

# CENTERS OF DISCONTINUOUS PIECEWISE SMOOTH QUASI-HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

HEBAI CHEN<sup>A,B</sup>, JAUME LLIBRE<sup>C</sup> AND YILEI TANG<sup>D</sup>

ABSTRACT. In this paper we investigate the center problem for the discontinuous piecewise smooth quasi-homogeneous but non-homogeneous polynomial differential systems.

First, we provide sufficient and necessary conditions for the existence of a center in the discontinuous piecewise smooth quasi-homogeneous polynomial differential systems. Moreover, these centers are global, and the period function of their periodic orbits is monotonic.

Second, we characterize the centers of the discontinuous piecewise smooth quasi-homogeneous cubic and quartic polynomial differential systems.

## 1. INTRODUCTION

When all the orbits of a differential system in  $\mathbb{R}^2$  in a punctured neighborhood of an equilibrium point  $p$  of the system are periodic, we say that the  $p$  is a *center* of the differential system. A center  $p$  of a differential system is *global* when all the orbits of the system in  $\mathbb{R}^2 \setminus \{p\}$  are periodic.

In the qualitative theory of planar smooth differential systems, the center problem is a classical problem, which consist in determining the existence of a center, i.e. give necessary and sufficient conditions in order that an equilibrium point of a differential system in the plane  $\mathbb{R}^2$  be a center. The study of the centers goes back to Poincaré [29] and Dulac [9], and in the present days many questions about them remain open.

It is known that there are three kind of centers for the analytic differential systems in  $\mathbb{R}^2$ . The *linear type centers* or simply *linear centers* are the centers whose linear part have purely imaginary eigenvalues. The *nilpotent centers* are the centers such that the eigenvalues of their linear part are both zero, but the linear part is not identically zero. Finally, the *degenerate centers* are the centers whose linear part is identically zero. For more details on these three kind of centers see for instance [24] and the references quoted there.

The linear centers can be studied computing their Lyapunov constants, and the study of these centers has been stimulated considerably by the use of computer algebra systems [10], but in any case these computations are in general huge and sometimes non-realistic because they need too much time of computation and too much memory of the computer. Moreover, in a neighborhood of a linear center of an analytic system always exist a local analytic first integral, as it was proved by

---

2010 *Mathematics Subject Classification.* 37J35, 34A05, 34A34, 34C14.

*Key words and phrases.* Global center, quasi-homogeneous polynomial systems, piecewise smooth systems, dynamics.

Poincaré [30] and Lyapunov [21]. In general do not exist local analytic first integrals in the neighborhoods of the nilpotent or degenerate centers, see for instance [24]. But any center always has a local  $C^\infty$  first integral, see [26].

For analytic and in particular for polynomial differential systems, there are some methods for studying the nilpotent centers, see [13, 17, 24, 18, 28]. But the determination of the degenerate centers of the analytical differential systems is more difficult and only very partial results have been obtained.

The centers of the quasi-homogeneous polynomial differential systems have been classified, and all are global centers see [19] and [16]. The normal forms of the quasi-homogeneous polynomial differential systems of degree  $n$  having a center at the origin were obtained in [35].

In these last decades discontinuous piecewise smooth differential systems have been widely used in a natural way for modeling real processes and different phenomena, see for instance [4, 5, 25]. The study of the discontinuous piecewise differential systems started with Andronov *et al* in [2], they mainly analyzed the discontinuous piecewise linear differential systems. Recently many works paid attention to different aspects (as their limit cycles, bifurcations, integrability, ...) of the linear or nonlinear discontinuous piecewise smooth differential systems, see for instance [1, 6, 7, 8, 12, 34].

In this paper we are interested in the centers of the discontinuous piecewise smooth quasi-homogeneous polynomial differential systems. It is known that the quasi-homogeneous polynomial differential systems of even degree has no centers, see [3, 19, 32, 35]. However, as we shall see the discontinuous piecewise smooth quasi-homogeneous polynomial differential systems of even degree can have centers.

A polynomial differential system

$$(1) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

in  $\mathbb{R}^2$  is a *quasi-homogeneous polynomial differential system* if there exist constants  $s_1, s_2, m \in \mathbb{Z}_+$  such that

$$P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1+m-1}P(x, y) \quad \text{and} \quad Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2+m-1}Q(x, y),$$

for all  $\alpha \in \mathbb{R}^+$ , with  $P(x, y), Q(x, y) \in \mathbb{R}[x, y] \setminus \{0\}$ . We say that the quasi-homogeneous polynomial differential system (1) or its associated vector field has *weight vector*  $w = (s_1, s_2, m)$ , and *degree*  $n$  if  $n$  is the maximum of the degrees of  $P$  and  $Q$ . When  $s_1 = s_2 = 1$ , system (1) is a *homogeneous system* of degree  $m$ .

Clearly a quasi-homogeneous polynomial differential system (1) has a unique *minimal weight vector* (MWV for short)  $\tilde{w} = (\tilde{s}_1, \tilde{s}_2, \tilde{m})$  satisfying that  $\tilde{s}_1 \leq s_1, \tilde{s}_2 \leq s_2$  and  $\tilde{m} \leq m$  for any other weight vector  $(s_1, s_2, m)$  of system (1). In what follows we assume without loss of generality that  $P$  and  $Q$  in system (1) have not a non-constant common factor.

Here we shall work with discontinuous piecewise smooth polynomial differential system in  $\mathbb{R}^2$  formed by two quasi-homogeneous polynomial differential systems separated by the straight line  $y = 0$ . More precisely, we deal with the discontinuous

piecewise smooth quasi-homogeneous polynomial differential system of degree  $n$

$$(2) \quad \begin{aligned} \dot{x} = P^+(x, y) &= \sum_{0 \leq i+j \leq n} a_{i,j}^+ x^i y^j, & \dot{y} = Q^+(x, y) &= \sum_{0 \leq i+j \leq n} b_{i,j}^+ x^i y^j, & \text{in } y \geq 0 \\ \dot{x} = P^-(x, y) &= \sum_{0 \leq i+j \leq n} a_{i,j}^- x^i y^j, & \dot{y} = Q^-(x, y) &= \sum_{0 \leq i+j \leq n} b_{i,j}^- x^i y^j, & \text{in } y \leq 0. \end{aligned}$$

We assume that  $(P^+, Q^+)$  and  $(P^-, Q^-)$  are quasi-homogeneous polynomial vector fields with the same MWV.

In this paper first we characterize the centers of the discontinuous piecewise smooth quasi-homogeneous but non-homogeneous polynomial differential systems, see Theorems 1 and 2. And after we characterize the global centers of discontinuous piecewise smooth quasi-homogeneous but non-homogeneous cubic and quartic polynomial vector fields.

This article is organized as follows. In section 2 we present sufficient and necessary conditions for the existence of a center in a discontinuous piecewise smooth quasi-homogeneous polynomial differential system. These centers will be global and the period function of their periodic orbits will be monotonic. Sections 3-4 are dedicated to analyze the center problem for discontinuous piecewise smooth quasi-homogeneous but non-homogeneous cubic and quartic polynomial differential systems, respectively.

## 2. CENTERS OF PIECEWISE SMOOTH QUASI-HOMOGENEOUS POLYNOMIAL DIFFERENTIAL SYSTEMS

According to [14, Proposition 10], if a smooth quasi-homogeneous but non-homogeneous polynomial differential system is of degree  $n$  with the weight vector  $(s_1, s_2, m)$  and  $m > 1$ , then the system has the MWV

$$(3) \quad \tilde{w} = \left( \frac{\varsigma + \kappa}{s}, \frac{\kappa}{s}, 1 + \frac{(p-1)\varsigma + (n-1)\kappa}{s} \right),$$

with  $p \in \{0, 1, \dots, n-1\}$ ,  $\varsigma \in \{1, 2, \dots, n-p\}$  and  $\kappa \in \{1, \dots, n-p-\varsigma+1\}$  satisfying

$$s_1 = \frac{(\varsigma + \kappa)(m-1)}{D}, \quad s_2 = \frac{\kappa(m-1)}{D},$$

where  $D = (p-1)\varsigma + (n-1)\kappa$  and  $s = \gcd(\varsigma, \kappa)$ . Furthermore, the integers  $(\varsigma + \kappa)/s$  and  $\kappa/s$  are coprime and at least one of them is odd, see [33, 35]. From the algorithm described in subsection 3.1 of [14] the quasi-homogeneous but non-homogeneous polynomial differential system (1) of degree  $n$  with weight vector  $(s_1, s_2, m)$  can be written as

$$(4) \quad X_{p\varsigma\kappa} = X_n^p + X_{n-\varsigma}^{p\varsigma\kappa} + V_{p\varsigma\kappa},$$

where

$$X_n^p = (a_{p,n-p} x^p y^{n-p}, b_{p-1,n-p+1} x^{p-1} y^{n-p+1})^T$$

is the homogeneous part of degree  $n$  with coefficients not all zero,

$$V_{p\leq\kappa} = \sum_{\substack{\varsigma_1 \in \{1, \dots, n-p\} \setminus \{\varsigma\} \\ \kappa_{\varsigma_1} \varsigma = \kappa_{\varsigma_1} \text{ and} \\ \kappa_{\varsigma_1} \in \{1, \dots, n-\varsigma_1-p+1\}}} X_{n-\varsigma_1}^{p\kappa_{\varsigma_1}},$$

$$X_{n-\varsigma}^{p\kappa} = (a_{p+\kappa, n-\varsigma-p-\kappa} x^{p+\kappa} y^{n-\varsigma-p-\kappa}, b_{p+\kappa-1, n-\varsigma-p-\kappa+1} x^{p+\kappa-1} y^{n-\varsigma-p-\kappa+1})^T,$$

and  $X_{n-\varsigma_1}^{p\kappa_{\varsigma_1}}$ 's having the same expressions as that of  $X_{n-\varsigma}^{p\kappa}$ . In order that  $X_{p\leq\kappa}$  to be of degree  $n$  we must have  $X_n^p \neq 0$ , and in order that it does not be an homogeneous system at least one of the monomials which are not in  $X_n^p$  must be different from zero.

We shall study the dynamics of the generic discontinuous piecewise smooth quasi-homogeneous polynomial differential systems (2) close to the discontinuous line, i.e., the  $x$ -axis, by using the Filippov convection method, see [4, 11, 22, 23]. The discontinuous line

$$\mathcal{L} := \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = y = 0\}$$

separates the plane into two open non-overlapping regions

$$Y^+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \text{ and } Y^- = \{(x, y) \in \mathbb{R}^2 \mid y < 0\}.$$

Suppose that

$$\sigma(x, y) = \langle (F_x, F_y), (P^+, Q^+) \rangle \cdot \langle (F_x, F_y), (P^-, Q^-) \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product. From (2) and (4), the *crossing set* of system (2) is

$$\begin{aligned} (5) \quad \mathcal{L}_c &= \{(x, y) \in \mathcal{L} \mid \sigma(x, y) > 0\} \\ &= \{(x, y) \in \mathcal{L} \mid b_{p+\kappa-1,0}^+ b_{p+\kappa-1,0}^- x^{2(p+\kappa-1)} > 0\} \\ &= \begin{cases} \{(x, y) \in \mathcal{L} \mid x \neq 0\} & \text{if } b_{p+\kappa-1,0}^+ b_{p+\kappa-1,0}^- > 0, \\ \emptyset & \text{if } b_{p+\kappa-1,0}^+ b_{p+\kappa-1,0}^- \leq 0. \end{cases} \end{aligned}$$

The orbits of the (2) which reach any point  $(x, y) \in \mathcal{L}_c$  crosses the line of discontinuity  $\mathcal{L}$ , i.e. the vectors  $(P^+(x, y), Q^+(x, y))$  and  $(P^-(x, y), Q^-(x, y))$  point to the ‘‘same’’ direction and are transverse to  $\mathcal{L}_c$ .

The *sliding set*  $\mathcal{L}_s$  is the complement of  $\mathcal{L}_c$  in  $\mathcal{L}$ , which is given by

$$\begin{aligned} (6) \quad \mathcal{L}_s &= \{(x, y) \in \mathcal{L} \mid \sigma(x, y) \leq 0\} \\ &= \{(x, y) \in \mathcal{L} \mid b_{p+\kappa-1,0}^+ b_{p+\kappa-1,0}^- x^{2(p+\kappa-1)} \leq 0\} \\ &= \begin{cases} \mathcal{L} & \text{if } b_{p+\kappa-1,0}^+ b_{p+\kappa-1,0}^- \leq 0, \\ \{(x, y) \in \mathcal{L} \mid x = 0\} & \text{if } b_{p+\kappa-1,0}^+ b_{p+\kappa-1,0}^- > 0. \end{cases} \end{aligned}$$

Moreover, in  $\mathcal{L}_s$  the points satisfying the equation

$$(7) \quad \langle (F_x, F_y), (P^- - P^+, Q^- - Q^+) \rangle = 0$$

are the *sliding equilibrium points*.

The equilibrium points of a discontinuous piecewise smooth quasi-homogeneous polynomial differential system are the equilibria contained in the open half-planes  $\{y > 0\}$  and  $\{y < 0\}$ , together with the sliding equilibrium points.

If all periodic orbits inside the period annulus of a center have the same period, the center is *isochronous*.

Making a quasi-homogeneous blow-up  $x = r^{s_1} \cos \theta$ ,  $y = r^{s_2} \sin \theta$ , which was also used in [16], we transform system (2) into the system

$$(8) \quad \frac{dr}{d\theta} = \begin{cases} = \frac{r^d H^+(\theta)/(s_1 \cos^2 \theta + s_2 \sin^2 \theta)}{r^{d-1} G^+(\theta)/(s_1 \cos^2 \theta + s_2 \sin^2 \theta)} = \frac{r H^+(\theta)}{G^+(\theta)} & \text{if } y \geq 0, \\ = \frac{r^d H^-(\theta)/(s_1 \cos^2 \theta + s_2 \sin^2 \theta)}{r^{d-1} G^-(\theta)/(s_1 \cos^2 \theta + s_2 \sin^2 \theta)} = \frac{r H^-(\theta)}{G^-(\theta)} & \text{if } y < 0, \end{cases}$$

where

$$\begin{aligned} H^\pm(\theta) &= P^\pm(\cos \theta, \sin \theta) \cos \theta + Q^\pm(\cos \theta, \sin \theta) \sin \theta, \\ G^\pm(\theta) &= s_1 Q^\pm(\cos \theta, \sin \theta) \cos \theta - s_2 P^\pm(\cos \theta, \sin \theta) \sin \theta. \end{aligned}$$

According to [31, Chapter 2], a necessary condition for  $\theta = \theta_0$  to be an exceptional direction is  $G^+(\theta) = 0$  (resp.  $G^-(\theta) = 0$ ) on the half plane  $y \geq 0$  (resp.  $y < 0$ ), and no orbits connect with the origin  $O$  along other directions.

**Theorem 1.** *A discontinuous piecewise smooth quasi-homogeneous polynomial differential system (2) with a MWV  $(s_1, s_2, m)$  has a center at the origin  $O(0, 0)$  if and only if  $\mathcal{L}_c = \{(x, y) \in \mathcal{L} \mid x \neq 0\}$  and*

$$(9) \quad \int_0^\pi \frac{H^+(\theta)}{G^+(\theta)} d\theta + \int_{-\pi}^0 \frac{H^-(\theta)}{G^-(\theta)} d\theta = 0,$$

where  $G^+(\theta) \neq 0$  if  $\theta \in [0, \pi]$  and  $G^-(\theta) \neq 0$  if  $\theta \in [-\pi, 0]$ . The center is global and the period function of periodic orbits is monotonic.

*Proof.* Notice that no equilibria of piecewise smooth systems exist in the regions  $Y^\pm$ . From the aforementioned analysis of sliding sets and crossing sets in the discontinuous line  $\mathcal{L}$ , the crossing set  $\mathcal{L}_c$  of system (2) is either  $\emptyset$  or the  $x$ -axis except the origin. Thus, system (2) has a center at the origin only if  $\mathcal{L}_c = \{(x, y) \in \mathcal{L} \mid x \neq 0\}$ .

When the crossing set  $\mathcal{L}_c$  is the  $x$ -axis except the origin, let  $p^+(r, \theta)$  (resp.  $p^-(r, \theta)$ ) be the solution of piecewise smooth system (2) in polar coordinates  $(x, y) = (\rho \cos \theta, \rho \sin \theta)$  for  $0 \leq \theta < \pi$  (resp.  $-\pi \leq \theta < 0$ ), satisfying that the initial condition  $p^+(r, 0) = r$  (resp.  $p^-(r, -\pi) = r$ ) holds, which is well defined in the region  $\mathbb{R}^2 \setminus \mathcal{L}_s$ . We define the positive Poincaré half-return map as

$$\mathcal{P}^+(r) := \lim_{\theta \rightarrow \pi} p^+(r, \theta) = r e^{\int_0^\pi \frac{H^+(\theta)}{G^+(\theta)} d\theta}$$

and the negative Poincaré half-return map as

$$\mathcal{P}^-(r) := \lim_{\theta \rightarrow 0} p^-(r, \theta) = r e^{\int_{-\pi}^0 \frac{H^-(\theta)}{G^-(\theta)} d\theta}$$

according to system (8), as shown in Figure 1. Here, without loss of generality, we suppose that the orbits surrounding the origin rotate anti-clockwise. If the direction is clockwise, we can make a time rescaling. In the next step of this proof, we will show that the orbits around the origin are spirals with a period  $2\pi$  under the conditions of this theorem.

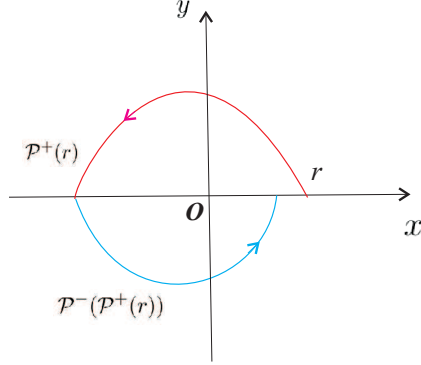


FIGURE 1. Existence of closed orbits.

The Poincaré return map associated to piecewise smooth system (2) is given by the composition of these two maps

$$(10) \quad \mathcal{P}(r) := \mathcal{P}^-(\mathcal{P}^+(r)) = r e^{\int_0^\pi \frac{H^+(\theta)}{G^+(\theta)} d\theta} e^{\int_{-\pi}^0 \frac{H^-(\theta)}{G^-(\theta)} d\theta}.$$

In order to obtain the existence of a center and further a global center at the origin, we need to present  $\mathcal{P}(r) - r \equiv 0$  for  $r > 0$ , which is equivalent to (9).

Note that the conditions  $G^+(\theta) \neq 0$  if  $\theta \in [0, \pi]$  and  $G^-(\theta) \neq 0$  if  $\theta \in [-\pi, 0]$  can not only guarantee the integrability in (9), but also the non-existence of invariant curves passing through the origin. Otherwise, if  $G^+(\theta_0) = 0$  for  $\theta_0 \in [0, \pi]$  or  $G^-(\theta_0) = 0$  for  $\theta_0 \in [-\pi, 0]$ , we obtain the invariant

$$(11) \quad \mathcal{C}_1 := \cos^{s_2}(\theta_0) y^{s_1} - \sin^{s_1}(\theta_0) x^{s_2} = 0$$

when  $\theta_0 \neq \pm\pi/2$ . If  $\theta_0 = \pm\pi/2$ , the half  $y$ -axis is invariant according to the expression of  $G^\pm(\theta)$  in (8) and the origin cannot be a center. Because at least one of  $s_1$  and  $s_2$  is odd, we can always solve a real branch connecting with the origin from the above equation (11), which is just an invariant curve of system (2). Actually, if  $p$  is odd and  $\pi/2 \neq \theta_0 \in [0, \pi]$ , from (11) we have the invariant curve

$$y = \frac{\sin(\theta_0)}{\cos^{s_2/s_1}(\theta_0)} x^{s_2/s_1}$$

on the half plane  $y \geq 0$ . If  $p$  is even and  $\pi/2 \neq \theta_0 \in [0, \pi]$ , again from (11) we have the invariant curve

$$x = \frac{\cos(\theta_0)}{\sin^{s_1/s_2}(\theta_0)} y^{s_1/s_2}$$

on the half plane  $y \geq 0$ , because  $s_2$  must be odd. On the half plane  $y < 0$ , we can discuss in a similar way and can always find an real invariant curve if  $G^-(\theta) = 0$  for  $\theta \in [-\pi, 0]$ . Remark that the factor  $y$  divides  $Q^+(x, y)$  or  $Q^-(x, y)$  if  $G^\pm(0) = 0$  or  $G^\pm(\pi) = 0$ , respectively. Thus,  $\sigma(x, y) = 0$  for  $(x, y) \in \mathcal{L}$  according to (6) and  $\mathcal{L}_s = \mathcal{L}$ , yielding that the origin cannot be a center in this case. Thus, the conditions  $G^+(\theta) \neq 0$  if  $\theta \in [0, \pi]$  and  $G^-(\theta) \neq 0$  if  $\theta \in [-\pi, 0]$  are necessary in order that the origin be a center.

Moreover, we actually get  $\mathcal{P}(r) = r$  for all  $r > 0$ , implying that the solution curve of system (8) through  $(r, 0)$  is a closed orbit and the origin  $O$  is a global center.

Finally, we shall prove that the period annulus of the center  $O$  is monotonic with respect to the initial value and then it cannot be isochronous. Assuming that  $\Gamma_{r_0}$  is the closed trajectory through  $(r_0, 0)$  inside the periodic annulus of the center  $O$ , we can define the positive half-period function as  $T^+(r_0) := \int_{\Gamma_{r_0}^+} dt$  and the negative half-period function as  $T^-(r_0) := \int_{\Gamma_{r_0}^-} dt$ , where  $r_0 > 0$ ,  $\Gamma_{r_0}^+ = \{(\rho, \theta) \in \mathbb{R}^2 \mid \rho = p^+(r_0, \theta)\}$  and  $\Gamma_{r_0}^- = \{(\rho, \theta) \in \mathbb{R}^2 \mid \rho = p^-(r_0, \theta)\}$ . Thus, the complete period function associated to system (8) is given by the sum of these two functions

$$\begin{aligned}
(12) \quad T(r_0) &= \oint_{\Gamma_{r_0}} dt = T^+(r_0) + T^-(r_0) \\
&= \int_{\Gamma_{r_0}^+} \frac{s_1 \cos^2 \theta + s_2 \sin^2 \theta}{r^{m-1} G^+(\theta)} d\theta + \int_{\Gamma_{r_0}^-} \frac{s_1 \cos^2 \theta + s_2 \sin^2 \theta}{r^{m-1} G^-(\theta)} d\theta \\
&= \int_0^\pi \frac{s_1 \cos^2 \theta + s_2 \sin^2 \theta}{(p^+(r_0, \theta))^{m-1} G^+(\theta)} d\theta + \int_{-\pi}^0 \frac{s_1 \cos^2 \theta + s_2 \sin^2 \theta}{(p^-(r_0, \theta))^{m-1} G^-(\theta)} d\theta.
\end{aligned}$$

Notice that the functions  $p^\pm(r_0, \theta)$  and  $G^\pm(\theta)$  are bounded and nonzero with respect to  $\theta$  for arbitrary fixed  $r_0 > 0$ , because the origin is a center and  $p^-(p^+(r_0, \theta), \theta)$  has period  $2\pi$  in  $\theta$ . Therefore, the two integrand functions in (12) are integrable. By (8),

$$p^+(r_0, \theta) = r_0 e^{\int_0^\theta \frac{H^+(\theta)}{G^+(\theta)} d\theta} \quad \text{and} \quad p^-(r_0, \theta) = r_0 e^{\int_{-\pi}^\theta \frac{H^-(\theta)}{G^-(\theta)} d\theta},$$

it follows that

$$\frac{dT(r_0)}{dr_0} = \frac{(1-m)}{r_0} T(r_0) < 0,$$

which implies that the period  $T(r_0)$  of the periodic orbits inside the period annulus of the center  $O$  is monotonic in  $r_0$ . Clearly, the center cannot be isochronous. This completes the proof of the theorem.  $\square$

Remark that we can also obtain the results of Theorem 1 using the properties of quasi-homogeneous functions

$$\eta^\pm(x, y) = s_1 x Q^\pm(x, y) - s_2 y P^\pm(x, y).$$

According to [27, Proposition 4], if  $\eta^+(0, 1) = 0$  (or  $\eta^-(0, 1) = 0$ ), then the half  $y$ -axis is invariant for the flow of the vector  $(P^+, Q^+)$  (or  $(P^-, Q^-)$ ). Moreover, if there are real values  $\lambda_\pm$  such that  $\eta^\pm(1, \lambda_\pm) = 0$  respectively, then there is an invariant curve from the real branch of the curve  $y^{s_1} - \lambda_\pm^{s_1} x^{s_2} = 0$  for system (2), where  $\lambda_+ \geq 0$  and  $\lambda_- \leq 0$  if  $s_2$  is even. Thus, the origin cannot be a center.

Notice that the smooth quasi-homogeneous polynomial differential system (1) with a MWV  $(s_1, s_2, d)$  is Liouvillian integrable. Moreover, we can find its inverse integrating factors  $s_1 x Q(x, y) - s_2 y P(x, y)$  or  $(s_1 x Q(x, y) - s_2 y P(x, y)) / \tilde{\mathcal{H}}(x, y)$  according to [19], where

$$\tilde{\mathcal{H}}(x, y) = (s_1 x^{2s_2} + s_2 y^{2s_1})^{\frac{1}{2s_1 s_2}} e^{-\int_0^{\arctan(y^{s_1}/x^{s_2})} \frac{F(\theta)}{G(\theta)} d\theta}$$

is a first integral of system (1),

$$\begin{aligned}
F(\theta) &= C s^{2s_2-1} \theta P(C s \theta, S n \theta) + S n^{2s_1-1} \theta Q(C s \theta, S n \theta), \\
G(\theta) &= s_1 C s \theta Q(C s \theta, S n \theta) - s_2 S n \theta P(C s \theta, S n \theta),
\end{aligned}$$

$$Tn(\theta) = \frac{Sn^{s_1}\theta}{Cs^{s_2}\theta}$$

and  $Cs\theta$  and  $Sn\theta$  are the  $(s_1, s_2)$ -trigonometric functions. Thus, we can always suppose that system (2) has a first integral  $\mathcal{H}^+(x, y)$  for  $y \geq 0$  and a first integral  $\mathcal{H}^-(x, y)$  for  $y < 0$ . With the analysis of Poincaré return map, we can present another sufficient and necessary conditions for the existence of center as follows, which is equivalent to the conditions in Theorem 1.

**Theorem 2.** *Piecewise smooth quasi-homogeneous polynomial differential system (2) with a MWV  $(s_1, s_2, m)$  has a center at the origin  $O(0, 0)$  if and only if  $\mathcal{L}_c = \{(x, y) \in \mathcal{L} \mid x \neq 0\}$ ,  $\eta^\pm(0, 1) \neq 0$ ,  $\eta^\pm(1, \lambda)$  have not real zeros for odd  $s_2$  and  $\eta^+(1, \lambda)$  (resp.  $\eta^-(1, \lambda)$ ) has not non-negative (resp. non-positive) zeros for even  $s_2$ , and there exists  $r_1 < 0$  such that  $\mathcal{H}^+(r_0, 0) = \mathcal{H}^+(r_1, 0)$  and  $\mathcal{H}^-(r_0, 0) = \mathcal{H}^-(r_1, 0)$  for arbitrary  $r_0 > 0$ . The center is global and the period function of periodic orbits is monotonic.*

### 3. GLOBAL CENTER OF PIECEWISE SMOOTH CUBIC QUASI-HOMOGENEOUS SYSTEMS

Due to Proposition 19 of García, Llibre and Pérez del Río [14], a smooth quasi-homogeneous but non-homogeneous cubic polynomial differential system without common factors has one of the following seven forms:

- (i) :  $\dot{x} = y(a_{11}x + a_{12}y^2)$ ,  $\dot{y} = b_{11}x + b_{12}y^2$   
with MWV  $(2, 1, 2)$  and  $a_{12}b_{11} \neq 0$ ,  $a_{11}b_{12} - a_{12}b_{11} \neq 0$ ;
- (ii) :  $\dot{x} = a_{21}x^2 + a_{22}y^3$ ,  $\dot{y} = b_2xy$  with MWV  $(3, 2, 4)$  and  $a_{21}a_{22}b_2 \neq 0$ ;
- (iii) :  $\dot{x} = a_3y^3$ ,  $\dot{y} = b_3x^2$  with MWV  $(4, 3, 6)$  and  $a_3b_3 \neq 0$ ;
- (iv) :  $\dot{x} = x(a_{41}x + a_{42}y^2)$ ,  $\dot{y} = y(b_{41}x + b_{42}y^2)$   
with MWV  $(2, 1, 3)$  and  $a_{41}b_{42} \neq 0$ ,  $a_{41}b_{42} - a_{42}b_{41} \neq 0$ ;
- (v) :  $\dot{x} = a_5xy^2$ ,  $\dot{y} = b_{51}x^2 + b_{52}y^3$  with MWV  $(3, 2, 5)$  and  $a_5b_{51}b_{52} \neq 0$ ;
- (vi) :  $\dot{x} = a_6xy^2$ ,  $\dot{y} = b_{61}x + b_{62}y^3$  with MWV  $(3, 1, 3)$  and  $a_6b_{61}b_{62} \neq 0$ ;
- (vii) :  $\dot{x} = a_{71}x + a_{72}y^3$ ,  $\dot{y} = b_7y$  with MWV  $(3, 1, 1)$  and  $a_{71}a_{72}b_7 \neq 0$ .

We have the following piecewise smooth quasi-homogeneous but non-homogeneous cubic polynomial differential systems directly from the smooth systems (i)-(vii).



**Lemma 3.** *Every planar piecewise smooth quasi-homogeneous but non-homogeneous cubic polynomial differential system is one of the following seven systems:*

$$\begin{aligned}
(I) : \quad & \dot{x} = y(a_{11}x + a_{12}y^2), \quad \dot{y} = b_{11}x + b_{12}y^2 && \text{if } y \geq 0, \\
& \dot{x} = y(\tilde{a}_{11}x + \tilde{a}_{12}y^2), \quad \dot{y} = \tilde{b}_{11}x + \tilde{b}_{12}y^2 && \text{if } y < 0; \\
(II) : \quad & \dot{x} = a_{21}x^2 + a_{22}y^3, \quad \dot{y} = b_2xy && \text{if } y \geq 0, \\
& \dot{x} = \tilde{a}_{21}x^2 + \tilde{a}_{22}y^3, \quad \dot{y} = \tilde{b}_2xy && \text{if } y < 0; \\
(III) : \quad & \dot{x} = a_3y^3, \quad \dot{y} = b_3x^2 && \text{if } y \geq 0, \\
& \dot{x} = \tilde{a}_3y^3, \quad \dot{y} = \tilde{b}_3x^2 && \text{if } y < 0; \\
(IV) : \quad & \dot{x} = x(a_{41}x + a_{42}y^2), \quad \dot{y} = y(b_{41}x + b_{42}y^2) && \text{if } y \geq 0, \\
& \dot{x} = x(\tilde{a}_{41}x + \tilde{a}_{42}y^2), \quad \dot{y} = y(\tilde{b}_{41}x + \tilde{b}_{42}y^2) && \text{if } y < 0; \\
(V) : \quad & \dot{x} = a_5xy^2, \quad \dot{y} = b_{51}x^2 + b_{52}y^3 && \text{if } y \geq 0, \\
& \dot{x} = \tilde{a}_5xy^2, \quad \dot{y} = \tilde{b}_{51}x^2 + \tilde{b}_{52}y^3 && \text{if } y < 0; \\
(VI) : \quad & \dot{x} = a_6xy^2, \quad \dot{y} = b_{61}x + b_{62}y^3 && \text{if } y \geq 0, \\
& \dot{x} = \tilde{a}_6xy^2, \quad \dot{y} = \tilde{b}_{61}x + \tilde{b}_{62}y^3 && \text{if } y < 0; \\
(VII) : \quad & \dot{x} = a_{71}x + a_{72}y^3, \quad \dot{y} = b_7y && \text{if } y \geq 0, \\
& \dot{x} = \tilde{a}_{71}x + \tilde{a}_{72}y^3, \quad \dot{y} = \tilde{b}_7y && \text{if } y < 0;
\end{aligned}$$

where all parameters satisfy the same inequality conditions as in the smooth systems (i)-(vii).

The piecewise smooth systems (II), (IV) and (VII) have the same sliding set  $\mathcal{L}$ , yielding that their origins cannot be a center. Moreover, both systems (V) and (VI) have the invariant straight line  $x = 0$ , so their origins cannot be a center. Thus, only the piecewise smooth systems (I) and (III) may have a center at the origin.

After taking linear changes together with scalings  $(x, y, dt) \rightarrow (b_{11}x/a_{11}, y, a_{11}dt)$  if  $y \geq 0$ , and  $(x, y, dt) \rightarrow (a_{11}x/\tilde{a}_{12}, y, a_{11}dt)$  if  $y < 0$ , system (I) becomes

$$(13) \quad \begin{aligned}
\dot{x} &= y(x + a_{12}y^2), \quad \dot{y} = x + b_{12}y^2 && \text{if } y \geq 0, \\
\dot{x} &= y(\tilde{a}_{11}x + y^2), \quad \dot{y} = \tilde{b}_{11}x + \tilde{b}_{12}y^2 && \text{if } y < 0.
\end{aligned}$$

Moreover, with the scaling  $(x, y, dt) \rightarrow ((b_3/a_3)^{\frac{1}{3}}x, y, a_3^{\frac{2}{3}}b_3^{\frac{1}{3}}dt)$ , system (III) becomes

$$(14) \quad \begin{aligned}
\dot{x} &= y^3, \quad \dot{y} = x^2 && \text{if } y \geq 0, \\
\dot{x} &= \tilde{a}_3y^3, \quad \dot{y} = \tilde{b}_3x^2 && \text{if } y < 0.
\end{aligned}$$

Here, we still write the new parameters as  $a_{ij}$ ,  $b_{ij}$ ,  $\tilde{a}_{ij}$  and  $\tilde{b}_{ij}$  for simpler notations.

From the definitions (5) and (6) we compute for system (13) its crossing and sliding sets in  $\mathcal{L}$ :

$$(15) \quad \mathcal{L}_c^I = \{(x, y) \in \mathcal{L} \mid \tilde{b}_{11}x^2 > 0\} = \begin{cases} \{(x, y) \in \mathcal{L} \mid x \neq 0\} & \text{if } \tilde{b}_{11} > 0, \\ \emptyset & \text{if } \tilde{b}_{11} < 0, \end{cases}$$

and

$$(16) \quad \mathcal{L}_s^I = \{(x, y) \in \mathcal{L} \mid \tilde{b}_{11}x^2 \leq 0\} = \begin{cases} \{(x, y) \in \mathcal{L} \mid x = 0\} & \text{if } \tilde{b}_{11} > 0, \\ \mathcal{L} & \text{if } \tilde{b}_{11} < 0, \end{cases}$$

respectively. Then we find that the only solution of (7) for system (13) in  $\mathcal{L}_s^I$  is the origin, which is a singular sliding point and also a boundary equilibrium because the vector fields vanish at the origin.

By an analogous analysis of system (13) we obtain for piecewise smooth system (14) the crossing and the sliding sets in  $\mathcal{L}$ :

$$(17) \quad \mathcal{L}_c^{III} = \{(x, y) \in \mathcal{L} \mid \tilde{b}_3 x^2 > 0\} = \begin{cases} \{(x, y) \in \mathcal{L} \mid x \neq 0\} & \text{if } \tilde{b}_3 > 0, \\ \emptyset & \text{if } \tilde{b}_3 < 0, \end{cases}$$

$$(18) \quad \mathcal{L}_s^{III} = \{(x, y) \in \mathcal{L} \mid \tilde{b}_3 x^2 \leq 0\} = \begin{cases} \{(x, y) \in \mathcal{L} \mid x = 0\} & \text{if } \tilde{b}_3 > 0, \\ \mathcal{L} & \text{if } \tilde{b}_3 < 0. \end{cases}$$

We also find that the origin of system (14) in  $\mathcal{L}_s^{III}$  is a unique singular sliding point, which is a boundary equilibrium.

**Theorem 4.** *Piecewise smooth quasi-homogeneous cubic polynomial differential systems (II)-(VII) have no centers. Reduced piecewise smooth quasi-homogeneous cubic polynomial differential system (13) from (I) has a center at the origin if and only if  $\tilde{b}_{11} > 0$ ,  $(2b_{12} - 1)^2 + 8a_{12} < 0$  (or  $a_{12} > 0$ ,  $-1 + 2b_{12} < 0$ , or  $a_{12} < 0$ ,  $\sqrt{-8a_{12}} = 2b_{12} - 1$ ),  $(A_{11} - 2B_{12})^2 + 8B_{11} < 0$  (or  $A_{11} - 2B_{12} \geq \sqrt{-8B_{11}}$ ) and  $B_{11} < 0$ . The center of system (13) is global and the period function of the periodic orbits is monotonic.*

*Proof.* Obviously, systems (II), (IV)-(VII) have no centers. We consider the reduced systems (13) and (14) in place of (I) and (III). From the above mentioned analysis of the sliding sets and singular sliding points on the discontinuous line  $\mathcal{L}$ , and by Theorem 1 systems (13) (respectively (14)) have a center at the origin only if  $\tilde{b}_{11} > 0$  (respectively  $\tilde{b}_3 > 0$ ).

For system (14) we calculate that

$$\begin{aligned} G_{III}^+(\theta) &= s_1 Q^+(\cos \theta, \sin \theta) \cos \theta - s_2 P^+(\cos \theta, \sin \theta) \sin \theta \\ &= 4 \cos^3 \theta - 3 \sin^4 \theta. \end{aligned}$$

Then  $G_{III}^+(\theta)$  has real roots for  $\theta \in [0, \pi]$  satisfying  $\cos(\theta) \approx 0.639697327225298$  or  $\cos(\theta) \approx 2.16002341801127$ . Thus the origin of system (14) cannot be a center according to Theorem 1.

For system (13) with  $y \geq 0$  we compute

$$\eta_I^+(0, 1) := s_1 x Q^+(0, 1) - s_2 y P^+(0, 1) = -a_{12} \neq 0,$$

and

$$\eta_I^+(1, \lambda) = s_1 x Q^+(1, \lambda) - s_2 y P^+(1, \lambda) = 2 + (2b_{12} - 1)y^2 - a_{12}y^4.$$

Clearly,  $\eta_I^+(0, 1) \neq 0$ . Besides  $\eta_I^+(1, \lambda)$  has not real zeros if and only if  $(2b_{12} - 1)^2 + 8a_{12} < 0$ , or  $(2b_{12} - 1)^2 + 8a_{12} \geq 0$ ,  $a_{12}(-1 + 2b_{12} + \sqrt{4b_{12}^2 + 8a_{12} - 4b_{12} + 1}) < 0$  and  $a_{12}(2b_{12} - 1 - \sqrt{(2b_{12} - 1)^2 + 8a_{12}}) < 0$ , which can be reduced to  $(2b_{12} - 1)^2 + 8a_{12} < 0$ , or  $a_{12} > 0$ ,  $-1 + 2b_{12} < 0$ , or  $a_{12} < 0$ ,  $\sqrt{-8a_{12}} = 2b_{12} - 1$ .

For system (13) with  $y < 0$  we compute

$$\eta_I^-(0, 1) := s_1 x Q^-(0, 1) - s_2 y P^-(0, 1) = -1 \neq 0$$

and

$$\eta_I^-(1, \lambda) = s_1 x Q^-(1, \lambda) - s_2 y P^-(1, \lambda) = 2B_{11} + (-A_{11} + 2B_{12})y^2 - y^4.$$

Clearly  $\eta_I^-(1, \lambda)$  has not real zeros if and only if  $(A_{11} - 2B_{12})^2 + 8B_{11} < 0$ ; or  $(A_{11} - 2B_{12})^2 + 8B_{11} \geq 0$ ,  $-A_{11} + 2B_{12} + \sqrt{(A_{11} - 2B_{12})^2 + 8B_{11}} < 0$  and  $-A_{11} +$

$2B_{12} - \sqrt{(A_{11} - 2B_{12})^2 + 8B_{11}} < 0$ , which can be reduced to  $(A_{11} - 2B_{12})^2 + 8B_{11} < 0$ ; or  $A_{11} - 2B_{12} \geq \sqrt{-8B_{11}}$  and  $B_{11} < 0$ .

According to Theorem 2, if  $\eta^+(1, \lambda)$  or  $\eta^-(1, \lambda)$  has a real zero, there is an invariant curve passing through the origin of system (13) and the origin cannot be a center.

Note that system (13) has a first integral

$$\tilde{H}_1^+(x, y) = (a_{12}y^4 - 2b_{12}xy^2 + xy^2 - 2x^2) e^{\frac{-2\operatorname{arctanh}\left(\frac{2a_{12}y^2 - 2b_{12}x + x}{x\sqrt{4b_{12}^2 + 8a_{12} - 4b_{12} + 1}}\right)(1+2b_{12})}{\sqrt{4b_{12}^2 + 8a_{12} - 4b_{12} + 1}}}$$

if  $y \geq 0$ , and a first integral

$$\tilde{H}_1^-(x, y) = (A_{11}xy^2 - 2B_{12}xy^2 + y^4 - 2B_{11}x^2) e^{\frac{2\operatorname{arctan}\left(\frac{A_{11}x - 2x B_{12} + 2y^2}{x\sqrt{-A_{11}^2 + 4A_{11}B_{12} - 4B_{12}^2 - 8B_{11}}}\right)(A_{11} + 2B_{12})}{\sqrt{-A_{11}^2 + 4A_{11}B_{12} - 4B_{12}^2 - 8B_{11}}}}$$

if  $y < 0$ . In addition we have

$$\tilde{H}_1^+(x, 0) = -2x^2 e^{\frac{-2\operatorname{arctanh}\left(\frac{-2b_{12} + 1}{\sqrt{4b_{12}^2 + 8a_{12} - 4b_{12} + 1}}\right)(1+2b_{12})}{\sqrt{4b_{12}^2 + 8a_{12} - 4b_{12} + 1}}},$$

and

$$\tilde{H}_1^-(x, 0) = -2B_{11}x^2 e^{\frac{2\operatorname{arctan}\left(\frac{A_{11} - 2B_{12}}{\sqrt{-A_{11}^2 + 4A_{11}B_{12} - 4B_{12}^2 - 8B_{11}}}\right)(A_{11} + 2B_{12})}{\sqrt{-A_{11}^2 + 4A_{11}B_{12} - 4B_{12}^2 - 8B_{11}}}},$$

yielding that  $\tilde{H}_1^+(r_0, 0) = \tilde{H}_1^+(r_1, 0)$  and  $\tilde{H}_1^-(r_0, 0) = \tilde{H}_1^-(r_1, 0)$  for arbitrary  $r_1 = -r_0 < 0$ . Therefore the center of the piecewise smooth quasi-homogeneous cubic polynomial differential system (13) at the origin is global. This completes the proof of the theorem for the system (13).

In the case  $a_1 < 0$ ,  $b_1 > 0$  and  $\tilde{a}_1 > 0$ , we prove that the center  $O_1$  at the origin of piecewise smooth system (I) is not isochronous. For  $r_0 > 0$  we assume that  $\Gamma_{r_0}$  is the closed trajectory through  $(r_0, 0)$  inside the periodic annulus of the center  $O_1$ , we can define the positive half-period function as  $T^+(r_0) := \int_{\Gamma_{r_0}^+} dt$  and the negative half-period function as  $T^-(r_0) := \int_{\Gamma_{r_0}^-} dt$ , where

$$\Gamma_{r_0}^+ = \left\{ (\rho, \theta) \in \mathbb{R}^2 \mid \rho = p^+(r_0, \theta) \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid y = \left( \frac{3b_1}{2a_1} (x^2 - r_0^2) \right)^{\frac{1}{3}} \right\}$$

and

$$\Gamma_{r_0}^- = \left\{ (\rho, \theta) \in \mathbb{R}^2 \mid \rho = p^-(r_0, \theta) \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid y = \left( \frac{3}{2\tilde{a}_1} (x^2 - r_0^2) \right)^{\frac{1}{3}} \right\}.$$

Thus the complete period function associated to the piecewise smooth system (I) is given by the sum of these two functions

$$\begin{aligned} T_I(r_0) &= \oint_{\Gamma_{r_0}} dt = T^+(r_0) + T^-(r_0) = \int_{\Gamma_{r_0}^+} \frac{dx}{a_1 y^2} + \int_{\Gamma_{r_0}^-} \frac{dx}{\tilde{a}_1 y^2} \\ &= \int_{r_0}^{-r_0} \frac{dx}{a_1 \left( \frac{3b_1}{2a_1} (x^2 - r_0^2) \right)^{\frac{2}{3}}} + \int_{-r_0}^{r_0} \frac{dx}{\tilde{a}_1 \left( \frac{3}{2\tilde{a}_1} (x^2 - r_0^2) \right)^{\frac{2}{3}}} \\ &= \beta_0 r_0^{-\frac{1}{3}}, \end{aligned}$$

where

$$\beta_0 = \frac{\left(\frac{2}{3}\right)^{\frac{5}{3}} \pi^{\frac{3}{2}} \sqrt{3}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)} \left(-a_1^{-\frac{1}{3}} b_1^{-\frac{2}{3}} + \tilde{a}_1^{-\frac{1}{3}}\right) > 0$$

and  $\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds$  is the Gamma function. Clearly the period  $T_I(r_0)$  of the periodic orbits inside the period annulus of the center  $O_1$  is monotonic in  $r_0$ . This completes the proof of the theorem.  $\square$

Notice that all smooth quasi-homogeneous quadratic polynomial differential systems (i) – (iii) have no centers, because there exists an invariant straight line or an invariant curve passing through the origin for such systems. However for the piecewise smooth quasi-homogeneous quadratic polynomial differential systems we will find the existence of a center at the origin under some conditions on the parameter.

#### 4. GLOBAL CENTER FOR PIECEWISE SMOOTH QUASI-HOMOGENEOUS QUARTIC POLYNOMIAL DIFFERENTIAL SYSTEMS

According to Proposition 1 of [20] there exist 17 quasi-homogeneous but non-homogeneous quartic polynomial differential systems. Fourteen of them either have the invariant straight line  $x = 0$  or  $y = 0$ . From the proof of Theorem 1 these fourteen classes of piecewise smooth quasi-homogeneous but non-homogeneous quartic polynomial differential systems cannot have a center at the origin. We consider the following three remainder piecewise smooth quartic polynomial differential systems

$$(19) \quad \begin{cases} \dot{x} = c_1 y^4, & \dot{y} = x^3 & \text{if } y \geq 0, \\ \dot{x} = C_1 y^4, & \dot{y} = D_1 x^3 & \text{if } y < 0, \end{cases} \quad \text{with } c_1 C_1 D_1 \neq 0, \quad MWV = (5, 4, 12),$$

$$(20) \quad \begin{cases} \dot{x} = c_2 y^4, & \dot{y} = x & \text{if } y \geq 0, \\ \dot{x} = C_2 y^4, & \dot{y} = D_2 x & \text{if } y < 0, \end{cases} \quad \text{with } c_2 C_2 D_2 \neq 0, \quad MWV = (5, 2, 4),$$

and

$$(21) \quad \begin{cases} \dot{x} = c_3 y^4, & \dot{y} = x^2 & \text{if } y \geq 0, \\ \dot{x} = C_3 y^4, & \dot{y} = D_2 x^2 & \text{if } y < 0, \end{cases} \quad \text{with } c_3 C_3 D_3 \neq 0, \quad MWV = (5, 3, 8).$$

After time rescalings and linear transformations  $(x, y, dt) \rightarrow (x, y, d_1 dt)$ ,  $(x, y, dt) \rightarrow (x, y, d_2 dt)$  and  $(x, y, dt) \rightarrow ((d_3 c_3^2)^{1/3} x / c_3, y, d_3 dt / c_1^2)$ , respectively.

**Theorem 5.** *Only two classes of piecewise smooth quasi-homogeneous quartic polynomial differential systems have a center at the origin. In particular reduced piecewise smooth quasi-homogeneous quartic polynomial differential system (19) has a center at the origin if and only if  $D_1 > 0$ ,  $c_1 < 0$  and  $C_1 > 0$ . Reduced piecewise*

smooth quartic quasi-homogeneous system (20) has a center at the origin if and only if  $D_2 > 0$ ,  $c_2 < 0$  and  $C_2 > 0$ . Both centers of system (19) are global and the period function of periodic orbits is monotonic.

*Proof.* Obviously we only need to consider reduced systems (19)-(21) for the existence of centers at the origin. For system (21) we compute

$$G_3^+(\theta) = s_1 \dot{y} \cos \theta - s_2 \dot{x} \sin \theta = 5 \cos^3 \theta - 3 \sin^5 \theta,$$

according to (8). It follows that  $G_3^+(\theta)$  has real roots  $\theta \approx 0.9394265760$  and  $\theta \approx 2.201353768$  for  $\theta \in [0, \pi]$ . Therefore, by Theorem 1 the origin of system (21) cannot be a center.

From above mentioned analysis of sliding sets and singular sliding points and Theorem 1, on the discontinuous line  $\mathcal{L}$  systems (19) (or (20)) have a center at the origin only if  $D_1 > 0$  (or  $D_2 > 0$ ).

For system (19) and  $y \geq 0$  we compute

$$\eta_1^+(0, 1) := s_1 x \dot{y} - s_2 y \dot{x}|_{(x,y)=(0,1)} = -4c_1 \neq 0$$

and

$$\eta_1^+(1, \lambda) = s_1 x \dot{y} - s_2 y \dot{x}|_{(x,y)=(1,\lambda)} = -4c_1 y^5 + 5.$$

Clearly,  $\eta_1^+(0, 1) \neq 0$ . Besides  $\eta_1^+(1, \lambda)$  only has negative zeros if and only if  $c_1 < 0$ . For system (19) and  $y < 0$  we compute

$$\eta_1^-(0, 1) := s_1 x \dot{y} - s_2 y \dot{x}|_{(x,y)=(0,1)} = -4C_1 \neq 0$$

and

$$\eta_1^-(1, \lambda) := s_1 x \dot{y} - s_2 y \dot{x}|_{(x,y)=(1,\lambda)} = -4C_1 y^5 + 5D_1.$$

Clearly  $\eta_1^-(1, \lambda)$  only has positive zeros if and only if  $C_1 D_1 > 0$ . From Theorem 2 if  $\eta_1^+(1, \lambda)$  has a non-negative zero or  $\eta_1^-(1, \lambda)$  has a non-positive zero, there is an invariant curve passing through the origin of system (19) and the origin cannot be a center.

Note that system (19) has the first integral

$$\phi_1^+(x, y) = 4c_1 y^5 - 5x^4$$

if  $y \geq 0$ , and the first integral

$$\phi_1^-(x, y) = 4C_1 y^5 - 5D_1 x^4$$

if  $y < 0$ . Consequently we have

$$\phi_1^+(x, 0) = -5x^4$$

and

$$\phi_1^-(x, 0) = -5D_1 x^4,$$

yielding that  $\phi_1^+(r_0, 0) = \phi_1^+(r_1, 0)$  and  $\phi_1^-(r_0, 0) = \phi_1^-(r_1, 0)$  for arbitrary  $r_1 = -r_0 < 0$ . Therefore system (19) has a center at the origin if and only if  $D_1 > 0$ ,  $c_1 < 0$  and  $C_1 > 0$ , and this center is global.

In the case  $D_1 > 0$ ,  $c_1 < 0$  and  $C_1 > 0$  we prove that the center  $O_1$  at the origin of system (19) is not isochronous and the period function of its periodic orbits is monotonic. Assuming that  $\Gamma_{r_0}$  is the closed trajectory through  $(r_0, 0)$  inside the

periodic annulus of the center  $O_1$ , we can define the positive half-period function as  $T_1^+(r_0) := \int_{\Gamma_{r_0}^+} dt$  and the negative half-period function as  $T_1^-(r_0) := \int_{\Gamma_{r_0}^-} dt$ , with  $r_0 > 0$ . Thus we have

$$\Gamma_{r_0}^+ = \left\{ (\rho, \theta) \in \mathbb{R}^2 \mid \rho = p^+(r_0, \theta) \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid y = \left( \frac{5}{4c_1}(x^4 - r_0^4) \right)^{\frac{1}{5}} \right\}$$

and

$$\Gamma_{r_0}^- = \left\{ (\rho, \theta) \in \mathbb{R}^2 \mid \rho = p^-(r_0, \theta) \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid y = \left( \frac{5D_1}{4C_1}(x^4 - r_0^4) \right)^{\frac{1}{5}} \right\}.$$

Then the complete period function associated to system (19) is given by the sum of these two functions

$$\begin{aligned} T_1(r_0) &= \oint_{\Gamma_{r_0}} dt = T_1^+(r_0) + T_1^-(r_0) = \int_{\Gamma_{r_0}^+} \frac{dx}{c_1 y^4} + \int_{\Gamma_{r_0}^-} \frac{dx}{C_1 y^4} \\ &= \int_{r_0}^{-r_0} \frac{dx}{c_1 \left( \frac{5}{4c_1}(x^4 - r_0^4) \right)^{\frac{4}{5}}} + \int_{-r_0}^{r_0} \frac{dx}{C_1 \left( \frac{5D_1}{4C_1}(x^4 - r_0^4) \right)^{\frac{4}{5}}} \\ &= \int_1^{-1} \frac{dx}{c_1 \left( \frac{5}{4c_1} \right)^{\frac{4}{5}} r_0^{\frac{11}{5}} (x^4 - 1)^{\frac{4}{5}}} + \int_{-1}^1 \frac{dx}{C_1 \left( \frac{5D_1}{4C_1} \right)^{\frac{4}{5}} r_0^{\frac{11}{5}} (x^4 - 1)^{\frac{4}{5}}} \\ &= \beta_1 r_0^{-\frac{11}{5}}, \end{aligned}$$

where

$$\beta_1 = \frac{\sqrt{2}\pi \csc\left(\frac{\pi}{5}\right) \sin\left(\frac{9\pi}{20}\right) \Gamma\left(\frac{11}{20}\right)}{2\Gamma\left(\frac{4}{5}\right)\Gamma\left(\frac{3}{4}\right)} \left( \frac{1}{C_1 \left( \frac{5D_1}{4C_1} \right)^{\frac{4}{5}}} - \frac{1}{c_1 \left( \frac{5}{4c_1} \right)^{\frac{4}{5}}} \right) > 0,$$

where  $\Gamma(z) = \int_0^\infty e^{-s} s^{z-1} ds$  is the Gamma function. Clearly the period  $T_1(r_0)$  of the periodic orbits inside the period annulus of the center  $O_1$  is monotonic in  $r_0$  and then it cannot be isochronous.

For system (20) we compute

$$\begin{aligned} \eta_2^-(0, 1) &:= s_1 x \dot{y} - s_2 y \dot{x}|_{(x,y)=(0,1)} = -2c_2 \neq 0, \\ \eta_2^-(1, \lambda) &:= s_1 x \dot{y} - s_2 y \dot{x}|_{(x,y)=(1,\lambda)} = 5 - 2c_2 y^5, \end{aligned}$$

if  $y \geq 0$ , and

$$\begin{aligned} \eta_2^-(0, 1) &:= s_1 x \dot{y} - s_2 y \dot{x}|_{(x,y)=(0,1)} = -2C_2 \neq 0, \\ \eta_2^-(1, \lambda) &:= s_1 x \dot{y} - s_2 y \dot{x}|_{(x,y)=(1,\lambda)} = 5D_2 - 2C_2 y^5, \end{aligned}$$

if  $y < 0$ . Using a similar analysis as the one done for system (19), we have that  $\eta_2^+(1, \lambda)$  only has negative zeros if and only if  $c_2 < 0$ , and  $\eta_2^-(1, \lambda)$  only has positive zeros if and only if  $C_2 D_2 > 0$ . Moreover system (19) has the first integral  $\phi_2^+(x, y) = 2c_2 y^5 - 5x^2$  if  $y \geq 0$ , and the first integral  $\phi_2^-(x, y) = 2C_2 y^5 - 5D_2 x^2$  if  $y < 0$ , satisfying that  $\phi_2^+(x, 0) = -5x^2$  and  $\phi_2^-(x, 0) = -5D_2 x^2$ , yielding that  $\phi_2^+(r_0, 0) = \phi_2^+(r_1, 0)$  and  $\phi_2^-(r_0, 0) = \phi_2^-(r_1, 0)$  for arbitrary  $r_1 = -r_0 < 0$ . Therefore system (20) has a global center at the origin if and only if  $D_2 > 0$ ,  $c_2 < 0$  and  $C_2 > 0$ .

In the case  $D_2 > 0$ ,  $c_2 < 0$  and  $C_2 > 0$  we assume that  $\zeta_{r_0}$  is the closed trajectory through  $(r_0, 0)$  inside the periodic annulus of the center  $O_2$  at the origin of system

(20), then we define the positive half-period function as  $T_2^+(r_0) := \int_{\zeta_{r_0}^+} dt$  and the negative half-period function as  $T_2^-(r_0) := \int_{\zeta_{r_0}^-} dt$  with  $r_0 > 0$ . Then we have

$$\zeta_{r_0}^+ = \left\{ (\rho, \theta) \in \mathbb{R}^2 \mid \rho = p^+(r_0, \theta) \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid y = \left( \frac{5}{2c_2} (x^2 - r_0^2) \right)^{\frac{1}{5}} \right\}$$

and

$$\zeta_{r_0}^- = \left\{ (\rho, \theta) \in \mathbb{R}^2 \mid \rho = p^-(r_0, \theta) \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid y = \left( \frac{5D_2}{2C_2} (x^2 - r_0^2) \right)^{\frac{1}{5}} \right\}.$$

Therefore the complete period function associated to system (20) is given by the sum of these two functions

$$\begin{aligned} T_2(r_0) &= \oint_{\zeta_{r_0}} dt = T_2^+(r_0) + T_2^-(r_0) = \int_{\zeta_{r_0}^+} \frac{dx}{c_2 y^4} + \int_{\zeta_{r_0}^-} \frac{dx}{C_2 y^4} \\ &= \int_{r_0}^{-r_0} \frac{dx}{c_2 \left( \frac{5}{2c_2} (x^2 - r_0^2) \right)^{\frac{4}{5}}} + \int_{-r_0}^{r_0} \frac{dx}{C_2 \left( \frac{5D_2}{2C_2} (x^2 - r_0^2) \right)^{\frac{4}{5}}} \\ &= \int_1^{-1} \frac{dx}{c_2 \left( \frac{5}{2c_2} \right)^{\frac{4}{5}} r_0^{\frac{3}{5}} (x^2 - 1)^{\frac{4}{5}}} + \int_{-1}^1 \frac{dx}{C_2 \left( \frac{5D_2}{2C_2} \right)^{\frac{4}{5}} r_0^{\frac{3}{5}} (x^2 - 1)^{\frac{4}{5}}} \\ &= \beta_2 r_0^{-\frac{3}{5}}, \end{aligned}$$

where

$$\beta_2 = \frac{\pi^{\frac{3}{2}} \csc\left(\frac{\pi}{5}\right)}{\Gamma\left(\frac{4}{5}\right)\Gamma\left(\frac{7}{10}\right)} \left( \frac{1}{C_2 \left( \frac{5D_2}{2C_2} \right)^{\frac{4}{5}}} - \frac{1}{c_2 \left( \frac{5}{2c_2} \right)^{\frac{4}{5}}} \right) > 0.$$

Clearly the period  $T_2(r_0)$  of the periodic orbits inside the period annulus of the center  $O_2$  is monotonic in  $r_0$ . This completes the proof of the theorem.  $\square$

#### ACKNOWLEDGMENTS

The first author is supported by the National Natural Science Foundation of China (No. 11801079).

The second author is supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and MDM-2014-0445, the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

The third author is supported by the National Natural Science Foundations of China (No. 11871041, 11431008).

#### REFERENCES

- [1] A. Algaba, E. Gamero, C. García, The integrability problem for a class of planar systems, *Nonlinearity* **22** (2009), 396–420.
- [2] A. Andronov, A. Vitt, S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.
- [3] W. Aziz, J. Llibre, C. Pantazi, Centers of quasi-homogeneous polynomial differential equations of degree three, *Adv. Math.* **254** (2014), 233–250.

- [4] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, *Piecewise-smooth Dynamical Systems: Theory and Applications*, Springer-Verlag, London, 2008.
- [5] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, A. Nordmark, G. Tost, P. Piiroinen, Bifurcations in nonsmooth dynamical systems, *SIAM Review* **50** (2008), 629-701.
- [6] X. Cen, C. Liu, L. Yang, M. Zhang, Limit cycles by perturbing quadratic isochronous centers inside piecewise polynomial differential systems, *J. Differential Equations* **265** (2018), 6083–6126.
- [7] H. Chen, S. Duan, Y. Tang, J. Xie, Global dynamics of a mechanical system with dry friction, *J. Differential Equations* **265** (2018), 5490–5519.
- [8] H. Chen, Y. Tang, At most two limit cycles in a piecewise linear differential system with three zones and asymmetry, *Physica D*, <https://doi.org/10.1016/j.physd.2018.08.004>.
- [9] H. Dulac, Détermination et intégration d'une certaine classe d'équations différentielles ayant par point singulier un centre, *Bull. Sci. Math. Sér. (2)* **32** (1908), 230–252.
- [10] F. Dumortier, J. Llibre, J.C. Artés, *Qualitative theory of planar differential systems*, Springer-Verlag, Berlin, 2006.
- [11] A.F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic, Dordrecht, 1988.
- [12] E. Freire, E. Ponce, F. Torres, Canonical discontinuous planar piecewise linear system, *SIAM J. Appl. Dyn. Syst.* **11** (2012), 181-211.
- [13] I. Garcia, H. Giacomini, J. Giné, J. Llibre, Analytic nilpotent centers as limits of nondegenerate centers revisited, *J. Math. Anal. and Appl.* **441** (2016), 893–899.
- [14] B. García, J. Llibre, J.S. Pérez del Río, Planar quasihomogeneous polynomial differential systems and their integrability, *J. Differential Equations* **255** (2013), 3185–3204.
- [15] J. Giné, On the centers of planar analytic differential systems, *Int. J. Bifurcation and Chaos* **17** (2007), 3061–3070.
- [16] J. Giné, M. Grau, J. Llibre, Limit cycles bifurcating from planar polynomial quasihomogeneous centers, *J. Differential Equations* **259** (2015), 7135–7160.
- [17] J. Giné, J. Llibre, A method for characterizing nilpotent centers, *J. Math. Anal. Appl.* **413** (2014), 537–545.
- [18] R. Moussu, Symétrie et forme normale des centres et foyers dégénérés, *Ergodic Theory Dynam. Systems* **2** (1982), 241–251.
- [19] W. Li, J. Llibre, J. Yang, Z. Zhang, Limit cycles bifurcating from the period annulus of quasi-homogeneous centers, *J. Dyn. Diff. Equat.* **21** (2009), 133–152.
- [20] H. Liang, J. Huang, Y. Zhao, Classification of global phase portraits of planar quartic quasihomogeneous polynomial differential systems, *Nonlinear Dynam.* **78** (2014), 1659–1681.
- [21] M. A. Lyapunov, *Problème Général de la Stabilité du Mouvement*, Ann. of Math. Stud., Vol. 17, Princeton University Press, 1947
- [22] M. Kunze, *Non-Smooth Dynamical Systems*, Springer-Verlag, Berlin-Heidelberg, 2000.
- [23] Yu.A. Kuznetsov, S. Rinaldi, A. Gragnani, One-parameter bifurcations in planar Filippov systems, *Internat. J. Bifur. Chaos* **13** (2003), 2157-2188.
- [24] J. Llibre, Centers: their integrability and relations with the divergence, *Appl. Math. and Nonl. Sci.* **1** (2016), 79–86.
- [25] O. Makarenkov, J.S.W. Lamb, Dynamics and bifurcations of nonsmooth systems: A survey, *Physica D* **241** (2012), 1826–1844.
- [26] L. Mazzi, M. Sabatini, A characterization of centres via first integrals, *J. Differential Equations* **76** (1988), 222–237.
- [27] R. Oliveira, Y. Zhao, Structural stability of planar quasihomogeneous vector fields, *Qual. Theory Dyn. Syst.*, **13** (2014), 39–72.
- [28] J.M. Pearson, N.G. Lloyd, C.J. Christopher, Algorithmic derivation of centre conditions, *SIAM Rev.* **38** (1996), 619–636.
- [29] H. Poincaré, Mémoire sur les courbes définies par les équations différentielles, *J. de Mathématiques* **37** (1881), 375–422; *Oeuvres de Henri Poincaré*, vol. I, Gauthier-Villars, Paris, 1951, pp 3–84.



- [30] H. Poincaré, Sur l'intégration des équations différentielles du premier ordre et du premier degré I and II, *Rendiconti del Circolo Matematico di Palermo* **5** (1891), 161–191; **11** (1897), 193–239.
- [31] G. Sansone, R. Conti, *Non-Linear Differential Equations*, Pergamon Press, New York, 1964.
- [32] Y. Tang, L. Wang, X. Zhang, Center of planar quintic quasi-homogeneous polynomial differential systems, *Discrete Contin. Dyn. Syst.* **35** (2015), 2177–2191.
- [33] Y. Tang, L. Wang, X. Zhang, Global dynamics of planar quasi-homogeneous differential systems, preprint.
- [34] L. Wei, X. Zhang, Limit cycle bifurcations near generalized homoclinic loop in piecewise smooth differential systems, *Discrete Contin. Dyn. Syst.* **36** (2016), 2803–2825.
- [35] J. Yu and L. Zhang, Center of planar quasi-homogeneous polynomial differential systems, *Bull. Sci. Math.* **147** (2018), 7–25.

<sup>a</sup> COLLEGE OF MATHEMATICS AND COMPUTER SCIENCE, FUZHOU UNIVERSITY, FUZHOU, FUJIAN 350108, P. R. CHINA

*E-mail address:* chen\_hebai@sina.com

<sup>b</sup> KEY LABORATORY OF OPERATIONS RESEARCH AND CONTROL OF UNIVERSITY IN FUJIAN, FUZHOU UNIVERSITY, FUZHOU, FUJIAN 350116, P. R. CHINA

<sup>c</sup> DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BEL-LATERRA, BARCELONA, CATALONIA, SPAIN

*E-mail address:* jllibre@mat.uab.cat

<sup>d</sup> SCHOOL OF MATHEMATICAL SCIENCES, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI, 200240, P.R. CHINA

*E-mail address:* mathtyl@sjtu.edu.cn