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HIGHEST WEAK FOCUS ORDER FOR TRIGONOMETRIC LIÉNARD EQUATIONS

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ABSTRACT. Given a planar analytic differential equation with a critical point which is a weak focus of order k, it is well-known that at most k limit cycles can bifurcate from it. Moreover, in case of analytic Liénard differential equations this order can be computed as one half of the multiplicity of an associated planar analytic map. By using this approach, we can give an upper bound of the maximum order of the weak focus of pure trigonometric Liénard equations only in terms of the degrees of the involved trigonometric polynomials. Our result extends to this trigonometric Liénard case a similar result known for polynomial Liénard equations.

1. INTRODUCTION AND MAIN RESULTS

Recall that a critical point of a planar analytic vector field is called a *focus* if the eigenvalues of its linear approximation at the point are not real, i.e. $\alpha \pm \mathrm{i}\beta$, $\beta \neq 0$. Moreover, when $\alpha \neq 0$ the point is called a *strong focus* and, otherwise, it is called a *weak focus*. The complex Poincaré's normal form of its associated differential equation at this weak focus point is

$$\dot{z} = iz\Big(\beta + \sum_{j=1}^{\infty} c_j (z\bar{z})^{2j}\Big), \quad c_j = \alpha_j + \mathrm{i}\beta_j \in \mathbb{C}.$$

The values α_j give the so called *Lyapunov quantities* and can be computed in many other ways, see for instance [1, 4, 12, 16, 17, 18, 19, 23]. When all $\alpha_j = 0$ the weak focus is a *center*, otherwise, if $\alpha_k \neq 0$, is the first non-zero α_j then it is said that the origin is a *weak focus of order k*. It is well known that k is the maximum number of limit cycles (isolated periodic orbits) that bifurcate from this type of points, and that this amount of limit cycles is attained for some analytic perturbations. Therefore, given an analytic family, $\mathcal{F}$, of planar analytic differential equations depending on finitely many parameters, it is very interesting to know which is the maximum order of the weak focus inside this family, $\sigma(\mathcal{F})$, see [24]. This number is known to exist when the Lyapunov quantities are polynomials on the parameters of the system, because of the Hilbert's basis Theorem.

Unfortunately Hilbert's result is not constructive and, in general, an explicit bound of the number of needed Lyapunov quantities is not known. In fact, even for cubic vector fields this number is nowadays unknown.

The most important family $\mathcal{F}$ of systems of arbitrary degree for which an explicit upper bound of $\sigma(\mathcal{F})$ is known is the family of polynomial Liénard equations. This

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family is the one formed for the planar vector fields with a weak focus at the origin

$$\begin{cases} \dot x = y, \\ \dot y = g(x) + yf(x), \end{cases}$$

where f and g are polynomials with given degrees, satisfying $f(0) = g(0) = 0$ and $g'(0) < 0$. This upper bound (which is not sharp in general) depends on these degrees and it is given in [6, 7, 8, 9, 15]. This proof relies on two main facts: a relation between the order of a weak focus for analytic Liénard equations with the multiplicity at the origin of a planar polynomial map, and on Bezout's Theorem.

The same tools can be applied to solve the same question for trigonometric Liénard systems. Notice that, as well as polynomial systems, trigonometric systems are differential systems of current interest, see for instance [3, 13, 14, 20, 21, 22, 25]. We will say that

$$\begin{cases} \dot \theta = y, \\ \dot y = G'(\theta) + yF'(\theta), \end{cases} \tag{1}$$

is a *pure trigonometric Liénard system* if F and G are 2π-trigonometric polynomials satisfying $F(0) = F'(0) = 0$, $G(0) = G'(0) = 0$, $G''(0) < 0$.

We denote by $\mathcal{L}_{m,n}$ the family of all pure trigonometric Liénard systems where F and G satisfy the above properties and their degrees are at most m and n, respectively. Because of the lack of symmetry between F and G we also introduce the subclass formed by the F's that also satisfy $F''(0) < 0$. We denote it by $\mathcal{L}^*_{m,n} \subset \mathcal{L}_{m,n}$ Recall that if a real 2π-trigonometric polynomial p is such that its Fourier series satisfies

$$p(\theta) = \sum_{k=-\ell}^{\ell} a_k e^{ki\theta}, \quad a_{-k} = \overline{a}_k, \quad \text{with} \quad a_\ell \neq 0, \tag{2}$$

then its degree is ℓ.

$n\backslash m$	1	2	3	4
1	0	1	2	3
2	1	2	6	7
3	2	6	7	-
4	3	7	-	-

TABLE 1. Some values of $\sigma(\mathcal{L}_{m,n})$.

Our main result is:

Theorem 1. *Let $\sigma(\mathcal{L}_{m,n})$ (resp. $\sigma(\mathcal{L}^*_{m,n})$) be highest weak focus order for systems* (1) *inside family $\mathcal{L}_{m,n}$ (resp. $\sigma(\mathcal{L}^*_{m,n})$). Then,*

(i) For all positive m and n, $\sigma(\mathcal{L}_{1,n}) = n-1$ and $\sigma(\mathcal{L}_{m,1}) = m-1$. Similarly, $\sigma(\mathcal{L}^*_{1,n}) = n-1$ and $\sigma(\mathcal{L}^*_{m,1}) = m-1$.

(ii) For all positive $m \geq n$,

$$\sigma(\mathcal{L}_{m,n}) = \sigma(\mathcal{L}^*_{m,n}) = \sigma(\mathcal{L}^*_{n,m}) \leq \sigma(\mathcal{L}_{n,m}).$$

(iii) For n and $m \geq 2$,

$$m+n-2 \leq \sigma(\mathcal{L}_{m,n}) \leq 6mn-3(m+n)+1.$$

(iv) For small m and n, some values of $\sigma(\mathcal{L}_{m,n})$ are given in Table 1.

Although, comparing with the results of Table 1, the upper bounds given in item (iii) of the theorem are not sharp, the more important fact is that they are explicit.

If in system (1), instead of F and G we consider $F(\theta) = \widetilde{F}(\theta) + \alpha\theta$ and $G(\theta) = \widetilde{F}(\theta) + \beta\theta$, with $\widetilde{F}$ and $\widetilde{G}$ trigonometric polynomials, we will say that (1) is a (non pure) trigonometric Liénard system. In Section 4 we give some partial results for this case.

2. Preliminary results

Let $(P,Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic function at $(0,0)$. As usual we denote by $\mu_0[P,Q]$ its multiplicity at this point. Recall that when $(P(0,0),Q(0,0)) \neq (0,0)$ then $\mu_0[P,Q] = 0$ and, otherwise, $\mu_0[P,Q]$ is the number of (P,Q)-complex preimages around $(0,0)$ of any regular point near the origin, see [2]. When $(0,0)$ is a non isolated zero then it is said that the multiplicity is infinity. Notice also that $\mu_0[P,Q] = \mu_0[Q,P]$.

In text proposition we collect some useful properties to compute multiplicities, see again [2]. As usual, $O(k)$ denotes terms of order at least k.

Proposition 2. *Let $(P,Q)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic map at the origin with finite multiplicity and let $R\colon \mathbb{R}^2 \to \mathbb{R}$ and $S\colon \mathbb{R} \to \mathbb{R}$ be also analytic at the origin. Then:*

(a) The multiplicity of (P,Q) at the origin does not depend on the choice of coordinates;
(b) It holds that $\mu_0[RP,Q] = \mu_0[R,Q] + \mu_0[P,Q]$. In particular, if $R(0,0) \neq 0$ then $\mu_0[RP,Q] = \mu_0[P,Q]$;
(c) Write $P = P_j + O(j+1)$ and $Q = Q_k + O(k+1)$, with P_j and Q_k homogeneous with respective degrees j and k. Then $\mu_0[P,Q] \geq jk$, and the equality holds if and only if the system $P_j = 0$, $Q_k = 0$ has only the trivial solution $(0,0)$ in $\mathbb{C}^2$.
(d) It holds that $\mu_0[P+RQ,Q] = \mu_0[P,Q]$.
(e) If $P(x,y) = (y - S(x))R(x,y)$, with $S(0) = 0$ and $R(0,0) \neq 0$, and $Q(x,S(x)) = ax^k + O(k+1)$, with $a \neq 0$, then $\mu_0[P,Q] = k$.

In [5] the authors proved the following nice theorem which is based on previous results of [6]. Our results will be strongly based on it.

Theorem 3. ([5]) *Consider the Liénard system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = g(x) + yf(x), \end{cases} \tag{3}$$

where f and g are analytic at the origin and satisfy $f(0) = g(0) = 0$ and $g'(0) < 0$. Define

$$\mu(f,g) = \mu_0\left[\frac{F(x)-F(y)}{x-y}, \frac{G(x)-G(y)}{x-y}\right],$$

where $F(x) = \int_0^x f(s)\,ds$ and $G(x) = \int_0^x g(s)\,ds$. Then

(i) if $\mu(f,g) = \infty$, the origin of (3) is a center,

(ii) if $\mu(f,g)<\infty$, the origin of (3) *is a weak focus of order $\mu(f,g)/2$.*

We will use a different characterization of trigonometrical polynomials of degree ℓ. Consider the ring of trigonometric polynomials, $\mathbb{R}_t[\theta]=\mathbb{R}[\sin\theta,\cos\theta]$. It is well known that its quotient field, $\mathbb{R}_t(\theta)$, is isomorphic to $\mathbb{R}(x)$ by means of the morphism $\Phi\colon \mathbb{R}_t(\theta)\to\mathbb{R}(x)$ defined by

$$\Phi(\sin\theta)=\frac{2x}{1+x^2}\quad\text{and}\quad\Phi(\cos\theta)=\frac{1-x^2}{1+x^2}. \tag{4}$$

Note that

$$\Phi(\tan(\theta/2))=\Phi\Big(\frac{\sin\theta}{1+\cos\theta}\Big)=x\quad\text{and}\quad\Phi(\theta)=\arctan\Big(\frac{2x}{1-x^2}\Big).$$

Observe also that Φ is a well-defined change of variables around the origin.

If p is a trigonometric polynomial of degree ℓ, as in (2), it holds that

$$\Phi(p(\theta))=\frac{P(x)}{(1+x^2)^\ell},\quad\text{with}\quad P\in\mathbb{R}[x],\ \deg(P)\le 2\ell\quad\text{and}\quad\gcd(P(x),1+x^2)=1. \tag{5}$$

Moreover, the converse is also true: for each P under the above hypotheses, there exists a trigonometric polynomial, p, of degree at most ℓ, such that $\Phi(p(\theta))=\frac{P(x)}{(1+x^2)^\ell}$, see [10, 11].

Consider now a pair f,g of 2π-periodic trigonometric polynomials of degrees m and n, respectively and define

$$F(x)=\int_0^x f(\theta)\,d\theta,\quad G(x)=\int_0^x g(\theta)\,d\theta. \tag{6}$$

Then $F(\theta)=\alpha\theta+\widetilde{F}(\theta)$ and $G(\theta)=\beta\theta+\widetilde{G}(\theta)$ where $\alpha=\frac{F(2\pi)}{2\pi}$ and $\beta==\frac{G(2\pi)}{2\pi}$ and $\widetilde{F}$ and $\widetilde{G}$ are also trigonometric polynomials of degrees n and m, respectively. Using the change of variables Φ given in (4) we have that

$$\begin{aligned} F(\theta)&=F\Big(\arctan\Big(\frac{2x}{1-x^2}\Big)\Big)=\alpha\arctan\Big(\frac{2x}{1-x^2}\Big)+\frac{M(x)}{(1+x^2)^m},\\ G(\theta)&=G\Big(\arctan\Big(\frac{2x}{1-x^2}\Big)\Big)=\beta\arctan\Big(\frac{2x}{1-x^2}\Big)+\frac{N(x)}{(1+x^2)^n}, \end{aligned} \tag{7}$$

where M is the polynomial of degree less than or equal to $2m$ associated to $\widetilde{F}$ and N is the polynomial of degree less than or equal to $2n$ associated to $\widetilde{G}$.

As a corollary of the previous results we prove the following proposition.

Proposition 4. *Let f and g be 2π-trigonometric polynomials with degrees m and n, and such that $f(0)=g(0)=0$. Then, following the notation introduced in* (6) *and* (7)*, it holds that*

$$\mu_0\left[\frac{F(\theta)-F(\psi)}{\theta-\psi},\frac{G(\theta)-G(\psi)}{\theta-\psi}\right]=\mu_0\left[P_\alpha(x,y),Q_\beta(x,y)\right],$$

where

$$\begin{aligned} P_\alpha(x,y)=&\big(\alpha\Delta(x,y)(1+x^2)^m(1+y^2)^m+M(x)(1+y^2)^m-M(y)(1+x^2)^m\big)/(x-y),\\ Q_\beta(x,y)=&\big(\beta\Delta(x,y)(1+x^2)^n(1+y^2)^n+N(x)(1+y^2)^n-N(y)(1+x^2)^n\big)/(x-y) \end{aligned}$$

and $\Delta(x,y) = 2\big(\arctan(x) - \arctan(y)\big)$.

Proof. Notice that given any analytic map r it holds that $R(x,y) = (r(x) - r(y)/(x-y)$ is analytic as well at $(0,0)$. Moreover at the origin $R(0,0) = r'(0)$. Thus, taking $r(\theta) = \tan(\theta/2)$ we have that $R(0,0) = 1/2$ and R and $1/R$ are analytic at zero. Hence, by using property (b) of Proposition 2, we get that

$$\begin{aligned}\mu_0\left[\frac{F(\theta)-F(\psi)}{\theta-\psi}, \frac{G(\theta)-G(\psi)}{\theta-\psi}\right] &= \mu_0\left[\frac{F(\theta)-F(\psi)}{\theta-\psi}\frac{1}{R(\theta,\psi)}, \frac{G(\theta)-G(\psi)}{\theta-\psi}\frac{1}{R(\theta,\psi)}\right] \\ &= \mu_0\left[\frac{F(\theta)-F(\psi)}{\tan(\theta/2)-\tan(\psi/2)}, \frac{G(\theta)-G(\psi)}{\tan(\theta/2)-\tan(\psi/2)}\right]\end{aligned}$$

Notice that by (4), if we take $(x,y) = (\tan(\theta/2), \tan(\psi/2))$ it holds that

$$\begin{aligned}\frac{F(\theta)-F(\psi)}{\tan(\theta/2)-\tan(\psi/2)} &= \frac{1}{x-y}\Big(\alpha\widetilde{\Delta}(x,y) + \frac{M(x)}{(1+x^2)^m} - \frac{M(y)}{(1+y^2)^m}\Big), \\ \frac{G(\theta)-G(\psi)}{\tan(\theta/2)-\tan(\psi/2)} &= \frac{1}{x-y}\Big(\alpha\widetilde{\Delta}(x,y) + \frac{N(x)}{(1+x^2)^n} - \frac{N(y)}{(1+y^2)^n}\Big),\end{aligned}$$

where

$$\widetilde{\Delta}(x,y) = \arctan\Big(\frac{2x}{1-x^2}\Big) - \arctan\Big(\frac{2y}{1-y^2}\Big).$$

Observe also that for $|x| < 1$ and $|y| < 1$,

$$\widetilde{\Delta}(x,y) = \arctan\Big(\frac{2x}{1-x^2}\Big) - \arctan\Big(\frac{2y}{1-y^2}\Big) = 2\big(\arctan(x) - \arctan(y)\big) = \Delta(x,y).$$

Hence, by property (a) of Proposition 2 we obtain that

$$\mu_0\left[\frac{F(\theta)-F(\psi)}{\tan(\theta/2)-\tan(\psi/2)}, \frac{G(\theta)-G(\psi)}{\tan(\theta/2)-\tan(\psi/2)}\right] = \mu_0\left[\frac{P_\alpha(x,y)}{(1+x^2)^m(1+y^2)^m}, \frac{Q_\alpha(x,y)}{(1+x^2)^n(1+y^2)^n}\right].$$

Finally, by using again property (b) of the same proposition, we can remove in each component the factor $(1+x^2)^{-m}(1+y^2)^{-m}$ and the factor $(1+x^2)^{-n}(1+y^2)^{-n}$, without changing the multiplicity, because they do not vanish at $(0,0)$, giving the desired result. □

3. Proof of Theorem 1

Proof of Theorem 1. (i) We will prove that $\sigma(\mathcal{L}_{m,1}) = m-1$. The proof that $\sigma(\mathcal{L}_{1,n}) = n-1$ follows similarly. We will use Theorem 3 and Proposition 4. Note that following the notation of Proposition 4, $\mu(F', G') = \mu_0\left[P_0(x,y), Q_0(x,y)\right]$, because $\alpha = \beta = 0$, where

$$P_0(x,y) = \frac{M(x)(1+y^2)^m - M(y)(1+x^2)^m}{x-y},$$

and

$$Q_0(x,y) = \frac{bx^2(1+y^2) - by^2(1+x^2)}{x-y} = b(x+y).$$

By property (e) of Proposition 2, to know the multiplicity $\mu_0[P_0, Q_0]$ is suffices to study the function

$$H(x) = P_0(x,-x) = \frac{M(x) - M(-x)}{2x}(1+x^2)^m = \frac{M^{\mathrm{odd}}(x)}{x}(1+x^2)^m,$$

where M^{odd} is the odd part of M. Clearly, when $H \neq 0$, its highest order term at the origin is $x^{2(m-1)}$. Hence, by Theorem 3, $\sigma(\mathcal{L}_{m,1}) = m-1$, as we wanted to prove. The proofs for $\sigma(\mathcal{L}^*_{m,1})$ and $\sigma(\mathcal{L}^*_{1,n})$ are similar.

(ii) Simply because $\mathcal{L}^*_{n,m} \subset \mathcal{L}_{n,m}$ it holds that $\sigma(\mathcal{L}^*_{n,m}) \le \sigma(\mathcal{L}_{n,m})$. Moreover, by Theorem 3, and due to the symmetry between F and G in $\mathcal{L}^*_{n,m}$ it holds that $\sigma(\mathcal{L}^*_{m,n}) = \sigma(\mathcal{L}^*_{n,m})$.

Hence, we only need to prove that $\sigma(\mathcal{L}_{m,n}) = \sigma(\mathcal{L}^*_{m,n})$. To prove this equality, notice that if the trigonometric polynomials F and G that give the maximum weak focus order correspond to an F such that for it corresponding expression as in (7) it holds that $M(x) = a_2x^2 + 0(3)$, with $a_2 \neq 0$, then it is clear that the maximum highest order in $\mathcal{L}_{m,n}$ is also taken for an element that is in $\mathcal{L}^*_{m,n}$ and the equality follows. Otherwise, the maximum highest order is reached for an F such that its corresponding M satisfies $M(x) = 0(3)$. Let us prove in this situation how to construct another F such that the order of the origin is the same but this new M, say $\widehat{M}$, is such that $\widehat{M}(x) = b_2x^2 + 0(3)$, with $b_2 \neq 0$.

Let M and N such that $m \ge n$ and $2\sigma(\mathcal{L}_{m,n}) = \mu_0[P_0, Q_0]$, where recall that

$$(P_0, Q_0) = \left(\frac{M(x)(1+y^2)^m - M(y)(1+x^2)^m}{x-y}, \frac{N(x)(1+y^2)^n - N(y)(1+x^2)^n}{x-y}\right).$$

Consider, as new F, a trigonometric polynomial of degree m, $\widehat{F}$, such that its corresponding M according to (5) is the polynomial of degree $2m$, $\widehat{M}(x) = N(x)(1+x^2)^{m-n} + M(x)$. Notice that

$$\begin{aligned}\widehat{P}_0(x,y) &= \frac{\widehat{M}(x)(1+y^2)^m - \widehat{M}(y)(1+x^2)^m}{x-y}\\ &= \frac{\big(N(x)(1+x^2)^{m-n} + M(x)\big)(1+y^2)^m - \big(N(y)(1+y^2)^{m-n} + M(y)\big)(1+x^2)^m}{x-y}\\ &= (1+x^2)^{m-n}(1+y^2)^{m-n}Q_0(x,y) + P_0(x,y).\end{aligned}$$

Hence, by property (d) of Proposition 2,

$$\mu_0\left[\widehat{P}_0, Q_0\right] = \mu_0\left[P_0, Q_0\right] = 2\sigma(\mathcal{L}_{m,n})$$

and $\widehat{M}(x) = N(x)(1+x^2)^{m-n} + M(x) = x^2 + 0(3)$, as we wanted to prove.

(iii) We start with the upper bound. Recall that if a polynomial map (P,Q) has only isolated (real or complex) singularities and $\mathcal{Z}$ denotes the set formed for all of them, then by Bezout's theorem

$$\sum_{z\in\mathcal{Z}} \mu_z[P,Q] \le \deg(P)\deg(Q).$$

Recall that in our situation, $\alpha=\beta=0$ and by Proposition 4,

$$\mu(F',G') = \mu_0\left[\frac{F(\theta)-F(\psi)}{\theta-\psi}, \frac{G(\theta)-G(\psi)}{\theta-\psi}\right] = \mu_0\left[P_0(x,y),Q_0(x,y)\right],$$

where last two functions are polynomials and

$$\deg\left(P_0(x,y)\right) = 4m-2 \quad \text{and} \quad \deg\left(Q_0(x,y)\right) = 4n-2,$$

because the term of degree $4m-1$ (resp. $4n-1$) of P_0 (resp. Q_0) vanishes. Moreover, the four points $(\pm\mathrm{i},\pm\mathrm{i})$ are also singularities of (P_0,Q_0). By using property (c) of Theorem 2 it is not difficult to prove that

$$\begin{aligned}\mu_{(\mathrm{i},\mathrm{i})}[P_0,Q_0] = \mu_{(-\mathrm{i},-\mathrm{i})}[P_0,Q_0] &\ge (m-1)(n-1),\\ \mu_{(\mathrm{i},-\mathrm{i})}[P_0,Q_0] = \mu_{(-\mathrm{i},\mathrm{i})}[P_0,Q_0] &\ge mn.\end{aligned}$$

Hence, by the above inequalities,

$$\begin{aligned}\mu(F',G') &\le \deg(P_0)\deg(Q_0) - \sum_{z\in\mathcal{Z}\setminus\{(0,0)\}} \mu_z[P_0,Q_0]\\ &\le 4(2m-1)(2n-1) - 2mn - 2(m-1)(n-1).\end{aligned}$$

Finally, by Theorem 3,

$$\sigma(\mathcal{L}_{m,n}) \le 2(2m-1)(2n-1) - mn - (m-1)(n-1) = 6mn - 3(m+n) + 1.$$

Now we compute the lower bound. We consider F' and G' such that their corresponding expressions, as rational functions following (4) and (5), are

$$M(x) = \frac{x^{2m}}{(1+x^2)^m} \quad \text{and} \quad N(x) = \frac{x^2+x^{2n-1}}{(1+x^2)^n}.$$

For each $i,\ell\in\mathbb{N}$ introduce the following polynomials

$$R_{i,\ell}(x,y) = \frac{x^i(1+y^2)^\ell - y^i(1+x^2)^\ell}{x-y} \quad \text{and} \quad S_{2i,\ell}(x,y) = \frac{x^{2i}(1+y^2)^\ell - y^{2i}(1+x^2)^\ell}{x^2-y^2}.$$

Then,

$$\begin{aligned}P_0(x,y) &= \frac{x^{2m}(1+y^2)^m - y^{2m}(1+x^2)^m}{x-y} = S_{2m,m}(x,y)(x+y),\\ Q_0(x,y) &= \frac{(x^2+x^{2n-1})(1+y^2)^n - (y^2+y^{2n-1})(1+x^2)^n}{x-y}\\ &= S_{2,n}(x,y)(x+y) + R_{2n-1,n}(x,y),\end{aligned}$$

where P_0 and Q_0 are the polynomials appearing in Proposition 4. Notice that

$$S_{2,n}(x,y) = 1 + O(1) \quad \text{and} \quad S_{2m,m}(x,y) = \frac{x^{2m}-y^{2m}}{x^2-y^2} + O(2m-1).$$

By Proposition 4 and Theorem 3,

$$\begin{aligned}\mu(F',G') =&\mu_0\left[S_{2m,m}(x,y)(x+y),S_{2,n}(x,y)(x+y)+R_{2n-1,n}(x,y)\right]\\ =&\mu_0\left[S_{2m,m}(x,y),S_{2,n}(x,y)(x+y)+R_{2n-1,n}(x,y)\right]\\ &+\mu_0\left[x+y,S_{2,n}(x,y)(x+y)+R_{2n-1,n}(x,y)\right],\end{aligned}$$

where in the last equality we have used property (d) of Proposition 2. We consider separately each of the terms of the sum.

By using again the properties of Proposition 4, since $S_{2,n}(0,0)\neq 0$,

$$\mu_0\left[S_{2m,m}(x,y),S_{2,n}(x,y)(x+y)+R_{2n-1,n}(x,y)\right]=\mu_0\left[S_{2m,m}(x,y),x+y+O(2)\right]=$$
$$\mu_0\left[\frac{x^{2m}-y^{2m}}{x^2-y^2}+O(2m-1),x+y+O(2)\right]=\mu_0\left[\frac{x^{2m}-y^{2m}}{x^2-y^2},x+y\right]=2(m-1).$$

Similarly, by property (e) of Proposition 4, the second term coincides with the degree of the lowest term at the origin of $R_{2n-1,n}(x,-x)=(1+x^2)^n x^{2n-2}$. Hence,

$$\mu_0\left[x+y,S_{2,n}(x,y)(x+y)+R_{2n-1,n}(x,y)\right]=2(n-1).$$

Putting all the results together

$$\mu(F',G')=2(m+n-2).$$

Hence, by Theorem 3, the order of the corresponding weak focus in $m+n-2$ and the lower bound of the theorem follows.

(iv) We only will give the full details of two cases of Table 1, $(m,n)\in\{(2,3),(3,2)\}$. The others follow similarly.

When $(m,n)=(2,3)$,

$$P_0(x,y)=\frac{M(x)(1+y^2)^2-M(y)(1+x^2)^2}{x-y},\,Q_0(x,y)=\frac{N(x)(1+y^2)^3-N(y)(1+x^2)^3}{x-y},$$

where $M(x)=b_2x^2+b_3x^3+b_4x^4$ and $N(x)=x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6$ because $G(0)=G'(0)=0$ and $G''(0)\neq 0$ (the coefficient of x^2 is normalized to one for the sake of simplicity).

Since $Q_0(x,y)=x+y+O(2)$, we have that $\partial Q_0(0,0)/\partial y\neq 0$, and by the Weierstrass Preparation Theorem it holds that $Q_0(x,y)=(y-S(x))R(x,y)$ for some analytic functions such that $R(0,0)\neq 0$ and $S(x)=-x+O(2)=-x+\sum_{i=2}^{\infty}a_ix^i$. Moreover, $Q_0(x,S(x))\equiv 0$. Hence, by property (e) of Proposition 2, we can compute the maximum multiplicity by taking $P_0(x,S(x))$ and vanishing this power series to the highest possible order by using the free parameters b_i of M and c_i of N. The first non-zero term is $-4(c_3b_2-b_3)x^2$ which forces $b_3=c_3b_2$ to have order bigger than 2. The next order is $4(-c_3b_2+2c_3c_4b_2-c_5b_2-2c_3b_4)x^4$. It can be seen that if $c_3=0$ we obtain lower vanishing order. So, we assume that $c_3\neq 0$ we take $b_4=(-c_3b_2+2c_3c_4b_2-c_5b_2)/(2c_3)$ to go on. The next power is

$$\frac{1}{2c_3}(3c_3^4-2c_3^2c_4+4c_3c_5-3c_3^3c_5-4c_3c_4c_5+2c_5^2+6c_3^2c_6)b_2x^6.$$

From here we have that $c_6 = (-3c_3^4+2c_3^2c_4-4c_3c_5+3c_3^3c_5+4c_3c_4c_5-2c_5^2)/(6c_3^2)$ to arrive to order bigger than 6. The next power is

$$-\frac{1}{6}c_3(-21c_3+22c_3c_4-11c_5)(c_3-c_5)b_2x^8.$$

If $(c_3-c_5)b_2=0$ we have that multiplicity infinity (or in other words that the corresponding Liénard system has a center at the origin). Then we must take $c_5 = c_3(-21+22c_4)/11$. The next power is

$$\frac{2}{1331}\big(c_3^3(-4+143c_3^2)(-16+11c_4)b_2\big)x^{10}.$$

The case $(-16+11c_4)b_2=0$ gives again a case of multiplicity infinity. Hence we take $c_3 = \pm 2/\sqrt{143}$ and we have that the next power is

$$\pm\frac{65536}{353829047\sqrt{143}}((-16+11c_4)b_2)x^{12}. \tag{8}$$

Therefore the highest multiplicity is 12 which implies that $\sigma(\mathcal{L}_{2,3})=6$.

To get $\sigma(\mathcal{L}_{3,2})$, notice first that by item (ii), $\sigma(\mathcal{L}_{3,2}) \le \sigma(\mathcal{L}_{2,3}) = 6$. Moreover, since the cases giving rise to order of the weak focus 6 satisfy that $b_2 \neq 0$, see (8), taking one of them and as a new M as M/b_2, we have that $\sigma(\mathcal{L}_{3,2}) \ge 6$. Thus $\sigma(\mathcal{L}_{3,2}) = 6$, as we wanted to show. □

4. Some results for the non pure trigonometric case

For the case of non pure trigonometric polynomials a table similar to Table 1, but for the values $\sigma(\mathcal{L}^{\alpha,\beta}_{m,n})$, can be done. We present some cases in Table 2, where the numbers with a star mean a lower bound for the highest weak focus order and simply correspond to the values $\sigma(\mathcal{L}_{m,n})$.

$n\backslash m$	1	2	3	4
1	1	3	5	7
2	3	4	6*	7*
3	5	6*	7*	-
4	7	7*	-	-

Table 2. Some values of $\sigma(\mathcal{L}^{\alpha,\beta}_{m,n})$.

We only will give some details for the case $(m,n)=(2,2)$. For these values

$$P_\alpha = \frac{2\alpha\,(\arctan(x)-\arctan(y))\,(1+x^2)^2(1+y^2)^2+M(x)(1+y^2)^2-M(y)(1+x^2)^2}{x-y},$$

$$Q_\beta = \frac{2\beta\,(\arctan(x)-\arctan(y))\,(1+x^2)^2(1+y^2)^2+N(x)(1+y^2)^2-N(y)(1+x^2)^2}{x-y},$$

where $M(x) = -2\alpha x+b_2x^2+b_3x^3+b_4x^4$ and $N(x) = -2\beta x+x^2+c_3x^3+c_4x^4$. We proceed as in the proof of item (iv) of Theorem 1 by using property (e) of Proposition 2. Hence we have to find the highest order at zero of $P_\alpha(x,S(x))$ where S is the analytic function that satisfies $S(0)=0$ and $Q_\beta(x,S(x)) \equiv 0$. The first non-zero order is $(-3c_3b_2+3b_3-$

$10b_2\beta+10\alpha)x^2/3$ which yields $b_3=(3c_3b_2+10b_2\beta-10\alpha)/3$ if we want to arrive to higher order. The next power is

$$\frac{2}{15}(15c_3c_4b_2-15c_3b_4-8b_2\beta+50c_4b_2\beta-50b_4\beta+8\alpha)x^4.$$

From it, to go on, we impose that $b_4=(15c_3c_4b_2-8b_2\beta+50c_4b_2\beta+8\alpha)/(5(3c_3+10\beta))$ because, otherwise, if we take $3c_3+10\beta=0$ it can be seen that we arrive to a lower vanishing order. The next power is

$$\frac{8(\alpha-b_2\beta)}{1575(3c_3+10\beta)}\big(-720c_3+945c_3^3+1260c_3c_4-3072\beta+9450c_3^2\beta+4200c_4\beta$$
$$+31500c_3\beta^2+35000\beta^3\big)x^6.$$

If $\alpha-b_2\beta=0$ we have that multiplicity infinity, that is, we obtain that $P_\alpha(x,S(x))\equiv 0$. Then we must take

$$c_4=\frac{720c_3-945c_3^3+3072\beta-9450c_3^2\beta-31500c_3\beta^2-35000\beta^3}{420(3c_3+10\beta)}.$$

The next power is

$$\frac{4(\alpha-b_2\beta)}{59535}(432+7(3c_3+10\beta)^2(360+77(3c_3+10\beta)^2))x^8$$

Hence we must impose that $432+7(3c_3+10\beta)^2(360+77(3c_3+10\beta)^2)=0$ to obtain higher multiplicity. Doing the reparametrization $3c_3+10\beta=k_1$, this second term does not vanish because it corresponds to $432+2520k_1^2+539k_1^4$, which has no real roots. Hence the maximum multiplicity is 8 and by Theorem 3, $\sigma(\mathcal{L}_{2,2}^{\alpha,\beta})=4$.

Remark 5. *In general, in this work we have not addressed the question of knowing if the highest order cyclicity gives rise to the corresponding number of limit cycles inside the Liénard trigonometric family. In general, the easiest way to ensure that this happens is to prove that the gradients of the Lyapunov quantities (the coefficient of the even orders in the above procedure) have the maximum rank at zero. For instance, it can be seen that this is the case in the above example.*

We end this section with a particular result refereed to a subfamily of non pure trigonometric Liénard systems.

Proposition 6. *Let $\sigma(\mathcal{L}_{m,1}^{\alpha,0})$ be highest weak focus order for systems* (1) *inside the family non pure trigonometric Liénard systems with $\alpha\in\mathbb{R}$ and $\beta=0$. Then $\sigma(\mathcal{L}_{m,1}^{\alpha,0})=\sigma(\mathcal{L}_{m,1})+1=m$.*

Proof. Arguing as in the proof of item (i) of Theorem 1 we know that $\mu(F',G')=\mu_0\left[P_\alpha(x,y),Q_0(x,y)\right]$, where $Q_0(x,y)=b(x+y)$, $b\neq 0$, and $P_\alpha(x,y)$ is as in the statement of Proposition 4. Hence, by property (e) of Proposition 2, to know the above multiplicity it suffices to know the highest order at the origin of the map

$$K(x)=P_\alpha(x,-x)=(1+x^2)^m\left[\frac{2\alpha(1+x^2)^m\arctan(x)}{x}+\frac{M^{\text{odd}}(x)}{x}\right],$$

where M^{odd} is the odd part of M, which recall that is a polynomial of degree at most $2m$. Clearly the coefficients of M can be chosen in such a way that K starts at the origin with terms of order at least $2m$. Hence, to prove that the maximum order of $K \neq 0$ at the origin is $2m$ we need to prove that the coefficient of order $2m+1$ at the origin of the function $(1+x^2)^m \arctan(x)$ is not null.

With this aim, we fix m and consider

$$H(x) = (1+x^2)^m \arctan(x) = \sum_{k=0}^{\infty} h_k x^{2k+1}, \quad \text{for} \quad |x| < 1,$$

where, for the sake of simplicity, we have removed the dependence on m of the function and on its Taylor series. We will prove that for all $k \in \mathbb{N}$, $h_k \neq 0$. It is not difficult to check that

$$(1+x^2)H'(x) = 2m\, xH(x) + (1+x^2)^m.$$

As a consequence, $h_0 = 1$, and equating the terms with x^{2k} in the above expression, we obtain that for all $k \geq 1$,

$$h_k = \frac{(2(m-k)+1)h_{k-1} + \binom{m}{k}}{2k+1},$$

where $\binom{m}{k} = 0$ for $k > m$. The above recurrence implies that $h_k > 0$ for all $k \leq m$ and $(-1)^{k-m} h_k > 0$ for all $k > m$, because $2(m-k)+1$ is positive for $k \leq m$ and negative otherwise. Hence $h_k \neq 0$ for all $k \geq 0$, and in particular h_m, the coefficient of x^{2m+1}, is not zero, as we wanted to prove. Thus, $H[F', G'] \leq 2m$ and this upper bound is attained. Hence, by Theorem 3 the upper bound of the order of the weak focus for the family considered is m and this values is reached. $\square$

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