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CYCLICITY OF $(1,3)$-SWITCHING FF TYPE EQUILIBRIA

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Abstract. Hilbert's 16th Problem suggests a concern to the cyclicity of planar polynomial differential systems, but it is known that a key step to the answer is finding the cyclicity of center-focus equilibria of polynomial differential systems (even of order 2 or 3). Correspondingly, the same question for polynomial discontinuous differential systems is also interesting. Recently, it was proved that the cyclicity of $(1,2)$-switching FF type equilibria is at least 5. In this paper we prove that the cyclicity of $(1,3)$-switching FF type equilibria with homogeneous cubic nonlinearities is at least 3.

1. Introduction and the main result

A differential system of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where the dot denotes derivative with respect to an independent variable t, and P and Q are both polynomials in the real variables x and y, is called a *polynomial differential system* on the plane $\mathbb{R}^2$. The maximum of the degrees of P and Q is referred to the *degree* of system (1). Thus, a planar polynomial differential system of degree one is a *linear differential system*, and a planar polynomial differential system of degree two (or three) is called a *quadratic* (or *cubic*) *differential system*.

A periodic orbit of a differential system which is isolated in the set of all periodic orbits of the system is called a *limit cycle*, which is one of main topics in the qualitative theory of differential equations in the plane (see [5, 10, 17, 19]). The rise of limit cycles near an equilibrium caused by the changes of its stability is called *Hopf bifurcation* (see [15]). The *cyclicity* of that equilibrium is the maximum number of limit cycles which can be bifurcated from that equilibrium with Hopf bifurcation in a given family of differential systems. Usually, we also call it *Hopf cyclicity*. Clearly, the cyclicity of linear differential systems is 0 because they do not have limit cycles. Bautin [1] proved that the cyclicity of a center-focus equilibrium in quadratic systems is 3. It is proved in [26] that the cyclicity of a center-focus equilibrium in cubic systems is at least 12, but the exact cyclicity of a center-focus equilibrium is still unknown for general polynomial differential systems of degree ≥ 3.

In contrast to the above continuous differential systems, in recent decades switching differential systems became attractive because nonsmooth dynamics and sliding

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mode control were considered in mechanics, electronics and economics as shown in books by Bernardo *et al* [2], Filippov [12], Kunze [18] and Simpson [25], the survey [23] by Makarenkov and Lamb, and references therein. Accordingly, the cyclicity of center-focus equilibria is also interesting for discontinuous differential systems but some new difficulties are involved (see [8, 11, 16]) because the well-known fact that the coefficient of even order is generated by coefficents of lower orders in the polynomial ring is not true again and therefore all coefficients in the displacement function are needed.

A general form of switching polynomial differential systems is the following

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \mathcal{X}_+(x,y) & \text{if } y > 0, \\ \mathcal{X}_-(x,y) & \text{if } y < 0. \end{cases} \tag{2}$$

This is a special class of discontinuous differential systems on $\mathbb{R}^2$ with a single switching line $y = 0$. We are interested in the case that the origin $O : (0,0)$ is an FF type equilibrium (see [11]), i.e., both systems $(\dot{x},\dot{y}) = \mathcal{X}_+(x,y)$ and $(\dot{x},\dot{y}) = \mathcal{X}_-(x,y)$ have a center-focus equilibrium at O. Since the transformation $y \to -y$ in (2) does not change the number of limit cycles, for convenience we simply call (2) an (m,n)-*switching differential system* if both $\mathcal{X}_+$ and $\mathcal{X}_-$ are of polynomial forms and

$$m = \min\{\deg(\mathcal{X}_+), \deg(\mathcal{X}_-)\}, \ n = \max\{\deg(\mathcal{X}_+), \deg(\mathcal{X}_-)\}.$$

It is known from [22] that the cyclicity of $(1,1)$-switching FF type equilibria is 2. For $(2,2)$-switching FF type equilibria, the cyclicity was proved to be at least 5 and 9 in the case of weak focus and the case of center respectively in [6, 8]. In 2001 Coll, Gasull and Prohens [11] considered a special $(1,2)$-switching FF type equilibrium, which appears in (2) with $\mathcal{X}_+(x,y) := (\lambda + i)z + p_{20}z^2 + p_{11}z\bar{z} + p_{02}\bar{z}^2$ and $\mathcal{X}_-(x,y) := iz$, where $z = x + i\,y$,

$$p_{20} = -\frac{11}{6} + 2b\,\pi - \frac{15}{32}c + \frac{6-3a}{8}i, \ p_{11} = \frac{11}{12} + \frac{5}{8}c - 4b\,\pi + i,$$
$$p_{02} = \frac{37}{48} + 2b\,\pi - \frac{5}{32}c - \frac{6-3a}{8}i,$$

and $\lambda, a, b, c \in \mathbb{R}$, and showed that the Hopf cyclicity is at least 4. In 2003 Gasull and Torregrosa in [13] generally considered the $(1,2)$-switching FF type equilibrium O of system (2) with

$$\mathcal{X}_+(x,y) := \begin{pmatrix} w_1x - y + x^2 + (\alpha + 7/5)xy + p_{02}y^2 \\ x + w_1y + x^2 + q_{11}xy + q_{02}y^2 \end{pmatrix}, \ \mathcal{X}_-(x,y) := \begin{pmatrix} -y \\ x \end{pmatrix},$$

where

$$p_{02} = -\frac{17}{50} + \frac{3}{20}\alpha - \frac{99}{40}w_2 + \frac{32}{25}w_5 + \frac{16}{5}\alpha w_5 + \frac{3}{2}w_4 - \frac{3}{2}\alpha w_2 + 24w_2w_5 - 8w_3,$$
$$q_{11} = \frac{13}{10} + 2\alpha - 32w_3, \ q_{02} = -\frac{6}{5} - \frac{1}{2}\alpha + \frac{3}{4}w_2,$$

and proved by using the Liapunov constants that the Hopf cyclicity is at least 5, Recently, the authors in [7] applied the averaging method up to 6-th order to the more general $(1,2)$–switching FF type equilibrium of system (2) with

$$\mathcal{X}_+(x,y) := \begin{pmatrix} \lambda_1x - y - \lambda_3x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6y^2 \\ x + \lambda_1y + \lambda_2x^2 + (2\lambda_3 + \lambda_4)xy + (\lambda_7 - \lambda_2)y^2 \end{pmatrix}, \ \mathcal{X}_-(x,y) := \begin{pmatrix} -y \\ x \end{pmatrix},$$

where

$$\lambda_1 = \sum_{j=1}^{7} \lambda_{1j}\epsilon^j,\ \lambda_i = \sum_{j=0}^{7} \lambda_{ij}\epsilon^j, \quad i = 2, ..., 7,$$

and also obtained that the Hopf cyclicity is at least 5.

In this paper, as an effort to higher degree, we further investigate the cyclicity problem for the system (1.2) with

$$\mathcal{X}_+(x, y) := \begin{pmatrix} \lambda_1 x - y + \lambda_2 x^3 + \lambda_3 x^2 y + \lambda_4 xy^2 + \lambda_5 y^3 \\ x + \lambda_1 y + \lambda_6 x^3 + \lambda_7 x^2 y + \lambda_8 xy^2 + \lambda_9 y^3 \end{pmatrix},\ \mathcal{X}_-(x, y) := \begin{pmatrix} -y \\ x \end{pmatrix}.$$

It is a $(1, 3)$-switching system (2) with homogeneous cubic nonlinear terms and obviously has an FF type equilibrium at O. For convenience we use HC(1,3) to label this system. We will prove the following results.

Theorem 1. *The averaging method up to k-th order provides an upper bound of cyclicity of system* HC(1,3) *at O to be at most k.*

Theorem 2. *The averaging method up to* 6*-th order provides a lower bound of cyclicity of system* HC(1,3) *at O to be at least 3. The method up to* 7*-th order does not increase the bound.*

Theorems 1 and 2, providing an upper bound and a lower bound respectively, will be proved in sections 2 and 3 respectively. Remark that the result of lower bound, given in Theorem 2 for special cubic systems, actually implies a general result: the cyclicity of a general (1,3)-switching FF type equilibrium is at least 3.

2. Prelimilaries and proof of Theorem 1

In polar coordinates $(r, \theta) \in \mathbb{R} \times \mathbb{S}^1$, where $x = r\cos\theta$ and $y = r\sin\theta$, system HC(1,3) can be rewritten as

$$\begin{pmatrix} \dot r \\ \dot\theta \end{pmatrix} = \begin{cases} \mathcal{Y}_+(r, \theta, \lambda) & \text{if } \theta \in [0, \pi], \\ \mathcal{Y}_-(r, \theta, \lambda) & \text{if } \theta \in [\pi, 2\pi], \end{cases} \tag{3}$$

where $\lambda = (\lambda_1, ..., \lambda_9) \in \mathbb{R}^9$, $\mathcal{Y}_-(r, \theta, \lambda) = (0, 1)^T$, $\mathcal{Y}_+(r, \theta, \lambda) = (\mathcal{H}(r, \theta, \lambda), \mathcal{G}(r, \theta, \lambda))^T$, and

$$\begin{aligned} \mathcal{H}(r, \theta, \lambda) = \ & \lambda_1 r + r^3(\lambda_2\cos^4\theta + (\lambda_3 + \lambda_6)\cos^3\theta\sin\theta + (\lambda_4 + \lambda_7)\cos^2\theta\sin^2\theta \\ & + (\lambda_5 + \lambda_8)\cos\theta\sin^3\theta + \lambda_9\sin^4\theta), \\ \mathcal{G}(r, \theta, \lambda) = \ & 1 + r^2(\lambda_6\cos^4\theta - (\lambda_2 - \lambda_7)\cos^3\theta\sin\theta \\ & - (\lambda_3 - \lambda_8)\cos^2\theta\sin^2\theta - (\lambda_4 - \lambda_9)\cos\theta\sin^3\theta - \lambda_5\sin^4\theta). \end{aligned}$$

Since we want to study the Hopf bifurcation at the origin of an FF-type equilibrium of system (3), introduce a parameter ε, make the rescaling $r \to r\varepsilon$, and express λ_i s in the form

$$\lambda_i := \sum_{j=1}^{9} \lambda_{ij}\varepsilon^j, \quad i = 1, ..., 9,$$

with $\lambda_{10} = 0$. Then system (3) reduces to the differential equation

$$\frac{dr}{d\theta} = \begin{cases} \dfrac{\mathcal{H}(r,\theta,\lambda,\varepsilon)}{\mathcal{G}(r,\theta,\lambda,\varepsilon)} = \displaystyle\sum_{i=1}^{\infty} \varepsilon^i F_i(\theta, r, \lambda) & \text{if } \theta \in [0, \pi], \\ 0 & \text{if } \theta \in (\pi, 2\pi], \end{cases} \tag{4}$$

where each F_i is a polynomial in the variables r, $\sin\theta$, $\cos\theta$ and λ. Then the problem on the cyclicity of the Hopf bifurcation at the origin of system (3) becomes the problem on the cyclicity of the Hopf bifurcation at the origin of the differential system (4), which is written in the normal form for applying the averaging theory of arbitrary order for studying its periodic orbits.

The classical averaging theory (see for instance [3, 4, 24]) has been extended recently for computing periodic orbits of analytical differential equations of one variable with arbitrary order in the small parameter ε by Giné, Grau and Llibre [14]. Later on this theory was extended by Llibre, Novaes and Teixeira [20] to arbitrary order in ε for continuous differential systems in n variables. Using Llibre, Novaes and Teixeira's arguments (see [21]) the formulas obtained for the averaged functions of arbitrary order in [14] or [20] also work for the discontinuous differential systems.

Let $\varphi(\cdot, z) : [0, t_z] \to \mathbb{R}^n$ [1] be the solution of the unperturbed differential equation $r'(\theta) = 0$ such that $\varphi(0, z) = z$. Then $\varphi(\theta, z) \equiv z$. Using the notations and results of [20, 21], we see that the *averaged function* $f_i : (0, \infty) \to \mathbb{R}$ of order $i = 1, 2, \ldots, k$ is given by

$$f_i(z) = \frac{w_i(\pi, z)}{i!}, \tag{5}$$

where functions $w_i : \mathbb{R} \times (0, \infty) \to \mathbb{R}$, $i = 1, 2, \ldots, k$, are defined recurrently by

$$w_i(t, z) = i! \int_0^t \Bigg(F_i\left(s, \varphi(s, z)\right) + \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \frac{\partial^L}{\partial r^L} F_{i-l}\left(s, \varphi(s, z)\right) \prod_{j=1}^{l} w_j(s, z)^{b_j} \Bigg) ds,$$

where $L = b_1 + b_2 + \cdots + b_l$ and S_l is the set of all l-tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$. Observe that in (5) we have $w_i(\pi, z)$ instead of $w_i(2\pi, z)$ this is due to the fact that the contribution of the linear vector field on the half-plane $y < 0$ to the averaged function is zero, see for more details [21]. In subsection 4.1 of [20] are provided all the expressions of the function $f_i(r)$ for $i = 1, \ldots, 5$.

The result stated in Theorem A of [20] and [21] implies that if the first averaged function $f_1(r)$ is not identically zero, then every positive simple zero of $f_1(r)$ provides a limit cycle of the discontinuous differential equation (4) on the cylinder $C = \{(r, \theta) \in (0, \infty) \times \mathbb{S}^1\}$. So it also provides a limit cycle of the discontinuous differential system HC(1,3) in $\mathbb{R}^2$. Furthermore if $f_1(r)$ is identically zero and $f_2(r)$ is not identically zero, then every positive simple zero of $f_2(r)$ provides a limit cycle of the discontinuous differential equation (4) on the cylinder C, and it also provides a limit cycle of the discontinuous differential system HC(1,3) in $\mathbb{R}^2$. If $f_1(r)$ and

[1] It seems that it should be $\to \mathbb{R}$.

$f_2(r)$ are identically zero and $f_3(r)$ is not identically zero, a similar result we have for the simple zeros of the function $f_3(r)$, and so on. In short, we need to compute the averaged functions $f_i(r)$, and to study the positive simple zeros of the function $f_i(r)$ when the functions $f_k(r)$ are identically zero for $k = 1, \ldots, i-1$.

Proof of Theorem 1. From (4) it follows easily that the functions $F_i(\theta, r, \lambda)$ is a polynomial in the variable r of degree $2i+1$ of the form $rG_i(r^2)$, where the coefficients of the polynomial G_i depend on θ and λ. Therefore looking at the definition of the averaged function of order i–th we obtain that $f_i(r)$ is also a polynomial of the form $rg_i(r^2)$ of degree at most $2i+1$. So the number of positive real roots of the polynomial $f_i(r)$ is at most i. Consequently the function $f_i(r)$ has at most i positive simple zeros, and it provides at most i limit cycles of system HC(1,3), and therefore from the averaging theory described just before this proves the theorem. □

3. Proof of Theorem 2

Using the software Mathematica, we compute

$$\begin{aligned} f_1(r) = {} & \lambda_{11}\pi r, \\ f_2(r) = {} & \lambda_{12}\pi r + \frac{1}{8}(3\lambda_{20} + \lambda_{40} + \lambda_{70} + 3\lambda_{90})\pi r^3, \\ f_3(r) = {} & \lambda_{13}\pi r + \frac{1}{8}(3\lambda_{21} + \lambda_{41} + \lambda_{71} + 3\lambda_{91})\pi r^3, \\ f_4(r) = {} & \lambda_{14}\pi r + \frac{1}{8}(3\lambda_{22} + \lambda_{42} + \lambda_{72} + 3\lambda_{92})\pi r^3 \\ & -\frac{1}{16}(\lambda_{20}\lambda_{30} + 3\lambda_{20}\lambda_{50} + 3\lambda_{20}\lambda_{60} + \lambda_{30}\lambda_{70} + \lambda_{50}\lambda_{70} + \lambda_{60}\lambda_{70} \\ & +\lambda_{20}\lambda_{80} + \lambda_{70}\lambda_{80} + 2\lambda_{30}\lambda_{90} + 2\lambda_{80}\lambda_{90})\pi r^5. \end{aligned} \tag{6}$$

Of course the above expression for the function $f_k(r)$ has been computed when $f_i(r) \equiv 0$ for all $i \le k-1$. Moreover it is easy to find that both $f_{2n}(r)$ and $f_{2n+1}(r)$ are polynomials with only monomials in r of odd degree and the degree of the polynomial $f_{2n+1}(r)$ is less than or equal to $2n+1$. Thus, in order to find ℓ limit cycles, we need to compute averaged functions of order 2ℓ. This is why we can find 5 limit cycles for $(1,2)$-switching system by the procedure of averaging up to order 6 but only get 3 limit cycles for our $(1,3)$-switching system HC(1,3) up to the same order 6.

We denote by $\#Z_+(f_i)$ the cardinal of the set $Z_+(f_i)$ consisting of all positive simple zeros of the averaged function $f_i(r)$.

$\#Z_+(f_1)$	$f_1{\equiv}0$	$\#Z_+(f_2)$	$f_2{\equiv}0$	$\#Z_+(f_3)$	$f_3{\equiv}0$	$\#Z_+(f_4)$	$f_4 \equiv 0$	$\#Z_+(f_5)$
0	$\lambda_{11}{=}0$	1	C_2	1	C_3	2	C_{41}	2
							C_{42}	2
							C_{43}	2

Table 1. Numbers of positive simple zeros of $f_1, \ldots, f_5$

Lemma 3. *The numbers* $\#Z_+(f_i)$ *for* $i = 1, ..., 5$ *are given in Table 1 under the conditions* $\lambda_{11} = 0$*,* C_2, C_3, C_{41}, C_{42} *and* C_{43}*, which are given in* (7), (8), (9), (10) *and* (11)*, respectively.*

Proof. Clearly from the expression of the polynomial $f_1(r)$ given in (6) it follows that it has no positive simple zeros. Note that $f_1(r) \equiv 0$ if and only if $\lambda_{11} = 0$. Furthermore from the expression of the polynomial $f_2(r)$ given in (6) we obtain that $f_2(r)$ has at most 1 positive simple zero, and that there are polynomials $f_2(r)$ with 1 positive simple zero.

Note that $f_2(r) \equiv 0$ if and only if

$$\lambda_{12} = 0, \text{ and } \lambda_{40} = -3\lambda_{20} - \lambda_{70} - 3\lambda_{90}. \tag{7}$$

These two conditions are denoted by C_2 in Table 1.

When (7) holds, we get that the expression of $f_3(r)$ is given in (6). We obtain that $f_3(r)$ has at most 1 positive simple zero, and that there are polynomials $f_3(r)$ with 1 positive simple zero.

Note that $f_3(r) \equiv 0$ if and only if

$$\lambda_{13} = 0, \text{ and } \lambda_{41} = -3\lambda_{21} - \lambda_{71} - 3\lambda_{91}. \tag{8}$$

These two conditions are denoted by C_3 in Table 1.

When (8) holds, we get that the expression of $f_4(r)$ is given in (6). We obtain that $f_4(r)$ has at most 2 positive simple zeros, and that there are polynomials $f_4(r)$ with 2 positive simple zeros.

Note that $f_4(r) \equiv 0$ if and only if either

$$\begin{aligned} &\lambda_{14} = 0, \ \lambda_{42} = -3\lambda_{22} - \lambda_{72} - 3\lambda_{92}, \ 3\lambda_{20} + \lambda_{70} \neq 0, \\ &\lambda_{50} = \frac{-1}{3\lambda_{20} + \lambda_{70}}(\lambda_{20}\lambda_{30} + 3\lambda_{20}\lambda_{60} + \lambda_{30}\lambda_{70} + \lambda_{60}\lambda_{70} \\ &\qquad + \lambda_{20}\lambda_{80} + \lambda_{70}\lambda_{80} + 2\lambda_{30}\lambda_{90} + 2\lambda_{80}\lambda_{90}), \end{aligned} \tag{9}$$

or

$$\begin{aligned} &\lambda_{14} = 0, \ \lambda_{42} = -3\lambda_{22} - \lambda_{72} - 3\lambda_{92}, \\ &\lambda_{20} = -\frac{1}{3}\lambda_{70}, \ \lambda_{30} = -\lambda_{80}, \end{aligned} \tag{10}$$

or

$$\begin{aligned} &\lambda_{14} = 0, \ \lambda_{42} = -3\lambda_{22} - \lambda_{72} - 3\lambda_{92}, \\ &\lambda_{20} = -\frac{1}{3}\lambda_{70}, \ \lambda_{90} = -\frac{1}{3}\lambda_{70}, \end{aligned} \tag{11}$$

denoted by C_{41}, C_{42}, C_{43} in Table 1, respectively.

When (9) holds, we get

$$
\begin{aligned}
f_5(r) =& \lambda_{15}\pi r + \frac{\pi}{8}(3\lambda_{23} + \lambda_{43} + \lambda_{73} + 3\lambda_{93})r^3 \\
& - \frac{\pi}{16(3\lambda_{20} + \lambda_{70})}(3\lambda_{20}^2\lambda_{31} + 9\lambda_{20}^2\lambda_{51} + 9\lambda_{20}^2\lambda_{61} - 6\lambda_{21}\lambda_{80}\lambda_{90} \\
& - 2\lambda_{21}\lambda_{30}\lambda_{70} + 4\lambda_{20}\lambda_{31}\lambda_{70} + 6\lambda_{20}\lambda_{51}\lambda_{70} + 6\lambda_{20}\lambda_{61}\lambda_{70} + \lambda_{31}\lambda_{70}^2 \\
& + \lambda_{51}\lambda_{70}^2 + \lambda_{61}\lambda_{70}^2 + 2\lambda_{20}\lambda_{30}\lambda_{71} - 2\lambda_{21}\lambda_{70}\lambda_{80} + 2\lambda_{70}\lambda_{81}\lambda_{90} \\
& + 2\lambda_{20}\lambda_{71}\lambda_{80} + 3\lambda_{20}^2\lambda_{81} + 4\lambda_{20}\lambda_{70}\lambda_{81} + \lambda_{70}^2\lambda_{81} - 6\lambda_{21}\lambda_{30}\lambda_{90} \\
& + 6\lambda_{20}\lambda_{31}\lambda_{90} + 2\lambda_{31}\lambda_{70}\lambda_{90} - 2\lambda_{30}\lambda_{71}\lambda_{90} - 2\lambda_{71}\lambda_{80}\lambda_{90} + 6\lambda_{20}\lambda_{81}\lambda_{90} \\
& + 6\lambda_{20}\lambda_{30}\lambda_{91} + 2\lambda_{30}\lambda_{70}\lambda_{91} + 6\lambda_{20}\lambda_{80}\lambda_{91} + 2\lambda_{70}\lambda_{80}\lambda_{91})r^5.
\end{aligned} \tag{12}
$$

We obtain that $f_5(r)$ has at most 2 positive simple zeros, and that there are polynomials $f_5(r)$ with 2 positive simple zeros.

Similarly, when (10) (resp. (11)) holds, we get

$$
\begin{aligned}
f_5(r) =& \lambda_{15}\pi r + \frac{\pi}{8}(3\lambda_{23} + \lambda_{43} + \lambda_{73} + 3\lambda_{93})r^3 \\
& - \frac{\pi}{48}(9\lambda_{21}\lambda_{50} + 9\lambda_{21}\lambda_{60} + 2\lambda_{31}\lambda_{70} + 3\lambda_{50}\lambda_{71} \\
& + 3\lambda_{60}\lambda_{71} + 2\lambda_{70}\lambda_{81} + 6\lambda_{31}\lambda_{90} + 6\lambda_{81}\lambda_{90})r^5.
\end{aligned} \tag{13}
$$

(resp.

$$
\begin{aligned}
f_5(r) =& \lambda_{15}\pi r + \frac{\pi}{8}(3\lambda_{23} + \lambda_{43} + \lambda_{73} + 3\lambda_{93})r^3 \\
& - \frac{\pi}{16}(\lambda_{21}\lambda_{30} + 3\lambda_{21}\lambda_{50} + 3\lambda_{21}\lambda_{60} + \lambda_{30}\lambda_{71} + \lambda_{50}\lambda_{71} \\
& + \lambda_{60}\lambda_{71} + \lambda_{21}\lambda_{80} + \lambda_{71}\lambda_{80} + 2\lambda_{30}\lambda_{91} + 2\lambda_{80}\lambda_{91})r^5.)
\end{aligned} \tag{14}
$$

So we obtain that $f_5(r)$ has at most 2 (resp. 2) positive simple zeros, and that there are polynomials $f_5(r)$ with 2 (resp. 2) positive simple zeros. This completes the proof of Table 1. □

condition for $f_4 \equiv 0$	condition for $f_5 \equiv 0$	$\#Z_+(f_6)$
C_{41}	C_{411}	3
C_{42}	C_{421}	2
	C_{422}	2
	C_{423}	2
C_{43}	C_{431}	3
	C_{432}	2
	C_{433}	3

TABLE 2. Number of positive simple zeros of f_6.

By the numbers $\#Z_+(f_i)$ for $i = 1, ..., 5$ given in Lemma 3 we get some lower bounds of the cyclicity of the Hopf bifurcation at the origin of the discontinuous differential system HC(1,3). In order to find a greater bound, we consider averaged functions of higher order.

Lemma 4. *The number* $\#Z_+(f_6)$ *is given in Table 2 under the conditions* C_{411}, C_{421}, C_{422}, C_{423}, C_{431}, C_{432} *and* C_{433}, *which are given in* (15), (16), (17), (18), (19), (20), (21), *respectively.*

Proof. When $\lambda_{11} = 0$ and conditions C_2, C_3, C_{41} hold, $f_1(r) \equiv f_2(r) \equiv f_3(r) \equiv f_4(r) \equiv 0$ and the expression of $f_5(r)$ is given in (12). It is easy to check that $f_5(r) \equiv 0$ if and only if

$$\begin{aligned}
&\lambda_{15} = 0,\ \lambda_{43} = -3\lambda_{23} - \lambda_{73} - 3\lambda_{93},\\
&\lambda_{51} = \frac{1}{(3\lambda_{20}+\lambda_{70})^2}(-3\lambda_{20}^2\lambda_{31} - 9\lambda_{20}^2\lambda_{61} + 2\lambda_{21}\lambda_{30}\lambda_{70}\\
&-4\lambda_{20}\lambda_{31}\lambda_{70} - 6\lambda_{20}\lambda_{61}\lambda_{70} - \lambda_{31}\lambda_{70}^2 - \lambda_{61}\lambda_{70}^2 - 2\lambda_{20}\lambda_{30}\lambda_{71}\\
&+2\lambda_{21}\lambda_{70}\lambda_{80} - 2\lambda_{20}\lambda_{71}\lambda_{80} - 3\lambda_{20}^2\lambda_{81} - 4\lambda_{20}\lambda_{70}\lambda_{81} - \lambda_{70}^2\lambda_{81}\\
&+6\lambda_{21}\lambda_{30}\lambda_{90} - 6\lambda_{20}\lambda_{31}\lambda_{90} - 2\lambda_{31}\lambda_{70}\lambda_{90} + 2\lambda_{30}\lambda_{71}\lambda_{90}\\
&+6\lambda_{21}\lambda_{80}\lambda_{90} + 2\lambda_{71}\lambda_{80}\lambda_{90} - 6\lambda_{20}\lambda_{81}\lambda_{90} - 2\lambda_{70}\lambda_{81}\lambda_{90}\\
&-6\lambda_{20}\lambda_{30}\lambda_{91} - 2\lambda_{30}\lambda_{70}\lambda_{91} - 6\lambda_{20}\lambda_{80}\lambda_{91} - 2\lambda_{70}\lambda_{80}\lambda_{91}),
\end{aligned} \tag{15}$$

denoted by C_{411} in Table 2.

Under C_{411}, we compute $f_6(r)$ and obtain

$$\begin{aligned}
f_6(r) \;=\; & \lambda_{16}\pi r + \frac{\pi}{8}(3\lambda_{24} + \lambda_{44} + \lambda_{74} + 3\lambda 94)r^3\\
&-\frac{\pi}{16(3\lambda_{20}+\lambda_{70})^2}(9\lambda_{20}^3\lambda_{32} + 27\lambda_{20}^3\lambda_{52} + 27\lambda_{20}^3\lambda_{62} + 6\lambda_{21}^2\lambda_{30}\lambda_{70}\\
&-6\lambda_{20}\lambda_{22}\lambda_{30}\lambda_{70} - 6\lambda_{20}\lambda_{21}\lambda_{31}\lambda_{70} + 15\lambda_{20}^2\lambda_{32}\lambda_{70} + 27\lambda_{20}^2\lambda_{52}\lambda_{70}\\
&+27\lambda_{20}^2\lambda_{62}\lambda_{70} - 2\lambda_{22}\lambda_{30}\lambda_{70}^2 - 2\lambda_{21}\lambda_{31}\lambda_{70}^2 + 7\lambda_{20}\lambda_{32}\lambda_{70}^2 + 9\lambda_{20}\lambda_{52}\lambda_{70}^2\\
&+\lambda_{20}\lambda_{62}\lambda_{70}^2 + \lambda_{32}\lambda_{70}^3 + \lambda_{52}\lambda_{70}^3 + \lambda_{62}\lambda_{70}^3 - 6\lambda_{20}\lambda_{21}\lambda_{30}\lambda_{71} + 2\lambda_{70}^2\lambda_{80}\lambda_{92}\\
&+6\lambda_{20}^2\lambda_{31}\lambda_{71} + 2\lambda_{21}\lambda_{30}\lambda_{70}\lambda_{71} + 2\lambda_{20}\lambda_{31}\lambda_{70}\lambda_{71} - 2\lambda_{20}\lambda_{30}\lambda_{71}^2\\
&+6\lambda_{20}^2\lambda_{30}\lambda_{72} + 2\lambda_{20}\lambda_{30}\lambda_{70}\lambda_{72} + 6\lambda_{21}^2\lambda_{70}\lambda_{80} - 6\lambda_{20}\lambda_{22}\lambda_{70}\lambda_{80}\\
&-2\lambda_{22}\lambda_{70}^2\lambda_{80} - 6\lambda_{20}\lambda_{21}\lambda_{71}\lambda_{80} + 2\lambda_{21}\lambda_{70}\lambda_{71}\lambda_{80} - 2\lambda_{20}\lambda_{71}^2\lambda_{80}\\
&+6\lambda_{20}^2\lambda_{72}\lambda_{80} + 2\lambda_{20}\lambda_{70}\lambda_{72}\lambda_{80} - 6\lambda_{20}\lambda_{21}\lambda_{70}\lambda_{81} - 2\lambda_{21}\lambda_{70}^2\lambda_{81}\\
&+6\lambda_{20}^2\lambda_{71}\lambda_{81} + 2\lambda_{20}\lambda_{70}\lambda_{71}\lambda_{81} + 9\lambda_{20}^3\lambda_{82} + 15\lambda_{20}^2\lambda_{70}\lambda_{82}7\lambda_{20}\lambda_{70}^2\lambda_{82}\\
&+\lambda_{70}^3\lambda_{82} + 18\lambda_{21}^2\lambda_{30}\lambda_{90} - 18\lambda_{20}\lambda_{22}\lambda_{30}\lambda_{90} - 18\lambda_{20}\lambda_{21}\lambda_{31}\lambda_{90}\\
&+18\lambda_{20}^2\lambda_{32}\lambda_{90} - 6\lambda_{22}\lambda_{30}\lambda_{70}\lambda_{90} - 6\lambda_{21}\lambda_{31}\lambda_{70}\lambda_{90} + 12\lambda_{20}\lambda_{32}\lambda_{70}\lambda_{90}\\
&+2\lambda_{32}\lambda_{70}^2\lambda_{90} + 12\lambda_{21}\lambda_{30}\lambda_{71}\lambda_{90} - 6\lambda_{20}\lambda_{31}\lambda_{71}\lambda_{90} - 2\lambda_{31}\lambda_{70}\lambda_{71}\lambda_{90}\\
&+2\lambda_{30}\lambda_{71}^2\lambda_{90} - 6\lambda_{20}\lambda_{30}\lambda_{72}\lambda_{90} - 2\lambda_{30}\lambda_{70}\lambda_{72}\lambda_{90} + 18\lambda_{21}^2\lambda_{80}\lambda_{90}\\
&-18\lambda_{20}\lambda_{22}\lambda_{80}\lambda_{90} - 6\lambda_{22}\lambda_{70}\lambda_{80}\lambda_{90} + 12\lambda_{21}\lambda_{71}\lambda_{80}\lambda_{90} + 2\lambda_{71}^2\lambda_{80}\lambda_{90}\\
&-6\lambda_{20}\lambda_{72}\lambda_{80}\lambda_{90} - 2\lambda_{70}\lambda_{72}\lambda_{80}\lambda_{90} - 18\lambda_{20}\lambda_{21}\lambda_{81}\lambda_{90} - 6\lambda_{21}\lambda_{70}\lambda_{81}\lambda_{90}\\
&-6\lambda_{20}\lambda_{71}\lambda_{81}\lambda_{90} - 2\lambda_{70}\lambda_{71}\lambda_{81}\lambda_{90} + 18\lambda_{20}^2\lambda_{82}\lambda_{90} + 12\lambda_{20}\lambda_{70}\lambda_{82}\lambda_{90}\\
&+2\lambda_{70}^2\lambda_{82}\lambda_{90} - 18\lambda_{20}\lambda_{21}\lambda_{30}\lambda_{91} + 18\lambda_{20}^2\lambda_{31}\lambda_{91} - 6\lambda_{21}\lambda_{30}\lambda_{70}\lambda_{91}\\
&+12\lambda_{20}\lambda_{31}\lambda_{70}\lambda_{91} + 2\lambda_{31}\lambda_{70}^2\lambda_{91} - 6\lambda_{20}\lambda_{30}\lambda_{71}\lambda_{91} - 2\lambda_{30}\lambda_{70}\lambda_{71}\lambda_{91}\\
&-18\lambda_{20}\lambda_{21}\lambda_{80}\lambda_{91} - 6\lambda_{21}\lambda_{70}\lambda_{80}\lambda_{91} - 6\lambda_{20}\lambda_{71}\lambda_{80}\lambda_{91} - 2\lambda_{70}\lambda_{71}\lambda_{80}\lambda_{91}\\
&+18\lambda_{20}^2\lambda_{81}\lambda_{91} + 12\lambda_{20}\lambda_{70}\lambda_{81}\lambda_{91} + 2\lambda_{70}^2\lambda_{81}\lambda_{91} + 18\lambda_{20}^2\lambda_{30}\lambda_{92}\\
&+12\lambda_{20}\lambda_{30}\lambda_{70}\lambda_{92} + 2\lambda_{30}\lambda_{70}^2\lambda_{92} + 18\lambda_{20}^2\lambda_{80}\lambda_{92} + 12\lambda_{20}\lambda_{70}\lambda_{80}\lambda_{92})r^5
\end{aligned}$$

$$-\frac{\pi}{64(3\lambda_{20}+\lambda_{70})}(\lambda_{20}+2\lambda_{70}+5\lambda_{90})(9\lambda_{20}^3+\lambda_{20}\lambda_{30}^2+3\lambda_{20}\lambda_{30}\lambda_{60}$$
$$+6\lambda_{20}^2\lambda_{70}+\lambda_{30}^2\lambda_{70}+\lambda_{20}\lambda_{70}^2-\lambda_{20}\lambda_{30}\lambda_{80}+\lambda_{30}\lambda_{60}\lambda_{70}+3\lambda_{20}\lambda_{60}\lambda_{80}$$
$$+\lambda_{30}\lambda_{70}\lambda_{80}+\lambda_{60}\lambda_{70}\lambda_{80}-2\lambda_{20}\lambda_{80}^2+9\lambda_{20}^2\lambda_{90}+6\lambda_{20}\lambda_{70}\lambda_{90}+\lambda_{70}^2\lambda_{90})r^7.$$

We obtain that $f_6(r)$ has at most 3 positive simple zeros, and that there are polynomials $f_6(r)$ with 3 positive simple zeros by the same method used in the proof of Lemma 3.2 of [7].

When $\lambda_{11}=0$ and conditions C_2, C_3, C_{42} hold, $f_1(r)\equiv f_2(r)\equiv f_3(r)\equiv f_4(r)\equiv 0$ and the expression of $f_5(r)$ is given in (13). It is easy to check that $f_5(r)\equiv 0$ if and only if either

$$\begin{aligned}&\lambda_{15}=0,\quad \lambda_{43}=-3\lambda_{23}-\lambda_{73}-3\lambda_{93},\\ &\lambda_{31}=\frac{-1}{2(\lambda_{70}+3\lambda_{90})}(9\lambda_{21}\lambda_{50}+9\lambda_{21}\lambda_{60}\\ &\qquad +3\lambda_{50}\lambda_{71}+3\lambda_{60}\lambda_{71}+2\lambda_{70}\lambda_{81}+6\lambda_{81}\lambda_{90}),\end{aligned} \tag{16}$$

or

$$\begin{aligned}&\lambda_{15}=0,\quad \lambda_{43}=-3\lambda_{23}-\lambda_{73}-3\lambda_{93},\\ &\lambda_{70}=-3\lambda_{90},\quad \lambda_{50}=-\lambda_{60},\end{aligned} \tag{17}$$

or

$$\begin{aligned}&\lambda_{15}=0,\quad \lambda_{43}=-3\lambda_{23}-\lambda_{73}-3\lambda_{93},\\ &\lambda_{70}=-3\lambda_{90},\quad \lambda_{50}=-3\lambda_{21},\end{aligned} \tag{18}$$

denoted by C_{421}, C_{422}, C_{423} in Table 2 respectively.

Under C_{421} we compute $f_6(r)$ and obtain

$$\begin{aligned}f_6(r) = {} & \lambda_{16}\pi r+\frac{\pi}{8}(3\lambda_{24}+\lambda_{44}+\lambda_{74}+3\lambda 94)r^3\\ &-\frac{\pi}{96(\lambda_{70}+3\lambda_{90})}(-27\lambda_{21}^2\lambda_{50}-27\lambda_{21}^2\lambda_{60}+18\lambda_{22}\lambda_{50}\lambda_{70}+18\lambda_{21}\lambda_{51}\lambda_{70}\\ &+18\lambda_{22}\lambda_{60}\lambda_{70}+18\lambda_{21}\lambda_{61}\lambda_{70}+4\lambda_{32}\lambda_{70}^2-36\lambda_{21}\lambda_{50}\lambda_{71}-36\lambda_{21}\lambda_{60}\lambda_{71}\\ &+6\lambda_{51}\lambda_{70}\lambda_{71}+6\lambda_{61}\lambda_{70}\lambda_{71}-9\lambda_{50}\lambda_{71}^2-9\lambda_{60}\lambda_{71}^2+6\lambda_{50}\lambda_{70}\lambda_{72}\\ &+6\lambda_{60}\lambda_{70}\lambda_{72}+4\lambda_{70}^2\lambda_{82}+54\lambda_{22}\lambda_{50}\lambda_{90}+54\lambda_{21}\lambda_{51}\lambda_{90}+54\lambda_{22}\lambda_{60}\lambda_{90}\\ &+54\lambda_{21}\lambda_{61}\lambda_{90}+24\lambda_{32}\lambda_{70}\lambda_{90}+18\lambda_{51}\lambda_{71}\lambda_{90}+18\lambda_{61}\lambda_{71}\lambda_{90}+18\lambda_{50}\lambda_{72}\lambda_{90}\\ &+18\lambda_{60}\lambda_{72}\lambda_{90}+24\lambda_{70}\lambda_{82}\lambda_{90}+36\lambda_{32}\lambda_{90}^2+36\lambda_{82}\lambda_{90}^2-54\lambda_{21}\lambda_{50}\lambda_{91}\\ &-54\lambda_{21}\lambda_{60}\lambda_{91}-18\lambda_{50}\lambda_{71}\lambda_{91}-18\lambda_{60}\lambda_{71}\lambda_{91})r^5.\end{aligned}$$

We obtain that $f_6(r)$ has at most 2 positive simple zeros, and that there are polynomials $f_6(r)$ with 2 positive simple zeros by the same method used in the proof of Lemma 3.2 of [7].

Under C_{422} we compute $f_6(r)$ and obtain

$$\begin{aligned}f_6(r) = {} & \lambda_{16}\pi r+\frac{\pi}{8}(3\lambda_{24}+\lambda_{44}+\lambda_{74}+3\lambda 94)r^3\\ &-\frac{\pi}{16}(\lambda_{21}\lambda_{31}+3\lambda_{21}\lambda_{51}+3\lambda_{21}\lambda_{61}+\lambda_{31}\lambda_{71}+\lambda_{51}\lambda_{71}+\lambda_{61}\lambda_{71}+\lambda_{21}\lambda_{81}\\ &+\lambda_{71}\lambda_{81}+2\lambda_{31}\lambda_{91}+2\lambda_{81}\lambda_{91})r^5.\end{aligned}$$

We obtain that $f_6(r)$ has at most 2 positive simple zeros, and that there are polynomials $f_6(r)$ with 2 positive simple zeros by the same method used in the proof of Lemma 3.2 of [7].

Under C_{423} we compute $f_6(r)$ and get

$$\begin{aligned} f_6(r) &= \lambda_{16}\pi r + \frac{\pi}{8}(3\lambda_{24} + \lambda_{44} + \lambda_{74} + 3\lambda 94)r^3 \\ &\quad + \frac{\pi}{16}(2\lambda_{21}\lambda_{31} - 3\lambda_{22}\lambda_{50} - 3\lambda_{22}\lambda_{60} - \lambda_{50}\lambda_{72} - \lambda_{60}\lambda_{72} \\ &\quad + 2\lambda_{21}\lambda_{81} - 2\lambda_{31}\lambda_{91} - 2\lambda_{81}\lambda_{91})r^5. \end{aligned}$$

We obtain that $f_6(r)$ has at most 2 positive simple zeros, and that there are polynomials $f_6(r)$ with 2 positive simple zeros by the same method used in the proof of Lemma 3.2 of [7].

When $\lambda_{11} = 0$ and conditions C_2, C_3, C_{43} hold, $f_1(r) \equiv f_2(r) \equiv f_3(r) \equiv f_4(r) \equiv 0$ and the expression of $f_5(r)$ is given in (14). It is easy to check that $f_5(r) \equiv 0$ if and only if either

$$\begin{aligned} &\lambda_{15} = 0, \quad \lambda_{43} = -3\lambda_{23} - \lambda_{73} - 3\lambda_{93}, \\ &\lambda_{50} = \frac{-1}{3\lambda_{21} + \lambda_{71})}(\lambda_{21}\lambda_{30} + 3\lambda_{21}\lambda_{60} + \lambda_{30}\lambda_{71} \\ &\qquad + \lambda_{60}\lambda_{71} + \lambda_{21}\lambda_{80} + \lambda_{71}\lambda_{80} + 2\lambda_{30}\lambda_{91} + 2\lambda_{80}\lambda_{91}), \end{aligned} \tag{19}$$

or

$$\begin{aligned} &\lambda_{15} = 0, \quad \lambda_{43} = -3\lambda_{23} - \lambda_{73} - 3\lambda_{93}, \\ &\lambda_{30} = -\lambda_{80}, \quad \lambda_{71} = -3\lambda_{21} \end{aligned} \tag{20}$$

or

$$\begin{aligned} &\lambda_{15} = 0, \quad \lambda_{43} = -3\lambda_{23} - \lambda_{73} - 3\lambda_{93}, \\ &\lambda_{21} = \lambda_{91}, \quad \lambda_{71} = -3\lambda_{21}, \end{aligned} \tag{21}$$

denoted by C_{431}, C_{432}, C_{433} in Table 2 respectively.

Under C_{431} we compute $f_6(r)$ and obtain

$$\begin{aligned} f_6(r) &= \lambda_{16}\pi r + \frac{\pi}{8}(3\lambda_{24} + \lambda_{44} + \lambda_{74} + 3\lambda 94)r^3 \\ &\quad - \frac{\pi}{16(3\lambda_{21} + \lambda_{71})}(3\lambda_{21}^2\lambda_{31} + 9\lambda_{21}^2\lambda_{51} + 9\lambda_{21}^2\lambda_{61} - 2\lambda_{22}\lambda_{30}\lambda_{71} \\ &\quad + 4\lambda_{21}\lambda_{31}\lambda_{71} + 6\lambda_{21}\lambda_{51}\lambda_{71} + 6\lambda_{21}\lambda_{61}\lambda_{71} + \lambda_{31}\lambda_{71}^2 + \lambda_{51}\lambda_{71}^2 \\ &\quad + \lambda_{61}\lambda_{71}^2 + 2\lambda_{21}\lambda_{30}\lambda_{72} - 2\lambda_{22}\lambda_{71}\lambda_{80} + 2\lambda_{21}\lambda_{72}\lambda_{80} + 3\lambda_{21}^2\lambda_{81} \\ &\quad + 4\lambda_{21}\lambda_{71}\lambda_{81} + \lambda_{71}^2\lambda_{81} - 6\lambda_{22}\lambda_{30}\lambda_{91} + 6\lambda_{21}\lambda_{31}\lambda_{91} + 2\lambda_{31}\lambda_{71}\lambda_{91} \\ &\quad - 2\lambda_{30}\lambda_{72}\lambda_{91} - 6\lambda_{22}\lambda_{80}\lambda_{91} - 2\lambda_{72}\lambda_{80}\lambda_{91} + 6\lambda_{21}\lambda_{81}\lambda_{91} \\ &\quad + 2\lambda_{71}\lambda_{81}\lambda_{91} + 6\lambda_{21}\lambda_{30}\lambda_{92} + 2\lambda_{30}\lambda_{71}\lambda_{92} + 6\lambda_{21}\lambda_{80}\lambda_{92} + 2\lambda_{71}\lambda_{80}\lambda_{92})r^5 \\ &\quad - \frac{\pi}{96(3\lambda_{21} + \lambda_{71})}(\lambda_{70}(\lambda_{30} + \lambda_{80})^2(\lambda_{21} + 2\lambda_{71} + 5\lambda_{91}))r^7. \end{aligned}$$

Therefore $f_6(r)$ has at most 3 positive simple zeros, and that there are polynomials $f_6(r)$ with 3 positive simple zeros by the same method used in the proof of Lemma 3.2 of [7].

Under C_{432} we compute $f_6(r)$ and have

$$\begin{aligned} f_6(r) &= \lambda_{16}\pi r + \frac{\pi}{8}(3\lambda_{24} + \lambda_{44} + \lambda_{74} + 3\lambda 94)r^3 \\ &\quad -\frac{\pi}{16}(-2\lambda_{21}\lambda_{31} + 3\lambda_{22}\lambda_{50} + 3\lambda_{22}\lambda_{60} + \lambda_{50}\lambda_{72} + \lambda_{60}\lambda_{72} \\ &\quad -2\lambda_{21}\lambda_{81} + 2\lambda_{31}\lambda_{91} + 2\lambda_{81}\lambda_{91})r^5. \end{aligned}$$

Hence $f_6(r)$ has at most 2 positive simple zeros, and that there are polynomials $f_6(r)$ with 2 positive simple zeros by the same method used in the proof of Lemma 3.2 of [7].

Under C_{433} we compute $f_6(r)$ and get

$$\begin{aligned} f_6(r) &= \lambda_{16}\pi r + \frac{\pi}{8}(3\lambda_{24} + \lambda_{44} + \lambda_{74} + 3\lambda 94)r^3 \\ &\quad -\frac{\pi}{16}(\lambda_{22}\lambda_{30} + 3\lambda_{22}\lambda_{50} + 3\lambda_{22}\lambda_{60} + \lambda_{30}\lambda_{72} + \lambda_{50}\lambda_{72} + \lambda_{60}\lambda_{72} \\ &\quad +\lambda_{22}\lambda_{80} + \lambda_{72}\lambda_{80} + 2\lambda_{30}\lambda_{92} + 2\lambda_{80}\lambda_{92})r^5 \\ &\quad +\frac{\pi}{192}(\lambda_{70}(\lambda_{30} + \lambda_{80})(\lambda_{30} + 5\lambda_{50} + 5\lambda_{60} + \lambda_{80}))r^7. \end{aligned}$$

So $f_6(r)$ has at most 3 positive simple zeros, and that there are polynomials $f_6(r)$ with 3 positive simple zeros by the same method used in the proof of Lemma 3.2 of [7]. □

Proof of Theorem 2. By Lemmas 3 and 4, the averaging method up to 6-th order provides a lower bound of cyclicity of system HC(1,3) at the origin to be at least 3. We continue applying the averaging method up to 7-th order and do not find a higher bound than 3 because $f_7(r)$ is odd and has degree less than or equal to 7. □

Finally we note that the results of switching normal forms obtained in [9], which imply that our system HC(1,3) does not represent the general system with (1,3)-switching FF type equilibrium at the origin. So, the lower bound 3 obtained in Theorem 2 is expected to be improved, but more difficulties will be involved in the computation.

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