Math. Model. Nat. Phenom. 14 (2019) 409
https://doi.org/10.1051/mmnp/2019012

Mathematical Modelling of Natural Phenomena
www.mmnp-journal.org

CANARDS EXISTENCE IN THE HINDMARSH–ROSE MODEL

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Abstract. In two previous papers we have proposed a new method for proving the existence of "canard solutions" on one hand for three and four-dimensional singularly perturbed systems with only one *fast* variable and, on the other hand for four-dimensional singularly perturbed systems with two *fast* variables [J.M. Ginoux and J. Llibre, *Qual. Theory Dyn. Syst.* **15** (2016) 381–431; J.M. Ginoux and J. Llibre, *Qual. Theory Dyn. Syst.* **15** (2015) 342010]. The aim of this work is to extend this method which improves the classical ones used till now to the case of three-dimensional singularly perturbed systems with two *fast* variables. This method enables to state a unique generic condition for the existence of "canard solutions" for such three-dimensional singularly perturbed systems which is based on the stability of *folded singularities* (*pseudo singular points* in this case) of the *normalized slow dynamics* deduced from a well-known property of linear algebra. Applications of this method to a famous neuronal bursting model enables to show the existence of "canard solutions" in the Hindmarsh-Rose model.

Mathematics Subject Classification. 34C23, 34C25, 37G10

Received October 23, 2018. Accepted February 13, 2019.

1. Introduction

The concept of "canard solutions" for three-dimensional singularly perturbed systems with two *slow* variables and one *fast* has been introduced in the beginning of the eighties by Benoît and Lobry [5], Benoît [4]. Their existence has been proved by Benoît ([4], p. 170) in the framework of "Non-Standard Analysis" according to a theorem which states that canard solutions exist in such systems provided that the *pseudo singular point* of the *slow dynamics*, *i.e.*, of the *reduced vector field* is of *saddle* type. Nearly twenty years later, while using the so-called "blow-up" technique they introduced, Dumortier and Roussarie [6] and then, Szmolyan and Wechselberger [31] provided a "standard version" of Benoît's theorem [4]. Recently, Wechselberger [37] generalized this theorem for n-dimensional singularly perturbed systems with k *slow* variables and m *fast* (where $n = k + m$). The method they used require to implement a "desingularization procedure" which can be summarized as follows: first, they compute the *normal form* of such singularly perturbed systems which is expressed according to some coefficients (a and b for dimension three and $\tilde{a}$, $\tilde{b}$ and $\tilde{c}_1$ for

Keywords and phrases: Hindmarsh–Rose model, singularly perturbed dynamical systems, canard solutions.

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dimension four) depending on the functions defining the original vector field and their partial derivatives with respect to the variables. Secondly, they project the "desingularized vector field" (originally called "normalized slow dynamics" by Eric Benoît [4], p. 166) of such a *normal form* on the tangent bundle of the critical manifold. Finally, they evaluate the Jacobian of the projection of this "desingularized vector field" at the *folded singularity* (originally called *pseudo singular points* by José Argémi [1], p. 336). This lead Szmolyan and Wechselberger ([31], p. 427) and Wechselberger ([37], p. 3298) to a "classification of *folded singularities* (*pseudo singular points*)". Thus, they showed that for three-dimensional (resp. four-dimensional) singularly perturbed systems such *folded singularity* is of *saddle type* if the following condition is satisfied: $a < 0$ (resp. $\tilde{a} < 0$).

In a first paper entitled: "Canards Existence in Memristor's Circuits" (see Ginoux and Llibre [16]) we presented a method enabling to state a unique "generic" condition for the existence of "canard solutions" for three and four-dimensional singularly perturbed systems with only one fast variable which is based on the stability of *folded singularities* of the *normalized slow dynamics* deduced from a well-known property of linear algebra. We proved that this unique condition is completely identical to that provided by Benoît [4], Szmolyan and Wechselberger [31] and Wechselberger [37].

In a second paper entitled: "Canards Existence in FitzHugh-Nagumo and Hodgkin-Huxley Neuronal Models" (see Ginoux and Llibre [15]) we extended this method to the case of four-dimensional singularly perturbed systems with $k = 2$ *slow* and $m = 2$ *fast* variables. Then, we stated that the provided condition for the existence of canards is "generic" since it is exactly the same for singularly perturbed systems of dimension three and four with one or two *fast* variables. The method we used led us to the following proposition: *If the normalized slow dynamics has a pseudo singular point of saddle type, i.e. if the sum* σ_2 *of all second-order diagonal minors of the Jacobian matrix of the normalized slow dynamics evaluated at the pseudo singular point is negative, i.e. if* $\sigma_2 < 0$ *then, the three-dimensional (resp. four-dimensional) singularly perturbed system exhibits a canard solution which evolves from the attractive part of the slow manifold towards its repelling part.* Then, we proved on one hand for three-dimensional singularly perturbed systems with only one fast variable that the condition for which the *pseudo singular point* is of saddle type, *i.e.* $\sigma_2 < 0$ is identical to that proposed by Benoît ([4], p. 171) in his theorem, *i.e.* $D < 0$ and also to that provided by Szmolyan and Wechselberger [31], *i.e.* $a < 0$. On the other hand, we proved for four-dimensional singularly perturbed systems with one or two fast variables that the condition for which the *folded singularity* (resp. the *pseudo singular point*) is of saddle type, *i.e.* $\sigma_2 < 0$ is identical to that proposed by Wechselberger ([37], p. 3298) in his theorem, *i.e.* $\tilde{a} < 0$.

Notice that there is no proof of the approximation. It is not established that the time-scaled reduced system holds on the approximation for the original system in the case of k slow variables ($k \geq 3$), m fast variables ($m \geq 2$). It was proved in the case $k = 2$ and $m = 1$ by Benoît; constructing a local model and obtaining its solution, and in the case $k = 2$ and $m = 2$ was also proved extensively by Tchizawa [33, 34]. For the case $k = 1$ and $m = 2$ (Hindmarsch–Rose model), we shall construct a local model again and we shall obtain their solutions, providing a constructive proof for the approximation (see also the seminal works of Belikov and Samborskii [3], Gol'dshtein *et al.* [17], Sobolev and Shchepakina [30] and Kolesov and Rozov [25]). Being the pseudo-singular point a saddle, or a node it does not ensure the existence of canards, because it may not satisfy the approximation.

The aim of this work is to extend this method to the case of three-dimensional singularly perturbed systems with one *slow* and two *fast* variables and to show that the provided condition for the existence of canards, *i.e.* $\sigma_2 < 0$ still holds and is consequently "generic".

The outline of this paper is as follows. In Section 2, definitions of singularly perturbed system, critical manifold, reduced system, "constrained system", canard cycles, folded singularities and pseudo singular points are recalled. The method proposed in this article is presented in Section 3 for the case of three-dimensional singularly perturbed systems with two *fast* variables. In Section 4, applications of this method to the famous Hindmarsh-Rose model enables to show the existence of "canard solutions" in such system.

2. Definitions

2.1. Singularly perturbed systems

According to Tikhonov [35], Jones [22] and Kaper [23] *singularly perturbed systems* (or *slow-fast*) are defined as:

$$\begin{aligned} \vec{x}' &= \varepsilon \vec{f}\left(\vec{x}, \vec{y}, \varepsilon\right), \\ \vec{y}' &= \vec{g}\left(\vec{x}, \vec{y}, \varepsilon\right), \end{aligned} \tag{2.1}$$

where $\vec{x} \in \mathbb{R}^k$, $\vec{y} \in \mathbb{R}^m$, $\varepsilon \in \mathbb{R}^+$, and the prime denotes differentiation with respect to the independent variable t'. The functions $\vec{f}$ and $\vec{g}$ are assumed to be C^∞ functions[1] of $\vec{x}$, $\vec{y}$ and ε in $U \times I$, where U is an open subset of $\mathbb{R}^k \times \mathbb{R}^m$ and I is an open interval containing $\varepsilon = 0$.

In the case when $0 < \varepsilon \ll 1$, *i.e.* ε is a small positive number, the variable $\vec{x}$ is called *slow* variable, and $\vec{y}$ is called *fast* variable. Using Landau's notation[2]: $O\left(\varepsilon^p\right)$ represents a function f of u and ε such that $f(u, \varepsilon)/\varepsilon^p$ is bounded for positive ε going to zero, uniformly for u in the given domain.

In general we consider that $\vec{x}$ evolves at an $O\left(\varepsilon\right)$ rate; while $\vec{y}$ evolves at an $O\left(1\right)$ *slow* rate. Reformulating system (2.1) in terms of the rescaled variable $t = \varepsilon t'$, we obtain

$$\begin{aligned} \dot{\vec{x}} &= \vec{f}\left(\vec{x}, \vec{y}, \varepsilon\right), \\ \varepsilon \dot{\vec{y}} &= \vec{g}\left(\vec{x}, \vec{y}, \varepsilon\right). \end{aligned} \tag{2.2}$$

The dot represents the derivative with respect to the new independent variable t. The independent variables t' and t are referred to the *fast* and *slow* times, respectively, and (2.1) and (2.2) are called the *fast* and *slow* systems, respectively. These systems are equivalent whenever $\varepsilon \neq 0$, and they are labeled *singular perturbation problems* when $0 < \varepsilon \ll 1$. The term "singular perturbation" was introduced by Friedrichs and Wasow [12] and the label "singular" stems in part from the discontinuous limiting behavior in system (2.1) as $\varepsilon \to 0$.

2.2. Reduced slow system

When $\varepsilon \to 0$ system (2.2) leads to a system of differential-algebraic equations (D.A.E.) called *reduced slow system* whose dimension decreases from $k + m = n$ to m. Then, the *slow* variable $\vec{x} \in \mathbb{R}^k$ partially evolves in the submanifold

$$M_0 := \left\{ \left(\vec{x}, \vec{y}\right) : \vec{g}\left(\vec{x}, \vec{y}, 0\right) = \vec{0} \right\}. \tag{2.3}$$

called the *critical manifold*[3]. The *reduced slow system* is

$$\begin{aligned} \dot{\vec{x}} &= \vec{f}\left(\vec{x}, \vec{y}, \varepsilon\right), \\ \vec{0} &= \vec{g}\left(\vec{x}, \vec{y}, \varepsilon\right). \end{aligned} \tag{2.4}$$

2.3. Slow invariant manifold

Such a normally hyperbolic invariant manifold (2.3) of the *reduced slow system* (2.4) persists as a locally invariant *slow manifold* of the full system (2.1) for ε sufficiently small. Following to Kaper [23]: "The definition of normal hyperbolicity for general systems involves verifying that the exponential growth rates of orbits in the

[1] In certain applications these functions will be supposed to be C^r, $r \geqslant 1$.

[2] Edmund Georg Hermann Landau (1877–1938) (not to be confused with Lev Landau (1908–1968), Nobel Prize in Physics in 1962) specifies having adopted in 1909 the symbol O introduced by Paul Bachmann (1837–1920) in 1894.

[3] It represents the approximation of the slow invariant manifold, with an error of $O(\varepsilon)$.

directions normal to M_0 are stronger than the growth rates in the tangential directions, as may be measured, for example, using Lyapunov type numbers." (For more details see Zadiraka [39, 40], Baris [2], Fenichel [7], Hirsch *et al.* [20], Knobloch [24] and Wiggins [38]). This locally *slow invariant manifold* is $O(\varepsilon)$ close to the *critical manifold*. When $D_{\vec{x}}\vec{f}$ is invertible, thanks to the Implicit Function Theorem, M_0 is given by the graph of a C^∞ function $\vec{x} = \vec{G}_0(\vec{y})$ for $\vec{y} \in D$, where $D \subseteq \mathbb{R}^k$ is a compact, simply connected domain and the boundary of D is a $(k-1)$–dimensional C^∞ submanifold[4].

According to Fenichel [7, 10] theory if $0 < \varepsilon \ll 1$ is sufficiently small, then there exists a function $\vec{G}(\vec{y}, \varepsilon)$ defined on D such that the manifold

$$M_\varepsilon := \left\{(\vec{x}, \vec{y}) : \vec{x} = \vec{G}(\vec{y}, \varepsilon)\right\}, \tag{2.5}$$

is locally invariant under the flow of system (2.1). Moreover, there exist perturbed local stable (or attracting) M^a and unstable (or repelling) M^r branches of the *slow invariant manifold* M_ε. Thus, normal hyperbolicity of M_ε is lost via a saddle-node bifurcation of the *reduced slow system* (2.4). Then, it gives rise to solutions of "canard" type.

2.4. Canards, singular canards and maximal canards

A *canard* is a solution of a singularly perturbed dynamical system (2.1) following the *attracting* branch M^a of the *slow invariant manifold*, passing near a bifurcation point located on the fold of this *slow invariant manifold*, and then following the *repelling* branch M^r of the *slow invariant manifold*.

A *singular canard* is a solution of a *reduced slow system* (2.4) following the *attracting* branch M_0^a of the *critical manifold*, passing near a bifurcation point located on the fold of this *critical manifold*, and then following the *repelling* branch M_0^r of the *critical manifold*.

A *maximal canard* corresponds to the intersection of the attracting and repelling branches $M_\varepsilon^a \cap M_\varepsilon^r$ of the slow manifold in the vicinity of a non-hyperbolic point.

According to Wechselberger ([37], p. 3302):

> "Such a maximal canard defines a family of canards nearby which are exponentially close to the maximal canard, *i.e.* a family of solutions of (2.1) that follow an attracting branch M_ε^a of the slow manifold and then follow, rather surprisingly, a repelling/saddle branch M_ε^r of the slow manifold for a considerable amount of slow time. The existence of this family of canards is a consequence of the non-uniqueness of M_ε^a and M_ε^r. However, in the singular limit $\varepsilon \to 0$, such a family of canards is represented by a unique singular canard."

Canards are a special class of solution of singularly perturbed dynamical systems for which normal hyperbolicity is lost. Canards in singularly perturbed systems with two or more slow variables ($\vec{x} \in \mathbb{R}^k$, $k \geqslant 2$) and one fast variable ($\vec{y} \in \mathbb{R}^m$, $m = 1$) are robust, since maximal canards generically persist under small parameter changes[5].

2.5. Constrained system

In order to characterize the "slow dynamics", *i.e.* the slow trajectory of the *reduced slow system* (2.4) (obtained by setting $\varepsilon = 0$ in (2.2)), Floris Takens [32] introduced the "constrained system" defined as follows:

$$\begin{aligned} \dot{\vec{x}} &= \vec{f}(\vec{x}, \vec{y}, 0), \\ D_{\vec{y}}\vec{g}.\dot{\vec{y}} &= -(D_{\vec{x}}\vec{g}.\vec{f})(\vec{x}, \vec{y}, 0), \\ \vec{0} &= \vec{g}(\vec{x}, \vec{y}, 0). \end{aligned} \tag{2.6}$$

[4]The set D is overflowing invariant with respect to (2.2) when $\varepsilon = 0$. See Kaper [23] and Jones [22].

[5]See Benoît [4], Szmolyan and Wechselberger [31] and Wechselberger [36, 37].

Since, according to Fenichel [7, 10], the *critical manifold* $\vec{g}\left(\vec{x},\vec{y},0\right)$ may be considered as locally invariant under the flow of system (2.1), we have:

$$\frac{\mathrm{d}\vec{g}}{\mathrm{d}t}\left(\vec{x},\vec{y},0\right)=0 \quad\Longleftrightarrow\quad D_{\vec{x}}\vec{g}.\dot{\vec{x}}+D_{\vec{y}}\vec{g}.\dot{\vec{y}}=\vec{0}.$$

By replacing $\dot{\vec{x}}$ by $\vec{f}\left(\vec{x},\vec{y},0\right)$ leads to:

$$D_{\vec{x}}\vec{g}.\vec{f}\left(\vec{x},\vec{y},0\right)+D_{\vec{y}}\vec{g}.\dot{\vec{y}}=\vec{0}.$$

This justifies the introduction of the *constrained system*.

Now, let $adj(D_{\vec{y}}\vec{g})$ denote the adjoint of the matrix $D_{\vec{y}}\vec{g}$ which is the transpose of the co-factor matrix $D_{\vec{y}}\vec{g}$, then while multiplying the left hand side of (2.6) by the inverse matrix $(D_{\vec{y}}\vec{g})^{-1}$ obtained by the adjoint method we have:

$$\begin{aligned}\dot{\vec{x}}&=\vec{f}\left(\vec{x},\vec{y},0\right),\\ det(D_{\vec{y}}\vec{g})\dot{\vec{y}}&=-\left(adj(D_{\vec{y}}\vec{g}).D_{\vec{x}}\vec{g}.\vec{f}\right)\left(\vec{x},\vec{y},0\right),\\ \vec{0}&=\vec{g}\left(\vec{x},\vec{y},0\right).\end{aligned}\tag{2.7}$$

2.6. Normalized slow dynamics

By rescaling the time by setting $t=-det(D_{\vec{y}}\vec{g})\tau$ in system (2.7) we obtain the following system which has been called by Eric Benoît ([4], p. 166) “normalized slow dynamics”:

$$\begin{aligned}\dot{\vec{x}}&=-det(D_{\vec{y}}\vec{g})\vec{f}\left(\vec{x},\vec{y},0\right),\\ \dot{\vec{y}}&=\left(adj(D_{\vec{y}}\vec{g}).D_{\vec{x}}\vec{g}.\vec{f}\right)\left(\vec{x},\vec{y},0\right),\\ \vec{0}&=\vec{g}\left(\vec{x},\vec{y},0\right).\end{aligned}\tag{2.8}$$

where the overdot now denotes the time derivation with respect to τ. We notice that Argémi [1] proposed to rescale the time by setting $t=-det(D_{\vec{y}}\vec{g})sgn(det(D_{\vec{y}}\vec{g}))\tau$ in order to keep the same flow direction in (2.8) as in (2.7).

2.7. Desingularized vector field

By application of the Implicit Function Theorem, we suppose that we can explicitly express from equation (2.3), say without loss of generality, x_1 as a function ϕ_1 of the other variables. This implies that M_0 is locally the graph of a function $\phi_1:\mathbb{R}^k\to\mathbb{R}^m$ over the base $U=(\vec{\chi},\vec{y})$ where $\vec{\chi}=(x_2,x_3,...,x_k)$. Thus, we can span the “normalized slow dynamics” on the tangent bundle at the *critical manifold* M_0 at the *pseudo singular point*. This leads to the so-called *desingularized vector field*:

$$\begin{aligned}\dot{\vec{\chi}}&=-det(D_{\vec{y}}\vec{g})\vec{f}\left(\vec{\chi},\vec{y},0\right),\\ \dot{\vec{y}}&=\left(adj(D_{\vec{y}}\vec{g}).D_{\vec{x}}\vec{g}.\vec{f}\right)\left(\vec{\chi},\vec{y},0\right).\end{aligned}\tag{2.9}$$

2.8. Pseudo singular points and folded singularities

As recalled by Guckenheimer and Haiduc ([18], p. 91), *pseudo-singular points* have been introduced by the late José Argémi [1] for low-dimensional singularly perturbed systems and are defined as singular points of the “normalized slow dynamics” (2.8). Twenty-three years later, Szmolyan and Wechselberger ([31], p. 428) called such *pseudo singular points*, *folded singularities*. In a recent publication entitled “A propos de canards” Wechselberger

([37], p. 3295) proposed to define such singularities for n-dimensional singularly perturbed systems with k *slow* variables and m *fast* as the solutions of the following system:

$$\begin{aligned} &det(D_{\vec{y}}\vec{g}) = 0, \\ &\left(adj(D_{\vec{y}}\vec{g}).D_{\vec{x}}\vec{g}.\vec{f}\right)(\vec{x}, \vec{y}, 0) = \vec{0}, \\ &\vec{g}\,(\vec{x}, \vec{y}, 0) = \vec{0}. \end{aligned} \tag{2.10}$$

Thus, for dimensions higher than three, his concept encompasses that of Argémi. Moreover, for $k \geqslant 2$), Wechselberger ([37], p. 3296) proved that *folded singularities* form a $(k-2)$-dimensional manifold. Thus, for $k = 2$ the *folded singularities* are nothing else than the *pseudo singular points* defined by Argémi [1]. Nevertheless, for the degenerate case $k = 1$, *folded singularities* still form a zero-dimensional manifold. So, we will see in Section 3 that the stability analysis of the *pseudo singular points* will give rise to a condition for the existence of canard solutions in the original system (2.1).

3. Three-dimensional singularly perturbed systems with two fast variables

A three-dimensional *singularly perturbed dynamical system* (2.2) with $k = 1$ *slow* variables and $m = 2$ *fast* may be written as:

$$\dot{x}_1 = f_1\left(x_1, y_1, y_2\right), \tag{3.1a}$$
$$\varepsilon\dot{y}_1 = g_1\left(x_1, y_1, y_2\right), \tag{3.1b}$$
$$\varepsilon\dot{y}_2 = g_2\left(x_1, y_1, y_2\right), \tag{3.1c}$$

where $x_1 \in \mathbb{R}$, $\vec{y} = (y_1, y_2)^t \in \mathbb{R}^2$, $0 < \varepsilon \ll 1$ and the functions f_i and g_i are assumed to be C^2 functions of (x_1, y_1, y_2).

3.1. Critical manifold

The critical manifold equation of system (3.1) is defined by setting $\varepsilon = 0$ in equations (3.1b) and (3.1c). Thus, we obtain:

$$g_1\left(x_1, y_1, y_2\right) = 0, \tag{3.2a}$$
$$g_2\left(x_1, y_1, y_2\right) = 0. \tag{3.2b}$$

By application of the Implicit Function Theorem, we suppose that we can explicitly express from equations (3.2a) and (3.2b), say without loss of generality, x_1 and y_1 as functions of the others variables:

$$y_1 = \phi_1\left(x_1, y_2\right), \tag{3.3a}$$
$$x_1 = \phi_2\left(y_1, y_2\right). \tag{3.3b}$$

3.2. Constrained system

The *constrained system* is obtained by equating to zero the time derivative of $g_{1,2}\left(x_1, y_1, y_2\right)$:

$$\frac{\mathrm{d}g_1}{\mathrm{d}t} = \frac{\partial g_1}{\partial x_1}\dot{x}_1 + \frac{\partial g_1}{\partial y_1}\dot{y}_1 + \frac{\partial g_1}{\partial y_2}\dot{y}_2 = 0, \tag{3.4a}$$

$$\frac{dg_2}{dt} = \frac{\partial g_2}{\partial x_1}\dot{x}_1 + \frac{\partial g_2}{\partial y_1}\dot{y}_1 + \frac{\partial g_2}{\partial y_2}\dot{y}_2 = 0. \tag{3.4b}$$

Equations (3.4a) and (3.4b) may be written as:

$$\frac{\partial g_1}{\partial y_1}\dot{y}_1 + \frac{\partial g_1}{\partial y_2}\dot{y}_2 = -\frac{\partial g_1}{\partial x_1}\dot{x}_1, \tag{3.5a}$$
$$\frac{\partial g_2}{\partial y_1}\dot{y}_1 + \frac{\partial g_2}{\partial y_2}\dot{y}_2 = -\frac{\partial g_2}{\partial x_1}\dot{x}_1. \tag{3.5b}$$

By solving the system of two equations (3.5a) and (3.5b) with two unknowns $(\dot{y}_1, \dot{y}_2)$ we find:

$$\dot{y}_1 = -\frac{\left(\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial y_2} - \frac{\partial g_1}{\partial y_2}\frac{\partial g_2}{\partial x_1}\right)\dot{x}_1}{det\left[J_{(y_1,y_2)}\right]}, \tag{3.6a}$$
$$\dot{y}_2 = -\frac{\left(\frac{\partial g_1}{\partial y_1}\frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial y_1}\right)\dot{x}_1}{det\left[J_{(y_1,y_2)}\right]}. \tag{3.6b}$$

So, we have the following constrained system:

$$\begin{aligned}
\dot{x}_1 &= f_1\left(x_1, y_1, y_2\right), \\
\dot{y}_1 &= -\frac{\left(\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial y_2} - \frac{\partial g_1}{\partial y_2}\frac{\partial g_2}{\partial x_1}\right)\dot{x}_1}{det\left[J_{(y_1,y_2)}\right]}, \\
\dot{y}_2 &= -\frac{\left(\frac{\partial g_1}{\partial y_1}\frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial y_1}\right)\dot{x}_1}{det\left[J_{(y_1,y_2)}\right]}, \\
0 &= g_1\left(x_1, y_1, y_2\right), \\
0 &= g_2\left(x_1, y_1, y_2\right).
\end{aligned} \tag{3.7}$$

3.3. Normalized slow dynamics

By rescaling the time by setting $t = -det\left[J_{(y_1,y_2)}\right]\tau$ we obtain the "normalized slow dynamics":

$$\begin{aligned}
\dot{x}_1 &= -f_1\left(x_1, y_1, y_2\right) det\left[J_{(y_1,y_2)}\right] = F_1\left(x_1, y_1, y_2\right), \\
\dot{y}_1 &= f_1\left(x_1, y_1, y_2\right)\left(\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial y_2} - \frac{\partial g_1}{\partial y_2}\frac{\partial g_2}{\partial x_1}\right) = G_1\left(x_1, y_1, y_2\right), \\
\dot{y}_2 &= f_1\left(x_1, y_1, y_2\right)\left(\frac{\partial g_1}{\partial y_1}\frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial y_1}\right) = G_2\left(x_1, y_1, y_2\right), \\
0 &= g_1\left(x_1, y_1, y_2\right), \\
0 &= g_2\left(x_1, y_1, y_2\right),
\end{aligned} \tag{3.8}$$

where the overdot now denotes the time derivation with respect to τ.

3.4. Desingularized system on the critical manifold

Since we have supposed that y_1 and y_2 may be explicitly expressed as functions of the others variables (3.3a) and (3.3b), they can be used to project the normalized slow dynamics (3.8) on the tangent bundle of the critical manifold. So, we have:

$$\begin{aligned} \dot{x}_1 &= -f_1\left(x_1, y_1, y_2\right) det\left[J_{(y_1,y_2)}\right] = F_1\left(x_1, y_2\right), \\ \dot{y}_2 &= f_1\left(x_1, y_1, y_2\right)\left(\frac{\partial g_1}{\partial y_1}\frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial y_1}\right) = G_2\left(x_1, y_2\right). \end{aligned} \tag{3.9}$$

3.5. Pseudo singular points

Pseudo-singular points are defined as singular points of the "normalized slow dynamics", *i.e.* as the set of points for which we have:

$$det\left[J_{(y_1,y_2)}\right] = 0, \tag{3.10a}$$

$$\left(\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial y_2} - \frac{\partial g_1}{\partial y_2}\frac{\partial g_2}{\partial x_1}\right) = 0, \tag{3.10b}$$

$$\left(\frac{\partial g_1}{\partial y_1}\frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial y_1}\right) = 0, \tag{3.10c}$$

$$g_1\left(x_1, y_1, y_2\right) = 0, \tag{3.10d}$$

$$g_2\left(x_1, y_1, y_2\right) = 0. \tag{3.10e}$$

Remark. We notice on the one hand that equations (3.10b) and (3.10c) are linearly dependent and on the other hand that contrary to previous works we do not use the "desingularized vector field" (3.9) but the "normalized slow dynamics" (3.8).

The Jacobian matrix of system (3.8) reads:

$$J_{(F_1,G_1,G_2)} = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} \\ \dfrac{\partial G_1}{\partial x_1} & \dfrac{\partial G_1}{\partial y_1} & \dfrac{\partial G_1}{\partial y_2} \\ \dfrac{\partial G_2}{\partial x_1} & \dfrac{\partial G_2}{\partial y_1} & \dfrac{\partial G_2}{\partial y_2} \end{pmatrix}. \tag{3.11}$$

3.6. Extension of Benoît's generic hypothesis

Without loss of generality, it seems reasonable to extend Benoît's generic hypotheses introduced for three-dimensional singularly perturbed systems with only one *fast* variable to the case of two *variables*. So, first, we suppose that by a "standard translation" the *pseudo singular point* can be shifted at the origin $O(0,0,0)$ and that by a "standard rotation" of y_1-axis that the *slow manifold* is tangent to (x_1, y_1)-plane, so we have

$$g_1\left(0,0,0\right) = g_2\left(0,0,0\right) = 0, \tag{3.12a}$$

$$\left.\frac{\partial g_1}{\partial y_1}\right|_{(0,0,0)} = \left.\frac{\partial g_2}{\partial y_1}\right|_{(0,0,0)} = 0, \tag{3.12b}$$

$$\left.\frac{\partial g_1}{\partial y_2}\right|_{(0,0,0)} = \left.\frac{\partial g_2}{\partial y_2}\right|_{(0,0,0)} = 0. \tag{3.12c}$$

Then, we make the following assumptions for the non-degeneracy of the *pseudo singular point*:

$$f_1\left(0,0,0\right) \neq 0. \tag{3.13}$$

Thus, we have the following Cayley-Hamilton eigenpolynomial associated with such a Jacobian matrix (3.11) evaluated at the *pseudo singular point*, *i.e.* at the origin:

$$\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = 0. \tag{3.14}$$

First, it can be proved that the sum of all first-order diagonal minors of $J_{(F_1,G_1,G_2)}$, *i.e.* $\sigma_1 = Tr(J_{(F_1,G_1,G_2)}) = 0$ due to circular permutations of the partial derivatives. Secondly, it can also be proved that $\sigma_3 = |J_{(F_1,G_1,G_2)}| = 0$ vanishes at a *pseudo singular point* provided that generic condition (3.12b) is satisfied. So, the eigenpolynomial (3.14) is reduced to

$$\lambda\left(\lambda^2 + \sigma_2\right) = 0 \tag{3.15}$$

Let λ_i be the eigenvalues of the eigenpolynomial (3.15) and we denote by $\lambda_3 = 0$ the obvious root of this polynomial. We have:

$$\sigma_2 = \sum_{i=1}^{3} \left| J^{ii}_{(F_1,G_1,G_2)} \right| = \lambda_1 \lambda_2. \tag{3.16}$$

where $\sigma_2 = \sum_{i=1}^{3} \left| J^{ii}_{(F_1,G_1,G_2)} \right| = q$ represents the sum of all second-order diagonal minors of $J_{(F_1,G_1,G_2)}$. Obviously, the *pseudo singular point* is of saddle-type if and only if $\sigma_2 < 0$. This leads to the the following condition:

$$C_1: \quad q < 0. \tag{3.17}$$

3.7. Canard existence in $\mathbb{R}^{1+2}$

In an article entitled "Systèmes lents-rapides dans $\mathbb{R}^3$ et leurs canards", Benoît ([4], p. 171) has stated in the framework of "non-standard analysis" a theorem that can be written as follows:

Benoît's theorem [1983]. *If the desingularized vector field* (3.7) *has a pseudo singular point of saddle type, then system* (3.1) *exhibits a canard solution which evolves from the attractive part of the slow manifold towards its repelling part.* A few years later, Szmolyan and Wechselberger [31] gave a "standard version" of Benoît's theorem [4] (see Benoît's theorem above) for three-dimensional singularly perturbed systems with $k = 2$ *slow* variables and $m = 1$ *fast*. While using "standard analysis" and blow-up technique they introduced, Dumortier and Roussarie [6], and then, Szmolyan and Wechselberger ([31], p. 427) stated in their Lemma 2.1, while using "a smooth change of coordinates" (see Ginoux *et al.* [15, 16]), that the original system can be transformed into a "normal form" (4.1) from which they deduced that the condition for the *pseudo singular point* to be of saddle type is $a < 0$. Then, they proved the existence of canard solutions for the original system according to their Theorem 4.1(a).

In our previous papers (see Ginoux *et al.* [15, 16]), we have established the following Proposition 3.4 for three and four-dimensional singularly perturbed systems with $k = 1$ *fast* variable and for four-dimensional singularly perturbed systems with $k = 2$ *fast* variables. In this work we show that this proposition still holds for three-dimensional singularly perturbed systems with $k = 2$ *fast* variables and $m = 1$ *slow* variable. (See also the seminal works of Belikov and Samborskii [3], Gol'dshtein *et al.* [17], Sobolev and Shchepakina [30] and Kolesov and Rozov [25].)

As we have pointed out the approximation regarding the reduced system is done constructing a local model in the case $k = 1$ and $m = 3$. Let the origin O be a saddle or a node. Blowing up the coordinates $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $y \in \mathbb{R}$ as follows: $x_1 = \alpha^2 u_1$, $x_2 = \alpha^2 u_2$, $x_3 = \alpha u_3$ and $y = \alpha^2 v$ with $\alpha \approx 0$, the system is reduced to

$$\begin{aligned} \varepsilon \frac{\mathrm{d}u_1}{\mathrm{d}t} &= \frac{1}{\alpha^2} h_1(x, y, \varepsilon), \\ \varepsilon \frac{\mathrm{d}u_2}{\mathrm{d}t} &= \frac{1}{\alpha^2} h_2(x, y, \varepsilon), \\ \varepsilon \frac{\mathrm{d}u_3}{\mathrm{d}t} &= \frac{1}{\alpha} h_3(x, y, \varepsilon), \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1}{\alpha^2} f(x, y, \varepsilon). \end{aligned} \tag{3.18}$$

Assume that the rank of the matrix (h_x) is 3, *i.e.* there exists a function $x = \xi(y)$ which is invertible. When $\varepsilon = 0$ the time-scaled reduced system is

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -\det(h_x)(h_x)^{-1}(h_y)\frac{\mathrm{d}y}{\mathrm{d}t}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= f(x, y, 0). \end{aligned}$$

Doing the scaling $t = \alpha^2 \tau$ and $\varepsilon/\alpha^2 \approx 0$ we obtain the local model

$$\begin{aligned} \delta \dot{u}_1 &= \frac{1}{\alpha^2} h_1(x, y, \varepsilon) \\ &= \frac{h_1(0)}{\alpha^2} + \frac{\partial h_1}{\partial x_1}(0) u_1 + \frac{\partial h_1}{\partial x_2}(0) u_2 + \frac{\partial h_1}{\partial x_3}(0) u_3 + \frac{\partial h_1}{\partial y}(0) v + \frac{1}{2}\frac{\partial^2 h_1}{\partial x_3^2}(0) u_3^2 + \cdots, \\ \delta \dot{u}_2 &= \frac{1}{\alpha^2} h_2(x, y, \varepsilon) \\ &= \frac{h_2(0)}{\alpha^2} + \frac{\partial h_2}{\partial x_1}(0) u_1 + \frac{\partial h_2}{\partial x_2}(0) u_2 + \frac{\partial h_2}{\partial x_3}(0) u_3 + \frac{\partial h_2}{\partial y}(0) v + \frac{1}{2}\frac{\partial^2 h_2}{\partial x_3^2}(0) u_3^2 + \cdots, \\ \delta \dot{u}_3 &= \frac{1}{\alpha} h_3(x, y, \varepsilon) \\ &= \frac{h_3(0)}{\alpha} + \alpha \frac{\partial h_3}{\partial x_1}(0) u_1 + \alpha \frac{\partial h_3}{\partial x_2}(0) u_2 + \frac{\partial h_3}{\partial x_3}(0) u_3 + \alpha \frac{\partial h_3}{\partial y}(0) v + \cdots, \\ \dot{v} &= f(x, y, \varepsilon) \\ &= f(0) + \alpha^2 \frac{\partial f}{\partial x_1}(0) u_1 + \alpha^2 \frac{\partial f}{\partial x_2}(0) u_2 + \alpha \frac{\partial f}{\partial x_3}(0) u_3 + \alpha^2 \frac{\partial f}{\partial y}(0) v + \cdots, \end{aligned} \tag{3.19}$$

where here the dot denotes derivative with respect to τ, and $\delta = \varepsilon/\alpha^2 \approx 0$.

Now we denote

$$a_{ij} = \frac{\partial h_i}{\partial x_j}(0) \quad \text{and} \quad b_{ij} = \frac{\partial^2 h_i}{\partial x_j^2}(0).$$

Then, under the assumptions

$$a_{13} = a_{23} = 0, \quad a_{33} < 0 \quad f(0) \neq 0, \quad \text{and} \quad h(0) = 0,$$

and taking $\delta = 0$ we have the system

$$\begin{aligned} a_{11}u_1 + a_{12}u_2 + \frac{a_{13}}{\alpha}u_3 + a_{14}v + \frac{1}{2}b_{13}u_3^2 &= 0, \\ a_{21}u_1 + a_{22}u_2 + \frac{a_{23}}{\alpha}u_3 + a_{24}v + \frac{1}{2}b_{23}u_3^2 &= 0, \\ a_{33}u_3 &= 0, \\ v &= f(0)t. \end{aligned}$$

Solving this system we get

$$\begin{aligned} u_1 &= u_1^0 = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}(a_{12}a_{24} - a_{14}a_{22})f(0)t, \\ u_2 &= u_2^0 = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}(a_{14}a_{21} - a_{11}a_{24})f(0)t, \\ u_3 &= u_3^0 = 0. \end{aligned}$$

Now the solution with $\delta \approx 0$ is of the form

$$\begin{aligned} u_1 &= u_1^0 + \mathcal{L}(\delta), \\ u_2 &= u_2^0 + \mathcal{L}(\delta), \\ u_3 &= u_3^0 + \mathcal{L}(\delta). \end{aligned}$$

Lemma 3.1. *The time-scaled-reduced system in R^{1+3} is an approximation of system (3.18). If the singular point of the reduced system is stable, $a_{33} < 0$ and the trace$[\partial h(0,0)/\partial x] < 0$, then there exists a canard in the system R^{1+3}.*

For the system in $\mathbb{R}^{1+2}$ taking $x_2 = 0$ and $h_2(x, y, \varepsilon) = 0$, then $x = (x_1, x_3) \in \mathbb{R}^2$ and $y \in \mathbb{R}$ and system (3.18) becomes

$$\begin{aligned} \varepsilon\frac{\mathrm{d}u_1}{\mathrm{d}t} &= \frac{1}{\alpha^2}h_1(x, y, \varepsilon), \\ \varepsilon\frac{\mathrm{d}u_3}{\mathrm{d}t} &= \frac{1}{\alpha}h_3(x, y, \varepsilon), \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1}{\alpha^2}f(x, y, \varepsilon), \end{aligned} \tag{3.20}$$

and consequently the local model given by system (3.19) now writes

$$\begin{aligned} \delta\dot{u}_1 &= \frac{1}{\alpha^2}h_1(x, y, \varepsilon) = a_{11}u_1 + \frac{a_{13}}{\alpha}u_3 + a_{14}v + \frac{1}{2}b_{13}u_3^2, \\ \delta\dot{u}_3 &= \frac{1}{\alpha}h_3(x, y, \varepsilon) = a_{33}u_3, \\ \dot{v} &= f(x, y, \varepsilon) = f(0). \end{aligned} \tag{3.21}$$

Therefore, under the assumptions

$$a_{13} = a_{23} = 0, \quad a_{33} < 0 \quad f(0) \neq 0, \quad \text{and} \quad h(0) = 0,$$

taking $\delta = 0$ we obtain the system

$$\begin{aligned} a_{11}u_1 + a_{14}v &= 0, \\ a_{33}u_3 &= 0, \\ v &= f(0)t. \end{aligned}$$

Solving this system we get the solution

$$\begin{aligned} u_1 &= u_1^0 = -a_{14}f(0)t, \\ u_3 &= u_3^0 = 0, \\ v &= f(0)t. \end{aligned}$$

Hence the solution with $\delta \approx 0$ is of the form

$$\begin{aligned} u_1 &= u_1^0 + \mathcal{L}(\delta), \\ u_3 &= u_3^0 + \mathcal{L}(\delta). \end{aligned}$$

Lemma 3.2. *The time-scaled-reduced system in R^{3+1} is an approximation for the system (3.20). If the pseudo singular point is a saddle (all the eigenvalues are negative in the slow vector field), or a node (all the eigenvalues are positive) and $a_{11} < 0$ (or $a_{11} > 0$) and $f(0) \neq 0$, then there exists a canard in the system R^{3+1}.*

Now we use Theorems 4.1 and 4.2 from [33] for the following model in $\mathbb{R}^{2+2}$:

$$\begin{aligned} \varepsilon\dot{x} &= h(x, y, \varepsilon), \\ \dot{y} &= f(x, y, \varepsilon), \end{aligned}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, $h = (h_1, h_2)$ and $f = (f_1, f_2)$. Assume that the origin of this system is a singular point, and that the slow variable $y_2 \equiv 0$ and the corresponding function $f_2 \equiv 0$. Then we obtain the following system in $\mathbb{R}^{1+2}$, *i.e.* with $k = 1$ and $m = 2$.

Blowing up the variables as $x_1 = \alpha^2 u_1$, $x_2 = \alpha u_2$ and $y_1 = \alpha^2 v_1$, we write the system as

$$\begin{aligned} \varepsilon\dot{u}_1 &= \frac{1}{\alpha^2}h_1(x, y, \varepsilon), \\ \varepsilon\dot{u}_2 &= \frac{1}{\alpha}h_2(x, y, \varepsilon), \\ \dot{v}_1 &= \frac{1}{\alpha^2}f_1(x, y, \varepsilon). \end{aligned} \tag{3.22}$$

Changing the time scale as $t = \alpha^2\tau$ this system becomes

$$\begin{aligned}\delta\dot{u}_1 &= \frac{1}{\alpha^2}h_1(x, y, \varepsilon),\\ \delta\dot{u}_2 &= \frac{1}{\alpha}h_2(x, y, \varepsilon),\\ \dot{v}_1 &= \frac{1}{\alpha^2}f_1(x, y, \varepsilon),\end{aligned}$$

where $\delta = \varepsilon/\alpha^2$ is very small and now the dot denotes derivative with respect to τ. Under the assumptions

(1) $a_{12} = 0$ and $a_{11} \neq 0$,
(2) $f_1(0, 0, 0) \neq 0$, and
(3) there exists $\varphi(x) = y_1$ such that the trace of the Jacobian matrix $h_x(0, \varphi(0), 0)$ is negative.

Then the last system can be written into the form

$$\begin{aligned}\delta\dot{u}_1 &= a_{11}u_1 + a_{12}u_2 + a_{13}v_1 + \frac{1}{2}b_{12}u_2^2 + \mathcal{L}(\delta),\\ \delta\dot{u}_2 &= a_{21}u_2 + \mathcal{L}(\delta),\\ \dot{v}_1 &= \frac{1}{\alpha^2}f_1(0, 0, 0),\end{aligned}$$

Taking $\delta = 0$ we get that

$$\begin{aligned}u_1 &= -\frac{a_{13}}{a_{11}}f_1(0, 0, 0)t,\\ u_2 &= 0,\\ v_1 &= f_1(0, 0, 0)t.\end{aligned}$$

Therefore, for $\delta \neq 0$ but sufficiently small we have

$$\begin{aligned}u_1 &= -\frac{a_{13}}{a_{11}}f_1(0, 0, 0)t + \mathcal{L}(\delta),\\ u_2 &= +\mathcal{L}(\delta),\\ v_1 &= f_1(0, 0, 0)t + \mathcal{L}(\delta).\end{aligned}$$

Lemma 3.3. *Using the case $\mathbb{R}^{2+2}$, and taking $y_2 = 0$ and $f_2 = 0$, the time-scaled reduced system has the approximation of system* (3.22).

Proposition 3.4. *If the normalized slow dynamics* (3.8) *has a pseudo singular point of saddle type, i.e. if the sum σ_2 of all second-order diagonal minors of the Jacobian matrix of the normalized slow dynamics* (3.8) *evaluated at the pseudo singular point is negative, i.e. if $\sigma_2 < 0$ then, system* (3.1) *exhibits a canard solution which evolves from the attractive part of the slow manifold towards its repelling part.*

Proof. The method used by Benoît [4], Szmolyan and Wechselberger [31] and Wechselberger [37] requires to implement a "desingularization procedure" which implies to project the "normalized slow dynamics" (3.8) on the tangent bundle of the *critical manifold* and, to evaluate the Jacobian of the projection of this "desingularized system" (3.9) at the *pseudo singular points*. As previously recalled, the method presented in this paper does not use the "desingularized system" (3.8) but the "normalized slow dynamics" (3.9). So, to prove the Proposition 3.4, we have just to show that the determinant of the Jacobian of the "desingularized system" (3.8) is identical to the sum σ_2 of all second-order diagonal minors of the Jacobian matrix of the "normalized slow dynamics" (3.9). According to equation (3.16):

$$\sigma_2 = \sum_{i=1}^{3} \left| J^{ii}_{(F_1,G_1,G_2)} \right| = \left| J^{11}_{(F_1,G_1,G_2)} \right| + \left| J^{22}_{(F_1,G_1,G_2)} \right| + \left| J^{33}_{(F_1,G_1,G_2)} \right|.$$

While using the generic hypotheses equations (3.12), it is easy to prove that:

$$\left| J^{11}_{(F_1,G_1,G_2)} \right| = \left| J^{33}_{(F_1,G_1,G_2)} \right| = 0$$

The remaining determinant $\left| J^{22}_{(F_1,G_1,G_2)} \right|$ is exactly that of the "desingularized system" (3.9). So, Proposition 3.4 can be also used to state the existence of canard solution for such systems. □

4. Canards existence in the Hindmarsh–Rose model

The Hindmarsh-Rose model [19] describes the basic properties of individual neurons and appears as a reduction of the conductance based in the Hodgkin-Huxley model for neural spiking, see for more details [21]. Thus, the three-dimensional Hindmarsh-Rose polynomial ordinary differential system was originally written as:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y - ax^3 + bx^2 - z + I, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= c - dx^2 - y, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= r\left[s\left(x - \alpha\right) - z\right], \end{aligned} \tag{4.1}$$

where x is a transmembrane neuron potential, y and z are the characteristics of ionic currents dynamic, I is ambient current. The other parameters (a, b, c, d, I, s, α and r) reflect the physical features of the neurons and the dot indicates derivative with respect to the time t. We notice that the parameter $r \ll 1$. Existence of canard solutions in such system (4.1) has been originally suspected by Shilnikov *et al.* ([29], p. 2149) and highlighted by Shchepakina [28]. Thus, by posing $x \to y_2$, $y \to y_1$, $z \to x_1$ and $t' \to \varepsilon t$ with $\varepsilon = r$ and according to the previous definitions, the Hindmarsh-Rose model may be written as a three-dimensional singularly perturbed system with $k = 1$ *slow* variable and $m = 2$ *fast* variables:

$$\dot{x}_1 = f_1\left(x_1, y_1, y_2\right) = s\left(y_2 - \alpha\right) - x_1, \tag{4.2a}$$

$$\varepsilon \dot{y}_1 = g_1\left(x_1, y_1, y_2\right) = c - dy_2^2 - y_1, \tag{4.2b}$$

$$\varepsilon \dot{y}_2 = g_2\left(x_1, y_1, y_2\right) = y_1 - ay_2^3 + by_2^2 - x_1 + I, \tag{4.2c}$$

where $x_1 \in \mathbb{R}$, $\vec{y} = (y_1, y_2)^t \in \mathbb{R}^2$, $0 < \varepsilon \ll 1$ and the functions f_i and g_i are assumed to be C^2 functions of (x_1, y_1, y_2) and the dot now indicates derivative with respect to the time t'.

4.1. Critical manifold

The critical manifold equation of system (4.2) is defined by setting $\varepsilon = 0$ in equations (4.2b) and (4.2c). Thus, we have:

$$g_1\left(x_1, y_1, y_2\right) = c - dy_2^2 - y_1 = 0, \tag{4.3a}$$

$$g_2\left(x_1, y_1, y_2\right) = y_1 - a y_2^3 + b y_2^2 - x_1 + I = 0. \tag{4.3b}$$

This leads to the following *critical manifold* equation:

$$y_1 = \phi\left(y_2\right) = c + I - a y_2^3 + \left(b - d\right) y_2^2. \tag{4.4}$$

4.2. Constrained system

According to equations (3.7), we have the following *constrained system*:

$$\begin{aligned}
\dot{x}_1 &= f_1\left(x_1, y_1, y_2\right) = s\left(y_2 - \alpha\right) - x_1, \\
\dot{y}_1 &= -\frac{-2 d y_2 f_1\left(x_1, y_1, y_2\right)}{3 a y_2^2 - 2\left(b - d\right) y_2}, \\
\dot{y}_2 &= -\frac{f_1\left(x_1, y_1, y_2\right)}{3 a y_2^2 - 2\left(b - d\right) y_2}, \\
0 &= c - d y_2^2 - y_1, \\
0 &= y_1 - a y_2^3 + b y_2^2 - x_1 + I.
\end{aligned} \tag{4.5}$$

4.3. Normalized slow dynamics

By rescaling the time by setting $t' = -(3ay_2^2 - 2\left(b-d\right)y_2)\tau$ we obtain the "normalized slow dynamics":

$$\begin{aligned}
\dot{x}_1 &= -\left[3 a y_2^2 - 2\left(b - d\right) y_2\right]\left[s\left(y_2 - \alpha\right) - x_1\right] = F_1\left(x_1, y_1, y_2\right), \\
\dot{y}_1 &= -2 d y_2\left[s\left(y_2 - \alpha\right) - x_1\right] = G_1\left(x_1, y_1, y_2\right), \\
\dot{y}_2 &= \left[s\left(y_2 - \alpha\right) - x_1\right] = G_2\left(x_1, y_1, y_2\right), \\
0 &= c - d y_2^2 - y_1, \\
0 &= y_1 - a y_2^3 + b y_2^2 - x_1 + I.
\end{aligned} \tag{4.6}$$

4.4. Desingularized system on the critical manifold

The projection of the normalized slow dynamics (4.5) on the tangent bundle of the critical manifold reads:

$$\begin{aligned}
\dot{x}_1 &= -\left[3 a y_2^2 - 2\left(b - d\right) y_2\right]\left[s\left(y_2 - \alpha\right) - x_1\right] = F_1\left(x_1, y_2\right), \\
\dot{y}_2 &= \left[s\left(y_2 - \alpha\right) - x_1\right] = G_2\left(x_1, y_2\right).
\end{aligned} \tag{4.7}$$

4.5. Pseudo singular points

According to equations (3.10) the *pseudo-singular points* of system (4.2) are:

$$\left(\tilde{x}_1, \tilde{y}_1, \tilde{y}_2\right) = \left(c + I, c, 0\right), \tag{4.8a}$$

$$\left(\tilde{x}_1, \tilde{y}_1, \tilde{y}_2\right) = \left(\tilde{y}_1 - a\tilde{y}_2^3 + b\tilde{y}_2^2 + I, c - d\tilde{y}_2^2, \frac{2}{3a}\left(b - d\right)\right). \tag{4.8b}$$

4.6. Canard existence

The Jacobian matrix of system (4.6) evaluated at the *pseudo singular points* (4.8a) reads:

$$J_{(F_1,G_1,G_2)} = \begin{pmatrix} 0 & 0 & -2\,(b-d)\,(c+I+s\alpha) \\ 0 & 0 & 2d\,(c+I+s\alpha) \\ -1 & 0 & s \end{pmatrix}. \tag{4.9}$$

According to equations (3.16) we find that:

$$\begin{aligned} p &= \sigma_1 = Tr\,[J] = s, \\ q &= \sigma_2 = -2\,(b-d)\,(c+I+s\alpha)\,. \end{aligned}$$

Thus, according to Proposition 3.4, the *pseudo singular point* is of saddle-type if:

$$-\,2\,(b-d)\,(c+I+s\alpha) < 0. \tag{4.10}$$

In her work Shchepakina [28] used the following parameter set: $a = 1$, $b = 3$, $c = 1$, $d = 0.275255$, $I = 2.7$ and $\alpha = -1.2$. She found a canard without head (see Fig. 1) for the "duck parameter" value $s = 3.08104454785581412 14$. According to equation (4.10) and with such a parameter set, *i.e.* $b - d > 0$, the *pseudo singular point* is of saddle-type if and only if:

$$s < \frac{c+I}{\alpha}.$$

With $c = 1$, $I = 2.7$ and $\alpha = -1.2$, we find that: $s < 3.0833$. Thus, Shchepakina highlighted a canard without head in the Hindmarsh-Rose model (see Fig. 1) for the "duck parameter" value $s = 3.0810445478558141214$ < 3.0833.

In the inset of Figure 1, the zoom in highlights a large distance between the canard solution and that of the *critical manifold* (4.4). This is due to the fact that this latter corresponds to zero-order approximation in ε of the *slow invariant manifold*. Nevertheless, while using the so-called *Flow Curvature Method* Ginoux and Rossetto [13] have already provided a second-order approximation in ε of the *slow invariant manifold* of the Hindmarsh-Rose model (4.1). The result is presented in Figure 2.

Doing the change of variables $(x_1, y_1, y_2) \to (y, x_1, x_3)$ system (4.2) writes

$$\begin{aligned} \dot{y} &= f(x_1, x_3, y) = s(x_3 - \alpha) - y, \\ \varepsilon\dot{x}_1 &= h_1(x_1, x_3, y) = c - dx_3^2 - x_1, \\ \varepsilon\dot{x}_3 &= h_3(x_1, x_3, y) = x_1 - \alpha x_3^3 + bx_3^2 - y + I. \end{aligned} \tag{4.11}$$

Using the previous notation we get

$$\begin{aligned} a_{11} &= \frac{\partial h_1}{\partial x_1}(\tilde{x}_1, \tilde{x}_3, \tilde{y}) = -1, \\ a_{14} &= \frac{\partial h_1}{\partial y}(\tilde{x}_1, \tilde{x}_3, \tilde{y}) = 0, \\ a_{33} &= \frac{\partial h_3}{\partial x_3}(\tilde{x}_1, \tilde{x}_3, \tilde{y}) = -3\alpha\tilde{x}_3^2 + 2b\tilde{x}_3 = 0, \end{aligned}$$

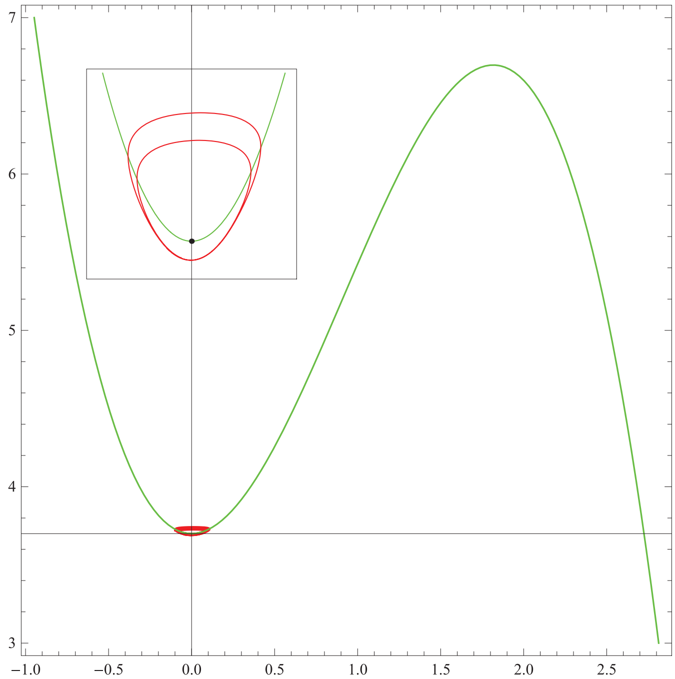

FIGURE 1. Singular canard solution of the Hindmarsh-Rose (4.1) model in the (x_3, y) plane phase with the following parameter set: $a = 1$, $b = 3$, $c = 1$, $d = 0.275255$, $I = 2.7$, $\alpha = -1.2$ and for the "duck parameter" value $s = 3.0810445478558141214$.

where $\tilde{x}_1 = c$, $\tilde{x}_3 = 0$ and $\tilde{y} = c + I$. The solution of the corresponding equations (3.21) for system (4.11) with $\delta = 0$ are

$$u_1 = u_1^0 = 0, \quad u_2 = u_2^0 = 0, \quad v = -(\alpha s + c + I)t.$$

So the solutions for the corresponding equations (3.21) for system (4.11) with $\delta \approx 0$ are

$$u_1 = u_1^0 = \mathcal{L}(\delta), \quad u_2 = u_2^0 = \mathcal{L}(\delta), \quad v = -(\alpha s + c + I)t + \mathcal{L}(\delta).$$

Remark. We notice on the one hand that the *Flow Curvature Method* provides a better of the approximation of the *slow invariant manifold* of such system and, on the other hand, that it could also be used to highlight the bifurcation leading to a canard solution as emphasized by Ginoux and Llibre [14] in the case of the Van der Pol model.

The Jacobian matrix of system (4.6) evaluated at the *pseudo singular points* (4.8b) reads:

$$J_{(F_1,G_1,G_2)} = \begin{pmatrix} 0 & 0 & -2\left(b-d\right)\left[s\left(\tilde{y}_2-\alpha\right)-\tilde{x}_1\right] \\ -2d\tilde{y}_2 & 0 & 2d\left[s\left(\alpha-2\tilde{y}_2\right)+\tilde{x}_1\right] \\ -1 & 0 & s \end{pmatrix}. \tag{4.12}$$

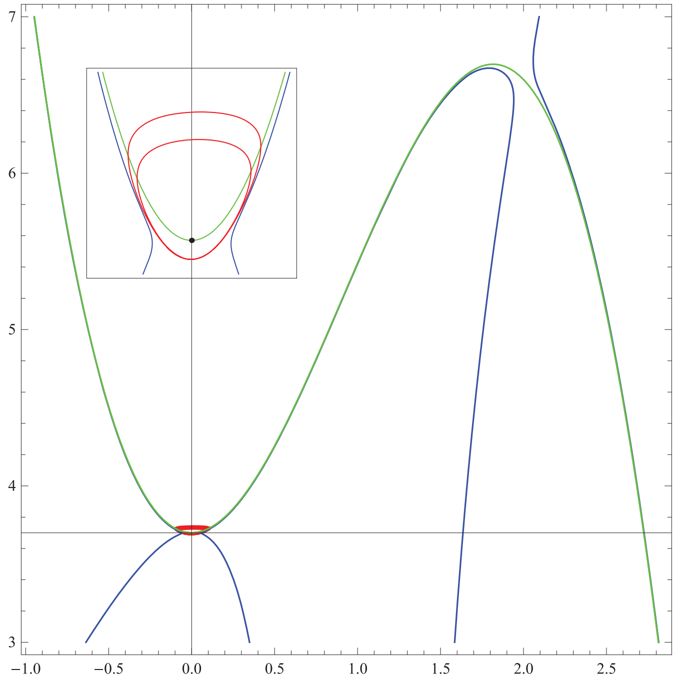

FIGURE 2. Singular canard solution of the Hindmarsh-Rose (4.1) model in the (x_3, y) plane phase, its *critical manifold* (in green) and the second-order approximation in ε of the *slow invariant manifold* (in blue) with the following parameter set: $a = 1$, $b = 3$, $c = 1$, $d = 0.275255$, $I = 2.7$, $\alpha = -1.2$ and for the "duck parameter" value $s = 3.08104454785581412 14$.

According to equations (3.16) we find that:

$$p = \sigma_1 = Tr\,[J] = s,$$
$$q = \sigma_2 = -2\,(b - d)\,[s\,(\tilde{y}_2 - \alpha) - \tilde{x}_1]\,.$$

Thus, according to Proposition 3.4, the *pseudo singular point* is of saddle-type if and only if:

$$-\,2\,(b - d)\,[s\,(\tilde{z} - \alpha) - \tilde{x}] < 0. \tag{4.13}$$

In her work Shchepakina [28] used the following parameter set: $a = 1$, $b = 3$, $c = 1$, $d = 0.275255$, $I = 2.7$ and $\alpha = -1.2$. According to equation (4.13) and with such a parameter set, *i.e.* $b - d > 0$, the *pseudo singular point* is of saddle-type if and only if:

$$s < \frac{4\,(b - d)^3 + 27a^2\,(c + I)}{9a\,[2\,(b - d) - 3a\alpha]}$$

With $c = 1$, $I = 2.7$ and $\alpha = -1.2$, we find that: $s < 2.2200954$. Thus, we have highlighted a canard with head in the Hindmarsh-Rose model (see Fig. 3) for the "duck parameter" value $s = 2.220095 < 2.2200954$. For this parameters set the second-order approximation in ε of the *slow invariant manifold* of the Hindmarsh-Rose model (4.1) can be provided while using the *Flow Curvature Method* introduced by Ginoux and Rossetto [13]. The result is presented in Figure 3.

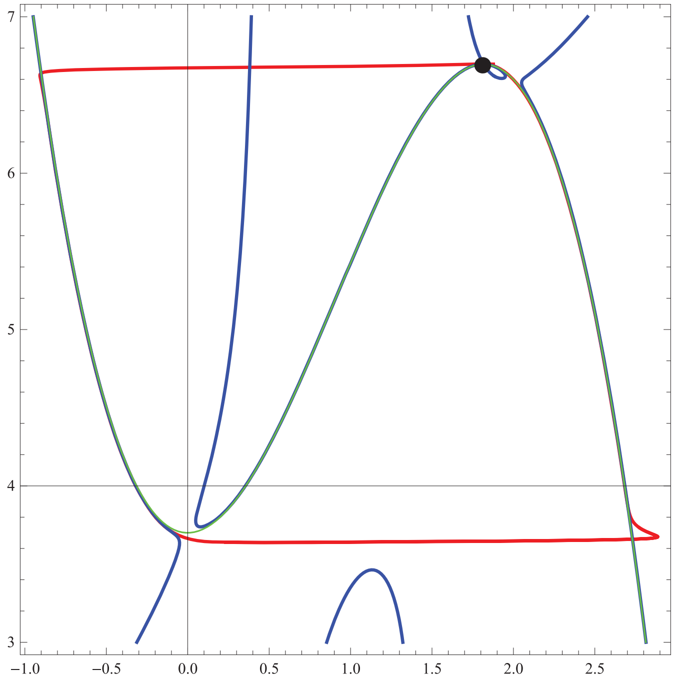

FIGURE 3. Maximal canard solution of the Hindmarsh-Rose (4.1) model in the (x_3, y) plane phase, its *critical manifold* (in green) and the second-order approximation in ε of the *slow invariant manifold* (in blue) with the following parameter set: $a = 1$, $b = 3$, $c = 1$, $d = 0.275255$, $I = 2.7$, $\alpha = -1.2$ and for the "duck parameter" value $s = 2.220095$.

5. Discussion

In a previous paper entitled: "Canards Existence in Memristor's Circuits" (see Ginoux & Llibre [16]) we have proposed a new method for proving the existence of "canard solutions" for three and four-dimensional singularly perturbed systems with only one *fast* variable which is based on the stability of *folded singularities* of the *normalized slow dynamics* (also called "desingularized vector field" in Llibre *et al.* [26]) deduced from a well-known property of linear algebra. Thus, we proved that this unique condition is completely identical to that provided by Benoît [4], Szmolyan and Wechselberger [31] and Wechselberger [37]. In a second paper entitled: "Canards Existence in FitzHugh-Nagumo and Hodgkin-Huxley Neuronal Models" (see Ginoux and Llibre [15]) we extended this method to the case of four-dimensional singularly perturbed systems with $k = 2$ *slow* and $m = 2$ *fast* variables. In this work we have extended this new method to the case of three-dimensional singularly perturbed systems with one *slow* and two *fast* variables and we have stated that the condition for the existence of "canard solutions" in such systems is exactly identical to those proposed in our previous paper. This result confirms the genericity of the condition ($\sigma_2 < 0$) that we have highlighted and provides a simple and efficient tool for testing the occurrence of "canard solutions" in any three or four-dimensional singularly perturbed systems with one or two *fast* variables. Applications of this method to the famous coupled Hindmarsh-Rose model has enabled to confirm the existence of "canard solutions" in such systems as already stated by Shchepakina [28]. However, in this paper, only the case of *pseudo singular points* or *folded singularities* of saddle-type has been analyzed. Of course, the case of of *pseudo singular points* or *folded singularities* of node-type and focus-type could be also studied with the same method.

Acknowledgements. Authors would like to thank the reviewers for their fruitful comments and suggestions which helped us to improve the presentation of this paper. This work is supported by the Ministerio de Economía, Industria y

Competitividad, Agencia Estatal de Investigación grant MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The first author would like to thank Pr. J. Llibre for is kind invitation at Universitat Autònoma de Barcelona.

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