$

and

$$\text{int}(C_2) \cap (\cosh \circ \lambda \sinh \circ \phi_0)^{-2}(\tilde{A}_2) = \emptyset.$$

What we really want is that

$$(5.1) \quad \tilde{\delta}_1 > \text{radius}(C'_2) \quad \text{and} \quad \tilde{\delta}_1(1 - \tilde{\mu}_1) < \text{radius}(C_2).$$

Indeed, roughly speaking, the first inequality will allow us to ensure that the critical values of  $f$  coming from the critical points in  $\tilde{D}_1$  lie sufficiently far from the outer boundary of  $f^{-2}(\tilde{A}_2)$ , while the second inequality will show that the size of  $f(\tilde{D}_{1, \tilde{\mu}_1})$  is smaller than a disk contained in  $f^{-2}(\tilde{D}_{2, \tilde{\mu}_2})$ . See Figure 5.

As mentioned in the caption of Figure 4, even if we know that  $(1 + \tilde{\mu}_2)/(1 - \tilde{\mu}_2) < (1 + \tilde{\mu}_1)/(1 - \tilde{\mu}_1)$ , it may not be the case that  $\text{radius}(C'_2)/\text{radius}(C_2) < (1 + \tilde{\mu}_1)/(1 - \tilde{\mu}_1)$  due to the distortion of  $f$ . But, as we presently justify, by Theorem 2.2 we have that

$$(5.2) \quad \frac{\text{radius}(C'_n)/\text{radius}(C_n)}{(1 + \tilde{\mu}_n)/(1 - \tilde{\mu}_n)} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

---

<sup>1</sup>Here we are using the notation  $\phi_{(d_n)}$  to emphasize that the correction map *depends* on a choice of the parameters  $(d_n)$ . Indeed the correction map also depends on a choice of  $(\delta_n), (w_n)$ , but that dependence is inessential for the purposes of this argument.

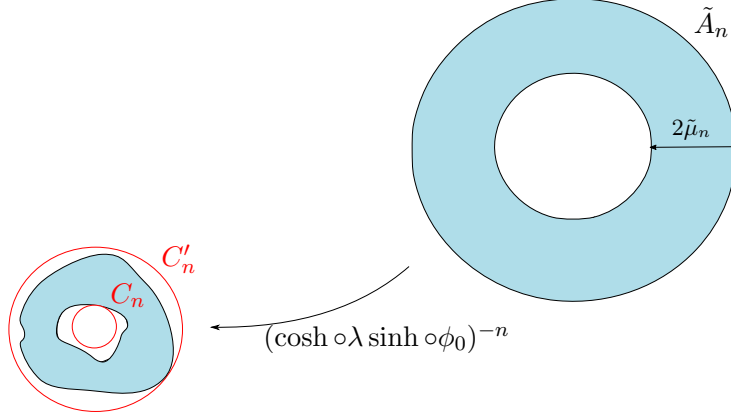


FIGURE 4. Here we see the definitions of the concentric circles  $C'_2, C_2$ . Note that the ratio  $\text{rad}(C'_2)/\text{rad}(C_2)$  may be larger than the ratio  $(1 + \tilde{\mu}_2)/(1 - \tilde{\mu}_2)$  pertaining to the annulus  $\tilde{A}_2$ . (If  $f$  were a Euclidean similarity, we would have  $\text{radius}(C'_2)/\text{radius}(C_2) = (1 + \tilde{\mu}_2)/(1 - \tilde{\mu}_2)$ ). In particular even if  $(1 + \tilde{\mu}_2)/(1 - \tilde{\mu}_2) < (1 + \tilde{\mu}_1)/(1 - \tilde{\mu}_1)$ , it may not be the case that  $\text{radius}(C'_2)/\text{radius}(C_2) < (1 + \tilde{\mu}_1)/(1 - \tilde{\mu}_1)$ . However, as we argue, the ratio  $(\text{radius}(C'_n)/\text{radius}(C_n))/((1 + \tilde{\mu}_n)/(1 - \tilde{\mu}_n)) \rightarrow 1$  as  $n \rightarrow \infty$ , so that if necessary we can just consider instead  $f^{-n}(\tilde{A}_n)$  for  $n > 2$ .

Indeed, consider disks  $B_{n-1}$  of a fixed unit size containing  $\lambda \sinh \phi_0(f^{n-1}(1/2))$ . By the expanding properties of  $\cosh$ , we have that  $\cosh : B_{n-1} \rightarrow \cosh(B_{n-1})$  is conformal,  $\tilde{A}_n \subset \cosh(B_{n-1})$ , and that as  $n \rightarrow \infty$ ,  $\text{dist}_n := \text{dist}(\tilde{A}_n, \partial(\cosh(B_{n-1}))) \rightarrow \infty$ . It follows from Theorem 2.2 (b), that the distortion of the branch  $F$  of  $f^{-n}$  mapping  $\cosh(B_{n-1})$  conformally to a neighborhood of  $1/2$ , tends to 0 as  $n$  increases. More precisely, for points  $z$  on the boundary of  $\tilde{A}_n$ , and  $\tilde{z}_n$  the center of  $\tilde{A}_n$ , the theorem yields

$$\frac{\text{radius}(C'_n)}{\text{radius}(C_n)} = \frac{\max_{|z-\tilde{z}_n|=1+\tilde{\mu}_n} |F(z) - F(\tilde{z}_n)|}{\min_{|z-\tilde{z}_n|=1-\tilde{\mu}_n} |F(z) - F(\tilde{z}_n)|} \leq \frac{1 + \tilde{\mu}_n}{1 - \tilde{\mu}_n} \frac{(\text{dist}_n + (1 - \tilde{\mu}_n))^2}{(\text{dist}_n - (1 + \tilde{\mu}_n))^2} \xrightarrow{n \rightarrow \infty} \frac{1 + \tilde{\mu}_n}{1 - \tilde{\mu}_n},$$

and also

$$\frac{\max_{|z-\tilde{z}_n|=1+\tilde{\mu}_n} |F(z) - F(\tilde{z}_n)|}{\min_{|z-\tilde{z}_n|=1-\tilde{\mu}_n} |F(z) - F(\tilde{z}_n)|} \geq \frac{1 + \tilde{\mu}_n}{1 - \tilde{\mu}_n} \frac{(\text{dist}_n - (1 - \tilde{\mu}_n))^2}{(\text{dist}_n + (1 + \tilde{\mu}_n))^2} \xrightarrow{n \rightarrow \infty} \frac{1 + \tilde{\mu}_n}{1 - \tilde{\mu}_n},$$

giving (5.2).

Thus we have shown that we can find some sufficiently large  $n_2$  so that  $\text{radius}(C'_{n_2})/\text{radius}(C_{n_2}) < (1 + \tilde{\mu}_1)/(1 - \tilde{\mu}_1)$ . We also require that  $\text{radius}(C'_{n_2})/\text{radius}(C_{n_2}) < 1/(1 - \tilde{\mu}_1)$ . Once we have found such an  $n_2$ , we fix

$$\tilde{\delta}_1 = \text{radius}(C'_{n_2}) \cdot \text{const}_1,$$

where

$$1 < \text{const}_1 < \frac{\text{radius}(C_{n_2})}{\text{radius}(C'_{n_2})} \frac{1}{1 - \tilde{\mu}_1}.$$

As a consequence

$$(1 - \tilde{\mu}_1) \cdot \tilde{\delta}_1 = (1 - \tilde{\mu}_1) \cdot \text{radius}(C'_{n_2}) \cdot \text{const}_1 < \text{radius}(C_{n_2})$$

and therefore the inequalities (5.1) are satisfied.

Up to this point, we have arranged for the image of  $\tilde{D}_{1, \tilde{\mu}_1}$  to be smaller than the  $n_2$ -preimage of  $\tilde{D}_{n_2, \tilde{\mu}_{n_2}}$ . In the next section we will perform surgery to move the image of  $\tilde{D}_{1, \tilde{\mu}_1}$  inside the preimage of  $\tilde{D}_{n_2, \tilde{\mu}_{n_2}}$ . On the other hand, since  $\text{const}_1 > 1$ , the critical values coming from  $\tilde{D}_1$  land at a distance greater than  $\text{radius}(C'_{n_2})$  from the center of the image of  $\tilde{D}_1$ . We will need this estimate to prove that there are no critical values in the orbit of our wandering domain.

Now fix  $d_1, d_2, \dots, \tilde{d}_1$  sufficiently large so that we have  $|\phi_{(d_n)} - \phi_0| < \varepsilon_1$  on some large compact disc centered at 0 and containing  $\tilde{A}_{n_2}$ . The quantity  $\varepsilon_1$  is chosen sufficiently small so that for  $k = n_2$ , we have that  $(\cosh \circ \lambda \sinh \circ \phi_{(d_n)})^{-k}(\tilde{A}_k)$  is  $\varepsilon'_1$  close to  $(\cosh \circ \lambda \sinh \circ \phi_0)^{-k}(\tilde{A}_k)$  in the Hausdorff metric, where  $\varepsilon'_1 \ll \text{modulus}((\cosh \circ \lambda \sinh \circ \phi_0)^{-k}(\tilde{A}_k))$ . Moreover we want this condition to hold true regardless of how we choose  $d_{p_1+1}, d_{p_1+2}, \dots, \tilde{d}_2 = d_{p_2}, \dots$  at further steps. This is possible because of uniform convergence of  $\phi_{(d_n)}$  to  $\phi_0$  on compact subsets.

Let us insist further on the choice of  $\varepsilon_1$ . We have a set of inequalities in step 1, for example (5.1), that are true if we replace  $\phi_{(d_n)}$  with the limiting map  $\phi_0$ . All of these inequalities have some “wiggle room”, and we are choosing  $\varepsilon_1$  to be much smaller than that wiggle room, so that the inequalities hold even if we replace  $\phi_0$  with  $\phi_{(d_n)}$ . We will also put a further condition on  $\tilde{d}_1$ . Recall that the (uniformly bounded) dilatation of  $g(z)$  on the disc  $\tilde{D}_1$  lives on a topological annulus we call  $\tilde{U}_1$  which tends in area to zero as  $\tilde{d}_1 \rightarrow \infty$ . We require  $\tilde{d}_1$  to be sufficiently large so that a quasiconformal map supported on  $\tilde{U}_1$  is  $\varepsilon_1$  close to the identity uniformly on  $S^+$ .

This ends the first step of the construction where we have fixed  $\tilde{\delta}_1$  and  $\tilde{d}_1$ , and have given lower bounds for further  $d_n$ . At step 2, we will determine how to fix  $\tilde{d}_{n_2}$ . Notice that as we have fixed  $\tilde{d}_1$ , if we increase  $d_{p_1+1}, d_{p_1+2}, \dots \rightarrow \infty$ , the map  $\phi_{(d_n)}$  no longer converges to  $\phi_0$  (since  $\tilde{d}_1$  now no longer tends to infinity), however  $\phi_{(d_n)}$  converges uniformly on compact subsets to some  $\phi_1$ , and  $\phi_1$  is  $\varepsilon_1$  close to  $\phi_0$  on the relevant compact disc containing 0,  $\tilde{A}_1, \tilde{A}_{n_2}$ .

We go through a similar procedure as in step 1, choosing some  $n_3 > n_2$  so that

$$\frac{\text{radius}(C'_{n_3})}{\text{radius}(C_{n_3})} < \frac{1}{1 - \tilde{\mu}_{n_2}} < \frac{1 + \tilde{\mu}_{n_2}}{1 - \tilde{\mu}_{n_2}}$$

Now, however, we look at  $\phi_1$  rather than  $\phi_0$ . Namely we are defining  $C_{n_3}, C'_{n_3}$  by considering  $(\cosh \circ \lambda \sinh \circ \phi_1)^{-n_3}(\tilde{A}_{n_3})$ . Then, as before, we fix

$$\tilde{\delta}_{n_2} = \text{radius}(C'_{n_3}) \cdot \text{const}_2,$$

where

$$1 < \text{const}_2 < \frac{\text{radius}(C'_{n_3})}{\text{radius}(C_{n_3})} \frac{1}{1 - \tilde{\mu}_{n_2}}.$$

Now we fix  $d_{p_1+1}, d_{p_1+2}, \dots, \tilde{d}_{n_2}$  sufficiently large so that  $|\phi_{(d_n)} - \phi_1| < \varepsilon_2$  on some large compact disc centered at 0 and containing  $\tilde{A}_{n_3}$ . The quantity  $\varepsilon_2$  is chosen so that, for example, for  $k = n_2, n_3$ , we have  $(\cosh \circ \lambda \sinh \circ \phi_{(d_n)})^{-k}(\tilde{A}_k)$  is  $\varepsilon'_2$  close to  $(\cosh \circ \lambda \sinh \circ \phi_1)^{-k}(\tilde{A}_k)$  in the Hausdorff metric, where  $\varepsilon'_2 \ll \text{modulus}((\cosh \circ \lambda \sinh \circ \phi_1)^{-k}(\tilde{A}_k))$ . We also specify  $\tilde{d}_{n_2}$  sufficiently large so that a quasiconformal map supported on  $\tilde{U}_{n_2}$  is  $\varepsilon_2$  close to the identity uniformly on  $S^+$ .

We continue similarly, hence defining  $d_n$  over all  $n$ . We note that at each step, the correction map  $\phi_{(d_n)}$  is  $\varepsilon_n$  close to  $\phi_n$ , where  $\phi_n$  is the limit obtained by fixing exponents already determined in previous steps and letting subsequent exponents tend to infinity. Moreover  $\phi_n$  is  $\varepsilon_1$ -close to  $\phi_0$ ,  $\varepsilon_2$ -close to  $\phi_1$ , ... and  $\varepsilon_n$ -close to  $\phi_{n-1}$ , at least on the relevant compact subsets. Thus, if we denote  $\phi$  as the correction map associated to the iteratively determined choice of  $(d_n), (\delta_n)$  as above, we know that  $\phi$  is  $\varepsilon_1$ -close to  $\phi_0$ ,  $\phi$  is  $\varepsilon_2$ -close to  $\phi_1$ , ..., on the relevant compact subsets of  $\mathbb{C}$ .

## 6. PERFORMING SURGERY TO YIELD A WANDERING DOMAIN

Having fixed the parameters  $\lambda, (d_n), (\delta_n)$ , we will now choose the parameters  $(w_n)$  iteratively so as to yield a wandering domain. The hard work was already done in the previous section. We start at what we call step 0, where we have the entire function  $f = g \circ \phi$ , where  $\phi$  is the correction map associated to the choice of parameters  $(d_n), (\delta_n)$  determined in the previous section, and for now we have set each  $w_n = 1/2$ .

Now in step 1 we consider the region  $f^{-n_2}(\tilde{A}_{n_2})$  and the corresponding circles  $C_{n_2}, C'_{n_2}$  pictured in Figure 4. We set  $\tilde{w}_1$  to be the common center of the circles  $C_{n_2}, C'_{n_2}$ . (Recall that  $C_{n_2}, C'_{n_2}$  were defined using  $\phi_0$  not  $\phi$ , but we know  $\phi_0$  is  $\varepsilon_1$  close to  $\phi$  on the relevant compact subset, which is good enough). In other words, we modify our quasiregular map<sup>2</sup>  $g$  only in  $\tilde{D}_1$ , by setting  $\tilde{\rho}_1(0) = \tilde{w}_1$  to be the center of the circles  $C_{n_2}, C'_{n_2}$ . We denote this new quasiregular map by  $g_1$ , so that  $g_1 = g$  except in  $\tilde{D}_1$ . In particular we know that now  $g_1 \circ \phi(\tilde{D}_{1, \tilde{\mu}_1}) \subset (g_1 \circ \phi)^{-n_2}(\tilde{D}_{n_2, \tilde{\mu}_{n_2}})$  by the estimates in the previous section, and also that the critical points under  $g_1 \circ \phi$  coming from  $\tilde{D}_1$  are sent *outside*  $C'_{n_2}$ . However notice that  $g_1 \circ \phi$  is no longer holomorphic, since there is a new dilatation concentrated on an annulus we called  $\tilde{U}_1$  near  $\partial \tilde{D}_1$ . We can find a correction map  $\eta_1$  so that  $f_1 := g_1 \circ \phi \circ \eta_1$  is holomorphic.

<sup>2</sup>recall that in a  $D$ -component  $D_n$ , we had the definition  $g(z) = \rho_n \circ \psi_n(z - z_n)$ .

Moreover we have already arranged for  $\tilde{d}_1$  to be sufficiently large so as to ensure  $\eta_1$  is close enough to the identity in the relevant region, so as to imply that we still have (see Figure 5)

$$f_1(\tilde{D}_{1,\tilde{\mu}_1}) \subset (f_1)^{-n_2}(\tilde{D}_{n_2,\tilde{\mu}_{n_2}}).$$

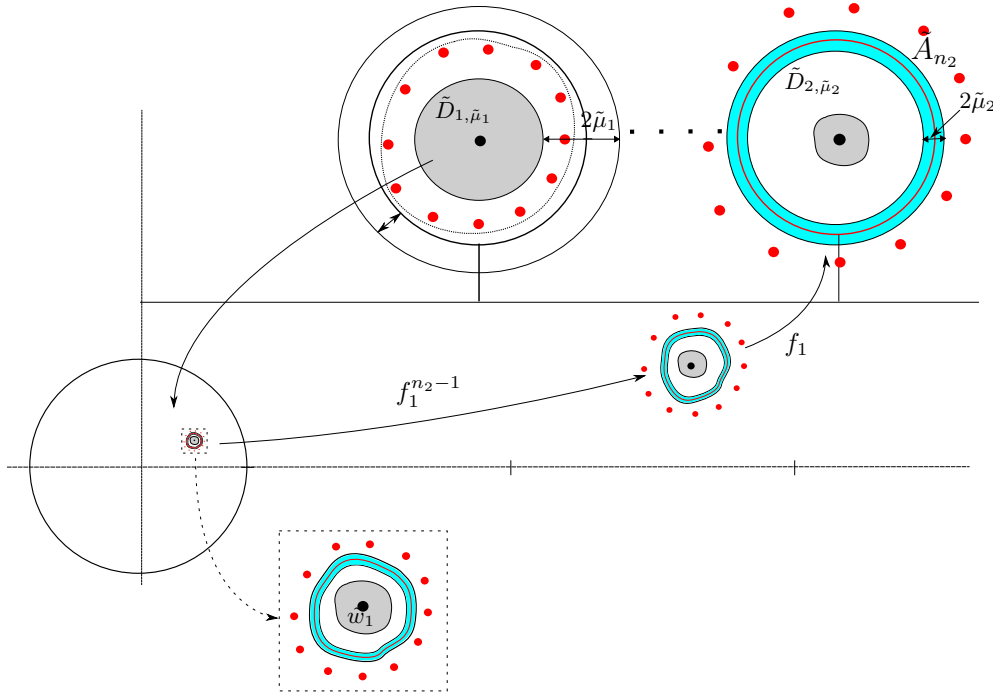


FIGURE 5. The first step in the construction of the univalent wandering domain, after adjusting the parameters so that  $f_1(\tilde{D}_{1,\tilde{\mu}_1}) \subset (f_1)^{-n_2}(\tilde{D}_{n_2,\tilde{\mu}_{n_2}})$ , and so that the critical values are outside  $(f_1)^{-n_2}(\tilde{A}_{n_2})$ .

Continuing to step 2, we now consider  $f_1^{-n_3}(A_{n_3})$  and the corresponding circles  $C_{n_3}, C'_{n_3}$ . (Again note that  $C_{n_3}, C'_{n_3}$  were defined using  $\phi_1$ , not the new correction map  $\phi \circ \eta_1$ , but we have already arranged for  $\phi \circ \eta_1$  to be  $\varepsilon_2$  close to  $\phi_1$  on the relevant compact subset). We set  $\tilde{w}_{n_2}$  to be the common center of the circles  $C_{n_3}, C'_{n_3}$ , i.e. we adjust our quasiregular map  $g_1$  only on the disc  $\tilde{D}_{n_2}$  and denote this new quasiregular map  $g_2$ . We know that  $g_2 \circ \phi \circ \eta_1(\tilde{D}_{n_2,\tilde{\mu}_{n_2}}) \subset (g_2 \circ \phi \circ \eta_1)^{-n_3}(\tilde{D}_{n_3,\tilde{\mu}_{n_3}})$  and that the critical points under  $g_2 \circ \phi \circ \eta_1$  coming from  $\tilde{D}_{n_2}$  are sent *outside*  $C'_{n_3}$ . However  $g_2 \circ \phi \circ \eta_1$  is no longer holomorphic. Thus we find a new quasiconformal correction map  $\eta_2$  so that  $f_2 := g_2 \circ \phi \circ \eta_1 \circ \eta_2$  is holomorphic. Moreover we have already arranged for  $\tilde{d}_{n_2}$  to be sufficiently large so that

$$f_2(\tilde{D}_{n_2,\tilde{\mu}_{n_2}}) \subset (f_2)^{-n_3}(\tilde{D}_{n_2,\tilde{\mu}_{n_2}}).$$



Now we also have to worry about whether it is still the case that  $f_2(\tilde{D}_{1,\tilde{\mu}_1}) \subset (f_2)^{-n_2}(\tilde{D}_{n_2,\tilde{\mu}_{n_2}})$ , however this was also built in to the definition of  $\tilde{d}_{n_2}$ . In the previous section we arranged for the errors coming from  $\eta_1, \eta_2, \dots$  to sum up so that continuing as above we will still have

$$f_k(\tilde{D}_{1,\tilde{\mu}_1}) \subset (f_k)^{-n_2}(\tilde{D}_{n_2,\tilde{\mu}_{n_2}})$$

for all  $k$ . Note that by construction, the errors  $\epsilon_k$  decrease exponentially fast with  $k$  so that summability is ensured.

We continue iteratively defining as above some sequence of entire maps  $f_k$  by adjusting the quasiregular map  $g_{k-1}$  from the previous step and then precomposing with a new correction map  $\eta_k$ . The parameters  $(d_n), (\delta_n)$  were chosen in the previous section so as to ensure that at step  $k$  we have  $\tilde{D}_{1,\tilde{\mu}_1}$  iterating after some number of steps into  $\tilde{D}_{n_k,\tilde{\mu}_{n_k}}$  under  $f_k$ . Taking the limit under this procedure we have a limiting map we call  $f$  which is our desired function in class  $\mathcal{B}$  with a wandering domain. In the next section we verify that there are no critical values in the orbit of our wandering domain.

## 7. VERIFYING THAT THE WANDERING DOMAIN IS INDEED UNIVALENT

We will prove that there are no critical values in the orbit of the Fatou component containing  $\tilde{D}_{1,\tilde{\mu}_1}$ . Since each component of the wandering domain is simply connected, this will imply that  $f$  acts univalently on the constructed wandering domain. Here it will be crucial that we have arranged for the critical points of  $f$  in  $\tilde{D}_{n_k}$  to land *outside*  $C'_{n_{k+1}}$ . This means that the critical points of  $f$  in  $\tilde{D}_{n_k}$  iterate into the unbounded component of  $\mathbb{C} \setminus \tilde{A}_{n_{k+1}}$ . Hence all we need to prove is that the unbounded component of  $\mathbb{C} \setminus \tilde{A}_{n_{k+1}}$  must have empty intersection with the Fatou component containing  $\tilde{D}_{n_{k+1},\tilde{\mu}_{n_{k+1}}}$ . To keep notation simple we will illustrate the argument by proving that the unbounded component of  $\mathbb{C} \setminus \tilde{A}_1$  has empty intersection with the Fatou component containing  $\tilde{D}_{1,\tilde{\mu}_1}$ .

The argument is similar to that given in [Laz17] where it is proven that the Fatou component containing  $\tilde{D}_{1,\tilde{\mu}_1}$  is bounded. We call this Fatou component  $U$ . Let  $u \in \tilde{D}_{1,\tilde{\mu}_1} \subset U$ , and suppose by way of contradiction that there exists a point  $v \in U \cap (\mathbb{C} \setminus \tilde{A}_1)$ . We may assume further that  $v$  lies in an  $R$ -component neighboring  $\tilde{D}_{1,\tilde{\mu}_1}$ . We call this  $R$ -component  $V$ . We note first that  $\mathbb{R}$  is in the escaping set of  $f$  (proven in [Laz17]) and hence, since  $f \in \mathcal{B}$ , that  $\mathbb{R}$  is in the Julia set [EL92]. That means that  $f^n(U) \subset \mathbb{H}$  over all  $n \in \mathbb{N}$ . Denote by  $d_{\mathbb{H}}, d_{f^n(U)}, d_U$  the hyperbolic distance in  $\mathbb{H}, f^n(U), U$  respectively. We have then that

$$(7.1) \quad d_{\mathbb{H}}(f^n(u), f^n(v)) \leq d_{f^n(U)}(f^n(u), f^n(v)) \leq d_U(u, v) \text{ for all } n \in \mathbb{N}$$

where both inequalities are consequences of Schwarz's lemma.

We argue that the left hand side of (7.1) must tend to infinity with  $n$ , at least in some subsequence. This will be our needed contradiction. Remember through the course of our construction that we have arranged for  $|f(u) - 1/2| < 1/4$  since  $u \in \tilde{D}_{1,\tilde{\mu}_1}$  and  $f(\tilde{D}_{1,\tilde{\mu}_1}) \subset$

$f^{-2}(\tilde{D}_{n_2, \tilde{\mu}_{n_2}})$ . What about  $f(v)$ ? We will argue that in fact we must have  $|f(v)| > 1$ . Indeed, consider a partition of  $\mathbb{H}_r$  into  $W_k := \{z : \pi k < \text{Im}(z) < (k+1)\pi, \text{Re}(z) > 0\}$  over  $k \in \mathbb{Z}$ , as illustrated to the left of Figure 6. In the course of the proof of the folding theorem, the map  $\tau : V \rightarrow \mathbb{H}_r$  was replaced by a locally quasiconformal  $\hat{\tau}$ , and the graph  $T$  was decorated with edges to form a graph  $T'$ , so that  $\hat{\tau} : V \setminus T' \rightarrow \mathbb{H}_r$  was quasiconformal and edges of  $T'$  were sent to the edges  $\{k < \text{Im}(z) < (k+1)\pi, \text{Re}(z) = 0\}$  (see [Bis15, Laz17, FGJ15] for details). Pulling the regions  $W_k := \{k < \text{Im}(z) < (k+1)\pi\}$  back under  $\hat{\tau}$  we have regions  $\hat{\tau}^{-1}(W_k)$ , and pulling back further by  $\phi \circ \eta_1 \circ \eta_2 \circ \dots$  we have regions as pictured in Figure 6. Now if  $\hat{\tau}^{-1}(W_k)$  neighbors a  $D$ -component, recall that for  $z \in \hat{\tau}^{-1}(W_k)$ , we have  $g(z) = \eta(\hat{\tau}(z))$  where  $\eta = \exp$  on  $\partial W_k$  interpolated with  $\eta = \cosh$  on  $\text{Re}(W_k) > \pi$ . In particular this means that as long as  $z \in \hat{\tau}^{-1}(W_k)$ , where  $\hat{\tau}^{-1}(W_k)$  neighbors a  $D$ -component, then  $|g(\hat{\tau}(z))| > 1$ .

Hence, we just need to argue that  $\phi \circ \eta_1 \circ \eta_2 \circ \dots(v) \in \hat{\tau}^{-1}(W_k)$  where  $\hat{\tau}^{-1}(W_k)$  neighbors a  $D$ -component (remember our entire function is just  $f = g \circ \hat{\tau} \circ (\phi \circ \eta_1 \circ \eta_2 \circ \dots)$ ). Indeed  $U$  can not intersect a region  $(\hat{\tau} \circ \phi \circ \eta_1 \circ \eta_2 \circ \dots)^{-1}(W_\ell) \subset (\phi \circ \eta_1 \circ \eta_2 \circ \dots)^{-1}(V)$  which does not neighbor  $(\phi \circ \eta_1 \circ \eta_2 \circ \dots)^{-1}(\tilde{D}_1)$ , the reason being that  $U$  also contains points inside  $(\phi \circ \eta_1 \circ \eta_2 \circ \dots)^{-1}(\tilde{D}_1)$  and therefore  $U$  would have to cross the boundary of some region  $\hat{\tau}^{-1}(W_l)$ . Namely  $U$  would have to cross  $(\hat{\tau} \circ \phi \circ \eta_1 \circ \eta_2 \circ \dots)^{-1}(\{y = (l+1)\pi\}) \cup (\hat{\tau} \circ \phi \circ \eta_1 \circ \eta_2 \circ \dots)^{-1}(\{y = l\pi\})$ , but  $(\hat{\tau} \circ \phi \circ \eta_1 \circ \eta_2 \circ \dots)^{-1}(\{y = (l+1)\pi\}) \cup (\hat{\tau} \circ \phi \circ \eta_1 \circ \eta_2 \circ \dots)^{-1}(\{y = l\pi\}) \subset \mathcal{J}(f)$ , since they are sent to  $\mathbb{R}$  by  $f$ , and this is a contradiction.

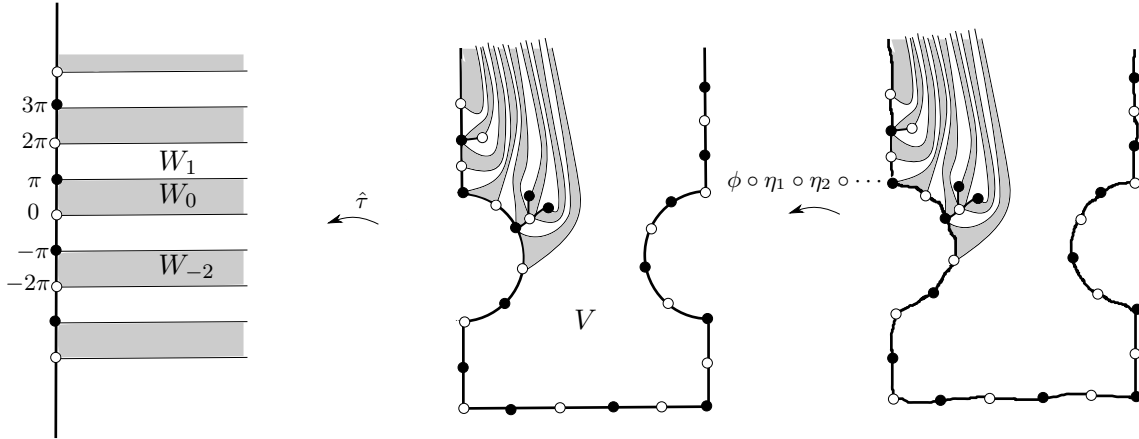


FIGURE 6. Rough sketch of the preimages of the semistrips  $W_k$  under  $\hat{\tau}$  and  $\phi \circ \eta_1 \circ \eta_2 \circ \dots$ , to show that the wandering domain needs to be contained in the  $D$ -components. For clarity's sake, only a few preimages are shown.

Now we have established that  $|f(u) - 1/2| < 1/4$  and  $|f(v)| > 1$ . Note that by the expanding properties of the exponential, upon subsequent applications of  $f$  we have that the distance between  $f(u), f(v)$  increases upon each iterate. Indeed  $f^k(u), f^k(v)$  must both lie in  $S^+$  for  $1 \leq k < n_2$  (otherwise the wandering domain would have to cross  $\partial S^+ \subset$

$J(f)$ ), and we have that  $f(z) = \cosh(\lambda \sinh(\phi(z)))$  in  $S^+$ , where  $\phi$  is uniformly close to the identity. Note, for example, that  $|f^{n_2}(u) - f^{n_2}(v)| > |\exp^{n_2}(3/4) - \exp^{n_2}(1)|$ , and that  $|\exp^n(3/4) - \exp^n(1)| \rightarrow \infty$  as  $n \rightarrow \infty$ . Next note that we know from the construction that  $f^{n_2}(u) \in \tilde{D}_{n_2, \tilde{\mu}_{n_2}}$ . This means that  $|f^{n_2+1}(u) - 1/2| < 1/4$ . What about  $f^{n_2+1}(v)$ ? Well as observed in [Laz17], by considering preimages of  $\mathbb{R}$  one sees that there is a ray belonging to  $\mathcal{J}(f)$  connecting  $z_n - i$  to  $\infty$  over all  $n$  (remember  $z_n$  is the center of the  $D$ -component  $D_n$ ). This means that  $f^{n_2}(v)$  must lie in an  $R$ -component neighboring  $D_{n_2}$ . Hence, as before, we know that  $|f^{n_2+1}(v)| > 1$ .

Now we start over, only that we have  $n_3 > n_2$  more iterations of  $f(z) = \cosh(\lambda \sinh(\phi(z)))$  in  $S^+$  before the points  $f^{n_2+1}(v), f^{n_2+1}(u)$  leave  $S^+$ . Hence again the expanding properties of  $\exp$ , together with the fact that  $|f^{n_2+1}(v)| > 1, |f^{n_2+1}(u) - 1/2| < 1/4$ , imply that  $|f^{n_2+n_3}(v) - f^{n_2+n_3}(u)| > |\exp^{n_3}(3/4) - \exp^{n_3}(1)|$ . Hence since  $n_k \rightarrow \infty$  as  $k \rightarrow \infty$ , we know that  $|f^{n_2+n_3+\dots+n_k}(v) - f^{n_2+n_3+\dots+n_k}(u)| \rightarrow \infty$  as  $k \rightarrow \infty$ . Moreover since by construction we know that  $f^{n_2+n_3+\dots+n_k}(u) \in \tilde{D}_{n_k}$ , we have that  $d_{\mathbb{H}}(f^{n_2+n_3+\dots+n_k}(u), f^{n_2+n_3+\dots+n_k}(v)) \rightarrow \infty$ , which is our needed contradiction.

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