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LINEAR TYPE GLOBAL CENTERS OF LINEAR SYSTEMS WITH CUBIC HOMOGENEOUS NONLINEARITIES

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Abstract. A center p of a differential system in $\mathbb{R}^2$ is global if $\mathbb{R}^2 \setminus \{p\}$ is filled of periodic orbits. It is known that a polynomial differential system of degree 2 has no global centers. Here we characterize the global centers of the differential systems

$$\dot{x} = ax + by + P_3(x, y), \quad \dot{y} = cx + dy + Q_3(x, y),$$

with P_3 and Q_3 homogeneous polynomials of degree 3, and such that the center has purely imaginary eigenvalues, i.e. a linear type center.

1. Introduction and statement of the main results

The notion of center goes back to Poincaré and Dulac, see [10, 6]. They defined a *center* for a vector field on the real plane as a singular point having a neighborhood filled of periodic orbits with the exception of the singular point. The problem of distinguishing when a monodromic singular point is a focus or a center, known as the focus-center problem started precisely with Poincaré and Dulac and is still active nowadays with many questions still unsolved. These last years also the centers are perturbed for studying the limit cycles bifurcating from their periodic solutions, see for instance [1, 3].

If an analytic system has a center, then it is known that after an affine change of variables and a rescaling of the time variable, it can be written in one of the following three forms:

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x + Q(x, y),$$

called *linear type center*, which has a pair of purely imaginary eigenvalues,

$$\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y)$$

called *nilpotent center*

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

called *degenerated center*, where $P(x, y)$ and $Q(x, y)$ are real analytic functions without constant and linear terms defined in a neighborhood of the origin.

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We recall that a *global center* for a vector field on the plane is a singular point p having $\mathbb{R}^2$ filled of periodic orbits with the exception of the singular point. The easiest global center is the linear center $\dot{x} = -y$, $\dot{y} = x$. It is known (see [11, 2]) that quadratic polynomial differential systems have no global centers. The global degenerated centers of homogeneous or quasi-homogeneous polynomial differential systems were characterized in [4] and [8], respectively. However the characterization of the global centers in the cases that the center is nilpotent or of linear-type has not been done. This is the first paper in which such classification is done for the linear-type centers for the systems having a linear part at the origin with purely imaginary eigenvalues and cubic homogeneous nonlinearities.

A polynomial differential system can be extended in a unique analytic way to infinity using the Poincaré compactification, for more details see Chapter 5 of [7].

We now state our main results. We first provide normal forms for the differential systems to be studied.

Theorem 1. *Any vector field having at the origin of coordinates a singular point with purely imaginary eigenvalues of the form linear plus cubic homogeneous terms and no infinite singular points in the Poincaré disc after a linear change of variables and a rescaling of the independent variable can be written as*

$$\begin{aligned} \dot{x} &= -dx - \frac{d^2+\omega^2}{c}y + p_1x^3 + (p_2 - 3\alpha\mu)x^2y + p_3xy^2 - \alpha y^3, \\ \dot{y} &= cx + dy + \alpha x^3 + p_1x^2y + (p_2 + 3\alpha\mu)xy^2 + p_3y^3, \end{aligned} \tag{1}$$

where $\alpha = \pm 1$ *and* $c, d, \omega, \mu, p_1, p_2, p_3 \in \mathbb{R}$ *with* $c \neq 0$, $\omega > 0$ *and* $\mu > -1/3$.

The proof of Theorem 1 is given in section 2.

A singular point p of a planar system is called *hyperbolic* if both eigenvalues of the Jacobian matrix at p have real part different from zero. It is called *semi-hyperbolic* if only one of the eigenvalues of the Jacobian matrix at p is zero, and if both eigenvalues of the Jacobian matrix at p are zero but this matrix is not identically zero it is called *nilpotent*. Finally, if the Jacobian matrix at p is identically zero then p is said to be *linearly zero*.

Let q be an infinite singular point and let h be a hyperbolic sector of q. We say that h is *degenerated* if its two separatrices are contained at infinity, that is, are contained in the boundary of the Poincaré disc.

It follows from Theorem 2.15 (for hyperbolic singular points), Theorem 2.19 (for semi-hyperbolic singular points) and Theorem 3.5 (for nilpotent singular points) in [7] that a singular point which is either hyperbolic, semi-hyperbolic or nilpotent cannot be formed by two degenerated hyperbolic sectors. So in order that an infinite singular point q can be formed by two degenerated hyperbolic sectors it must be linearly zero.

Theorem 2. *Any vector field having at the origin of coordinates a singular point with purely imaginary eigenvalues of the form linear plus cubic homogeneous terms such that all infinite singular points in the Poincaré disc are linearly zero after a linear change of variables and a rescaling of the independent variable can be written as one of the following three systems:*

$$\begin{aligned}\dot{x} &= -dx - \frac{d^2+\omega^2}{c}y + p_1x^3 + p_2x^2y,\\ \dot{y} &= cx + dy + \alpha x^3 + p_1x^2y + p_2xy^2;\end{aligned} \tag{2}$$

$$\begin{aligned}\dot{x} &= -dx - \frac{d^2+\omega^2}{c}y + (p_2-3\alpha)x^2y,\\ \dot{y} &= cx + dy + (p_2+3\alpha)xy^2;\end{aligned} \tag{3}$$

$$\begin{aligned}\dot{x} &= -dx - \frac{d^2+\omega^2}{c}y + (p_2-3\alpha)x^2y + p_3xy^2 - \alpha y^3,\\ \dot{y} &= cx + dy + (p_2+3\alpha)xy^2 + p_3y^3;\end{aligned} \tag{4}$$

where $\alpha = \pm 1$ *and* $c, d, \omega, p_1, p_2, p_3 \in \mathbb{R}$ *with* $c \neq 0$, $\omega > 0$.

The proof of Theorem 2 is given in section 3.

Theorem 3. *Systems* (1) *have a linear type center at the origin if and only if they can be written as one of the following five systems:*

$$\begin{aligned}\dot{x} &= -dx - \frac{d^2+\omega^2}{c}y - 3\alpha\mu x^2y - \alpha y^3,\\ \dot{y} &= cx + dy + \alpha x^3 + 3\alpha\mu xy^2,\end{aligned} \tag{I}$$

where $\alpha = \pm 1$ *and* $d, \omega, \mu \in \mathbb{R}$ *with* $c \neq 0$, $\omega > 0$ *and* $\mu > -1/3$. *These systems are Hamiltonian with*

$$H(x,y) = \frac{c}{2}x^2 + dxy + \frac{d^2+\omega^2}{2c}y^2 + \frac{\alpha}{4}x^4 + \frac{3}{2}\alpha\mu x^2y^2 + \frac{\alpha}{4}y^4.$$

$$\begin{aligned}\dot{x} &= -\frac{\omega^2}{c}y + (p_2-3\mu\alpha)x^2y - \alpha y^3,\\ \dot{y} &= cx + \alpha x^3 + (p_2+3\mu\alpha)xy^2,\end{aligned} \tag{II}$$

where $\alpha = \pm 1$ *and* $c, \omega, \mu \in \mathbb{R}$ *with* $c \neq 0$, $\omega > 0$ *and* $\mu > -1/3$;

$$\begin{aligned}\dot{x} &= -dx - cy + p_1x^3 - 3\alpha\mu x^2y - p_1xy^2 - \alpha y^3,\\ \dot{y} &= cx + dy + \alpha x^3 + p_1x^2y + 3\alpha\mu xy^2 - p_1y^3,\end{aligned} \tag{III}$$

where $\alpha = \pm 1$ *and* $c, d, \mu \in \mathbb{R}$ *with* $|c| > |d|$, $p_1 \neq 0$ *and* $\mu > -1/3$;

(IV)

$$\begin{aligned}\dot{x} &= -dx - \frac{d^2+\omega^2}{c}y - p_1x^3 - \frac{\alpha cd + p_1(c^2-d^2-\omega^2)}{cd}x^2y - p_1xy^2 - \alpha y^3,\\ \dot{y} &= cx + dy + \alpha x^3 + p_1x^2y + \frac{\alpha cd + p_1(d^2+\omega^2-c^2)}{cd}xy^2 - p_1y^3,\end{aligned}$$

where $\alpha = \pm 1$ *and* $c, d, \omega \in \mathbb{R}$ *with* $cd \neq 0$*,* $p_1 \neq 0$ *and* $\omega > 0$*;*

$$\text{(V)} \qquad \begin{aligned} \dot{x} &= -cy + p_1x^3 + (p_2 - \alpha)x^2y - p_1xy^2 - \alpha y^3, \\ \dot{y} &= cx + \alpha x^3 + p_1x^2y + (\alpha + p_2)xy^2 - p_1y^3, \end{aligned}$$

where $\alpha = \pm 1$*,* $c \in \mathbb{R} \setminus \{0\}$ *and* $p_1p_2 \neq 0$*.*

The proof of Theorem 3 is given in section 4. In order to state the following theorem we introduce the notation

(5)

$$\begin{aligned} R &= (2d(p_1 + \alpha) + c(2p_1 + \alpha(1 - 3\mu)))(2d(p_1 - \alpha) - c(2p_1 - \alpha(1 - 3\mu))), \\ S_1 &= -c^3d^4p_1^4(d\alpha((c^2 + d^2)p_1 + cd\alpha) + p_1(cp_1 + d\alpha)\omega^2), \\ S_2 &= c^4 + 2c^2(d^2 - \omega^2) + (d^2 + \omega^2)^2. \end{aligned}$$

Theorem 4. *Under the assumptions of Theorem 3 the following statements hold.*

(a) *A system (I) has a global center at the origin if and only if* $c\alpha > 0$*.*
(b) *A system (II) has a global center at the origin if and only if* $c\alpha > 0$ *and either* $((p_2 + 3\mu\alpha)\omega^2 - c^2\alpha)c(p_2^2 + \alpha^2(1 - 9\mu^2)) \leq 0$*, or* $(c^2(p_2 - 3\mu\alpha) + \alpha\omega^2)c(p_2^2 + \alpha^2(1 - 9\mu^2)) \geq 0$*.*
(c) *A system (III) has a global center at the origin if and only if* $(c \pm d)\alpha > 0$ *and if* $p_1 = -d\alpha/c$ *then* $c\alpha > 0$*, but if* $p_1 \neq -d\alpha/c$ *then either* $R < 0$*, or* $R > 0$ *and* $((2dp_1 + c\alpha(1-3\mu)) \pm \sqrt{R})(2p_1^2 + \alpha^2(3\mu - 1)) \leq 0$*.*
(d) *A system (IV) has a global center at the origin if and only if* $c\alpha((d^2 + \omega^2)\sqrt{S_2} + (c^2d^2 + d^4 - c^2\omega^2 + 2d^2\omega^2 + \omega^4)) \leq 0$ *and if* $p_1 = -d\alpha/c$ *then* $c\alpha > 0$*, but if* $p_1 \neq -d\alpha/c$ *then either* $S_1 < 0$*, or* $S_1 > 0$ *and* $-2\sqrt{S_1} + cdp_1^2(c^3p_1 + c^2d\alpha + cp_1(d^2 - \omega^2) - d\alpha(d^2 + \omega^2)) \leq 0$*.*
(e) *A system (V) has a global center at the origin if and only if* $c\alpha(p_2 + \sqrt{4p_1^2 + p_2^2}) \geq 0$ *and* $c\alpha(p_2 - \sqrt{4p_1^2 + p_2^2}) \leq 0$*.*

The proof of Theorem 4 is given in section 6.

Theorem 5. *Systems* (2) *have a linear type center at the origin and no more finite singular points with all infinite singular points formed by two degenerated hyperbolic sectors if and only if they can be written as one of the following systems*

$$\text{(VI)} \qquad \dot{x} = -dx - \frac{d^2 + \omega^2}{c}y, \quad \dot{y} = cx + dy + \alpha x^3,$$

with $\alpha = \pm 1$*,* $d, c, \omega \in \mathbb{R}$*,* $c \neq 0$*,* $\omega > 0$ *and* $c\alpha > 0$*. Note that these systems are Hamiltonian with*

$$H(x, y) = \frac{c}{2}x^2 + dxy + \frac{d^2 + \omega^2}{2c}y^2 + \frac{\alpha}{4}x^4;$$

(VII)
$$x' = -dx - \frac{d^2+\omega^2}{c}y + p_1x^3 + \frac{d^2+\omega^2}{cd}p_1x^2y,$$
$$y' = cx + dy + \alpha x^3 + p_1x^2y + \frac{d^2+\omega^2}{cd}p_1xy^2.$$

with $\alpha = \pm 1$, $d, c, \omega \in \mathbb{R}$, $\omega > 0$, $c\alpha > 0$ *and* $dp_1 < 0$*;*

(VIII)
$$x' = -\frac{\omega^2}{c}y + p_2x^2y, \quad y' = cx + \alpha x^3 + p_2xy^2.$$

with $\alpha = \pm 1$, $c, \omega \in \mathbb{R}$, $\omega > 0$, $c\alpha > 0$ *and* $cp_2 < 0$*;*

(IX)
$$x' = -dx - \frac{d^2+\omega^2}{c}y + \frac{\alpha d(d^2+\omega^2)}{c(d^2-\omega^2)}x^3 + \frac{\alpha(d^2+\omega^2)^2}{c^2(d^2-\omega^2)}x^2y,$$
$$y' = cx + dy + \alpha x^3 + \frac{\alpha(d^2+\omega^2)}{c(d^2-\omega^2)}x^2y + \frac{\alpha(d^2+\omega^2)}{c^2(d^2-\omega^2)}xy^2.$$

with $\alpha = \pm 1$, $d, c, \omega \in \mathbb{R}$, $\omega > 0$, $|d| > |w|$ *and* $c\alpha > 0$.

The proof of Theorem 5 is given in section 6.

Theorem 6. *Systems* (3) *have a linear type center at the origin and no more finite singular points with all infinite singular points formed by two degenerated hyperbolic sectors if and only if they can be written as one of the following systems*

(X)
$$\dot{x} = -dx - \frac{d^2+\omega^2}{c}y - 3\alpha x^2y, \quad \dot{y} = cx + dy + 3\alpha xy^2,$$

with $\alpha = \pm 1$, $c, d \in \mathbb{R}$, $c \neq 0$, $\omega > 0$ *and* $c\alpha > 0$. *These systems are Hamiltonian with*

$$H(x,y) = \frac{c}{2}x^2 + dxy + \frac{d^2+\omega^2}{2c}y^2 + \frac{3}{2}\alpha x^2y^2;$$

(XI)
$$\dot{x} = -\frac{\omega^2}{c}y + (p_2 - 3\alpha)x^2y, \quad \dot{y} = cx + (p_2 + 3\alpha)xy^2,$$

with $\alpha = \pm 1$, $c \in \mathbb{R} \setminus \{0\}$, $p_2 \neq 0$, $\omega > 0$, $c\alpha > 0$ *and* $p_2 \in [-3, 3]$.

The proof of Theorem 6 is given in section 7.

Theorem 7. *Systems* (4) *have a linear type center at the origin and no more finite singular points with all infinite singular points formed by two degenerated hyperbolic sectors if and only if they can be written as one of the following systems*

(XII)
$$\dot{x} = -dx - \frac{d^2+\omega^2}{c}y - 3\alpha x^2y - \alpha y^3, \quad \dot{y} = cx + dy + 3\alpha x^2,$$

with $\alpha = \pm 1$, $c, d \in \mathbb{R}$, $c \neq 0$, $\omega > 0$ *and* $c\alpha > 0$. *These systems are Hamiltonian with*

$$H(x,y) = \frac{c}{2}x^2 + dxy + \frac{d^2+\omega^2}{2}y^2 + \frac{3\alpha}{2}x^2y^2 + \frac{\alpha}{4}y^4;$$

$$\text{(XIII)} \qquad \dot{x} = -\frac{\omega^2}{c}y + (p_2 - 3\alpha)x^2y - \alpha y^3, \quad \dot{y} = cx + (p_2 + 3\alpha)xy^2,$$

with $\alpha = \pm 1$, $c \in \mathbb{R} \setminus \{0\}$, $p_2 \neq 0$, $\omega > 0$, $c\alpha > 0$ *and* $c(p_2 + 3\alpha) \geq 0$.

The proof of Theorem 7 is given in section 8.

An immediate consequence of Theorems 4, 5, 6 and 7 is the following.

Corollary 8. *Any polynomial vector field having at the origin of coordinates a singular point with purely imaginary eigenvalues of the form linear plus cubic homogeneous terms has a global center at the origin if and only if it satisfies the assumptions of Theorems 4, or 5, or 6, or 7.*

2. Proof of Theorem 1

Doing a linear change of variables and a rescaling of the independent variable, planar cubic homogeneous differential systems can be classified into the following ten classes, see [4]:

$$\text{(i)} \qquad \begin{aligned} \dot{x} &= x(p_1x^2 + p_2xy + p_3y^2), \\ \dot{y} &= y(p_1x^2 + p_2xy + p_3y^2), \end{aligned}$$

whose infinity in the Poincaré disc is formed by singular points;

$$\text{(ii)} \qquad \begin{aligned} \dot{x} &= p_1x^3 + p_2x^2y + p_3xy^2, \\ \dot{y} &= \alpha x^3 + p_1x^2y + p_2xy^2 + p_3y^3, \end{aligned}$$

where $\alpha = \pm 1$ and whose infinite singular points in the Poincaré disc are the real solutions of $\alpha x^4 = 0$ at infinity.

$$\text{(iii)} \qquad \begin{aligned} \dot{x} &= (p_1 - 1)x^3 + p_2x^2y + p_3xy^2, \\ \dot{y} &= (p_1 + 3)x^2y + p_2xy^2 + p_3y^3, \end{aligned}$$

whose infinite singular points in the Poincaré disc are the real solutions of $4x^3y = 0$ at infinity.

$$\text{(iv)} \qquad \begin{aligned} \dot{x} &= p_1x^3 + (p_2 - 3\alpha)x^2y + p_3xy^2, \\ \dot{y} &= p_1x^2y + (p_2 + 3\alpha)xy^2 + p_3y^3, \end{aligned}$$

where $\alpha = \pm 1$ and whose infinite singular points in the Poincaré disc are the real solutions of $6\alpha x^2y^2 = 0$ at infinity.

$$\text{(v)} \qquad \begin{aligned} \dot{x} &= p_1x^3 + (p_2 - \alpha)x^2y + p_3xy^2 - \alpha y^3, \\ \dot{y} &= \alpha x^3 + p_1x^2y + (p_2 + \alpha)xy^2 + p_3y^3, \end{aligned}$$

where $\alpha = \pm 1$ and without infinite singular points;

$$\text{(vi)} \qquad \begin{aligned} \dot{x} &= p_1x^3 + (p_2 - 3\alpha)x^2y + p_3xy^2 + y^3, \\ \dot{y} &= p_1^2y + (p_2 + 3\alpha)xy^2 + p_3y^3, \end{aligned}$$

whose infinite singular points in the Poincaré disc are the real solutions of $y^2(6x^2 - y^2)^2 = 0$ at infinity;

(vii)
$$\begin{aligned}\dot{x} &= p_1x^3 + (p_2 - 3\alpha)x^2y + p_3xy^2 - \alpha y^3,\\ \dot{y} &= p_1x^2y + (p_2 + 3\alpha)xy^2 + p_3y^3,\end{aligned}$$

where $\alpha = \pm 1$ and whose infinite singular points in the Poincaré disc are the real solutions of $\alpha y^2(6x^2 + y^2) = 0$ at infinity;

(viii)
$$\begin{aligned}\dot{x} &= p_1x^3 + (p_2 - 3\mu)x^2y + p_3xy^2 + y^3,\\ \dot{y} &= x^3 + p_1x^2y + (p_2 + 3\mu)xy^2 + p_3y^3,\end{aligned}$$

with $\mu \in \mathbb{R}$ and whose infinite singular points in the Poincaré disc are the real solutions of $x^4 + 6\mu x^2y^2 - y^4 = 0$ at infinity;

(ix)
$$\begin{aligned}\dot{x} &= p_1x^3 + (p_2 - 3\alpha\mu)x^2y + p_3xy^2 - \alpha y^3,\\ \dot{y} &= \alpha x^3 + p_1x^2y + (p_2 + 3\alpha\mu)xy^2 + p_3y^3,\end{aligned}$$

with $\alpha = \pm 1$, $\mu > -1/3$, $\mu \neq 1/3$ and without infinite singular points;

(x)
$$\begin{aligned}\dot{x} &= p_1x^3 + (p_2 - 3\mu)x^2y + p_3xy^2 - y^3,\\ \dot{y} &= x^3 + p_1x^2y + (p_2 + 3\mu)xy^2 + p_3y^3,\end{aligned}$$

with $\mu < -1/3$ and whose infinite singular points in the Poincaré disc are the solutions of $x^4 + 6\mu x^2y^2 + y^4 = 0$ at infinity.

Note that when $\mu = 1/3$ system (v) becomes system (ix) and from now on we consider system (ix) with $\mu > -1/3$ and forgot system (v).

In this theorem we are interested in systems without infinite singular points. The only classes of systems that have no infinite singular points are systems (ix) with $\mu > -1/3$.

For studying the cubic planar polynomial vector fields having linear and cubic terms being the origin a singular point, it is sufficient to add to the above family in (ix) a linear part. This is due to the fact that the linear changes of variables that are done to obtain the classes (i)–(x) are not affine, they are strictly linear. So a linear plus a cubic vector field being the origin a singular point with no infinite singular points in the Poincaré disc can be written as

(6)
$$\begin{aligned}\dot{x} &= ax + by + p_1x^3 + (p_2 - 3\alpha\mu)x^2y + p_3xy^2 - \alpha y^3,\\ \dot{y} &= cx + dy + \alpha x^3 + p_1x^2y + (p_2 + 3\alpha\mu)xy^2 + p_3y^3,\end{aligned}$$

for some real constants a, b, c, d and with $\alpha = \pm 1$, $\mu > -1/3$. The eigenvalues of the linear part of system (6) at the origin are

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}.$$

In order to have a linear type center at the origin, these eigenvalues must be $\pm\omega i$, for some $\omega > 0$. So $a = -d$ and $d^2 + bc = -\omega^2$. We see that $bc \neq 0$, otherwise the left hand side would be non-negative. Then we can solve for b and we get $b = -(d^2 + \omega^2)/c$. So system (6) becomes

$$
\begin{aligned}
\dot{x} &= -dx - \frac{d^2 + \omega^2}{c} y + p_1 x^3 + (p_2 - 3\alpha\mu) x^2 y + p_3 x y^2 - \alpha y^3, \\
\dot{y} &= cx + dy + \alpha x^3 + p_1 x^2 y + (p_2 + 3\alpha\mu) x y^2 + p_3 y^3.
\end{aligned} \tag{7}
$$

This completes the proof of the theorem.

3. Proof of Theorem 2

The proof of Theorem 2 is the same as the proof of Theorem 1 taking into account that system (ii) has only the origin of the local chart U_2 as the infinite singular points which is linearly zero if and only if $p_3 = 0$. This gives system (2) in Theorem 2.

Moreover, system (iv) has only the origin of the local charts U_1 and U_2 as infinite singular points and they are both linearly zero if and only if $p_1 = p_3 = 0$. This gives system (3) in Theorem 2.

System (vii) has an infinite singular point which is the origin of the local chart U_1 and is linearly zero if and only if $p_1 = 0$. This gives system (4) in Theorem 2.

Any other of the systems (i)–(x) either the infinite is formed by singular points, or they do not provide infinite singular points in the Poincaré disc, or they provide at least one infinite singular point which is not linearly zero. After this, the proof is exactly the same as in the proof of Theorem 1 (that is, adding the linear part).

4. Proof of Theorem 3

We will use the following result, proved in [9].

Theorem 9. *Any planar vector field of the form: linear plus cubic homogeneous terms being the origin a singular point with purely imaginary eigenvalues can be written in the form*

$$
\begin{aligned}
\dot{x} &= y + Fx^3 + Gx^2 y + (H - 3P) x y^2 + K y^3, \\
\dot{y} &= -x + Lx^3 + (M - H - 3F) x^2 y + (N - G) x y^2 + P y^3.
\end{aligned} \tag{8}
$$

The origin of system (8) *is a center if and only if one of the following conditions hold:*

(a) $H = M = N = 0$;
(b) $M = 0$, $HL = -2FN - HK - HN + 2NP$ *and* $H^2 F = GHN - HKN + 2N^2 P - H^2 P - HN^2$;
(c) $M = 0$, $H = 5(P - F)/2$, $L = -G - 4K$, $N = 5(G + 3K)$, *and* $3F^2 + 10FP + 16GK + 64K^2 + 3P^2 = 0$.

First we write the linear part of system (7) into its real Jordan normal form doing the change of variables $x = (\omega Y - dX)/c$, $y = X$ and then we do the rescaling $\tau = \omega t$. Doing so, the new system can be written as in (8) with

$$\begin{aligned}
F &= \frac{c^3 p_3 - 3\alpha c^2 d\mu - c^2 dp_2 + cd^2 p_1 - \alpha d^3}{c^3 \omega},\\
G &= \frac{3\alpha c^2 \mu + c^2 p_2 - 2cdp_1 + 3\alpha d^2}{c^3}, \quad H = \frac{4p_1\omega}{c^2}, \quad P = \frac{\omega(cp_1 + \alpha d)}{c^3},\\
K &= \frac{\alpha\omega^2}{c^3}, \quad L = -\frac{\alpha\left(c^4 + 6c^2 d^2 \mu + d^4\right)}{c^3\omega^2},\\
M &= \frac{4\left(c^2 p_3 - cdp_2 + (d^2 + \omega^2)p_1\right)}{c^2\omega}, \quad N = \frac{2(cp_2 - 2dp_1)}{c^2}.
\end{aligned}$$

Condition (a) in Theorem 9 yields $p_1 = p_2 = p_3 = 0$ and so we obtain system (I) in the statement of the theorem. Note that in this case system (I) is Hamiltonian with the Hamiltonian stated in the statement of the theorem.

Condition (b) in Theorem 9 yields the following real solutions in which $c\alpha \neq 0$, $\omega \neq 0$ and $\mu > -1/3$:

- $d = p_1 = p_3 = 0$. This condition yields system (II) in the statement of the theorem;
- $p_3 = -p_1$, $p_2 = 0$, $\omega = \sqrt{c^2 - d^2}$ (and so $|c| > |d|$). Moreover $p_1 \neq 0$ because otherwise the system becomes a system (I). This condition yields system (III) in the statement of the theorem;
- $\mu = 1/3$, $p_3 = -p_1$, $p_2 = p_1(d^2 + \omega^2 - c^2)/(cd)$ (and so $cd \neq 0$). Moreover $p_1 \neq 0$ because otherwise the system becomes a system (I). This condition yields system (IV) in the statement of the theorem;
- $\mu = 1/3$, $p_3 = -p_1$, $\omega = \pm c$, $d = 0$. Note that $p_1 p_2 \neq 0$ because if $p_2 = 0$ then the system becomes a system (III), and if $p_1 = 0$ then the system becomes a system (II). This condition yields system (V) in the statement of the theorem.

We will show now that condition (c) is not satisfied. Indeed conditions (c) yield that the unique possible solution with $\alpha \neq 0$ is

$$p_1 = -\frac{d\alpha}{c}, \quad p_2 = \frac{2c^3 dp_3 + \alpha(d^4 - 3c^4)}{2c^2 d^2}, \quad \mu = -\frac{c^4 + d^4}{6c^2 d^2}.$$

Note that $\mu < -1/3$ and so this condition is not possible. This completes the proof of the theorem.

5. Proof of Theorem 4

We will study each of the systems (I)–(V) separately.

First note that in case of system (I) we get that the system is Hamiltonian. It was proved in [5] that in this case there unique condition so that the unique finite singular point is the origin is $c\alpha > 0$. Hence statement (a) is proved.

The singular points of system (II) are

$$(0,0), \quad \Big(0, \pm i\frac{\omega}{\sqrt{c\alpha}}\Big), \quad \Big(\pm i\sqrt{\frac{c}{\alpha}}, 0\Big), \quad (\pm\overline{x}, \pm\overline{y})$$

with

$$\overline{x} = \sqrt{\frac{(p_2+3\mu\alpha)\omega^2 - c^2\alpha}{c(p_2^2+\alpha^2(1-9\mu^2))}}, \quad \overline{y} = i\sqrt{\frac{c^2(p_2-3\mu\alpha)+\alpha\omega^2}{c(p_2^2+\alpha^2(1-9\mu^2))}}$$

(note that if $c(p_2^2+\alpha^2(1-9\mu^2)) = 0$ the points $(\pm\overline{x}, \pm\overline{y})$ do not exist). So in order that the candidates to be singular points different from the origin do not exist we must have $c\alpha > 0$ and either $((p_2+3\mu\alpha)\omega^2 - c^2\alpha)c(p_2^2 + \alpha^2(1-9\mu^2)) \le 0$, or $(c^2(p_2-3\mu\alpha)+\alpha\omega^2)c(p_2^2+\alpha^2(1-9\mu^2)) \ge 0$. We recall that if $(p_2+3\mu\alpha)\omega^2 - c^2\alpha = 0$, i.e., $p_2 = \alpha(c^2-3\mu\omega^2)/\omega^2$, then $\overline{x} = 0$ and $\overline{y} = i\omega/\sqrt{\alpha c}$ which is non real because $c\alpha > 0$. On the other hand, if $c^2(p_2-3\mu\alpha)+\alpha\omega^2 = 0$, that is, $p_2 = \alpha(3c^2\mu-\omega^2)/c^2$ then $\overline{y} = 0$ and $\overline{x} = i\sqrt{c/\alpha}$ which is also non real. So statement (b) is proved.

The singular points of system (III) (which has the condition $|c| > |d|$ and $p_1 \ne 0$) are,

$$(0,0), \quad \pm i\sqrt{\frac{c+d}{\alpha(1+3\mu)}}(1,1), \quad \pm i\sqrt{\frac{c-d}{\alpha(1+3\mu)}}(1,1), \quad \pm(\overline{x}_\pm, -\overline{y}_\pm)$$

where

$$\overline{x}_\pm = \sqrt{\frac{2dp_1 + c\alpha(1-3\mu) \pm \sqrt{R}}{2(2p_1^2+\alpha^2(3\mu-1))}}, \quad \overline{y}_\pm = -\frac{2dp_1 + c\alpha(1-3\mu) \mp \sqrt{R}}{2(cp_1+d\alpha)}\overline{x}_\pm,$$

with R introduced in (5). Note that if $2p_1^2 + \alpha^2(3\mu-1) = 0$ then the points $\pm(\overline{x}_\pm, -\overline{y}_\pm)$ do not exist. On the other hand, if $cp_1 + d\alpha = 0$, i.e., $p_1 = -d\alpha/c$ then the points $\pm(\overline{x}_\pm, -\overline{y}_\pm)$ become

$$\Big(\pm i\sqrt{\frac{c}{\alpha}}, 0\Big) \quad \Big(0, \pm i\sqrt{\frac{c}{\alpha}}\Big).$$

So in order that all the candidates to be singular points different from the origin do not exist, we must have (besides the condition $|c| > |d|$, $p_1 \ne 0$), $(c \pm d)\alpha > 0$ and if $p_1 = -d\alpha/c$ then $c\alpha > 0$, but if $p_1 \ne -d\alpha/c$ then either $R < 0$, or $R > 0$ and $((2dp_1 + c\alpha(1-3\mu)) \pm \sqrt{R})(2p_1^2 + \alpha^2(3\mu-1)) \le 0$. Hence statement (c) is proved.

The singular points of system (IV) (which has the condition $cd \ne 0$ and $p_1 \ne 0$) are, besides the origin,

$$\pm\Big(-\frac{c^2d^3p_1^3 \pm \sqrt{S_1}}{c^2d^2p_1^2(cp_1+d\alpha)}\overline{y}, \overline{y}\Big), \quad \pm\Big(-\frac{d^2+\omega^2-c^2 \pm \sqrt{S_2}}{2cd}\hat{y}, \hat{y}\Big)$$

where

$$\overline{y} = -\sqrt{\frac{-2\sqrt{S_1} + cdp_1^2(c^3p_1 + c^2d\alpha + cp_1(d^2-\omega^2) - d\alpha(d^2+\omega^2))}{p_1^4 S_2}},$$

$$\hat{y} = -\sqrt{\frac{(d^2+\omega^2)\sqrt{S_2} + (c^2d^2 + d^4 - c^2\omega^2 + 2d^2\omega^2 + \omega^4)}{2c\alpha\sqrt{S_2}}},$$

and S_1, S_2 were introduced in (5). Note that if $cp_1 + d\alpha = 0$ that is $p_1 = -d\alpha/c$ then the singular points

$$\pm\Big(-\frac{c^2d^3p_1^3 \pm \sqrt{S_1}}{c^2d^2p_1^2(cp_1+d\alpha)}\overline{y}, \overline{y}\Big)$$

become

$$\Big(\pm i\sqrt{\frac{c}{\alpha}}, 0\Big), \quad \pm i\sqrt{\frac{c}{\alpha}}\frac{1}{\sqrt{S_2}}\big(d^2+\omega^2-c^2, -2dc\big).$$

So in order that all the candidates to be singular points different from the origin do not exist, we must have (besides the condition $cd \neq 0$ and $p_1 \neq 0$), $c\alpha((d^2+\omega^2)\sqrt{S_2} + (c^2d^2 + d^4 - c^2\omega^2 + 2d^2\omega^2 + \omega^4)) \leq 0$ and if $p_1 = -d\alpha/c$ then $c\alpha > 0$, but if $p_1 \neq -d\alpha/c$ then either $S_1 < 0$, or $S_1 > 0$ and $-2\sqrt{S_1} + cdp_1^2(c^3p_1 + c^2d\alpha + cp_1(d^2-\omega^2) - d\alpha(d^2+\omega^2)) \leq 0$. This completes the proof of statement (d).

The singular points of system (V) are, besides the origin,

$$\pm\frac{\sqrt{c}}{\sqrt{2p_1 - ip_2}}\Big(\frac{1-i}{\sqrt{2}}, -\frac{1+i}{\sqrt{2}}\Big), \quad \mp\frac{\sqrt{c}}{\sqrt{2p_1 + ip_2}}\Big(\frac{1+i}{\sqrt{2}}, -\frac{1-i}{\sqrt{2}}\Big),$$

$$\pm\Big(\frac{T_1}{\sqrt{2}}, -\frac{\sqrt{2}p_1T_1}{p_2 + \sqrt{4p_1^2+p_2^2}}\Big), \quad \pm\Big(\frac{T_2}{\sqrt{2}}, \frac{(p_2 + \sqrt{4p_1^2+p_2^2})T_2}{2\sqrt{2}p_1}\Big)$$

where

$$T_1 = \sqrt{-\frac{c(p_2 + \sqrt{4p_1^2+p_2^2})}{\alpha\sqrt{4p_1^2+p_2^2}}} \quad \text{and} \quad T_2 = \sqrt{\frac{c(p_2 - \sqrt{4p_1^2+p_2^2})}{\alpha\sqrt{4p_1^2+p_2^2}}}.$$

So taking into account that $p_1p_2 \neq 0$, in order that all the candidates to be singular points different from the origin do not exist, we must have $c\alpha(p_2 + \sqrt{4p_1^2+p_2^2}) \geq 0$ and $c\alpha(p_2 - \sqrt{4p_1^2+p_2^2}) \leq 0$. This concludes the proof of the theorem.

6. Proof of Theorem 5

First we write the linear part of system (2) into its real Jordan normal form doing the change of variables $x = (\omega Y - dX)/c$, $y = X$ and then we do the rescaling $\tau = \omega t$. Doing so the new system can be written as in (8) with

$$F = \frac{d}{c^3\omega}(cdp_1 - c^2p_2 - \alpha d^2), \quad G = \frac{c^2p_2 + 3\alpha d^2 - 2cdp_1}{c^3},$$
$$K = \frac{\alpha\omega^2}{c^3}, \quad L = -\frac{d^4\alpha}{c^3\omega^2}, \quad P = \frac{(cp_1 + d\alpha)\omega}{c^3},$$
$$H = \frac{4p_1\omega}{c^2}, \quad N = \frac{2(cp_2 - 2dp_1)}{c^2}, \quad M = \frac{4}{c^2\omega}(p_1(d^2+\omega^2) - cdp_2).$$

Condition (a) in Theorem 9 yields $p_1 = p_2 = 0$ and so we obtain system (VI). Note that system (VI) is Hamiltonian with Hamiltonian

$$H = \frac{c}{2}x^2 + dxy + \frac{d^2+\omega^2}{2c}y^2 + \frac{\alpha}{4}x^4.$$

It was proved in [5] that in this case the unique infinite singular point which is the origin of the local chart U_2 is formed by two degenerated hyperbolic sectors if and only if $c\alpha > 0$ and that in this case there are no more finite singular points besides the origin.

Condition (b) in Theorem 9 with $p_1^2 + p_2^2 + p_3^2 \neq 0$ yields the conditions

(b.1) $p_2 = p_1(d^2+\omega^2)/(cd)$ with $cd \neq 0$ and $p_1 \neq 0$ because otherwise this system becomes a system (VI);
(b.2) $p_1 = 0$, $d = 0$ and $p_2 \neq 0$ because otherwise this system becomes a system (VI);
(b.3) $p_1 = \dfrac{d\alpha(d^2+\omega^2)}{c(d^2-\omega^2)}$, $p_2 = \dfrac{\alpha(d^2+\omega^2)^2}{c^2(d^2-\omega^2)}$ with $|d| \neq |\omega|$.

System (2) with the conditions (b.1) becomes

$$(9) \qquad \begin{aligned} x' &= -dx - \frac{d^2+\omega^2}{c}y + p_1x^3 + \frac{d^2+\omega^2}{cd}p_1x^2y, \\ y' &= cx + dy + \alpha x^3 + p_1x^2y + \frac{d^2+\omega^2}{cd}p_1xy^2. \end{aligned}$$

The singular points of system (9) are

$$(0,0), \quad \pm\left(\sqrt{\frac{d}{p_1}}, -\frac{cd^{3/2}p_1^{1/2} \pm i\sqrt{cd(d^3\alpha + (cp_1 + d\alpha)\omega^2)}}{p_1(d^2+\omega^2)}\right),$$
$$\pm\, i\sqrt{\frac{c}{\alpha(d^2+\omega^2)}}\Big(\omega, \frac{d\omega c}{d^2+\omega^2}\Big).$$

In order that all the candidates to be singular points different from the origin do not exist, and taking into account that condition $c\alpha > 0$, $dp_1 > 0$ and $cd(d^3\alpha + (cp_1 + d\alpha)\omega^2) < 0$ never hold, we must have $c\alpha > 0$ and $dp_1 < 0$.

Now we study the infinite singular points. We already know that the unique infinite singular point is the origin of the local chart U_2 that is linearly zero. On the local chart U_2 we have

$$\begin{aligned}\dot{u} &= -\frac{d^2+\omega^2}{c}v^2 - 2duv^2 - \alpha u^4 - cu^2v^2,\\ \dot{v} &= -v\Big(\frac{d^2+\omega^2}{cd}p_1u + p_1u^2 + dv^2 + \alpha u^3 + cuv^2\Big).\end{aligned}$$

Doing two horizontal blow ups and one vertical blow up and taking into account that $c\alpha > 0$ and $dp_1 < 0$ we conclude that the origin of the local chart U_2 is formed by two degenerated hyperbolic sectors. So we obtain system (VII).

System (2) with the conditions (b.2) becomes

$$\begin{aligned}x' &= -\frac{\omega^2}{c}y + p_2x^2y,\\ y' &= cx + \alpha x^3 + p_2xy^2.\end{aligned} \tag{10}$$

The singular points of system (10) are

$$(0,0), \quad \pm i\Big(\sqrt{\frac{c}{\alpha}}, 0\Big), \quad \Bigg(\pm\frac{\omega}{\sqrt{cp_2}}, \pm i\sqrt{\frac{c^2p_2+\alpha\omega^2}{cp_2^2}}\Bigg).$$

In order that all the candidates to be singular points different from the origin do not exist, and taking into account that condition $c\alpha > 0$, $cp_2 > 0$ and $c(c^2p_2 + \alpha\omega^2) < 0$ is null, we must have $c\alpha > 0$ and $cp_2 < 0$.

Now we study the infinite singular points. We already know that the unique infinite singular point is the origin of the local chart U_2 that is linearly zero. On the local chart U_2 we have

$$\begin{aligned}\dot{u} &= -\frac{\omega^2}{c}v^2 - \alpha u^4 - cu^2v^2,\\ \dot{v} &= uv(p_2 + \alpha u^2 + cv^2).\end{aligned}$$

Doing one horizontal and one vertical blow up, taking into account that $c\alpha > 0$ and $cp_2 < 0$ we conclude that the origin of the local chart U_2 is formed by two degenerated hyperbolic sectors. So we obtain system (VIII).

System (2) with the conditions (b.3) becomes

$$\begin{aligned}x' &= -dx - \frac{d^2+\omega^2}{c}y + \frac{\alpha d(d^2+\omega^2)}{c(d^2-\omega^2)}x^3 + \frac{\alpha(d^2+\omega^2)^2}{c^2(d^2-\omega^2)}x^2y,\\ y' &= cx + dy + \alpha x^3 + \frac{\alpha d(d^2+\omega^2)}{c(d^2-\omega^2)}x^2y + \frac{\alpha(d^2+\omega^2)}{c^2(d^2-\omega^2)}xy^2.\end{aligned} \tag{11}$$

The singular points of system (11) are

$$(0,0), \quad \pm i\sqrt{\frac{c\omega^2}{\alpha(d^2+\omega^2)}}\Big(1, -\frac{dc}{d^2+\omega^2}\Big), \quad \sqrt{\frac{c(d^2-\omega^2)}{\alpha(d^2+\omega^2)}}\Big(\pm 1, -\frac{cd(1\pm i)}{d^2+\omega^2}\Big).$$

In order that all the candidates to be singular points different from the origin do not exist we must have $c\alpha > 0$.

Now we study the infinite singular points. We already know that the unique infinite singular point is the origin of the local chart U_2 that is linearly zero. On the local chart U_2 we have

$$\dot{u} = -\frac{\omega^2 + d^2}{c} v^2 - 2duv^2 - \alpha u^4 - cu^2v^2,$$
$$\dot{v} = -\frac{v\left(c^2\left(d^2-\omega^2\right)\left(v^2(cu+d)+\alpha u^3\right)+\alpha u\left(d^2+\omega^2\right)\left(cdu+d^2+\omega^2\right)\right)}{c^2\left(d^2-\omega^2\right)}.$$

Doing one horizontal and one vertical blow up, taking into account that $c\alpha > 0$ we see that if $|d| > |\omega|$ the origin of the local chart U_2 is formed by two degenerated hyperbolic sectors while if $|d| < |\omega|$ there are parabolic sectors arriving to the origin of the local chart U_2 and so it is not formed by two degenerated hyperbolic sectors. In short, for having the origin as the unique finite singular point and that the origin of the local chart U_2 is formed by two degenerated hyperbolic sectors we must have $c\alpha > 0$ and $|d| > |\omega|$. So we obtain system (IX).

For condition (c) of Theorem 9 with $p_1^2 + p_2^2 + p_3^2 \neq 0$ systems (2) have no linear type center at the origin. This completes the proof of Theorem 5.

7. Proof of Theorem 6

First we write the linear part of system (3) into its real Jordan normal form doing the change of variables $x = (\omega Y - dX)/c$, $y = X$ and then we do the rescaling $\tau = \omega t$. Doing so, the new system can be written as in (8) with

$$F = -\frac{d(p_2+3\alpha)}{c\omega}, \quad G = \frac{p_2+3\alpha}{c}, \quad K = 0, \quad L = -\frac{6d^2\alpha}{c\omega^2},$$
$$P = 0, \quad H = 0, \quad N = \frac{2p_2}{c}, \quad M = -\frac{4dp_2}{c\omega}.$$

Condition (a) in Theorem 9 yields $p_2 = 0$ and so we obtain system (X). Note that system (X) is Hamiltonian with Hamiltonian

$$H(x,y) = \frac{c}{2}x^2 + dxy + \frac{d^2+\omega^2}{2c}y^2 + \frac{3}{2}\alpha x^2 y^2.$$

It was proved in [5] that in this case the unique infinite singular points are the origin of the local charts U_1 and U_2 and they are formed by two degenerated hyperbolic sectors if and only if $c\alpha > 0$ and that in this case there are no more finite singular points besides the origin.

Condition (b) in Theorem 9 with $p_2 \neq 0$ yields the condition $d = 0$. So system (3) becomes

$$\dot{x} = -\frac{\omega^2}{c}y + (p_2 - 3\alpha)x^2y, \quad \dot{y} = cx + (p_2 + 3\alpha)xy^2. \tag{12}$$

First we study the finite singular points. They are

$$(0,0), \qquad \left(\pm\frac{\omega}{\sqrt{c(p_2-3\alpha)}}, \pm i\sqrt{\frac{c}{p_2+3\alpha}}\right),$$

where they do not exist if and only if either $c(p_2 - 3\alpha) \leq 0$, or $c(p_2 - 3\alpha) > 0$ and $c(p_2 + 3\alpha) \geq 0$.

Now we study the infinite singular points of this system. We already know that they are the origins of the local charts U_1 and U_2. On the local chart U_1 system (12) becomes

$$\dot{u} = 6\alpha u^2 + cv^2 + \frac{\omega^2}{c}u^2v^2, \quad \dot{v} = uv\Big(3\alpha - p_2 + \frac{\omega^2}{c}v^2\Big).$$

The only infinite singular point of this system is the origin which is linearly zero. Doing one horizontal and one vertical blow up, we get that if $c(p_2 + 3\alpha) > 0$ and $\alpha(p_2+3\alpha) > 0$ or $p_2+3\alpha = 0$ and $c\alpha > 0$, then the origin of the local chart U_1 is formed by two degenerated hyperbolic sectors. Otherwise there are parabolic sectors arriving at the origin of the local chart U_1 and so it is not formed by two degenerated hyperbolic sectors. These conditions are equivalent to: either $p_2 + 3\alpha > 0$, $c > 0$ and $\alpha = 1$; or $p_2 + 3\alpha < 0$, $c < 0$ and $\alpha = -1$; or $p_2 + 3\alpha = 0$ and $c\alpha > 0$.

On the local chart U_2 system becomes

$$\dot{u} = -6\alpha u^2 - \frac{\omega^2}{c}v^2 - cu^2v^2, \quad \dot{v} = -uv(p_2 + 3\alpha + cv^2).$$

Doing one horizontal and one vertical blow up, we get that if $c(p_2 - 3\alpha) < 0$ and $\alpha(p_2 - 3\alpha) < 0$, or $p_2 - 3\alpha = 0$ and $c\alpha > 0$, then the origin of the local chart U_2 is formed by two degenerated hyperbolic sectors. Otherwise, there are parabolic sectors arriving to the origin of the local chart U_2 and so it is not formed by two degenerated hyperbolic sectors. These conditions are equivalent to: either $p_2 - 3\alpha > 0$, $c < 0$ and $\alpha = -1$; or $p_2 - 3\alpha < 0$, $c > 0$ and $\alpha = 1$; or $p_2 - 3\alpha = 0$ and $c\alpha > 0$.

In short, for having the origin as the unique finite singular point and that both the origins of the local charts U_1 and U_2 are formed by two degenerated hyperbolic sectors we must have $c\alpha > 0$ and $p_2 \in [-3, 3]$. Hence we obtain system (XI).

For condition (c) of Theorem 9 with $p_2 \neq 0$ systems (3) have no linear type center at the origin. This completes the proof of Theorem 6.

8. Proof of Theorem 7

First we write the linear part of system (4) into its real Jordan normal form doing the change of variables $x = (\omega Y - dX)/c$, $y = X$ and then we do the rescaling $\tau = \omega t$. Doing so the new system can be written as in (8) with

$$F = \frac{cp_3 - dp_2 - 3d\alpha}{c\omega}, \quad G = \frac{p_2 + 3\alpha}{c}, \quad K = 0,$$
$$L = -\frac{\alpha(c^2 + 6d^2)}{c\omega^2}, \quad P = 0, \quad H = 0, \quad N = \frac{2p_2}{c}, \quad M = \frac{4(cp_3 - dp_2)}{c\omega}.$$

Condition (a) in Theorem 9 yields $p_1 = p_2 = 0$ and so we obtain system (XII). Note that system (XII) is Hamiltonian with Hamiltonian

$$H(x, y) = \frac{c}{2}x^2 + dxy + \frac{d^2 + \omega^2}{2c}y^2 + \frac{3\alpha}{2}x^2y^2 + \frac{\alpha}{4}y^4.$$

It was proved in [5] that in this case the unique infinite singular point is the origin of the local chart U_1 which is formed by two degenerated hyperbolic sectors if and only if $c\alpha > 0$ and that in this case there are no more finite singular points besides the origin.

Condition (b) in Theorem 9 with $p_2^2 + p_3^2 \neq 0$ and $\alpha \neq 0$ yields the condition $p_3 = d = 0$. So we get the system

$$\dot{x} = -\frac{\omega^2}{c}y + (p_2 - 3\alpha)x^2y - \alpha y^3, \quad \dot{y} = cx + (p_2 + 3\alpha)xy^2, \tag{13}$$

whose finite singular points are

$$(0, 0), \quad \Big(0, \pm\frac{i\omega}{\sqrt{c\alpha}}\Big), \quad \left(\pm\sqrt{\frac{(p_2 + 3\alpha)\omega^2 - c^2\alpha}{c(p_2^2 - 9\alpha^2)}}, \pm i\sqrt{\frac{c}{p_2 + 3\alpha}}\right).$$

Note that if $p_2 = \pm 3\alpha$ the last four points do not exist. So in order that they do not exist we must have $c\alpha > 0$ and either $c(p_2+3\alpha) \geq 0$, or $c(p_2+3\alpha) < 0$ and $(p_2 - 3\alpha)((p_2 + 3\alpha)\omega^2 - c^2\alpha) > 0$.

Now we study the infinite singular points of system (13). We already know that it is the origin of the local chart U_1. On the local chart U_1 system (13) becomes

$$\dot{u} = 6\alpha u^2 + cv^2 + \alpha u^4 + \frac{\omega^2}{c}u^2v^2, \quad \dot{v} = uv\Big(3\alpha - p_2 + \alpha u^2 + \frac{\omega^2}{c}v^2\Big).$$

Doing one horizontal and one vertical blow up and taking into account that $c\alpha > 0$, we get that if $c(p_2 + 3\alpha) \geq 0$ and $\alpha(p_2 + 3\alpha) \geq 0$, then after the blow-down the origin of the local chart U_1 is formed by two degenerated hyperbolic sectors. On the other hand, if the above conditions do not hold, then the origin of the local chart U_1 is not formed by two degenerated hyperbolic sectors. In short, in order that system (13) has no more finite singular points besides the origin and the origin of the local chart U_1 is formed by two degenerated hyperbolic sectors we must have $c(p_2 + 3\alpha) \geq 0$ and $c\alpha > 0$. Hence we get system (XIII).

For condition (c) of Theorem 9 with $p_2^2 + p_3^2 \neq 0$ systems (4) have no linear type center at the origin. This completes the proof of Theorem 7.

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