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EQUILIC QUADRILATERAL CENTRAL CONFIGURATIONS

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ABSTRACT. An equilic quadrilateral is a quadrilateral with one pair of opposite sides having the same length, which has angles of inclination whose sum is $2\pi/3$. We show that there are some region in which equilic quadrilateral configuration of four bodies is possible for positive masses.

1. Introduction and statement of the main results

In the n-body problem a central configuration occurs when the position vector of each particle with respect to the center of mass is a common scalar multiple of its acceleration vector, and the scalar difference is the same for all particles [36].

There are several aspects of the n-body problem that motivate the study of central configurations, one is that they allow to obtain the *homographic solutions* of the n–body problem, which are the unique explicit solutions in function of the time known until now of that problem, for such solutions the ratios of the mutual distances between the bodies remain constant. Also, the central configurations appear as a key in the bifurcations of the surfaces of constant energy and angular momentum, for more details see Meyer [29] and Smale [35].

However, although there is an extensive literature concerning the number of classes of planar central configurations of the n-body problem for an arbitrary given set of positive masses, it has been only solved for $n = 3$, where there are three collinear and the two equilateral triangle central configurations; see among others [36], [3], [6], [22], [25], [28].

The equations of motion of the planar n–body problem are

$$\ddot{x}_i = \sum_{j=1,\, j\neq i}^{n} \frac{m_j(x_j - x_i)}{r_{ij}^3}, \quad \text{for} \quad i = 1, \ldots, n.$$

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where $x_i \in \mathbb{R}^2$ are the position vectors of the bodies, $r_{ij} = |x_i - x_j|$ are their mutual distances, and m_i are their masses. Here the unit of time is taken in order that the Newtonian gravitational constant be equal to one.

The *configuration* of the system formed by the n bodies is given by the vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n}$. The differential equations of motion are well–defined when there is no collisions between the bodies, namely, when $r_{ij} \neq 0$ for $i \neq j$.

We define the *total mass* and the *center of mass* of the n bodies as

$$M = m_1 + \ldots + m_n, \qquad c = \frac{1}{M}\left(m_1 x_1 + \cdots + m_n x_n\right),$$

respectively. A configuration x is *central* when the acceleration vectors of the bodies satisfy

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_j(x_j - x_i)}{r_{ij}^3} = \lambda(x_i - c), \quad \text{for} \quad i = 1, \ldots, n,$$

where λ is a constant.

We say that two central configurations are *equivalent* if we can be pass from one to the other doing a rotation around the center of mass and a homothecy. This defines an equivalent relation in the set of all central configurations. From now on when we say a central configuration, we mean a class of central configurations.

The aim of the present article is to improve the knowledge of the planar central configurations of the 4–body problem. First we provide some research cites of planar 4-body central configurations, which shows that it is still an interesting and attracting field of research.

A numerical study on the classes of central configurations for the 4–body problem for arbitrary masses was done by Simó in [34]. In 2006 Hampton and Moeckel [22] used a computer assisted proof to show that the 4–body problem has finitely many classes of central configurations with any given positive masses. Later on Albouy and Kaloshin [6] proved analytically this result. Llibre [24] showed all the planar central configurations of the 4–body problem with equal masses by studying the intersection points of two planar curves and assuming that such central configurations have an axis of symmetry. In [1, 2] Albouy gave an analytic proof of this result, showing that there are exactly four classes of central configurations with equal masses. Pedersen [30], Gannaway [20], Arenstorf [9] and Barros and Leandro [10, 11] obtained numerically and analytically the central configurations of the 4–body problem when one of the four masses has infinitesimal mas. A central configuration of the 4–body problem having an axis of symmetry passing through two non–adjacent bodies is called *kite*. The non–collinear classes of kite central configurations with three equal masses were classified by Bernat et al. in [12], and also by Leandro [23].

A planar configuration of the 4–body problem is *convex* if none of the bodies is located in the interior of the triangle formed by the others, otherwise it is *concave*. MacMillan and Bartky [27] showed that for any four positive masses and any assigned order, there is a convex planar central configuration of the four bodies. Later on Xia [37] gave a simple proof of this result.

In 2010 Piña and Lonngi [32] numerically explored the new properties of symmetric and non-symmetric central configurations of the 4-body problem. Long and Sun [26] showed that any convex non-collinear central configurations of the planar 4-body problem with masses $m_1 = m_2 < m_3 = m_4$, such that the equal masses are located at opposite vertices of a quadrilateral and the diagonal corresponding to the mass m_1 is larger than or equal to the one corresponding to the mass m_3, must possess a symmetry and forms a rhombus. Pérez–Chavela and Santoprete [31] extended this result to the case where two of the masses are equal and at most, only one of the remaining mass is larger than the equal masses. Moreover, they shown that there exists a unique convex central configuration when the opposite masses are equal and it is a rhombus. Albouy et. al. [5] later on proved that a convex central configuration is symmetric with respect to one diagonal if and only if the masses of the two particles on the other diagonal are equal. In addition to, if these two masses are unequal, then the less massive one is closer to the former diagonal.

Álvarez–Ramírez and Llibre [7] discussed the convex and concave central configurations with an axis of symmetry of the 4–body problem when the masses satisfy that $m_1 = m_2 \neq m_3 = m_4$. In the same vein, Érdi and Czirják [18] derived a complete solution in a symmetric case of the planar four-body central configurations, when two bodies are on an axis of symmetry, and the other two bodies have equal masses and are situated symmetrically with respect to the axis of symmetry.

Albouy and Fu in [4] (see also [27, 31]) conjectured: *For the planar 4–body problem there is a unique convex central configuration for each ordering of the masses in the boundary of the convex hull of their positions.* A particular case was considered before by MacMillan and Bartky [27], they proved that there is a unique isosceles trapezoid central configuration of the 4–body when two pairs of equal masses are located at adjacent vertices. Later on Xie in [38] reproved this result. In fact there was also the following subconjecture: *For the planar 4–body problem there is a unique convex central configuration having two pairs of equal masses located at the adjacent vertices of the configuration and it is an isosceles trapezoid.*

Using these previous results on the symmetries Corbera and Llibre [13] gave a complete description of the families of central configurations with two pairs of equals masses and two equal masses sufficiently small, proving for these masses the convex conjecture and the subconjecture. Recently Fernandes et al. [19] have proved the subconjecture for arbitrary masses.

The central configurations when the four masses are on a circle have been classified by Cors and Roberts [16], and also by Álvarez–Ramírez et al. [8]. Moreover, Corbera et al. [15] have shown that any 4-body convex central configuration with perpendicular diagonals must be a kite configuration. Recently, Santoprete [33] proved that for a given ordering of the mutual distances, a trapezoidal central configuration must have a certain partial ordering of the masses. He also showed that if opposite masses of a four-body trapezoidal central configuration are equal, then the configuration has a line of symmetry and it must be a kite. Later on, Corbera et al. [14] classified all planar 4-body central configurations where two pairs of the bodies are on parallel lines, that is, all trapezoidal central configurations.

The paper is organized as follows. In Section 2 we present the system of equations for Dziobek's equations of the equilic central configurations of the 4-body problem here studied. Finally, in Section 3 we prove our main result, see Theorem 1.

2. Dziobek's equations of the equilic central configurations

A quadrilateral ABCD is *equilic* if $AD = BC$ and if angle $A +$ angle $B = \alpha + \beta = 2\pi/3$, see Figure 1. If α or β is equal to $\pi/2$, the equilic quadrilateral is called *right equilic*. For more details about these kind of quadrilaterals consult [21].

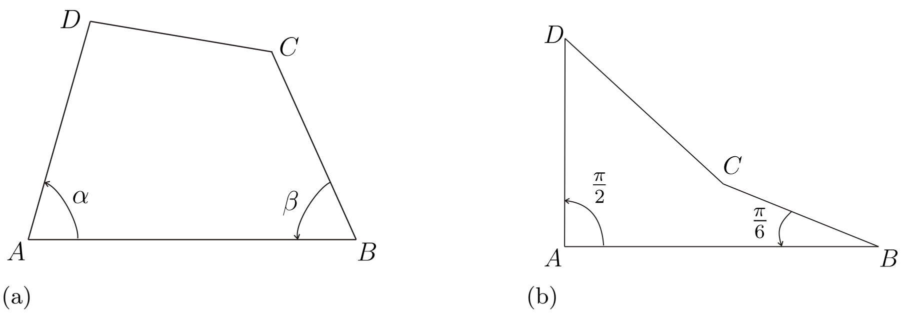

Figure 1. Equilic quadrilateral: (a) $\alpha + \beta = 2\pi/3$ and (b) right equilic.

In this section we give a derivation of the 4-body central configuration where four particles are at the vertices of an equilic quadrilateral, with m_1, m_2, m_3 and m_4 in clockwise order. Since we are interested in convex quadrilateral central configurations, we will not be considering the right equilic.

It is well-known that the set of central configurations are invariant under rotations and homotheties. So without loss of generality, we can assume that the positions of the vertices of the equilic quadrilateral are

$$\begin{array}{ll} x_1 = (0,0), & x_2 = (1,0), \\ x_3 = (1 - a\cos(2\pi/3 - \alpha), a\sin(2\pi/3 - \alpha)), & x_4 = (a\cos\alpha, a\sin\alpha), \end{array}$$

where a is a positive constant and $\alpha \in (0, \pi/3)$. Note that $\alpha < \pi/3$, because for $\alpha = \pi/3$ we should have a triangle instead of a quadrilateral.

Let $m_k > 0$ be the mass of the particle located at x_k for $k = 1, 2, 3, 4$. See Figure 2.

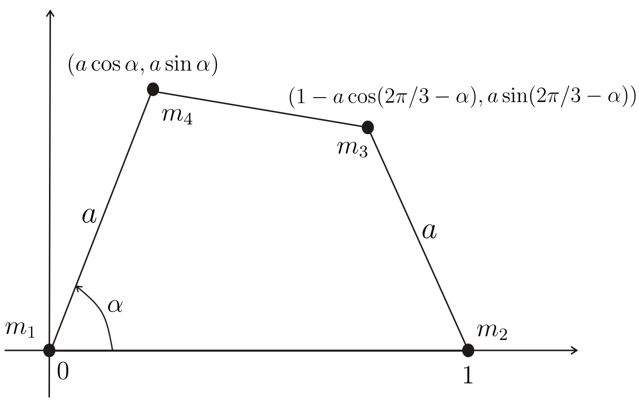

Figure 2. Coordinates of the 4-body configuration forming an equilic quadrilateral.

Let $x = (x_1, \dots, x_4) \in (\mathbb{R}^2)^4$ be the configuration vector and we associate to each x a 4×4 matrix

$$X = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_4 \\ 0 & \cdots & 0 \end{pmatrix}.$$

We define the 3×3 matrix X_k as the matrix obtained from the matrix X by deleting the k–th column and the last row. Then let $D_k = (-1)^{k+1}\det(X_k)$ be for $k = 1, \dots, 4$. D_k is twice the signed area of the triangle whose vertices contain all bodies except for the k-th body. It is easy to check that $D_1, D_3 > 0$ and $D_2, D_4 < 0$, satisfying the equation $D_1 + D_2 + D_3 + D_4 = 0$.

In [17] Dziobek studied shapes of central configurations for the planar 4-body problem using mutual distances as variables and obtained algebraic equations for determining the central configurations. By restricting the center of mass to the origin of coordinates, the Dziobek's equations are

$$\frac{1}{r_{ij}^3} = c_1 + c_2\frac{D_iD_j}{m_im_j},$$
$$t_i - t_j = 0,$$

for $1 \le i < j \le 4$, with

$$t_i = \sum_{j=1,\, j\neq i}^{4} D_j\, r_{ij}^2,$$

There are 12 unknowns, namely, 6 mutual distances r_{ij}, the 4 variables D_i, and the 2 constants c_1 and c_2.

Writing the first six Dziobek's equations (2) explicitly we obtain

$$\begin{aligned}
m_1m_2\left(r_{12}^{-3} - c_1\right) &= c_2D_1D_2,\\
m_1m_3\left(r_{13}^{-3} - c_1\right) &= c_2D_1D_3,\\
m_2m_3\left(r_{23}^{-3} - c_1\right) &= c_2D_2D_3,\\
m_1m_4\left(r_{14}^{-3} - c_1\right) &= c_2D_1D_4,\\
m_2m_4\left(r_{24}^{-3} - c_1\right) &= c_2D_2D_4,\\
m_3m_4\left(r_{34}^{-3} - c_1\right) &= c_2D_3D_4.
\end{aligned}$$

We multiply the equations (2) by row in order that the product of the right–hand side be simply $c_2^2D_1D_2D_3D_4$, and considering the fact that the masses are positive, we get the so called *Dziobeck relation*

$$(r_{12}^{-3} - c_1)(r_{34}^{-3} - c_1) = (r_{13}^{-3} - c_1)(r_{24}^{-3} - c_1) = (r_{14}^{-3} - c_1)(r_{23}^{-3} - c_1).$$

Combining any two of (2) we get

$$c_1 = \frac{r_{12}^{-3}\, r_{34}^{-3} - r_{13}^{-3}\, r_{24}^{-3}}{r_{12}^{-3} + r_{34}^{-3} - r_{13}^{-3} - r_{24}^{-3}} = \frac{r_{13}^{-3}\, r_{24}^{-3} - r_{14}^{-3}\, r_{23}^{-3}}{r_{13}^{-3} + r_{24}^{-3} - r_{14}^{-3} - r_{23}^{-3}} = \frac{r_{14}^{-3}\, r_{23}^{-3} - r_{12}^{-3}\, r_{34}^{-3}}{r_{14}^{-3} + r_{23}^{-3} - r_{12}^{-3} - r_{34}^{-3}}.$$

If we set

$$\begin{gathered}
s_1 = r_{12}^{-3} + r_{34}^{-3}, \qquad p_1 = r_{12}^{-3}r_{34}^{-3}, \qquad s_2 = r_{13}^{-3} + r_{24}^{-3},\\
p_2 = r_{13}^{-3}r_{24}^{-3}, \qquad s_3 = r_{14}^{-3} + r_{23}^{-3}, \qquad p_3 = r_{14}^{-3}r_{23}^{-3}.
\end{gathered}$$

Then equation (2) can be written as

$$c_1 \;=\; \frac{p_1 - p_2}{s_1 - s_2} \;=\; \frac{p_2 - p_3}{s_2 - s_3} \;=\; \frac{p_3 - p_1}{s_3 - s_1},$$

which means that (s_1, p_1), (s_2, p_2), (s_3, p_3) viewed as points in $(\mathbb{R}^+)^2$, must lie on the same line with slope c_1, or equivalently

$$\begin{vmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = 0.$$

Following Dziobek [17] it is customary to use the following planarity condition of the 4–body problem:

$$D = (r_{13}^3 - r_{12}^3)(r_{23}^3 - r_{34}^3)(r_{24}^3 - r_{14}^3) - (r_{12}^3 - r_{14}^3)(r_{24}^3 - r_{34}^3)(r_{13}^3 - r_{23}^3) = 0.$$

Now we are able to establish the main result of our article.

Theorem 1. *All points (α, a) of the open arc of the curve (2) in the variables (α, a) with endpoints $(\alpha, a) = (1, 0)$ and $(\alpha, a) = (\pi/6, 1/\sqrt{3})$ inside the region*

$$\mathcal{M} = \left\{ (\alpha, a) : 0 < \frac{1}{\cos\alpha + \sqrt{3}\sin\alpha} < a < 1 \quad \textit{with} \quad \alpha \in (0, \pi/3) \right\}.$$

correspond to all equilic quadrilateral central configurations.

3. Proof of Theorem 1

By convenience, but without loss of generality, from now on we will assume that $m_1 = 1$ changing the unit of mass. From (2) the masses m_2, m_3 and m_4 can be expressed in terms of the r_{ij} and the D_k as follows

$$\begin{aligned} m_2 &= \frac{D_2(r_{13}^3 - r_{14}^3)r_{23}^3 r_{24}^3}{D_1(r_{23}^3 - r_{24}^3)r_{13}^3 r_{14}^3}, \\ m_3 &= \frac{D_3(r_{12}^3 - r_{14}^3)r_{23}^3 r_{34}^3}{D_1(r_{23}^3 - r_{34}^3)r_{12}^3 r_{14}^3}, \\ m_4 &= \frac{D_4(r_{12}^3 - r_{13}^3)r_{24}^3 r_{34}^3}{D_1(r_{24}^3 - r_{34}^3)r_{12}^3 r_{13}^3}. \end{aligned}$$

According to MacMillan and Bartky [27] the longest and smaller sides of the quadrilateral correspond to opposite sides. Moreover, each side of the quadrilateral is shorter in length than both diagonals, i.e.

$$r_{13}, r_{24} > r_{12}, r_{14}, r_{23}, r_{34}.$$

It follows immediately from the geometry of the equilic quadrilateral, Figure 2, that the mutual distances and the areas can be written in terms of a and α as follows:

$$\begin{aligned} r_{12} &= 1, \quad r_{13} = \sqrt{1+a^2+a\cos\alpha-\sqrt{3}a\sin\alpha}, \\ r_{14} &= a, \quad r_{24} = \sqrt{1+a^2-2a\cos\alpha}, \\ r_{23} &= a, \quad r_{34} = \sqrt{1+a^2-a\cos\alpha-\sqrt{3}a\sin\alpha}, \end{aligned}$$

and

$$\begin{aligned} D_1 &= a(-\sqrt{3}a+\sqrt{3}\cos\alpha+\sin\alpha)/2, \quad & D_2 &= a(\sqrt{3}a-2\sin\alpha)/2, \\ D_3 &= a\sin\alpha, & D_4 &= -a\cos((\pi-6\alpha)/6)\,. \end{aligned}$$

Now it was already known that $D_1, D_3 > 0$ and $D_2, D_4 < 0$, and from (3) we get

$$r_{14} < r_{13}, \qquad r_{23} < r_{24}, \qquad r_{12} < r_{13}, \qquad r_{34} < r_{24}.$$

In order to become an equilic central configuration, the equilic quadrilateral must satisfy (2) and the masses (**??**) must be positive. From (3), it follows that m_2 and m_4 are positive for all $\alpha \in (0, \pi/3)$ and $a > 0$. Consequently we only need to check conditions for which m_3 is positive. This can happen only if

$$r_{12} > r_{14} \quad \text{and} \quad r_{23} > r_{34}$$

hold.

We remark that $r_{12} > r_{14}$ implies that $1 > a$. In addition, $m_3 = 0$ when $r_{12} = r_{14}$, and corresponds to $a = 1$. In fact m_2 and m_4 are also zero when $\alpha = 0$ and $a = 1$. Moreover, the condition that $m_3 = \infty$ is $r_{23} = r_{34}$, which can be expressed as

$$a = \frac{1}{\cos\alpha + \sqrt{3}\sin\alpha}.$$

This curve intersects the line $a = 1$ at the point $(\alpha, a) = (0, 1)$. Putting these observations together, we find that the inequalities (3) hold in the region $\mathcal{M}$. This region is the one shown in Figure 3, in which α is the abscissa and a is the ordinate.

For solving system (2) we replace the expressions of the masses (**??**), and taking only the numerators of these six equations because the denominators do not vanish, we obtain

$$\begin{aligned} e_1 = -D_2(c_1 r_{12}^3 r_{13}^3 r_{23}^3 r_{24}^3 - c_1 r_{12}^3 r_{14}^3 r_{23}^3 r_{24}^3 + c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^3 r_{23}^3 \\ - c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^3 r_{24}^3 - r_{13}^3 r_{23}^3 r_{24}^3 + r_{14}^3 r_{23}^3 r_{24}^3) = 0, \end{aligned}$$

$$\begin{aligned} e_2 = -D_3(c_1 r_{12}^3 r_{13}^3 r_{23}^3 r_{34}^3 - c_1 r_{13}^3 r_{14}^3 r_{23}^3 r_{34}^3 + c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^3 r_{23}^3 \\ - c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^3 r_{34}^3 - r_{12}^3 r_{23}^3 r_{34}^3 + r_{14}^3 r_{23}^3 r_{34}^3) = 0, \end{aligned}$$

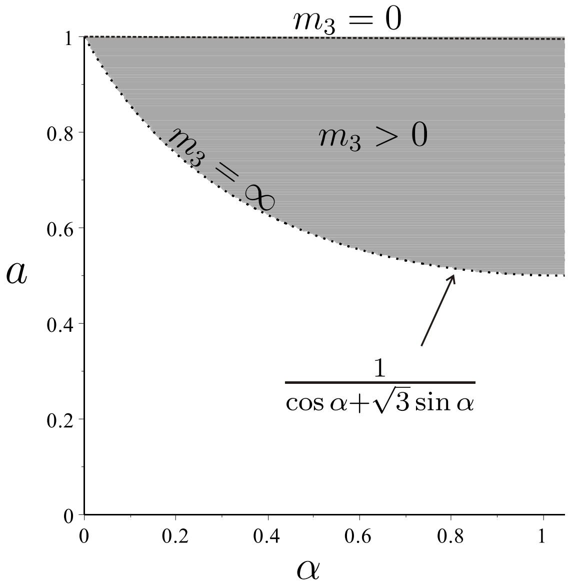

Figure 3. The gray shaded indicate the region in the (α, a)–plane where m_3 is positive.

$$\begin{aligned} e_3 = -D_2D_3(&c_1r_{12}^3r_{13}^3r_{23}^6r_{24}^3r_{34}^3 - c_1r_{12}^3r_{14}^3r_{23}^6r_{24}^3r_{34}^3 - c_1r_{13}^3r_{14}^3r_{23}^6r_{24}^3r_{34}^3 \\ &+ c_1r_{14}^6r_{23}^6r_{24}^3r_{34}^3 + c_2D_1^2r_{12}^3r_{13}^3r_{14}^6r_{23}^6 - c_2D_1^2r_{12}^3r_{13}^3r_{14}^6r_{23}^3r_{24}^3 \\ &- c_2D_1^2r_{12}^3r_{13}^3r_{14}^6r_{23}^3r_{34}^3 + c_2D_1^2r_{12}^3r_{13}^3r_{14}^6r_{24}^3r_{34}^3 - r_{12}^3r_{13}^3r_{23}^3r_{24}^3r_{34}^3 \\ &+ r_{12}^3r_{14}^3r_{23}^3r_{24}^3r_{34}^3 + r_{13}^3r_{14}^3r_{23}^3r_{24}^3r_{34}^3 - r_{14}^6r_{23}^3r_{24}^3r_{34}^3) = 0, \end{aligned}$$

$$\begin{aligned} e_4 = -D_4(&c_1r_{12}^3r_{14}^3r_{24}^3r_{34}^3 - c_1r_{13}^3r_{14}^3r_{24}^3r_{34}^3 + c_2D_1^2r_{12}^3r_{13}^3r_{14}^3r_{24}^3 \\ &- c_2D_1^2r_{12}^3r_{13}^3r_{14}^3r_{34}^3 - r_{12}^3r_{24}^3r_{34}^3 + r_{13}^3r_{24}^3r_{34}^3) = 0, \end{aligned}$$

$$\begin{aligned} e_5 = -D_2D_4(&c_1r_{12}^3r_{13}^3r_{23}^3r_{24}^6r_{34}^3 - c_1r_{12}^3r_{14}^3r_{23}^3r_{24}^6r_{34}^3 - c_1r_{13}^6r_{23}^3r_{24}^6r_{34}^3 \\ &+ c_1r_{13}^3r_{14}^3r_{23}^3r_{24}^6r_{34}^3 + c_2D_1^2r_{12}^3r_{13}^6r_{14}^3r_{23}^3r_{24}^3 - c_2D_1^2r_{12}^3r_{13}^6r_{14}^3r_{23}^3r_{34}^3 \\ &- c_2D_1^2r_{12}^3r_{13}^6r_{14}^3r_{24}^6 + c_2D_1^2r_{12}^3r_{13}^6r_{14}^3r_{24}^3r_{34}^3 - r_{12}^3r_{13}^3r_{23}^3r_{24}^3r_{34}^3 \\ &+ r_{12}^3r_{14}^3r_{23}^3r_{24}^3r_{34}^3 + r_{13}^6r_{23}^3r_{24}^3r_{34}^3 - r_{13}^3r_{14}^3r_{23}^3r_{24}^3r_{34}^3) = 0, \end{aligned}$$

$$\begin{aligned} e_6 = -D_3D_4(&c_1r_{12}^6r_{23}^3r_{24}^3r_{34}^6 - c_1r_{12}^3r_{13}^3r_{23}^3r_{24}^3r_{34}^6 - c_1r_{12}^3r_{14}^3r_{23}^3r_{24}^3r_{34}^6 \\ &+ c_1r_{13}^3r_{14}^3r_{23}^3r_{24}^3r_{34}^6 + c_2D_1^2r_{12}^6r_{13}^3r_{14}^3r_{23}^3r_{24}^3 - c_2D_1^2r_{12}^6r_{13}^3r_{14}^3r_{23}^3r_{34}^3 \\ &- c_2D_1^2r_{12}^6r_{13}^3r_{14}^3r_{24}^3r_{34}^3 + c_2D_1^2r_{12}^6r_{13}^3r_{14}^3r_{34}^6 - r_{12}^6r_{23}^3r_{24}^3r_{34}^3 \\ &+ r_{12}^3r_{13}^3r_{23}^3r_{24}^3r_{34}^3 + r_{12}^3r_{14}^3r_{23}^3r_{24}^3r_{34}^3 - r_{13}^3r_{14}^3r_{23}^3r_{24}^3r_{34}^3) = 0. \end{aligned}$$

We known that all the D_ℓ are non-zero, from $e_1 = 0$ and $e_2 = 0$ we isolate c_1 and c_2, and we substitute these two values into the other four equations obtaining the system

$$
\begin{aligned}
e_{31} &= \frac{D}{d} D_2 D_3 r_{23}^6 r_{24}^3 r_{34}^3 (r_{12} - r_{14})(r_{12}^2 + r_{12}r_{14} + r_{14}^2)(r_{14} - r_{13})(r_{13}^2 + r_{13}r_{14} + r_{14}^2) = 0,\\
e_{41} &= 0,\\
e_{51} &= \frac{D}{d} D_2 D_4 r_{23}^3 r_{24}^6 r_{34}^3 (r_{12} - r_{13})(r_{12}^2 + r_{12}r_{13} + r_{13}^2)(r_{13} - r_{14})(r_{13}^2 + r_{13}r_{14} + r_{14}^2) = 0,\\
e_{61} &= \frac{D}{d} D_3 D_4 r_{23}^3 r_{24}^3 r_{34}^6 (r_{12} - r_{13})(r_{12}^2 + r_{12}r_{13} + r_{13}^2)(r_{12} - r_{14})(r_{12}^2 + r_{12}r_{14} + r_{14}^2) = 0
\end{aligned}
$$

where $D = 0$ is the Dziobek equation (2), and

$$
d = -r_{12}^3 r_{13}^3 r_{23}^3 r_{24}^3 + r_{12}^3 r_{13}^3 r_{23}^3 r_{34}^3 + r_{12}^3 r_{14}^3 r_{23}^3 r_{24}^3 - r_{12}^3 r_{14}^3 r_{24}^3 r_{34}^3 - r_{13}^3 r_{14}^3 r_{23}^3 r_{34}^3 + r_{13}^3 r_{14}^3 r_{24}^3 r_{34}^3
$$

From the geometry of the equilic quadrilateral configuration we have that the areas D_1, D_2, D_3, D_4 are different from zero, as well as $r_{12} = r_{13}$ and $r_{13} = r_{14}$ are not allowed, because binary collisions have been excluded, while $r_{12} = r_{14}$ corresponds to square configuration. Thus, $e_{31} = 0$, $e_{51} = 0$ and $e_{61} = 0$ are satisfied if and only if $D = 0$.

On substituting the values (3) and (3) in Dziobek equation (2), this is expressed in the variables (α, a) as

$$
\begin{aligned}
D = (1 - a^3)&((a^2 - 2a\cos\alpha + 1)^{3/2} - (a^2 - \sqrt{3}a\sin\alpha - a\cos\alpha + 1)^{3/2})\\
&(a^3 - (a^2 - \sqrt{3}a\sin\alpha + a\cos\alpha + 1)^{3/2})\\
&+ (a^3 - (a^2 - 2a\cos\alpha + 1)^{3/2})(1 - (a^2 - \sqrt{3}a\sin\alpha + a\cos\alpha + 1)^{3/2})\\
&(a^3 - (a^2 - a(\sqrt{3}\sin\alpha + \cos\alpha) + 1)^{3/2}) = 0.
\end{aligned}
$$

We have only found some necessary conditions for the existence of 4-body central configurations of the prescribed type. We now have to see if such configurations actually exist. In Figure 4 we provide numerically the graphic of the curve $D = 0$. This figure shows that the curve $D = 0$ has a non-empty intersection with the region $\mathcal{M}$ of positive masses.

From the previous analysis, $a = 1$ is the unique solution of $m_3 = 0$ for $\alpha \in [0, \pi/3]$. Then substituting $a = 1$ in $D = 0$ we obtain

$$
\begin{aligned}
\left(1 - (2 - 2\cos\alpha)^{3/2}\right)&\left(1 - \left(-\sqrt{3}\sin\alpha - \cos\alpha + 2\right)^{3/2}\right)\\
&\left(1 - \left(-\sqrt{3}\sin\alpha + \cos\alpha + 2\right)^{3/2}\right) = 0,
\end{aligned}
$$

which has $\alpha = 0$ as the unique solution in $(0, \pi/3)$.

Now we note that when $a = \frac{1}{\cos\alpha + \sqrt{3}\sin\alpha}$ we have $D = 0$ only at $\alpha = \pi/6$. Thus the curves $m_3 = \infty$ and $D = 0$ intersect at one point, namely $(\alpha, a) =$

$(\pi/6, 1/\sqrt{3})$. Hence we have an equilic quadrilateral central configuration for each point (α, a) of the curve $D = 0$ contained in the region $\mathcal{M}$ defined in the statement of Theorem 1, excluding the endpoints $(0, 1)$ and $(\pi/6, 1/\sqrt{3}) = 0$ contained in the boundary of the region $\mathcal{M}$. This completes the proof of Theorem 1.

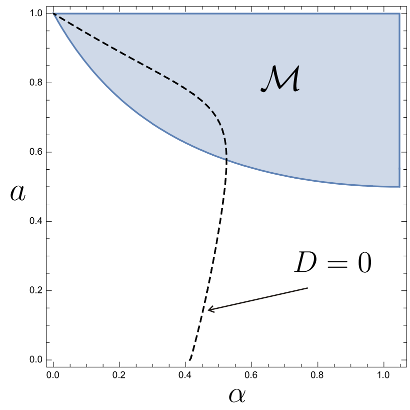

Figure 4. Curve $D = 0$ and the positive masses region $\mathcal{M}$.

In Figure 5 we show the degenerate quadrilateral configuration correspoinding to the values $(\alpha, a) = (0, 1)$ and $(\frac{\pi}{6}, \frac{1}{\sqrt{3}})$.

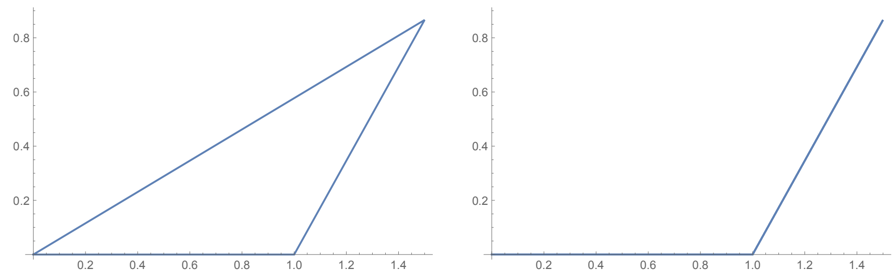

Figure 5. Degenerate configurations associated with $(\alpha, a) = (0, 1)$ and $(\frac{\pi}{6}, \frac{1}{\sqrt{3}})$, respectively.

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