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An algorithm for providing the normal forms of spatial quasi-homogeneous polynomial differential systems

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Abstract

Quasi-homogeneous systems, and in particular those 3-dimensional, are currently a thriving line of research. But a method for obtaining all fields of this class is not yet available. The weight vectors of a quasi-homogeneous system are grouped into families. We found the maximal spatial quasi-homogeneous systems with the property of having only one family with minimum weight vector. This minimum vector is unique to the system, thus acting as identification code. We develop an algorithm that provides all normal forms of maximal 3-dimensional quasi-homogeneous systems for a given degree. From these maximal systems can be trivially deduced all the other 3-dimensional quasi-homogeneous systems. We also list all the systems of this type of degree 2 using the algorithm. With this algorithm we make available to the researchers all 3-dimensional quasi-homogeneous systems.

Keywords: quasi-homogeneous, polynomial differential system, algorithm, weight vector.

1. Introduction

We deal with polynomial differential systems of the form

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z) \tag{1}$$

being $P, Q, R \in \mathbb{C}[x, y, z]$, with degrees n_1, n_2, n_3 respectively. As usual $\mathbb{C}[x, y, z]$ denotes the ring of all polynomials with coefficients in $\mathbb{C}$ and the complex variables x, y, z. The dot denotes derivative with respect to an independent variable t, which can be real or complex. We say that the *degree* of the system is $n = \max\{n_1, n_2, n_3\}$. From now on a polynomial differential system (1) will be denoted by $\mathsf{S}\,(P, Q, R)$, or by S when it does not lead to confusion.

Let $\mathbb{Z}^+$ denote the set of positive integers, and $\mathbb{R}^+$ the set of positive reals. A polynomial differential system $\mathsf{S}\,(P,Q,R)$ is *quasi–homogeneous* (from here on, simply QH) if there exist $s_1, s_2, s_3, d \in \mathbb{Z}^+$ such that for an arbitrary $\alpha \in \mathbb{R}^+$,

$$\begin{array}{l} P(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) = \alpha^{s_1-1+d}P(x,y,z), \\ Q(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) = \alpha^{s_2-1+d}Q(x,y,z), \\ R(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) = \alpha^{s_3-1+d}R(x,y,z). \end{array} \tag{2}$$

We call s_1, s_2 and s_3 *weight exponents* of S, and d the *weight degree* with respect to the weight exponents s_1, s_2 and s_3.

Suppose that a system S is QH, with weight exponents s_1, s_2 and s_3 and with weight degree d. In this case we state that $\mathbf{w} = (s_1, s_2, s_3, d)$ is a *weight vector* of the system S.

In the set of weight vectors of a QH system S it is possible to define a *partial order* relation as follows: given two weight vectors of S, $\mathbf{w} = (s_1, s_2, s_3, d)$ and $\mathbf{v} = (s_1^*, s_2^*, s_3^*, d^*)$, we write that $\mathbf{w} \leq \mathbf{v}$ when

$$s_1 \leq s_1^*,\ s_2 \leq s_2^*,\ s_3 \leq s_3^*,\ d \leq d^*. \tag{3}$$

We say that a weight vector $\mathbf{w}_m$ is the *minimum weight vector* of the QH system S if for any other weight vector $\mathbf{w}$ of the system S it is verified that $\mathbf{w}_m \leq \mathbf{w}$.

A QH system is called *maximal* if any new monomial added to its structure prevents it to be QH.

As an example consider the polynomial differential system

$$\begin{array}{l} \dot{x} = xyz + x^2, \\ \dot{y} = y^2z + xy, \\ \dot{z} = yz^2 + xz. \end{array} \tag{4}$$

It is a QH system with weight vector $(2,1,1,3)$, as can be seen from (2). But this system is not maximal, because it can be completed to

$$\begin{array}{l} \dot{x} = xyz + y^3 + x^2, \\ \dot{y} = y^2z + xz^2 + xy, \\ \dot{z} = yz^2 + y^2 + xz, \end{array} \tag{5}$$

which still is QH with the weight vector $(3,2,1,4)$.

We will focus our study on the maximal systems, considering the rest as particular cases of these in which some monomials are zero. A non-maximal system will possess all the weight vectors of those maximal systems to which it can be completed, and perhaps other new weight vectors. All the weight vectors of (5), i.e. the set $\{(3a, 2a, a, 3a+1) : a \in \mathbb{Z}^+\}$, are also weight vectors of (4), besides others as the mentioned $(2, 1, 1, 3)$.

The QH systems are a generalization of the homogeneous systems, to which they contain as a particular case. Several reasons motivate their study. For example, if a system S is QH with weight vector $\mathbf{v} = (s_1, s_2, s_3, d)$, being $d > 1$, then S is invariant under the changes of variable $x_i \to \alpha^{w_i} x_i$, $t \to \alpha^{-1} t$, for any $\alpha \in \mathbb{R}^+$, where $w_i = s_i / (d-1)$ for $i = 1, 2, 3$. In addition, the structural properties of QH systems allow to find their possible analytic first integrals through the Kowalevski exponents, see Yoshida, [15].

In the literature many authors have made contributions to this field, and in recent times it generates an increasing interest. The integrability has been studied extensively, highlighting the contributions of Llibre-Zhang [12], Kozlov [8], and García-Llibre-Pérez del Río [3]. In Liang-Huang-Zhao [10] are studied the phase portraits. The centers and limit cycles are discussed in Tang-Wang-Zhang [13], Geng-Lian [5], Li-Wu [9] and Xiong-Han-Wang [14]. Chiba [2] and Yoshida [15] have explored the Kowalevski exponents. Other topics such as the period function of the sum of two quasi-homogeneous [1], or the isochronicity and normal forms [6] have also been treated recently. On the other hand, García-Lombardero-Pérez del Río [4] have studied the classification and counting of this class of systems in dimension 2. Although the mentioned previous papers deal with systems in the plane, the area of QH systems in the space has recently begun to be explored, as shown by the works of Huang-Zhao [7], devoted to the limit set of trajectories, and Liang-Torregrosa [11], which studies the centers of a certain class of 3-dimensional QH systems.

The objective of this work is to develop an algorithm that provides, given a degree n supplied by the user, all normal forms of existing spatial QH polynomial differential systems of degree n. As we have said, we will restrict ourselves to maximal QH systems. A similar objective, but for systems in the plane, have been carried out in [3]. However, to the best of our knowledge, there is no work focused on supplying the complete set of 3-dimensional QH. We provide such algorithm in the present paper, which will be of valuable assistance for the development of future works in the field of study of polynomial differential systems

This work is organized as follows. In section 2 we present some properties about weight vectors of QH systems, besides provide some concepts like the weight vector family. Section 3 deals with the particular case of homogeneous QH systems. In section 4 we introduce the concept of brick, the unitary element with which we will later build the QH systems. Section 5 contains some of the most important theoretical results, such as Theorem 16, which states that the maximal QH systems have a single family of weight vectors, or the fact that the minimum weight vector can be constituted as a unique identifier in this type of systems. In section 6 are the main practical results used directly by the algorithm, which is provided in pseudocode in section 7. The work is closed with the list of all QH systems of degree $n = 2$, obtained by applying the algorithm, see section 8.

2. Some results on weight vectors

Given a QH system $\mathsf{S}(P,Q,R)$, where

$$P=\sum_{k=0}^{n_1} P_k, \quad Q=\sum_{k=0}^{n_2} Q_k, \quad R=\sum_{k=0}^{n_3} R_k, \tag{6}$$

we define its *homogeneous parts of degree* k, P_k, Q_k and R_k as:

$$P_k(x,y,z)=\sum_{p_1=0}^{k}\sum_{p_2=0}^{k-p_1} a_{p_1p_2k-p_1-p_2}x^{p_1}y^{p_2}z^{k-p_1-p_2} \quad (k=1,2,...,n_1)\,, \tag{7}$$

$$Q_k(x,y,z)=\sum_{q_1=0}^{k}\sum_{q_2=0}^{k-q_1} b_{q_1q_2k-q_1-q_2}x^{q_1}y^{q_2}z^{k-q_1-q_2} \quad (k=1,2,...,n_2)\,,$$

$$R_k(x,y,z)=\sum_{t_1=0}^{k}\sum_{t_2=0}^{k-t_1} c_{t_1t_2k-t_1-t_2}x^{t_1}y^{t_2}z^{k-t_1-t_2} \quad (k=1,2,...,n_3)\,.$$

Now we are going to obtain some properties of the coefficients of the QH systems. The equations included in the following result provide information of great relevance about the structure of the QH systems, because they are the key for determining when a monomial is present in the system or not.

Proposition 1. *Given a QH system* $\mathsf{S}(P,Q,R)$*, being* P_k*,* Q_k *and* R_k *its homogeneous parts of degree* k*, and*
p_1, p_2, q_1, q_2, t_1, t_2, p_1+p_2, q_1+q_2, $t_1+t_2 \in \{0,1,...,k\}$*, then*

$$a_{p_1p_2k-p_1-p_2}\neq 0 \Rightarrow (p_1-1)s_1+p_2s_2+(k-p_1-p_2)s_3=d-1, \tag{8}$$
$$b_{q_1q_2k-q_1-q_2}\neq 0 \Rightarrow q_1s_1+(q_2-1)s_2+(k-q_1-q_2)s_3=d-1, \tag{9}$$
$$c_{t_1t_2k-t_1-t_2}\neq 0 \Rightarrow t_1s_1+t_2s_2+(k-t_1-t_2-1)s_3=d-1, \tag{10}$$

for any weight vector $\mathbf{w}=(s_1,s_2,s_3,d)$ *of* S*. If* S *is a maximal system, then the three reciprocal implications are also true.*

Proof. We will do the proof for the coefficients of P_k, because the proofs for the coefficients of Q_k and R_k are identical. Let $\{\mathbf{w}_i\}_{i\in I}$ be the weight vector set of the QH system S.

Due to (6) and (7) we have that

$$P(x,y,z)=\sum_{k=0}^{n_1}\sum_{p_1=0}^{k}\sum_{p_2=0}^{k-p_1} a_{p_1p_2k-p_1-p_2}x^{p_1}y^{p_2}z^{k-p_1-p_2}$$

satisfies

$$P(\alpha^{s_1}x,\alpha^{s_2}y,\alpha^{s_3}z)=$$

$$= \sum_{k=0}^{n_1} \sum_{p_1=0}^{k} \sum_{p_2=0}^{k-p_1} a_{p_1 p_2 k-p_1-p_2} \alpha^{p_1 s_1 + p_2 s_2 + (k-p_1-p_2)s_3} x^{p_1} y^{p_2} z^{k-p_1-p_2}.$$

Then it follows from the fact that S is QH (see (2)) that

$$a_{p_1 p_2 k-p_1-p_2} \alpha^{p_1 s_1 + p_2 s_2 + (k-p_1-p_2)s_3} = a_{p_1 p_2 k-p_1-p_2} \alpha^{s_1 - 1 + d} \tag{11}$$

for all coefficients, for all $\mathbf{w} \in \{\mathbf{w}_i\}_{i\in I}$ and for any $\alpha \in \mathbb{R}^+$. Then we fix a coefficient $a_{p_1 p_2 k-p_1-p_2}$, corresponding to the k-degree monomial

$$a_{p_1 p_2 k-p_1-p_2} x^{p_1} y^{p_2} z^{k-p_1-p_2}. \tag{12}$$

Due to (11), if $a_{p_1 p_2 k-p_1-p_2} \neq 0$,

$$(p_1 - 1)s_1 + p_2 s_2 + (k - p_1 - p_2)s_3 = d - 1 \tag{13}$$

is necessarily fulfilled for all $\mathbf{w} \in \{\mathbf{w}_i\}_{i\in I}$, and thus the necessary condition ($\Rightarrow$) is proved.

We now study the sufficient condition ($\Leftarrow$) supposing that S is maximal. Fixed the values p_1, p_2, k, we have that (13) meets for any $\mathbf{w} \in \{\mathbf{w}_i\}_{i\in I}$. Suppose, by reductio ad absurdum, that $a_{p_1 p_2 k-p_1-p_2} = 0$, that is the monomial (12) is not present in P_k. Let us see that a new monomial can be added to P_k by maintaining the QH character of the system. The new system $\mathsf{S}'(P', Q', R')$ will be

$$P'(x,y,z) = P(x,y,z) + x^{p_1} y^{p_2} z^{k-p_1-p_2},$$
$$Q' = Q, \qquad R' = R.$$

Then if we take any weight vector of S, $\mathbf{w} \in \{\mathbf{w}_i\}_{i\in I}$, we have due to (13) and since S is QH, that

$$\begin{aligned} P'(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) &= \\ &= P(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) + \alpha^{p_1 s_1 + p_2 s_2 + (k-p_1-p_2)s_3} x^{p_1} y^{p_2} z^{k-p_1-p_2} \\ &= P(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) + \alpha^{s_1-1+d} x^{p_1} y^{p_2} z^{k-p_1-p_2} \\ &= \alpha^{s_1-1+d}\left[P(x,y,z) + x^{p_1} y^{p_2} z^{k-p_1-p_2}\right] \\ &= \alpha^{s_1-1+d} P'(x,y,z), \end{aligned}$$

for an arbitrary $\alpha \in \mathbb{R}^+$. As this same property is also fulfilled for Q' and R', we get that S' is also a QH system. As a consequence S was not maximal, a contradiction. □

Corollary 2. *If a QH system* S *has a monomial of degree* k*, then there exist* $x_1, x_2 \in \{-1, 0, ..., k\}$*,* $-1 \leq x_1 + x_2 \leq k$*, verifying*

$$x_1 s_1 + x_2 s_2 + (k - x_1 - x_2 - 1)s_3 = d - 1$$

for any weight vector (s_1, s_2, s_3, d) *of* S.

Proof. It is deduced from Proposition 1 taking into account that p_i, q_i, $t_i \in \{0, 1, ..., k\}$ for $i = 1, 2$. □

Corollary 3. *If a system* $\mathsf{S}(P, Q, R)$ *is QH, then* $P_0 = Q_0 = R_0 = 0$.

Proof. The equivalence (8) shows that if $P_0 = a_{0,0,0}$ is not null, then $d \le 0$, a contradiction. Also, if Q_0 or R_0 are different from zero, we obtain the same conclusion from (9) and (10). □

The following results are a direct consequence of Corollary 2, taking into account that a system must have some coefficient different from zero.

Remark 4. *(i) A necessary condition for a vector* $\mathbf{w} = (s_1, s_2, s_3, d) \in (\mathbb{Z}^+)^4$ *to be a weight vector of some QH system is that* $\gcd(s_1, s_2, s_3)$ *be a divisor of* $d - 1$.

(ii) Given a QH system S*, the weight degree* d *of any weight vector is uniquely determined by the weight exponents* s_1*,* s_2 *and* s_3.

The following result proves that the set of weight vectors of a QH system is infinite, and also provides a method for constructing new weight vectors from a given one.

Proposition 5. *Given a weight vector* (s_1, s_2, s_3, d) *of a QH system* S *and* $r = \frac{p}{q} \in \mathbb{Q}^+$ *with* p *and* q *coprime, the vector* (rs_1, rs_2, rs_3, d^*) *is also a weight vector of* S *if and only if* q *divides* $\gcd(s_1, s_2, s_3)$ *and* $d^* = r(d-1) + 1$.

Proof. Note first that $(rs_1, rs_2, rs_3, r(d-1) + 1)$ is a vector of positive integers if and only if q divides $\gcd(s_1, s_2, s_3, d-1)$. Taking int account that $\gcd(s_1, s_2, s_3)$ is a divisor of $d - 1$, it is enough that q divides $\gcd(s_1, s_2, s_3)$.

As (s_1, s_2, s_3, d) is a weight vector, for any $\alpha > 0$, we have that

$$P(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) = \alpha^{s_1+d-1}P(x, y, z).$$

On the other hand $r \in \mathbb{Q}^+$, whereby $\alpha^r > 0$, and consequently

$$\begin{aligned} P(\alpha^{rs_1}x, \alpha^{rs_2}y, \alpha^{rs_3}z) &= P((\alpha^r)^{s_1}x, (\alpha^r)^{s_2}y, (\alpha^r)^{s_3}z) \\ &= (\alpha^r)^{s_1+d-1} P(x, y, z) \\ &= \alpha^{rs_1+r(d-1)} P(x, y, z). \end{aligned}$$

Therefore (rs_1, rs_2, rs_3, d^*) will be weight vector if only if $d^* = r(d-1) + 1$. Similar conclusions for Q and R. □

The next result provides another way to build a new weight vector.

Proposition 6. *If* (s_1, s_2, s_3, d) *and* $(s_1^*, s_2^*, s_3^*, d^*)$ *are weight vectors of a QH system* S*, then* $(s_1 + s_1^*, s_2 + s_2^*, s_3 + s_3^*, d + d^* - 1)$ *is also a weight vector of* S*.*

Proof. As (s_1, s_2, s_3, d) and $(s_1^*, s_2^*, s_3^*, d^*)$ are weight vectors of S, we have that for every $\alpha \in \mathbb{R}^+$,

$$P(\alpha^{s_1} x, \alpha^{s_2} y, \alpha^{s_3} z) = \alpha^{s_1 - 1 + d} P(x, y, z),$$
$$P(\alpha^{s_1^*} x, \alpha^{s_2^*} y, \alpha^{s_3^*} z) = \alpha^{s_1^* - 1 + d^*} P(x, y, z).$$

Then,

$$\begin{aligned} P(\alpha^{s_1 + s_1^*} x, \alpha^{s_2 + s_2^*} y, \alpha^{s_3 + s_3^*} z) &= P(\alpha^{s_1} \left(\alpha^{s_1^*} x\right), \alpha^{s_2} \left(\alpha^{s_2^*} y\right), \alpha^{s_3} \left(\alpha^{s_3^*} z\right)) \\ &= \alpha^{s_1 - 1 + d} P(\alpha^{s_1^*} x, \alpha^{s_2^*} y, \alpha^{s_3^*} z) \\ &= \alpha^{s_1 - 1 + d} \alpha^{s_1^* - 1 + d^*} P(x, y, z) \\ &= \alpha^{(s_1 + s_1^*) - 1 + (d + d^* - 1)}. \end{aligned}$$

We get the same for Q and R, obtaining that $(s_1 + s_1^*, s_2 + s_2^*, s_3 + s_3^*, d + d^* - 1)$ is a weight vector of S. □

Given a QH system S and $\lambda,\ \mu \in \mathbb{Q}^+$, the *weight vector family of* S *with ratio* $(\lambda,\ \mu)$*,* $F_{\mathsf{S}}(\lambda,\ \mu)$, is defined as the set of weight vectors of S where the proportion between the exponents s_1 and s_2 is λ and the proportion between the exponents s_1 and s_3 is μ:

$$F_{\mathsf{S}}(\lambda,\ \mu) = \left\{ (s_1, s_2, s_3, d) \text{ weight vector of } \mathsf{S} \ : \ \frac{s_1}{s_2} = \lambda \text{ and } \frac{s_1}{s_3} = \mu \right\}.$$

Note that in this definition is not relevant the value that can take the weight degree d, which is uniquely determined by s_1, s_2 and s_3. Moreover, if we fix the system S and the family $F_{\mathsf{S}}(\lambda,\ \mu)$, the first weight exponent s_1 of a weight vector uniquely determines the rest of the vector, because $s_2 = s_1/\lambda$, $s_3 = s_1/\mu$ and, as we said before, the weight degree d depends functionally on s_1, s_2 and s_3.

However, fixed λ and μ, and given two systems S and T, it can happen that families $F_{\mathsf{S}}(\lambda,\ \mu)$ and $F_{\mathsf{T}}(\lambda,\ \mu)$ are different. In this case the weight exponents of the vectors match but not necessarily the weight degrees.

Given a weight vector family $F_{\mathsf{S}}(\lambda,\ \mu)$, a weight vector that minimizes the rest of vectors of the family in the sense of the order relation (3) is called the *family generator* and we denote it by $\mathbf{g}_{(\lambda,\mu)}$. Now we will prove that $\mathbf{g}_{(\lambda,\mu)}$ exists for every family. As it happens in two dimensions (see [4]), we have:

Proposition 7. *Given a weight vector family* $F_{\mathsf{S}}(\lambda,\ \mu)$ *of a QH system* S*, it is verified that:*

(i) The family generator $\mathbf{g}_{(\lambda,\mu)}$ *exists.*

(ii) Given $\mathbf{w} = (s_1^*, s_2^*, s_3^*, d^*) \in F_{\mathsf{S}}(\lambda,\ \mu)$*, then* $\mathbf{g}_{(\lambda,\mu)} = \mathbf{w}$ *if and only if* $\gcd(s_1^*, s_2^*, s_3^*) = 1$.

Proof. Let $\mathbf{w} = (s_1^*, s_2^*, s_3^*, d^*)$ be the only weight vector of the family $F_{\mathsf{S}}(\lambda,\ \mu)$ that verifies $s_1^* \leq s_1$ for every $(s_1, s_2, s_3, d) \in F_{\mathsf{S}}(\lambda,\ \mu)$. This weight vector always exists, because the weight exponents are positive integers; and it is unique, as two weight vectors of $F_{\mathsf{S}}(\lambda,\ \mu)$ with the same weight exponent s_1^* are identical. We will see that $\mathbf{w}$ is the generator of $F_{\mathsf{S}}(\lambda,\ \mu)$. Let $\mathbf{v} = (s_1, s_2, s_3, d)$ be any weight vector of $F_{\mathsf{S}}(\lambda,\ \mu)$. From $s_1^* \leq s_1$ we easily follow that $s_2^* = s_1^*/\lambda \leq s_1/\lambda = s_2$ and $s_3^* = s_1^*/\mu \leq s_1/\mu = s_3$. For proving $d^* \leq d$, we suppose $P \neq 0$. As $\mathbf{w}$ and $\mathbf{v}$ are weight vectors of $F_{\mathsf{S}}(\lambda,\ \mu)$, for any $\alpha,\ \beta > 0$ we have:

$$P\left(\alpha^{s_1^*} x, \alpha^{\frac{s_1^*}{\lambda}} y, \alpha^{\frac{s_1^*}{\mu}} z\right) = \alpha^{s_1^*-1+d^*} P(x,y,z) \tag{14}$$

and

$$P\left(\beta^{s_1} x, \beta^{\frac{s_1}{\lambda}} y, \beta^{\frac{s_1}{\mu}} z\right) = \beta^{s_1-1+d} P(x,y,z)\,. \tag{15}$$

Then, setting $\beta = \alpha^{\frac{s_1^*}{s_1}} > 0$, (15) becomes:

$$P\left(\alpha^{s_1^*} x, \alpha^{\frac{s_1^*}{\lambda}} y, \alpha^{\frac{s_1^*}{\mu}} z\right) = \alpha^{s_1^*+\frac{s_1^*}{s_1}(d-1)} P(x,y,z)\,. \tag{16}$$

So, by (14) and (16), and since P is not zero in the whole plane, we have

$$\alpha^{d^*-1} = \alpha^{\frac{s_1^*}{s_1}(d-1)}.$$

The exponential function is injective, hence $d^* - 1 = \frac{s_1^*}{s_1}(d-1)$. As a conclusion,

$$s_1^* \leq s_1 \Leftrightarrow d^* \leq d. \tag{17}$$

If $P = 0$, then $Q \neq 0$ or $R \neq 0$, and the result is proved in a similar way. Thus $\mathbf{w} \leq \mathbf{v}$ and accordingly $\mathbf{w} = \mathbf{g}_{(\lambda,\mu)}$.

Now let $\mathbf{g}_{(\lambda,\mu)} = (s_1^*, s_2^*, s_3^*, d^*)$ be the family generator, and we suppose that s_1, s_2 and s_3 share a common divisor $q > 1$. Making use of $r = 1/q$ in Proposition 5, $\mathbf{g}_{(\lambda,\mu)}$ would not be the family generator.

On the other hand, the fact that the weight exponents are coprime implies, also by Proposition 5, that they cannot be reduced more. Neither the weight degree, due to (17). □

The reason for calling such vector of $F_{\mathsf{S}}(\lambda,\ \mu)$ *family generator* is clear when we observe that the whole family can be constructed based on $\mathbf{g}_{(\lambda,\mu)}$ multiples.

It is clear from the above results that

$$F_{\mathsf{S}}(\lambda,\ \mu)=\left\{(as_1^*,as_2^*,as_3^*,a\,(d^*-1)+1)\ :\ a\in\mathbb{Z}^+\right\},$$

being $\mathbf{g}_{(\lambda,\mu)}=(s_1^*,s_2^*,s_3^*,d^*)$ the family generator of $F_{\mathsf{S}}(\lambda,\ \mu)$.

As a consequence a weight vector family is always contained in an unidimensional linear variety of $\mathbb{R}^4$ passing through the point $(0,0,0,1)$. The linear variety that contains the family generated by (s_1^*,s_2^*,s_3^*,d^*) is x_0+L, where $x_0=(0,0,0,1)$ and $L=<(s_1^*,s_2^*,s_3^*,d^*-1)>$ is the vector subspace with basis $(s_1^*,s_2^*,s_3^*,d^*-1)$. For the same reason, a dimension 1 variety of $\mathbb{R}^4$ cannot contain two different families, even if the families belong to distinct systems. Or it contains a unique whole family, or it does not contain any.

The next question is how many weight vector families can have a QH system. There are many examples of systems with more than one family of weight vectors. As an example, system (4) has $\{(a,b,a-b,a+1) : a,b\in\mathbb{Z}^+\ and\ a>b\}$ as a set of weight vectors, which means that $F\,(a/b,\ a/\,(a-b))$ is a weight vector family of (4) for every pair $a,b\in\mathbb{Z}^+$ verifying $a>b$. Therefore this is a case of infinite number of families.

The fact that a system S has more than one weight vector family implies that the existence of the minimum weight vector $\mathbf{w}_m$ of S is not guaranteed. If it exists, it should be the minimum of all family generators, and this minimum may not be reached. However in systems with a single family $F_{\mathsf{S}}(\lambda,\ \mu)$ we have that $\mathbf{g}_{(\lambda,\mu)}=\mathbf{w}_m$. We will show in Theorem 16 that the maximal QH systems, the main object of our study, always fulfill the property of having a unique family, and therefore they have minimum vector of the system.

3. The homogeneous maximal case

We say that a polynomial differential system $\mathsf{S}\,(P,Q,R)$ is *homogeneous of degree* n if

$$\begin{aligned}P(\alpha x,\alpha y,\alpha z)&=\alpha^n P(x,y,z),\\ Q(\alpha x,\alpha y,\alpha z)&=\alpha^n Q(x,y,z),\\ R(\alpha x,\alpha y,\alpha z)&=\alpha^n R(x,y,z)\end{aligned}$$

for every $\alpha\in\mathbb{R}$. This is equivalent to verify that all the monomials that constitute S are of degree n. All homogeneous polynomial differential systems belong to the set of QH systems. In order to verify this property, it is enough to take $(1,1,1,n)$, or any of its multiples, as weight vector. We will now see that the reciprocal is also true, whereby the homogeneous systems of degree n are totally determined as those having $(1,1,1,n)$ as weight vector.

Proposition 8. *If the system* $\mathsf{S}\,(P,Q,R)$ *is QH and* $\mathbf{w}=(s,s,s,d)$ *is a weight vector of* S*, then the system is homogeneous.*

Proof. If n is the degree of S, we can apply Corollary 2 to the weight vector $\mathbf{w}$

and $k = n$ to obtain $(n-1)s = d-1$, so $s - 1 + d = ns$. Therefore, from (2) we get

$$\begin{aligned} P(\alpha^s x, \alpha^s y, \alpha^s z) &= \alpha^{ns} P(x,y,z), \\ Q(\alpha^s x, \alpha^s y, \alpha^s z) &= \alpha^{ns} Q(x,y,z), \\ R(\alpha^s x, \alpha^s y, \alpha^s z) &= \alpha^{ns} R(x,y,z). \end{aligned} \tag{18}$$

Now we observe that any positive parameter β can be written in the form α^s just taking $\alpha = \beta^{\frac{1}{s}} > 0$ and therefore, by using (18), the system verifies $P(\beta x, \beta y, \beta z) = \beta^n P(x,y,z)$, $Q(\beta x, \beta y, \beta z) = \beta^n Q(x,y,z)$ and $R(\beta x, \beta y, \beta z) = \beta^n R(x,y,z)$, that is, the system is homogeneous. □

The system constructed with all possible monomials of degree n is the only maximal homogeneous system of degree n. We will denote this system by $\mathsf{H_n}$. The only weight vector family of $\mathsf{H_n}$ is

$$F_{\mathsf{H_n}}(1,1) = \left\{(a,a,a,a(n-1)+1) \,:\, a \in \mathbb{Z}^+\right\},$$

being $\mathbf{w}_m = (1,1,1,n)$ the minimum weight vector. Again non-maximal homogeneous systems are very varied and generally have more weight vectors than the maximal one.

4. Bricks and compatibility

We will use the weight vectors to list the set of maximal inhomogeneous QH systems. With the aim of reduce the total number of cases, in what follows we will consider only those weight vectors (s_1, s_2, s_3, d) that verify

$$s_1 \geq s_2 \geq s_3. \tag{19}$$

Also, taking into account Proposition 8, and with the aim of determine just the inhomogeneous systems, we impose that the condition

$$s_1 > s_3 \tag{20}$$

must be also verified. That is, we are going to construct the maximal systems that have among their weight vectors someone satisfying (19) and (20). In this way we simplify our study, significantly reducing the number of systems to study. The rest of systems are symmetrical to those obtained with these restrictions, without doing more than permutations on the variables x, y and z. As an example, if we find system (5), whose weight vectors, the set $\{(3a, 2a, a, 3a+1) : a \in \mathbb{Z}^+\}$, verify the restrictions (19) and (20), we are automatically finding five more systems, as there are six possible permutations of the variables x, y and z. One of these systems is obtained by permuting the variables x and y, and has as weight vectors the set $\{(2a, 3a, a, 3a+1) : a \in \mathbb{Z}^+\}$:

$$\begin{aligned} \dot{x} &= x^2z + yz^2 + xy, \\ \dot{y} &= xyz + x^3 + y^2, \\ \dot{z} &= xz^2 + x^2 + yz. \end{aligned}$$

In order to simplify the equations involved in the process, and taking into account the restrictions (19), we define the new variables $\bar{s_1}$, $\bar{s_2}$, $\bar{s_3}$ and $\bar{d}$ as follows:

$$\bar{s_1} = s_1 - s_3, \quad \bar{s_2} = s_2 - s_3, \quad \bar{s_3} = s_3, \quad \bar{d} = d - 1 + s_3. \tag{21}$$

It follows from (19), (20), (21) and from the fact that the weight vectors are made of positive integers, that these new variables verify the constraints

$$\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d} \in \mathbb{Z}, \tag{22}$$

$$\bar{s_1} \geq \bar{s_2} \geq 0, \tag{23}$$

$$\bar{s_1} > 0, \tag{24}$$

$$\bar{d} \geq \bar{s_3} > 0. \tag{25}$$

Given a weight vector $\mathbf{w} = (s_1, s_2, s_3, d)$ that satisfies (19) and (20), the new vector $\bar{\mathbf{w}} = (\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ obtained by the change of variables (21) is called the *transformed vector of* $\mathbf{w}$, and we denote the implicit bijection by $\bar{\mathbf{w}} = t(\mathbf{w})$. A transformed vector always verifies the conditions (22), (23), (24) and (25). Reciprocally, given a transformed vector $\bar{\mathbf{w}} = (\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$, its corresponding weight vector $\mathbf{w} = t^{-1}(\bar{\mathbf{w}}) = (s_1, s_2, s_3, d)$ can be obtained doing:

$$s_1 = \bar{s_1} + \bar{s_3}, \quad s_2 = \bar{s_2} + \bar{s_3}, \quad s_3 = \bar{s_3}, \quad d = \bar{d} - \bar{s_3} + 1. \tag{26}$$

The transformed vectors are also grouped into families. That is the weight vector family $F_{\mathsf{S}}(\lambda,\ \mu) = \{(as_1, as_2, as_3, a(d-1)+1)\ :\ a \in \mathbb{Z}^+\}$ is transformed by t into the set

$$\bar{F_{\mathsf{S}}}(\lambda,\ \mu) = \left\{\left(a\bar{s_1}, a\bar{s_2}, a\bar{s_3}, a\bar{d}\right)\ :\ a \in \mathbb{Z}^+\right\},$$

which is called the *transformed weight vector family of* $F_{\mathsf{S}}(\lambda,\ \mu)$. While a weight vector family was contained in a straight line of $\mathbb{R}^4$ passing through the point $(0,0,0,1)$, a transformed weight vector family exists within a more simple unidimensional subspace of $\mathbb{R}^4$. As previously, a subspace of $\mathbb{R}^4$ of dimension 1 cannot contain two different transformed weight vector families.

The following result, an improved and simplified version of Corollary 2, will be an important tool in this work, and shows the close relationship between the monomials of a QH system and a certain type of homogeneous linear equations.

Proposition 9. *A maximal QH system* S *has a monomial of degree* k *if and only if there exist* $x_1, x_2 \in \{-1, 0, ..., k\}$, $-1 \leq x_1 + x_2 \leq k$, *verifying*

$$x_1 \bar{s_1} + x_2 \bar{s_2} + k \bar{s_3} = \bar{d} \tag{27}$$

for any transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ *of* S .

Proof. Using of the change of variables (21), the equivalences of Proposition 1 can be rewritten for transformed vectors, as

$$\begin{array}{l} a_{p_1 p_2 k - p_1 - p_2} \neq 0 \Leftrightarrow (p_1 - 1)\bar{s_1} + p_2 \bar{s_2} + k\bar{s_3} = \bar{d}, \\ b_{q_1 q_2 k - q_1 - q_2} \neq 0 \Leftrightarrow q_1 \bar{s_1} + (q_2 - 1)\bar{s_2} + k\bar{s_3} = \bar{d}, \\ c_{t_1 t_2 k - t_1 - t_2} \neq 0 \Leftrightarrow t_1 \bar{s_1} + t_2 \bar{s_2} + k\bar{s_3} = \bar{d}. \end{array}$$

Therefore, the proof concludes simply by taking into account that S is a maximal system and p_i, q_i, $t_i \in \{0, 1, ..., k\}$ for $i = 1, 2$. □

As a consequence of Proposition 9, we have an interesting property of the maximal QH systems: certain monomials of the same degree k, belonging to each of the three homogeneous parts P_k, Q_k and R_k, are related. That is, if one of them appears in a given maximal system, the other two also appear.

Corollary 10. *Given a maximal QH system* $\mathsf{S}\,(P, Q, R)$ *of degree* n, $1 \leq k \leq n$, $x_1, x_2 \in \{0, 1, ..., k-1\}$, $0 \leq x_1 + x_2 \leq k - 1$, *then*

$$a_{x_1+1, x_2, k-x_1-x_2-1} \neq 0 \Leftrightarrow b_{x_1, x_2+1, k-x_1-x_2-1} \neq 0 \Leftrightarrow c_{x_1, x_2, k-x_1-x_2} \neq 0$$

Proof. By Proposition 9, the three inequalities of the statement are equivalents to the verification of equation (27) for every transformed vector of S. □

Fixed a degree k some monomials of P_k, Q_k and R_k are not related with other monomials. They appear freely within the QH maximal systems. This happens because they have zeros as exponents of the variables x, y or z. These are the following:

1. Monomials of the homogeneous part P_k with exponent 0 in the variable x, as
$$a_{0 p_2 k - p_2} y^{p_2} z^{k - p_2} \quad (0 \leq p_2 \leq k)\,,$$
which are present in the system when the equation $-\bar{s_1} + p_2 \bar{s_2} + k\bar{s_3} = \bar{d}$ is verified for any transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$.
2. Monomials of the homogeneous part Q_k with exponent 0 in the variable y, as
$$a_{p_1 0 k - p_1} x^{p_1} z^{k - p_1} \quad (0 \leq p_1 \leq k)\,,$$
which are present in the system when the equation $p_1 \bar{s_1} - \bar{s_2} + k\bar{s_3} = \bar{d}$ is verified for any transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$.

3. Monomials of the homogeneous part R_k with exponent 0 in the variable z, as
$$a_{p_1 k-p_1 0} x^{p_1} y^{k-p_1} \quad (0 \leq p_1 \leq k)\,,$$
which are present in the system when the equation $p_1\bar{s_1} + p_2\bar{s_2} + k\bar{s_3} = \bar{d}$, with $p_1 + p_2 = k$, is verified for any transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$.

From now on, when we speak of a *brick* of a system, we will refer to one of these sets of linked monomials, that are the simplest constituent elements of the maximal QH systems. We denote by $[x_1, x_2; k]$ the brick associated with the equation (27). Bricks can contribute to the system with three monomials, one in each of the components P, Q and R, or with only one if they are in any of the special situations studied before. The brick $[x_1, x_2; k]$ contributes to the component P with the monomial

$$a_{x_1+1,x_2,k-x_1-x_2-1} x^{x_1+1} y^{x_2} z^{k-x_1-x_2-1},$$

to the component Q with the monomial

$$b_{x_1,x_2+1,k-x_1-x_2-1} x^{x_1} y^{x_2+1} z^{k-x_1-x_2-1},$$

and to the component R with the monomial

$$c_{x_1,x_2,k-x_1-x_2} x^{x_1} y^{x_2} z^{k-x_1-x_2}.$$

Although as we said two of these monomials may be null. When this be necessary, we will summarize the contributions of $[x_1, x_2; k]$ with

$$(P, Q, R) = \left(x^{x_1+1} y^{x_2} z^{k-x_1-x_2-1}, x^{x_1} y^{x_2+1} z^{k-x_1-x_2-1}, x^{x_1} y^{x_2} z^{k-x_1-x_2}\right).$$

We denote by $\mathsf{B_k}$ the set of bricks of degree k, meaning *brick of degree* k those $[x_1, x_2; k]$ that contribute with monomials of such degree. From the constraints for x_1 and x_2 stated in Proposition 9 we deduce that given $k \in \mathbb{Z}^+$ we have

$$\mathsf{B_k} = \{[x_1, x_2; k] \,:\, x_{1,}x_2 \in \mathbb{Z} \quad \text{and} \quad -1 \leq x_1, x_2, x_1 + x_2 \leq k\}\,. \tag{28}$$

Table 1 shows the bricks of $\mathsf{B_1}$ together with their associated equations and the contributions to the maximal systems in which they are present.

Fixed a degree k the set of bricks $\mathsf{B_k}$ can be represented graphically in the plane without more than taking into account the restrictions on the integers x_1, x_2, which will act as abscissa and ordinate respectively in this graphic. In the region of the plane corresponding to $\mathsf{B_3}$, shown in Figure 1, it is observed that the bricks that contribute with a single monomial correspond to the border points of the region.

Proposition 11. *An inhomogeneous QH system of degree n can be constructed*

$[\mathbf{x_1}, \mathbf{x_2}; \mathbf{1}]$	$\mathbf{x_1}\bar{\mathbf{s_1}} + \mathbf{x_2}\bar{\mathbf{s_2}} + \bar{\mathbf{s_3}} = \bar{\mathbf{d}}$	$(\mathbf{P}, \mathbf{Q}, \mathbf{R})$
$[-1, 0; 1]$	$-\bar{s_1} + \bar{s_3} = \bar{d}$	$(z, 0, 0)$
$[0, -1; 1]$	$-\bar{s_2} + \bar{s_3} = \bar{d}$	$(0, z, 0)$
$[-1, 1; 1]$	$-\bar{s_1} + \bar{s_2} + \bar{s_3} = \bar{d}$	$(y, 0, 0)$
$[0, 0; 1]$	$\bar{s_3} = \bar{d}$	(x, y, z)
$[1, -1; 1]$	$\bar{s_1} - \bar{s_2} + \bar{s_3} = \bar{d}$	$(0, x, 0)$
$[0, 1; 1]$	$\bar{s_2} + \bar{s_3} = \bar{d}$	$(0, 0, y)$
$[1, 0; 1]$	$\bar{s_1} + \bar{s_3} = \bar{d}$	$(0, 0, x)$

Table 1: The set of bricks of degree 1 ($\mathsf{B_1}$).

with

$$\frac{n^3}{6} + 2n^2 + \frac{29}{6}n$$

different bricks.

Proof. An inhomogeneous QH system of degree n can be constructed with the bricks of the sets $\mathsf{B_k}$, $k \in \{1, ..., n\}$. The cardinal of $\mathsf{B_k}$, which matches the number of bricks of $\mathsf{H_k}$, is

$$2 + 3 + ... + (k+2) + (k+1) = \frac{(k+1)(k+6)}{2}.$$

Then, the total number of available bricks is

$$\sum_{k=1}^{n} \frac{(k+1)(k+6)}{2} = \frac{n^3}{6} + 2n^2 + \frac{29}{6}n.$$

□

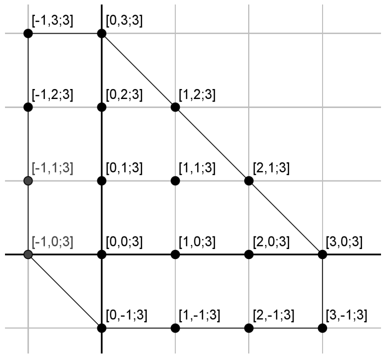

Figure 1: B_3 region in the plane

As an example, the maximal system (5) is splitted into the bricks that compose it. This QH system is built of five bricks with degrees running from 2 to 3.

$$
\begin{array}{rcl|c|c|c|c|c|}
\dot{x} & = & +xyz & & +y^3 & +x^2 & \\
\dot{y} & = & +y^2z & +xz^2 & & +xy & \\
\dot{z} & = & +yz^2 & & & +xz & +y^2 \\
 & & & & & & \\
\text{Brick:} & & [0,1;3] & [1,-1;3] & [-1,3;3] & [1,0;2] & [0,2;2]
\end{array}
$$

In the next two results we will study when two given bricks can coexist in the same QH system, that is, their compatibility. It is obvious that in this sense there must be restrictions. Otherwise, there would only be one possible maximal QH system of degree n, constituted by the $\frac{n^3}{6}+2n^2+\frac{29}{6}n$ bricks mentioned in Proposition 11. We will start by analyzing the compatibility of a pair of bricks, understanding for *compatible bricks* those that can coexist in a QH system, and *incompatible* those that cannot do so under any circumstances. Note that brick compatibility is not equivalent to the compatibility of their respective associated equations (27), because the equations can have common solutions $(\bar{s_1},\bar{s_2},\bar{s_3},\bar{d})$, but without satisfying the conditions (22), (23), (24) and (25).

Proposition 12. *Let $x_1,x_2,y_1,y_2,k,p\in\mathbb{Z}$ be with $-1\le x_1,x_2,x_1+x_2\le k$, $-1\le y_1,y_2,y_1+y_2\le p$ and $0<p<k$. The bricks $[x_1,x_2;k]$ and $[y_1,y_2;p]$ are compatible in an inhomogeneous QH system if and only if $Y_1>0$, or $Y_1+Y_2>0$, being $Y_i=y_i-x_i$, $i=1,2$.*

Proof. First we note that the compatibility of $[x_1,x_2;k]$ and $[y_1,y_2;p]$ within the same inhomogeneous QH system means that there is some transformed vector $(\bar{s_1},\bar{s_2},\bar{s_3},\bar{d})$ that satisfy the system

$$x_1\bar{s_1} + x_2\bar{s_2} + k\bar{s_3} - \bar{d} = 0,$$
$$y_1\bar{s_1} + y_2\bar{s_2} + p\bar{s_3} - \bar{d} = 0.$$

This system has the infinite set of solutions

$$\{(\alpha,\ \beta,\ (Y_1\alpha + Y_2\beta)/(k-p),\ (X_1\alpha + X_2\beta)/(k-p))\ :\ \alpha, \beta \in \mathbb{R}\},$$

where $Y_i = y_i - x_i$ and $X_i = ky_i - px_i$ for $i = 1, 2$. But the solutions that count for compatibility are those that are a transformed vector, i.e., those verifying (22), (23), (24) and (25).

We start proving that the compatibility implies $Y_1 > 0$ or $Y_1 + Y_2 > 0$, that is, if $Y_1 \leq 0$ and $Y_1 + Y_2 \leq 0$ then there are no values of α and β in the conditions of statements (22), (23), (24) and (25). Since $k - p$ is positive, it is sufficient to prove that $Y_1\alpha + Y_2\beta$ is negative or zero, so consequently statement (25) does not hold. If $Y_1 \leq 0$ and also $Y_2 \leq 0$, then $Y_1\alpha + Y_2\beta \leq 0$ for all $\alpha, \beta \in \mathbb{N}$. Otherwise, if $Y_1 \leq 0$ and $Y_2 > 0$, from condition $Y_1 + Y_2 \leq 0$ it is deduced that $Y_1 \leq -Y_2$ and since $\beta - \alpha \leq 0$, it follows that $Y_1\alpha + Y_2\beta \leq (-Y_2)\,\alpha + Y_2\beta = Y_2\,(\beta - \alpha) \leq 0$ for every $\alpha, \beta \in \mathbb{N}$.

To study the reciprocal implication we distinguish two cases:

Case $Y_1 > 0$. Let $\alpha = k - p$ and $\beta = 0$, so we obtain the solution $(k - p, 0, Y_1, X_1)$. It is clear that conditions (22), (23), (24) are verified with these values, and also that $\bar{s_3} = Y_1 > 0$. To prove $\bar{d} \geq \bar{s_3}$, note that $\bar{d} - \bar{s_3} = X_1 - Y_1 = (k-1)\,y_1 - (p-1)\,x_1$ and that $Y_1 > 0$ implies $y_1 > x_1 \geq -1$, so $y_1 \geq 0$. All this, together with $0 \leq p - 1 < k - 1$, means that $(k-1)\,y_1 \geq (p-1)\,x_1$, so $\bar{d} \geq \bar{s_3}$.

Case $Y_1 + Y_2 > 0$. Now we set $\alpha = \beta = k - p$, obtaining the solution $(k - p, k - p, Y_1 + Y_2, X_1 + X_2)$, and the proof is identical as in the previous case, although now $\bar{d} - \bar{s_3} = (X_1 + X_2) - (Y_1 + Y_2) = (k-1)\,(y_1 + y_2) - (p-1)\,(x_1 + x_2)$, and $Y_1 + Y_2 > 0$ implies $y_1 + y_2 > x_1 + x_2 \geq -1$, so $\bar{d} \geq \bar{s_3}$. □

Note that in addition to the transformed vector obtained in the proof there is not a single family of transformed vectors, but there are infinity families, contained in a dimension 2 vector subspace of $\mathbb{R}^4$.

We have studied the compatibility for two bricks of different degrees. On the other hand, two bricks of the same degree k are always compatible; even all bricks of degree k are mutually compatible, giving rise to the maximal homogeneous system $\mathsf{H_k}$. But this is a compatibility that, in the search for exclusively inhomogeneous systems, we are not interested in. The following result establishes the conditions of compatibility between bricks of the same degree in the case of inhomogeneous systems, those that consist of two or more different homogeneous parts. Note that previous papers ([3], [4]) have shown that the mentioned type of compatibility does not exist in two-dimensional QH systems.

Proposition 13. *Let* $x_1, x_2, y_1, y_2, k \in \mathbb{Z}$ *be with* $-1 \leq x_1, x_2, y_1, y_2, x_1 + x_2, y_1 +$

$y_2 \leq k$, and $k > 1$. Two different bricks $[x_1, x_2; k]$ and $[y_1, y_2; k]$ are compatible in an inhomogeneous QH system if and only if $Y_1 = 0$, or $-Y_2/Y_1 \geq 1$, being $Y_i = y_i - x_i$ for $i = 1, 2$.

Proof. First we note that the compatibility of $[x_1, x_2; k]$ and $[y_1, y_2; k]$ within the same inhomogeneous QH system means that there is some transformed vector $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ that satisfy the system

$$\begin{aligned} x_1\bar{s_1} + x_2\bar{s_2} + k\bar{s_3} - \bar{d} = 0, \\ y_1\bar{s_1} + y_2\bar{s_2} + k\bar{s_3} - \bar{d} = 0. \end{aligned}$$

The matrix expression of the system, after Gaussian transformations, adopts the form

$$\begin{pmatrix} x_1 & x_2 & k & -1 \\ Y_1 & Y_2 & 0 & 0 \end{pmatrix}.$$

The bricks are not the same, so it must be verified that $Y_1^2 + Y_1^2 \neq 0$. As a conclusion the rank of the system is 2, and the set of solutions is

$$\left\{ (Y_2\alpha,\ -Y_1\alpha,\ \beta,\ (Y_2x_1 - Y_1x_2)\,\alpha + k\beta) \ : \ \alpha, \beta \in \mathbb{R} \right\}, \tag{29}$$

where $Y_i = y_i - x_i$ for $i = 1, 2$.

We start by proving that the compatibility of the two bricks implies $Y_1 = 0$, or $-Y_2/Y_1 \geq 1$, or equivalently, that when $-Y_2/Y_1 < 1$ is verified, being $Y_1 \neq 0$, then the bricks are incompatible. The solutions $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ have to be of the form (29), so when $-Y_2/Y_1 < 1$ is verified we have $|\bar{s_2}| > |\bar{s_1}|$. In that case $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ cannot be a transformed vector, and therefore there is no compatibility between the bricks.

For proving the reciprocal implication we distinguish two cases:

Case $Y_1 = 0$. The solutions are in this case

$$\left\{ (Y_2\alpha,\ 0,\ \beta,\ Y_2x_1\alpha + k\beta) \ : \ \alpha, \beta \in \mathbb{R} \right\},$$

where $Y_2 \neq 0$. Setting $\alpha = 1/Y_2$ and $\beta = 1$ we obtain the solution $(1, 0, 1, x_1 + k)$. Taking into account that $x_1 \geq -1$ and that $k > 1$, the solution is a vector that satisfies the requirements (22), (23), (24) and (25), so it is a transformed vector.

Case $-Y_2/Y_1 \geq 1$. Note that in this case $Y_2 \neq 0$ and $0 < -Y_1/Y_2 \leq 1$. To find a transformed vector, we will take from the solutions (29) the case $\alpha = 1/Y_2$, $\beta = 1$:

$$(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d}) = \left(1, \frac{-Y_1}{Y_2}, 1, x_1 - \frac{Y_1}{Y_2}x_2 + k \right).$$

This vector trivially satisfies the conditions (23) and (24). Also fulfills $\bar{s_3} > 0$

of condition (25). Therefore we only need to prove $\bar{d} \geq \bar{s_3}$, that is,

$$x_1 - \frac{Y_1}{Y_2} x_2 + k \geq 1.$$

If $x_2 \geq 0$, as $x_1 \geq 1$ and $k \geq 2$, then $x_1 - Y_1 x_2 / Y_2 + k \geq 1 + (-Y_1/Y_2)\, x_2 \geq 1$.

In the case $x_2 = -1$, then $x_1 \geq 0$ because $x_1 + x_2 \geq -1$, and as well $k \geq 2$. Then $x_1 - Y_1 x_2 / Y_2 + k = x_1 - (-Y_1/Y_2) + k \geq 2 - (-Y_1/Y_2) \geq 1$.

Finally if the vector obtained is not formed by integers, it is suffice to multiply it by $|Y_2|$ to meet (22) and accordingly obtain a transformed vector. □

Remark 14. *Proposition 13 does not study the compatibility between pairs of bricks of degree 1, that is, the bricks of* $\mathsf{B_1}$. *If we study case by case all the possible compatibilities, we observe that here are some special situations that slightly modify the previous result. In general, Proposition 13 is true, but with the following exceptions:*

1. *The brick* $[-1, 0; 1]$ *has as associated equation* $-\bar{s_1} + \bar{s_3} = \bar{d}$, *which is incompatible with the requirements* (24) *and* (25). *Therefore, this brick can never exist in a QH system.*
2. *The compatibility between the bricks* $[0, -1; 1]$ *and* $[-1, 1; 1]$ *is not verified, contrary to what would be deduced from the application of Proposition 13.*

5. $\mathbf{w}_m$ as unique identifier of maximal QH systems

Lemma 15. *Given a brick* $[x_1, x_2; k]$ *belonging to* $\mathsf{B_k}$, *it is satisfied that:*

(i) $[x_1, x_2 + 1; k]$, $[x_1, x_2 - 1; k]$, *or both belong to* $\mathsf{B_k}$.

(ii) $[x_1 + 1, x_2 - 1; k]$, $[x_1 - 1, x_2 + 1; k]$, *or both belong to* $\mathsf{B_k}$.

Proof. Let $k \in \mathbb{Z}^+$ be and $[x_1, x_2; k] \in \mathsf{B_k}$.

From the definition (28) we know that the bricks of $\mathsf{B_k}$ are exactly those $[p, q; k]$ that verify the conditions

$$(a)\ p, q \in \mathbb{Z}, \quad (b)\ -1 \leq p \leq k, \quad (c)\ -1 \leq q \leq k, \quad (d)\ -1 \leq p + q \leq k.$$

(i) Let $[x_1, x_2; k] = [x_1, -1; k] \in \mathsf{B_k}$. Then (a) and (d) imply $0 \leq x_1 \leq k$, so $[x_1, x_2 + 1; k] = [x_1, 0; k] \in \mathsf{B_k}$.

On the other hand, if $[x_1, x_2; k] \in \mathsf{B_k}$, being $0 \leq x_2 \leq k$, then $[x_1, x_2 - 1; k]$ belongs to $\mathsf{B_k}$, except if $x_1 = -1$ and $x_2 = 0$. But in such a case $[x_1, x_2 + 1; k] = [-1, 1; k] \in \mathsf{B_k}$.

(ii) Let $[x_1, x_2; k] \in \mathsf{B_k}$ where $x_1 > x_2$. Then, by (b) and (c), $0 \leq x_1 \leq k$ and $-1 \leq x_2 \leq k - 1$. As a consequence $[x_1 - 1, x_2 + 1; k]$ verifies (a), (b), (c) and (d), so $[x_1 - 1, x_2 + 1; k] \in \mathsf{B_k}$.

In a similar way we can prove that $[x_1+1, x_2-1; k] \in \mathsf{B_k}$ when $x_1 < x_2$.

Now let $[x_1, x_2; k] \in \mathsf{B_k}$ being $x_1 = x_2$. By (d), we have that $-1 \leq 2x_1, 2x_2 \leq k$, so $0 \leq x_1,\ x_2 \leq k/2$, and being $k \geq 1$, we conclude $0 \leq x_1,\ x_2 \leq k-1$. Thus $[x_1+1, x_2-1; k]\,,\ [x_1-1, x_2+1; k] \in \mathsf{B_k}$. □

Theorem 16. *A maximal QH system has a unique weight vector family.*

Proof. The case of maximal homogeneous systems has already been studied in section 3. The maximal homogeneous system of degree n, $\mathsf{H_n}$, only has the weight vector family $F_{\mathsf{H_n}}(1,1)$.

We will therefore analyze the case of inhomogeneous maximal systems. An inhomogeneous QH system of degree n must have at least two compatible bricks: one of the degree of the system n, $[x_1, x_2; n]$, and another of degree $m < n$, $[y_1, y_2; m]$. Being compatible, we know from Proposition 12 that either $Y_1 > 0$, or $Y_1 + Y_2 > 0$ must occur, with $Y_i = y_i - x_i$ for $i = 1, 2$. Distinguishing these two cases we will show that there is always a third brick compatible with the two previous ones, so that it can be added to the system. In addition, we will see that any maximal system formed from these three bricks has a unique family of weight vectors.

Case $Y_1 > 0$: By Lemma 15 we know that $[x_1, x_2+1; n]$, $[x_1, x_2-1; n]$, or both, belong to $\mathsf{B_n}$. We suppose that $[x_1, x_2+1; n] \in \mathsf{B_n}$, because the other case is identical, and we will see that $[x_1, x_2+1; n]$ is compatible with $[x_1, x_2; n]$ and $[y_1, y_2; m]$. The system of equations associated with these three bricks is

$$\begin{array}{c} x_1\bar{s_1} + x_2\bar{s_2} + n\bar{s_3} - \bar{d} = 0, \\ y_1\bar{s_1} + y_2\bar{s_2} + m\bar{s_3} - \bar{d} = 0, \\ x_1\bar{s_1} + (x_2+1)\,\bar{s_2} + n\bar{s_3} - \bar{d} = 0. \end{array} \tag{30}$$

Solving it we observe that it is a system of rank 3 with the following infinite set of solutions of dimension 1:

$$\left\{ \left(\alpha,\ 0,\ \frac{Y_1}{n-m}\alpha,\ \left(\frac{n}{n-m} Y_1 + x_1 \right) \alpha \right)\ :\ \alpha \in \mathbb{R} \right\}.$$

Using the inverse transformation (26), we obtain the solution set

$$\left\{ \left(\left(\frac{Y_1}{n-m} + 1 \right) \alpha,\ \frac{Y_1}{n-m}\alpha,\ \frac{Y_1}{n-m}\alpha,\ \left(\frac{n-1}{n-m} Y_1 + x_1 \right) \alpha + 1 \right)\ :\ \alpha \in \mathbb{R} \right\},$$

of the corresponding system (30) in the variables s_1, s_2, s_3 and d.

This space of solutions has dimension 1, and therefore can contain at most one weight vector family. We will see that it contains that family: if we set $\alpha = n - m$, the solution of the obtained system is a weight vector. This solution is

$$(s_1, s_2, s_3, d) = (Y_1 + n - m,\ Y_1,\ Y_1,\ (n-1)\,Y_1 + (n-m)\,x_1 + 1)\,,$$

whose components are integers. In order to (s_1, s_2, s_3, d) be a weight vector, we have to check that $s_1 \geq s_2 \geq s_3 > 0$ and $d > 0$. Since $Y_1 > 0$ and $n > m$, the first is evident. To prove $d > 0$, we observe that being $n > m \geq 1$, $Y_1 > 0$ and $x_1 \geq -1$, the unique problematic case could appear when $x_1 = -1$. When it happens, we have that $d = (n-1)(y_1+1) - (n-m) + 1 = (n-1)\, y_1 + m > 0$, because $y_1 > x_1 \geq -1$.

It is possible to ask, since we deal with maximal systems, if adding a fourth brick, and therefore another equation to system (30), can reduce the number of solutions, thus losing the obtained family.

If we add the brick $[z_1, z_2; p]$ we have the system of equations in the variables $\bar{s_1}$, $\bar{s_2}$, $\bar{s_3}$ and $\bar{d}$

$$\begin{gathered} x_1\bar{s_1} + x_2\bar{s_2} + n\bar{s_3} - \bar{d} = 0, \\ y_1\bar{s_1} + y_2\bar{s_2} + m\bar{s_3} - \bar{d} = 0, \\ x_1\bar{s_1} + (x_2+1)\,\bar{s_2} + n\bar{s_3} - \bar{d} = 0, \\ z_1\bar{s_1} + z_2\bar{s_2} + p\bar{s_3} - \bar{d} = 0. \end{gathered}$$

If this new system of equations maintains the rank 3, it has the same solutions as (30) and therefore the corresponding QH system has a unique family of vectors.

If the system reaches rank 4 it would be a determined compatible system with single solution $(0,0,0,0)$, which cannot be a transformed vector because it does not satisfy (24) nor (25). Thus any system containing these four bricks can be QH. Therefore as we add bricks to the system while maintaining the quality of being QH, we maintain the rank 3 in the corresponding systems of equations, and also maintain the existence of a unique family of weight vectors.

Case $Y_1 + Y_2 > 0$: This case is proved almost identically to the case $Y_1 > 0$, so we will only point out the differences. Now the third brick will be $[y_1+1, y_2-1; m]$ or $[y_1-1, y_2+1; m]$, depending on which of the two exists (see Lemma 15). If we assume that the first one exists, the system of equations is

$$\begin{gathered} x_1\bar{s_1} + x_2\bar{s_2} + n\bar{s_3} - \bar{d} = 0, \\ y_1\bar{s_1} + y_2\bar{s_2} + m\bar{s_3} - \bar{d} = 0, \\ (y_1+1)\,\bar{s_1} + (y_2-1)\,\bar{s_2} + m\bar{s_3} - \bar{d} = 0, \end{gathered}$$

whose solutions are

$$\left\{ \left(\alpha,\ \alpha,\ \frac{Y_1+Y_2}{n-m}\alpha,\ \left(\frac{n}{n-m}\,(Y_1+Y_2) + x_1 + x_2 \right) \alpha \right)\ :\ \alpha \in \mathbb{R} \right\}.$$

Using the inverse transformation (26), we obtain the solutions in the variables s_1, s_2, s_3 and d:

$$\left\{ \left((A+1)\,\alpha,\ (A+1)\,\alpha,\ A\alpha,\ \left(\frac{n-1}{n-m}\,(Y_1+Y_2) + x_1 + x_2 \right) \alpha + 1 \right)\ :\ \alpha \in \mathbb{R} \right\},$$

where $A = (Y_1+Y_2)\,/\,(n-m)$.

As in the previous case if we take $\alpha = n-m$ we obtain the particular solution

$$s_1 = s_2 = Y_1 + Y_2 + n - m,$$

$$s_3 = Y_1 + Y_2,$$

$$d = (n-1)(Y_1 + Y_2) + (n-m)(x_1 + x_2) + 1.$$

This solution consists of integers, verifies $s_1 \geq s_2 \geq s_3 > 0$ and verifies $d > 0$ providing that $Y_1 + Y_2 > 0$ implies $y_1 + y_2 > x_1 + x_2 \geq -1$. It is, Therefore, it is a weight vector.

□

Corollary 17. *A maximal QH system is made up of at least three bricks.*

Proof. The homogeneous maximal system $\mathsf{H_n}$ has, as we have seen in Proposition 11, $(k+1)(k+6)/2$ bricks, which exceeds 2 for every $n > 0$.

An inhomogeneous maximal QH system must have at least two bricks of different degrees. In the proof of Theorem 16 it is showed that there is always another compatible brick that can be added to the system. □

Corollary 18. *A maximal QH system always has minimum weight vector of the system.*

Proof. Let S be the system, and $F_{\mathsf{S}}(\lambda,\ \mu)$ its unique family of weight vectors. Then, $\mathbf{w}_m$=$\mathbf{g}_{(\lambda,\mu)}$. □

Corollary 19. *Given a maximal QH system* S*, a weight vector* (s_1, s_2, s_3, d) *is the minimum weight vector of* S *if and only if* $\gcd(s_1, s_2, s_3) = 1$.

Proof. The proof is an easy consequence of Theorem 16 and Proposition 7. □

An important consequence of Theorem 16 is that a weight vector of a maximal QH system S verifies conditions (19) and (20) if and only if all other vectors of S verify them. Because of this we avoid the possibility that, when filtering with (19) and (20), we did not consider other vectors of S (for example (1,2,3,4)) whose presence would imply a modification in the algebraic structure of the system.

Proposition 20. *Two different maximal QH systems of degree* n *have no common weight vectors.*

Proof. Let S and T be two maximal QH systems of degree n that share the weight vector $\mathbf{w} = (s_1, s_2, s_3, d)$. Let $F_{\mathsf{S}}(\lambda,\ \mu)$ and $F_{\mathsf{T}}(\lambda,\ \mu)$ be the vector families to which $\mathbf{w}$ belongs in S and T respectively. We will show that S and T are the same system, proving that any monomial of S is contained in T and vice versa.

Let (12) be a monomial of S. Proposition 1 assure that $\mathbf{w}$ verifies the equation of the right side of (8). Since, by Theorem 16, all the weight vectors of T are those of $F_{\mathsf{T}}(\lambda, \mu)$, so being of the form $(rs_1, rs_2, rs_3, r(d-1)+1)$, with r a rational number, it is trivial that all of them also verify the equation (8). Therefore, due again to Proposition 1, the monomial (12) must be present in T. In the same way, we can show that all the monomials of T are in S, so S and T match. □

It should be noted that if we vary the degree n of the system, then the weight vectors could be repeated, and therefore also the families. For example, the weight vector $\mathbf{w} = (2, 2, 1, 2)$ and their corresponding family appears in the following maximal system of degree 2

$$\begin{aligned} \dot{x} &= xz + yz, \\ \dot{y} &= xz + yz, \\ \dot{z} &= z^2 + x + y, \end{aligned}$$

and also in the following maximal system of degree 3

$$\begin{aligned} \dot{x} &= z^3 + xz + yz, \\ \dot{y} &= z^3 + xz + yz, \\ \dot{z} &= z^2 + x + y. \end{aligned}$$

Remark 21. *As a consequence of the previous results we have that, given a maximal QH system* S *of degree* n*, its minimum weight vector* $\mathbf{w}_m$ *always exists and is a unique identifier of* S *within the set of maximal QH systems of degree* n*.*

6. Constructing the set of maximal inhomogeneous QH systems

By Corollary 17 we know that every inhomogeneous maximal QH system contains a minimum of three bricks. In addition, Theorem 16 assures that the system has a single family of weight vectors. Therefore given an n-degree system S of this type, we can always choose three bricks $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$ in such a way that:

(a) At least one of them is of the same degree as S, but they do not have all the same degree , i.e., $1 \leq m < n$, $1 \leq k \leq n$.

(b) Their respective associated equations form a system of rank 3, and therefore its solution space in $\mathbb{R}^4$ has dimension 1.

(c) The bricks are compatible. That is, some of the solutions of (b) verify the constraints (22), (23), (24) and (25).

When three bricks of S satisfy these requirements we say that they form a *seed of* S. With requirement (a) we force that the three bricks belong to an inhomogeneous system of degree n. From conditions (b) and (c) it follows that the compatibility between the three bricks is reduced to a single family of weight vectors. Consequently two distinct maximal systems cannot share a seed, although they may share three or more bricks that do not form a seed.

A maximal system always has at least one seed, although it usually owns more. In a later theorem we will show that if we have a seed of a system S, the rest of the system and its corresponding family of vectors are totally determined. Thus, all the information about the algebraic structure of a maximal inhomogeneous QH system is in any of its seeds. So we will start by identifying them in the following result.

Theorem 22. *The bricks $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$, with $1 \leq m < n$ and $1 \leq k \leq n$, form a seed of some n-degree maximal inhomogeneous QH system S if and only if the following three conditions hold:*

(i) $T_2 \neq 0$, $T_1 \cdot T_2 \leq 0$ *and* $|T_1| \leq |T_2|$,

(ii) $T_2 \begin{bmatrix} Y_1 & Y_2 \\ T_1 & T_2 \end{bmatrix} > 0$,

(iii) $\dfrac{Y_1 T_2 - Y_2 T_1}{(n-m)\, T_2} \geq \dfrac{x_2 T_1 - x_1 T_2}{(n-1)\, T_2}$,

where $Y_i = y_i - x_i$, $T_i = (k-m)\, x_i + (n-k)\, y_i + (m-n)\, z_i$, for $i = 1, 2$.

Proof. The equations associated with these three bricks form the homogeneous linear system

$$\begin{array}{c} x_1 \bar{s_1} + x_2 \bar{s_2} + n \bar{s_3} - \bar{d} = 0, \\ y_1 \bar{s_1} + y_2 \bar{s_2} + m \bar{s_3} - \bar{d} = 0, \\ z_1 \bar{s_1} + z_2 \bar{s_2} + k \bar{s_3} - \bar{d} = 0. \end{array} \tag{31}$$

After Gaussian transformations the system can be represented in its matrix form as

$$\begin{pmatrix} x_1 & x_2 & n & -1 \\ Y_1 & Y_2 & m-n & 0 \\ T_1 & T_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{s_1} \\ \bar{s_2} \\ \bar{s_3} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{32}$$

The set of solutions obtained solving this system is

$$\left(T_2 \alpha,\ -T_1 \alpha,\ \frac{Y_1 T_2 - Y_2 T_1}{n-m} \alpha,\ \left(\frac{n\,(Y_1 T_2 - Y_2 T_1)}{n-m} + x_1 T_2 - x_2 T_1 \right) \alpha \right), \tag{33}$$

where $\alpha \in \mathbb{R}$.

In order to prove the necessary condition ($\Rightarrow$), we suppose that the three bricks form a seed. Then, by (c), some of the solutions of the set (33) verify (22), (23), (24) and (25). Let $(\bar{s_1}, \bar{s_2}, \bar{s_3}, \bar{d})$ be one of these solutions, i.e., a transformed vector obtained by setting a particular $\bar{\alpha} \in \mathbb{R}$ in (33). Consequently, $\bar{s_1} \geq \bar{s_2} \geq 0$ and $\bar{s_1} > 0$, so (i) must be verified. Also, $\bar{s_1}$ and $\bar{s_3}$ have the same sign, so being $n - m > 0$, we get (ii). Finally, $\bar{d} \geq \bar{s_3}$, and then

$$\bar{\alpha}\left(\frac{n}{n-m}\left(Y_1T_2 - Y_2T_1\right) + x_1T_2 - x_2T_1\right) \geq \frac{\bar{\alpha}}{n-m}\left(Y_1T_2 - Y_2T_1\right).$$

As $\bar{\alpha}$ and T_2 are not null and have the same sign (because $\bar{s_1} > 0$), the previous inequality is equivalent to

$$\frac{n}{n-m} \cdot \frac{Y_1T_2 - Y_2T_1}{T_2} + \frac{x_1T_2 - x_2T_1}{T_2} \geq \frac{Y_1T_2 - Y_2T_1}{(n-m)\,T_2},$$

and from this by simple algebraic operations we obtain

$$\frac{Y_1T_2 - Y_2T_1}{(n-m)\,T_2} \geq \frac{x_2T_1 - x_1T_2}{(n-1)\,T_2},$$

so (iii) is proved.

To prove the sufficient condition ($\Leftarrow$) we must verify (a), (b) and (c) of the seed definition. Being $n > m$, (a) is fulfilled trivially. Due to $T_2 \neq 0$, and being $n \neq m$, the system has rank 3, as can be seen just checking the matrix of (32). Also, as a homogeneous system of equations, it has solutions. Therefore (b) is true. Now let $T_2 > 0$ be. Setting $\alpha = n - m$ we get the following particular solution of the set (33):

$$\begin{aligned}
\bar{s_1} &= T_2\,(n-m)\,,\\
\bar{s_2} &= -T_1\,(n-m)\,,\\
\bar{s_3} &= Y_1T_2 - Y_2T_1,\\
\bar{d} &= n\,(Y_1T_2 - Y_2T_1) + (n-m)\,(x_1T_2 - x_2T_1)\,.
\end{aligned}$$

This solution verifies (22), (23), (24) and (25), then (c) holds. If $T_2 < 0$ we set $\alpha = m - n$ with the same conclusion. □

Theorem 23. *If the bricks $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$, with $1 \leq m < n$ and $1 \leq k \leq n$, form a seed of an n-degree maximal inhomogeneous QH system S, then*

(i) The brick $[t_1, t_2; l]$ with $1 \leq l \leq n$, belongs to the system S if and only if $T_1R_2 = T_2R_1$.

(ii) The minimum weight vector of S is $\mathbf{w}_m = \left(\frac{\hat{s_1}}{G}, \frac{\hat{s_2}}{G}, \frac{\hat{s_3}}{G}, \frac{\tilde{d}}{G} + 1\right)$, where

$$Y_i = y_i - x_i,\ \textit{for}\ i = 1, 2,$$

$$T_i = (k - m)\, x_i + (n - k)\, y_i + (m - n)\, z_i,\ \textit{for}\ i = 1, 2,$$

$$R_i = (l - m)\, x_i + (n - l)\, y_i + (m - n)\, t_i,\ \textit{for}\ i = 1, 2,$$

$$\hat{s_1} = |Y_1 T_2 - Y_2 T_1| + (n - m)\, |T_2|,$$

$$\hat{s_2} = |Y_1 T_2 - Y_2 T_1| + (n - m)\, |T_1|,$$

$$\hat{s_3} = |Y_1 T_2 - Y_2 T_1|,$$

$$\tilde{d} = (n - 1)\, |Y_1 T_2 - Y_2 T_1| + \delta\, (n - m)\, (x_1 T_2 - x_2 T_1),$$

$$\delta = \operatorname{sgn}(T_2),$$

$$G = \gcd(\hat{s_1}, \hat{s_2}, \hat{s_3}).$$

Proof. (i) Since $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$ form a seed of S, we know that its associated system of equations (31) has rank 3. A fourth brick $[t_1, t_2; l]$ also belongs to S if and only if the increased system

$$\begin{array}{r} x_1 \bar{s_1} + x_2 \bar{s_2} + n \bar{s_3} - \bar{d} = 0, \\ y_1 \bar{s_1} + y_2 \bar{s_2} + m \bar{s_3} - \bar{d} = 0, \\ z_1 \bar{s_1} + z_2 \bar{s_2} + k \bar{s_3} - \bar{d} = 0, \\ t_1 \bar{s_1} + t_2 \bar{s_2} + l \bar{s_3} - \bar{d} = 0, \end{array} \tag{34}$$

which includes the equation associated to $[t_1, t_2; l]$ maintains the rank 3, because of the rank of (34) goes up to 4 the system becomes compatible determined with unique solution $(0, 0, 0, 0)$, and in this case S would not be QH because it would not have weight vectors. Then the matrix expression of system (34), after Gaussian transformations, adopts the form

$$\begin{pmatrix} x_1 & x_2 & n & -1 \\ Y_1 & Y_2 & m - n & 0 \\ T_1 & T_2 & 0 & 0 \\ R_1 & R_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{s_1} \\ \bar{s_2} \\ \bar{s_3} \\ \bar{d} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{35}$$

with Y_i, T_i and R_i being the values of the statement for $i = 1, 2$. Finally the matrix of (35) has rank 3 if and only if $T_1 R_2 = T_2 R_1$.

(ii) As seen in (i) the solution space of all the equations associated with bricks of S is the same as the solution space of (31). That is, it is the set defined in (33). Within (33) exists the transformed vectors of S. Applying to (33) the inverse transformation t^{-1} given in (26), we obtain the solutions corresponding to the equations in the variables s_1, s_2, s_3, and d, which are

$$s_1 = \alpha \left(\frac{Y_1 T_2 - Y_2 T_1}{n - m} + T_2 \right),$$

$$s_2 = \alpha \left(\frac{Y_1 T_2 - Y_2 T_1}{n-m} - T_1 \right), \tag{36}$$
$$s_3 = \alpha \left(\frac{Y_1 T_2 - Y_2 T_1}{n-m} \right),$$
$$d = \alpha \left((n-1) \frac{Y_1 T_2 - Y_2 T_1}{n-m} + x_1 T_2 - x_2 T_1 \right) + 1,$$

with $\alpha \in \mathbb{R}$ and Y_i, T_i being the values of the statement for $i = 1, 2$. The elements of (36) which are formed by positive integers are the weight vectors of the maximal QH system S. We now obtain a particular weight vector. For this we can take $\alpha = \operatorname{sgn}(T_2)(n-m)$. Taking into account that by Theorem 22 we know that $Y_1 T_2 - Y_2 T_1$ has the same sign of T_2, and also that when T_1 is not null, it has the opposite sign of T_2, we obtain the weight vector of S $\mathbf{w} = (\hat{s_1}, \hat{s_2}, \hat{s_3}, \hat{d})$, where

$$\hat{s_1} = |Y_1 T_2 - Y_2 T_1| + (n-m)\,|T_2|,$$

$$\hat{s_2} = |Y_1 T_2 - Y_2 T_1| + (n-m)\,|T_1|,$$

$$\hat{s_3} = |Y_1 T_2 - Y_2 T_1|,$$

$$\hat{d} = (n-1)\,|Y_1 T_2 - Y_2 T_1| + \delta\,(n-m)\,(x_1 T_2 - x_2 T_1) + 1,$$

being $\delta = \pm 1$ with the same sign as T_2, i.e., $\delta = \operatorname{sgn}(T_2)$. By Proposition 5 and Corollary 19 we know how to obtain the minimum weight vector $\mathbf{w}_m$ of a maximal system from one of its weight vectors $\mathbf{w} = (\hat{s_1}, \hat{s_2}, \hat{s_3}, \hat{d})$, namely

$$\mathbf{w}_m = \left(\frac{\hat{s_1}}{\gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})}, \frac{\hat{s_2}}{\gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})}, \frac{\hat{s_3}}{\gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})}, \frac{\hat{d}-1}{\gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})} + 1 \right).$$

In this way the result is proved without further action than setting $\tilde{d} = \hat{d} - 1$. □

Remark 24. *Note that if the minimum weight vector of a system* S *is known, we have an alternative way to Theorem 23 (i) to determine if a brick* $[t_1, t_2; l]$ *is in* S *or not. It is enough to check if the minimum weight vector of* S *verifies the equation associated with the brick, that is*

$$t_1 s_1 + t_2 s_2 + (l - t_1 - t_2 - 1) s_3 = d - 1.$$

In the same way that happens with the seeds, if we fix the degree of S *the* *whole information about the algebraic structure of* S *is in its minimum weight vector. Then when the minimum weight vector has previously been calculated this is a faster method than the one proposed in Theorem 23* (i).

7. The algorithm

The objective of our algorithm is to find all the maximal QH systems of a certain degree n. The only homogeneous maximal system is $\mathsf{H_n}$, so the algorithm is focused on the inhomogeneous. The algorithm is mainly based on Theorem 22, Theorem 23, and Remark 24. Briefly any system of this type must have among its constituent bricks one or more seeds, and detected a seed all information about the structure of the system can be obtained by using Theorem 23 and Remark 24. Therefore, our first goal is to get a list of all possible seeds of n-degree systems, for which we will use Theorem 22 with the help of Propositions 12 and 13. In this way we will obtain all the wanted systems, although there can be repetitions because two different seeds can generate the same system. We will avoid these repetitions using the minimum vector of each system as unique identifier of it.

It is an algorithm that requires quite computation, which also grows notably when the degree n of the required systems increases. Because of this, a basic principle of design has been to avoid unnecessary calculations. For this reason we have taken different steps that will be discussed later.

The algorithm has a modular structure, and is formed by a main process together with four auxiliary functions. The criterion that we followed to extract computation from the main body to the functions has been to isolate those calculations that are repeatedly executed.

In order to facilitate its practical implementation, but at the same time to provide a tool as general as possible, we present the algorithm written in pseudocode. The structure is highly detailed so that their later translation to any programming language will be simple.

Algorithm 1: Determines whether two bricks are compatible

Input: Two bricks: $B_i = [x_1^i, x_2^i; k^i]$, $B_j = [x_1^j, x_2^j; k^j]$
Output: $true$ if the bricks are compatible, $false$ if not

1 **Function** areCompat(B_i, B_j)
2     $Y_1 \leftarrow x_1^j - x_1^i$; $Y_2 \leftarrow x_2^j - x_2^i$
3   **if** $(k^i \neq k^j)$ *and* $(Y_1 > 0$ *or* $Y_1 + Y_2 > 0)$ **then**
4   **return** $true$
5   **else if** $(k^i = k^j)$ *and* $(Y_1 = 0$ *or* $-Y_2/Y_1 \geq 1)$ **then**
6   **return** $true$
7   **else**
8   **return** $false$

The first function, areCompat(B_i, B_j), receives two bricks and determines whether they are compatible or not. We try to avoid unnecessary computation by leaving the function as soon as possible. We do not consider the special cases of incompatibility stated in Remark 14, since these would fulfill their function of filtering in a very limited number of executions of the function, and in return we would have a remarkable computational cost. Besides, the few incompatible cases that are allowed to pass are subsequently filtered into other functions. We make use of Proposition 12 (Line 3) if both bricks are of different degree, and of Proposition 13 (Line 5) if the bricks are of the same degree. Note that in case of bricks of different degrees it is important the order of the inputs: the highest degree brick must be the first one.

Algorithm 2: Determines whether three bricks form a seed

Input: Three bricks:$B_i = [x_1^i, x_2^i; k^i]$, $B_j = [x_1^j, x_2^j; k^j]$, $B_p = [x_1^p, x_2^p; k^p]$
Output: $true$ if the bricks form a seed, $false$ if not

Function areSeed(B_i, B_j, B_p)
    $T_2 \leftarrow (k^p - k^j)x_2^i + (k^i - k^p)x_2^j + (k^j - k^i)x_2^p$
    if $T_2 = 0$ **then**
        return $false$
    $T_1 \leftarrow (k^p - k^j)x_1^i + (k^i - k^p)x_1^j + (k^j - k^i)x_1^p$
    if $T_1 \cdot T_2 > 0$ *or* $|T_1| > |T_2|$ **then**
        return $false$
    $Y_1 \leftarrow x_1^j - x_1^i$; $Y_2 \leftarrow x_2^j - x_2^i$
    if $T_2 \cdot (Y_1 T_2 - Y_2 T_1) \leq 0$ **then**
        return $false$
    if $\dfrac{Y_1 T_2 - Y_2 T_1}{(k^i - k^j)\, T_2} < \dfrac{x_2 T_1 - x_1 T_2}{(k^i - 1)\, T_2}$ **then**
        return $false$
    return $true$

Our second function, areSeed(B_i, B_j, B_p), determines whether the three bricks B_i, B_j, and B_p, form a seed or not. The function verifies the requirements stated in Theorem 22. It avoids unnecessary computation exiting the function at the moment in which one of these conditions is not verified. The calculation of the values of Y_1, Y_2, T_1 and T_2 only takes place when it is indispensable. Condition (i) of Theorem 22 is checked in Lines 3 and 6; condition (ii) in Line 9; and condition (iii) in Line 11 of the function.

Algorithm 3: Finds the minimum weight vector of a QH system

Input: Three bricks that form a seed: $B_i = [x_1^i, x_2^i; k^i]$, $B_j = [x_1^j, x_2^j; k^j]$, $B_p = [x_1^p, x_2^p; k^p]$

Output: The minimum weight vector $\mathbf{w}_m$ corresponding to that seed

Function calculateWm$(B_i,\ B_j,\ B_p)$
- $Y_1 \leftarrow x_1^j - x_1^i$
- $Y_2 \leftarrow x_2^j - x_2^i$
- $T_1 \leftarrow (k^p - k^j)x_1^i + (k^i - k^p)x_1^j + (k^j - k^i)x_1^p$
- $T_2 \leftarrow (k^p - k^j)x_2^i + (k^i - k^p)x_2^j + (k^j - k^i)x_2^p$
- $\delta \leftarrow sign(T_2)$
- $\hat{s_1} \leftarrow |Y_1T_2 - Y_2T_1| + (k^i - k^j)|T_2|$
- $\hat{s_2} \leftarrow |Y_1T_2 - Y_2T_1| + (k^i - k^j)|T_1|$
- $\hat{s_3} \leftarrow |Y_1T_2 - Y_2T_1|$
- $\tilde{d} \leftarrow (k^i - 1)|Y_1T_2 - Y_2T_1| + \delta(k^i - k^j)(x_1^iT_2 - x_2^iT_1)$
- $G \leftarrow gcd(\hat{s_1}, \hat{s_2}, \hat{s_3})$
- $\mathbf{w}_m \leftarrow (\frac{\hat{s_1}}{G}, \frac{\hat{s_2}}{G}, \frac{\hat{s_3}}{G}, \frac{\tilde{d}}{G} + 1)$
- **return** $\mathbf{w}_m$

On the other hand, calculateWm(B_i, B_j, B_p) function accurately reproduces the statement of section (ii) of Theorem 23. That is, it receives three bricks, which are known to form a seed of a maximal QH system S, and calculates the corresponding minimum weight vector of S.

Algorithm 4: Determines if a brick is in a maximal QH system with a given minimum vector

Input: The minimum weight vector $\mathbf{w}_m = (s_1, s_2, s_3, d)$ of a QH system and a brick $B_q = [x_1^q, x_2^q; k^q]$

Output: $true$ if the brick is in the system, $false$ if not

Function isBrickInSystem$(\mathbf{w}_m,\ B_q)$
- **if** $x_1^q s_1 + x_2^q s_2 + (k^q - x_1^q - x_2^q - 1)s_3 = d - 1$ **then**
  - **return** $true$
- **else**
  - **return** $false$

The last of the auxiliary functions is isBrickInSystem($\mathbf{w}_m, B_q$). As we have seen, the minimum vector of a maximal QH system stores all information about the algebraic structure of the system. This function receives the minimum vector $\mathbf{w}_m$ of a system S together with a brick B_q, and decides if B_q is present in S or not (Line 2). It is based on the result stated in Remark 24, because when we have the minimum vector, this method requires much less computation than the one exposed in section (i) of Theorem 23. This brief function has the last word in the process of construction of the maximal QH systems.

Algorithm 5: Provides all QH systems of a given degree

Input: Degree of the systems (n)
Output: List of all QH systems of degree n

1 $N \leftarrow \frac{(k+1)(k+6)}{2}$
2 $T \leftarrow \frac{n^3}{6} + 2n^2 + \frac{29}{6}n$
3 $Aux \leftarrow$ empty matrix of 4 columns
4 create ordered list of bricks $\{B_i\}_{i=1}^{T}$, where $B_i = \left[x_1^i, x_2^i; k^i\right]$
5 **for** $i \leftarrow 1$ **to** N **do**
6   **for** $j \leftarrow N+1$ **to** T **do**
7   **if** areCompat(B_i, B_j) **then**
8   **for** $p \leftarrow i+1$ **to** $j-1$ **do**
9   **if** areCompat(B_i, B_p) *and* areCompat(B_p, B_j) **then**
10   **if** areSeed(B_i, B_j, B_p) **then**
11                         $\mathbf{w}_m \leftarrow$ calculateWm(B_i, B_j, B_p)
12   **if** $\mathbf{w}_m$ *is not a row of* Aux **then**
13   add $\mathbf{w}_m$ as a new row of Aux
14   **for** $q \leftarrow 1$ **to** T **do**
15   **if** isBrickInSystem($\mathbf{w}_m$, B_q) **then**
16   add B_q to system S
17   **output** system S

The main process of the algorithm starts by asking the user for the only necessary input datum, that is, the degree n of the maximal QH systems that must to be listed. With this datum we apply Proposition 11 to obtain two values: N (Line 1) is the total number of n-degree available bricks, that is, the cardinal of $\mathsf{B_n}$; and T (Line 2) is the total number of bricks of degree less than or equal to n, that is, the cardinal of $\{\mathsf{B_k}\}_{k=1}^{n}$. To avoid repeating systems, we will store in the matrix Aux the minimum weight vector of each new system that we find.

In Line 4 is made the ordered list of the T available bricks. These must be ordered so that we can go over them one by one. The only condition for this order is to place first the N bricks of maximum degree. Once this is done, the rest of the ordination details are irrelevant to the operation of the algorithm.

Subsequently, there is a series of nested *for* loops (Lines 5, 6 and 8) which are intended to run, without repetitions, all possible trios of bricks made up of at least one brick of degree n and at least one brick of degree less than n. Thus we assure two things: the degree of the systems obtained from these trios is n, and the systems are inhomogeneous.

Obviously not all brick's trios are a seed, so we check each of them by calling the function $\textsc{areSeed}(B_i, B_j, B_p)$ (Line 10). In fact, most of trios do not form a seed. So with the aim of saving computation of unnecessary calls to this expensive function, we include several conditional structures (Lines 7 and 9) in which we check every required compatibilities of pairs of bricks. This is done by means of function $\textsc{areCompat}(B_i, B_j)$.

For each detected seed, we know that there is a maximal inhomogeneous QH system. But two different seeds can give rise to the same system, thus producing repetitions. To avoid them, we obtain the minimum weight vector associated with the seed by calling (Line 11) the function $\textsc{calculateWm}(B_i, B_j, B_p)$. The easiest way to identify this type of systems is through its minimum weight vectors. If the minimum weight vector obtained is already in the matrix Aux, it means that this system has already been found before, in which case we discard it. Otherwise we insert the vector into Aux and go to the process of building the system.

The function $\textsc{isBrickInSystem}(\mathbf{w}_m, B_q)$ is called for each of the T available Bricks (Line 15), even for the three bricks which, by forming the seed, we know that they are in the system. It is a function with little computation, so establishing filters to avoid calling it in these three particular cases would have more computational cost than leaving it that way.

Every time the function $\textsc{isBrickInSystem}(\mathbf{w}_m, B_q)$ returns *true* implies that the brick $B_q = [x_1^q, x_2^q; k^q]$ belongs to the system S we are building. This means (Line 16) that the monomial

$$a_{x_1^q+1, x_2^q, k^q-x_1^q-x_2^q-1} x^{x_1^q+1} y^{x_2^q} z^{k^q-x_1^q-x_2^q-1}$$

is added to the P component of S; the monomial

$$b_{x_1^q, x_2^q+1, k^q-x_1^q-x_2^q-1} x^{x_1^q} y^{x_2^q+1} z^{k^q-x_1^q-x_2^q-1}$$

is added to the Q component of S; and the monomial

$$c_{x_1^q, x_2^q, k^q-x_1^q-x_2^q} x^{x_1^q} y^{x_2^q} z^{k^q-x_1^q-x_2^q}$$

is added to the component R. We must take into account here the cases in which a brick contributes with a single monomial, studied previously in this work.

Finally being the construction process finished, the algorithm returns S in the format that is considered most appropriated (Line 17). This task is done recursively until the trios of bricks, and therefore the seeds and the possibility of forming new systems of these characteristics, are exhausted.

8. QH systems of degree 2

The following is the list of normal forms of 3-dimensional maximal inhomogeneous QH systems of degree 2, obtained by the algorithm described in the previous section. The minimum weight vector $\mathbf{w}_m$ of each system is also provided. Being $\mathbf{w}_m$ the generator of the unique weight vector family of each system, it is easy to obtain from it the remaining weight vectors. Since these are maximal systems, the coefficients a, b and c of the systems in the list can take any complex value other than zero.

$$\mathsf{S}_1: \begin{array}{l} \dot{x} = a_{020}y^2 + a_{011}yz + a_{002}z^2 + a_{100}x \\ \dot{y} = b_{010}y + b_{001}z \\ \dot{z} = c_{010}y + c_{001}z \\ \mathbf{w}_m = (2,1,1,1) \end{array}$$

$$\mathsf{S}_2: \begin{array}{l} \dot{x} = a_{002}z^2 + a_{100}x + a_{010}y \\ \dot{y} = b_{002}z^2 + b_{100}x + b_{010}y \\ \dot{z} = c_{001}z \\ \mathbf{w}_m = (2,2,1,1) \end{array}$$

$$\mathsf{S}_3: \begin{array}{l} \dot{x} = a_{020}y^2 + a_{011}yz + a_{002}z^2 \\ \dot{y} = b_{100}x \\ \dot{z} = c_{100}x \\ \mathbf{w}_m = (3,2,2,2) \end{array}$$

$$\mathsf{S}_4: \begin{array}{l} \dot{x} = a_{002}z^2 \\ \dot{y} = b_{002}z^2 \\ \dot{z} = c_{100}x + c_{010}y \\ \mathbf{w}_m = (3,3,2,2) \end{array}$$

$$\mathsf{S}_5: \begin{array}{l} \dot{x} = a_{002}z^2 \\ \dot{y} = b_{100}x \\ \dot{z} = c_{010}y \\ \mathbf{w}_m = (5,4,3,2) \end{array}$$

$$\mathsf{S}_6: \begin{array}{l} \dot{x} = a_{011}yz + a_{100}x \\ \dot{y} = b_{002}z^2 + b_{010}y \\ \dot{z} = c_{001}z \\ \mathbf{w}_m = (3,2,1,1) \end{array}$$

$$\mathsf{S}_7: \begin{array}{l} \dot{x} = a_{020}y^2 + a_{100}x \\ \dot{y} = b_{002}z^2 + b_{010}y \\ \dot{z} = c_{001}z \\ \mathbf{w}_m = (4,2,1,1) \end{array}$$

$$\mathsf{S}_8: \begin{array}{l} \dot{x} = a_{011}yz \\ \dot{y} = b_{002}z^2 + b_{100}x \\ \dot{z} = c_{010}y \\ \mathbf{w}_m = (4,3,2,2) \end{array}$$

$$\mathsf{S}_9: \begin{array}{l} \dot{x} = a_{110}xy + a_{101}xz \\ \dot{y} = b_{020}y^2 + b_{011}yz + b_{002}z^2 + b_{100}x \\ \dot{z} = c_{020}y^2 + c_{011}yz + c_{002}z^2 + c_{100}x \\ \mathbf{w}_m = (2,1,1,2) \end{array}$$

$$\mathsf{S}_{10}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{002}z^2 + b_{100}x \\ \dot{z} = 0 \\ \mathbf{w}_m = (6,4,3,3) \end{array}$$

$$\mathsf{S}_{11}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{002}z^2 \\ \dot{z} = c_{010}y \\ \mathbf{w}_m = (5,3,2,2) \end{array}$$

$$\mathsf{S}_{12}: \begin{array}{l} \dot{x} = a_{011}yz \\ \dot{y} = b_{002}z^2 \\ \dot{z} = c_{100}x \\ \mathbf{w}_m = (5,4,3,3) \end{array}$$

$$\mathsf{S}_{13}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{002}z^2 \\ \dot{z} = c_{100}x \\ \mathbf{w}_m = (7,5,4,4) \end{array}$$

$$\mathsf{S}_{14}: \begin{array}{l} \dot{x} = a_{101}xz + a_{011}yz \\ \dot{y} = b_{101}xz + b_{011}yz \\ \dot{z} = c_{002}z^2 + c_{100}x + c_{010}y \\ \mathbf{w}_m = (2,2,1,2) \end{array}$$

$$\mathsf{S}_{15}: \begin{array}{l} \dot{x} = a_{101}xz + a_{020}y^2 \\ \dot{y} = b_{011}yz + b_{100}x \\ \dot{z} = c_{002}z^2 + c_{010}y \\ \mathbf{w}_m = (3,2,1,2) \end{array}$$

$$\mathsf{S}_{16}: \begin{array}{l} \dot{x} = a_{101}xz + a_{020}y^2 \\ \dot{y} = b_{011}yz \\ \dot{z} = c_{002}z^2 + c_{100}x \\ \mathbf{w}_m = (4,3,2,3) \end{array}$$

$$\mathsf{S}_{17}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{101}xz \\ \dot{z} = c_{010}y \\ \mathbf{w}_m = (4,3,1,3) \end{array} \qquad \mathsf{S}_{19}: \begin{array}{l} \dot{x} = a_{110}xy \\ \dot{y} = b_{020}y^2 + b_{101}xz \\ \dot{z} = c_{011}yz + c_{100}x \\ \mathbf{w}_m = (3,2,1,3) \end{array}$$

$$\mathsf{S}_{18}: \begin{array}{l} \dot{x} = a_{020}y^2 \\ \dot{y} = b_{101}xz \\ \dot{z} = c_{100}x \\ \mathbf{w}_m = (5,4,2,4) \end{array} \qquad \mathsf{S}_{20}: \begin{array}{l} \dot{x} = 0 \\ \dot{y} = b_{101}xz \\ \dot{z} = c_{020}y^2 + c_{100}x \\ \mathbf{w}_m = (4,2,1,4) \end{array}$$

We could add to this list of inhomogeneous systems the only homogeneous maximal QH system of degree 2, H_2, with $\mathbf{w}_m = (1,1,1,2)$, and thus we obtain the complete list of maximal systems of degree 2.

In particular non-maximal QH systems are also represented in the above list. They are obtained from the maximals just canceling some of the coefficients, taking into account that we must always leave at least a monomial of degree 2, and that if we additionally want the system to be inhomogeneous, we must also leave some monomial of degree 1.

Another aspect to consider is that, as we have seen, in order to simplify our algorithm this provides only those systems whose weight vector family satisfies the restriction (19). This is just one of the six possible restrictions of this type that we can establish. But the remaining QH systems are again easy to obtain from the list we have. We could add to the 20 inhomogeneous systems of the list those systems obtained from making permutations of the variables x, y, and z. In this way we arrive at a total of 102 maximal inhomogeneous systems of degree 2, and not to 120 as one would expect of multiplying the number of systems by 3!. This is because there are systems, such as S_1 or S_3, which are symmetrical with respect to two of their variables, and therefore there are permutations that provide repeated systems. Thus, by adding the unique homogeneous system we have a total of 103 maximal QH systems of degree 2.

The number of systems increases notably with the degree. There are 137 inhomogeneous systems of degree 3 verifying the restriction (19), and we have found 643 systems of degree 4 and 2119 systems of degree 5 making use of our algorithm.

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