This is a preprint of: "Bicentric quadrilateral central configurations", Jaume Llibre, Pengfei Yuan, *Appl. Math. Comput.*, vol. 362, 124507: 1–7, 2019.
DOI: [10.1016/j.amc.2019.06.021]

BICENTRIC QUADRILATERAL CENTRAL CONFIGURATIONS

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ABSTRACT. A bicentric quadrilateral is a tangential cyclic quadrilateral. In a tangential quadrilateral the four sides are tangents to an inscribed circle, and in a cyclic quadrilateral the four vertices lie on a circumscribed circle. In this paper we classify all planar central configurations of the 4-body problem, where the four bodies are at the vertices of a bicentric quadrilateral.

1. Introduction and statement of the results

The well-known Newtonian n-body problem concerns with the motion of n mass points with positive mass m_i moving under their mutual attraction in $\mathbb{R}^d$ in accordance with Newton's law of gravitation.

The equations of the motion of the n-body problem are :

$$\ddot{x}_i = -\sum_{j=1, j\neq i}^{n} \frac{m_j(x_i - x_j)}{r_{ij}^3}, \quad 1 \leq i \leq n,$$

where we have taken the unit of time in such a way that the Newtonian gravitational constant be one, and $x_i \in \mathbb{R}^d (i = 1, \ldots, n)$ denotes the position vector of the i-body, $r_{ij} = |x_i - x_j|$ is the Euclidean distance between the i-body and the j-body.

Alternatively the equations of the motion can be written

$$m_i \ddot{x}_i = \nabla_i U(x), \quad 1 \leq i \leq n,$$

where $x = (x_1, \ldots, x_n)$, and

$$U(x) = \sum_{1\leq i<j\leq n} \frac{m_i m_j}{|x_i - x_j|},$$

is the potential of the system.

2010 *Mathematics Subject Classification.* 70F07,70F15.

Key words and phrases. Convex central configuration, four-body problem, bicentric quadrilateral.

The solutions of the 2-body problem (also called the Kepler problem) has been completely solved. Unfortunately the solutions for the n-body ($n \geq 3$) is still an open problem.

For the Newtonian n-body problem the simplest possible motions are such that the configuration is constant up to rotations and scaling. Only some special configurations of particles are allowed in such motions called homographic solutions. Wintner [47] called them central configurations.

More precisely, let

$$M = m_1 + \cdots + m_n, \quad c = \frac{m_1 x_1 + \cdots + m_n x_n}{M},$$

be the *total mass* and the *center of masses* of the n bodies, respectively.

A *configuration* x is called a *central configuration* if the acceleration vectors of the n bodies are proportional to their positions with respect to the center of masses with the same constant of proportionality, i.e.

$$\sum_{j=1, j\neq i}^{n} \frac{m_j (x_j - x_i)}{r_{ij}^3} = \lambda (x_i - c), \quad 1 \leq i \leq n, \tag{1}$$

where λ is the constant of proportionality.

Equations (1) are invariant under rotations, dilatations and translations on the plane. Two central configurations are related if we can pass from one to the other doing some of the mentioned transformations. This relation is of equivalence. When we talk about the number of central configurations we will talk about the number of classes of equivalence of central configurations.

Central configurations play an important role in Celestial Mechanics, for more details see([18, 23, 34, 36, 42, 45, 47].

There is an extensive literature on the study of central configurations, see Euler [20], Lagrange [27], Albouy and Chenciner [3], Albouy and Fu [4], Albouy and Kaloshin [6], Hampton and Moeckel [25], Llibre [30], Moulton [37] Palmore [38], Schmidt [43], Smale [45], Xia [48, 49], Xie [50], ...

In this paper we are interested in the planar 4-body problem. For the 4-body problem the exact number and classification of central configurations remain open, only some partial results are obtained. There is a good numerical study on the central configurations of the 4-body prolem, see Simó [44]. The finiteness for the general 4-body problem was settled by Hampton and Moeckel using an assisted proof by computer [25], and Albouy and Kaloshin [6] provided an analytical proof.

For $m_1 = m_2 = m_3 = m_4$ Llibre found all the planar central configurations assuming the central configurations have an axis of symmetry, see [29]. Later on Albouy proved the existence of such symmetry and provide a more analytical proof.

For the case of three equal masses Bernat at al. classified the non-collinear kite central configurations, see [13], also see [28].

A quadrilateral is *convex* if none of the vertices is located in the interior of the triangle formed by the other three vertices, otherwise it is *concave*.

For $m_1 = m_2$ and $m_3 = m_4$, Long and Sun [9] proved some symmetry of the central configurations. Perez-Chavela and Santoprete [40] proved that there is a unique convex non-collinear central configuration of planar 4-body problem when two equal masses are located at opposite vertices of a quadrilateral and, at most, only one of the remaining masses is larger than the equal masses.

When one of the 4 masses is sufficiently small, Pedersen [39], Barros and Leandro [11, 12] found the classes of central configurations of the 4-body, see also Gannaway [22] and Arenstorf [10].

For $m_1 = m_2 \neq m_3 = m_4$ Álvarez and Llibre [7] characterized the convex and concave central configurations with an axis of symmetry.

Corbera and Llibre [14] gave a complete description of the families of central configurations with two pairs of equal masses when two equal masses are sufficiently small.

Recently Álvarez and Llibre [7] classified Hjelmslev quadrilateral central configurations. A *Hjelmslev quadrilateral* is a quadrilateral with two right angles at opposite vertices.

A *bicentric quadrilateral* is a tangential cyclic quadrilateral. In a *tangential quadrilateral* the four sides are tangents to an inscribed circle, and in a *cyclic quadrilateral* the four vertices lie on a circumscribed circle, see Figure 1.

A *kite quadrilateral* is a quadrilateral that two pairs of adjacent sides are of equal length.

In [15] Cors and Robert studied the case when four masses are located at the vertices of a cyclic quadrilateral, see also [9].

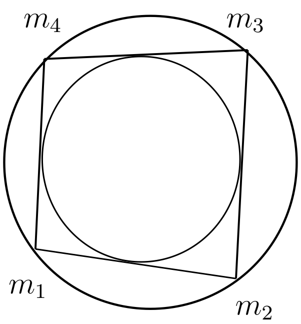

FIGURE 1. A bicentric quadrilateral.

In this paper we want to classify the bicentric quadrilateral central configurations, i.e. we will improve the classification of the cyclic quadrilateral central configurations provided by Cors and Robert showing which of these central configurations are also tangential quadrilaterals. We characterize all central configurations of the 4-body problem with the four bodies at the vertices of a bicentric quadrilateral.

Theorem 1. *Without loss of generality we can take positive masses for the* 4*-body problem with* $m_1 = 1$ *and* $r_{12} = 1$*, then*

(a) *we have a bicentric quadrilateral central configurations with positive masses satisfying* $m_1 = m_3 = 1$, $r_{12} = r_{23} = 1$, $r_{14} = r_{34}$, $r_{24} = \sqrt{1 + r_{14}^2}$, $r_{13} = 2r_{14}/\sqrt{1 + r_{14}^2}$, *and* $r_{14} \in (\frac{\sqrt{3}}{3}, 1]$,

(b) *the shape of a bicentric quadrilateral central configurations is a kite quadrilateral with* r_{14} *perpendicular to* r_{12}*, and* r_{23} *perpendicular to* r_{34}*, see Figure 2.*

Theorem 1 is proved in Section 3.

We note that from statement (b) of Theorem 1 any bicentric quadrilateral central configurations is a kite Hjelmslev quadrilateral central configuration.

2. Preliminaries

Let $x = (x_1, x_2, x_3, x_4) \in (\mathbb{R}^2)^4$. We associated with x the matrix:

$$X = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_4 \\ 0 & \cdots & 0 \end{pmatrix}.$$

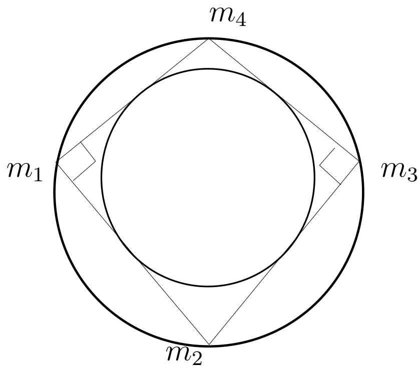

FIGURE 2. The shape of a bicentric quadrilateral central configuration is also a kite Hjelmslev quadrilateral.

X_k denotes the matrix obtained deleting from the matrix X its k-th column and its last row. Then let $D_k = (-1)^{k+1}\det(X_k)$ for $k = 1, \ldots, 4$.

D_k is twice the signed area of the triangle whose vertices contain all bodies except the k-th body, and from the quadrilateral of Figure 1 we have

$$D_1,\, D_3 > 0, \quad D_2,\, D_4 < 0, \qquad D_1 + D_2 + D_3 + D_4 = 0.$$

The equations for the central configurations (1) of the 4-body problem were written by Dziobek [16] (see also equations (8) and (16) of Moeckel [34] or [22]) as the following 12 equations with 12 unknowns.

$$\begin{aligned} \frac{1}{r_{ij}^3} &= c_1 + c_2 \frac{D_i D_j}{m_i m_j}, \\ t_i - t_j &= 0, \end{aligned} \tag{2}$$

for $1 \le i < j \le 4$, where

$$t_i = \sum_{j=1, j\neq i}^{4} D_j r_{ij}^2.$$

The unknowns of equations (3) are the mutual distances r_{ij}, the variables D_i, and the constants c_k $(k = 1, 2)$.

The first six Dziobek's equation (2) are

$$\begin{aligned} m_1 m_2 (r_{12}^{-3} - c_1) &= c_2 D_1 D_2, & m_3 m_4 (r_{34}^{-3} - c_1) &= c_2 D_3 D_4, \\ m_1 m_3 (r_{13}^{-3} - c_1) &= c_2 D_1 D_3, & m_2 m_4 (r_{24}^{-3} - c_1) &= c_2 D_2 D_4, \\ m_1 m_4 (r_{14}^{-3} - c_1) &= c_2 D_1 D_4, & m_2 m_3 (r_{23}^{-3} - c_1) &= c_2 D_2 D_3. \end{aligned} \tag{3}$$

Multiplying the two equations which appear in each of the three rows of equations (3) we obtain the same expression $c_2^2 D_1 D_2 D_3 D_4$, which provide the Dziobek relation:

$$(r_{12}^{-3}-c_1)(r_{34}^{-3}-c_1)=(r_{13}^{-3}-c_1)(r_{24}^{-3}-c_1)=(r_{14}^{-3}-c_1)(r_{23}^{-3}-c_1). \quad (4)$$

We can solve c_1 from the Dziobek relation (4) and we have

$$\begin{aligned} c_1 &= \frac{r_{12}^{-3}r_{34}^{-3}-r_{13}^{-3}r_{24}^{-3}}{r_{12}^{-3}+r_{34}^{-3}-r_{13}^{-3}-r_{24}^{-3}} \\ &= \frac{r_{13}^{-3}r_{24}^{-3}-r_{14}^{-3}r_{23}^{-3}}{r_{13}^{-3}+r_{24}^{-3}-r_{14}^{-3}-r_{23}^{-3}} \\ &= \frac{r_{14}^{-3}r_{23}^{-3}-r_{12}^{-3}r_{34}^{-3}}{r_{14}^{-3}+r_{23}^{-3}-r_{12}^{-3}-r_{34}^{-3}}. \end{aligned} \quad (5)$$

Defining

$$\begin{aligned} s_1 &= r_{12}^{-3}+r_{34}^{-3}, & p_1 &= r_{12}^{-3}r_{34}^{-3}, \\ s_2 &= r_{13}^{-3}+r_{24}^{-3}, & p_2 &= r_{13}^{-3}r_{24}^{-3}, \\ s_3 &= r_{14}^{-3}+r_{23}^{-3}, & p_3 &= r_{14}^{-3}r_{23}^{-3}, \end{aligned}$$

equation (5) become

$$c_1=\frac{p_1-p_2}{s_1-s_2}=\frac{p_2-p_3}{s_2-s_3}=\frac{p_3-p_1}{s_3-s_1}, \quad (6)$$

which imply that the points (s_1,p_1), (s_2,p_2), (s_3,p_3) of the plane are on the same straight line with slope c_1, i.e.

$$\begin{vmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = 0.$$

So we can write *Dziobek equation* as

$$D=(r_{13}^3-r_{12}^3)(r_{23}^3-r_{34}^3)(r_{24}^3-r_{14}^3)-(r_{12}^3-r_{14}^3)(r_{24}^3-r_{34}^3)(r_{13}^3-r_{23}^3)=0. \quad (7)$$

Lemma 2. *If the four masses are at the vertices of a bicentric quadrilateral as in Figure* 1*, then*

$$\begin{aligned} r_{23} &= 1-r_{14}+r_{34}, \\ r_{34} &= \frac{r_{14}^2+r_{13}r_{24}-r_{14}}{1+r_{14}}. \end{aligned} \quad (8)$$

Proof. For a bicentric quadrilateral as it is shown in Figure 1 the four sides are tangents to an inscribed circle, and the four vertices lie on a circumscribed circle.

According to the Pitot theorem [26], we have

$$r_{12} + r_{34} = r_{23} + r_{14}, \tag{9}$$

and by the Ptolemy's theorem [17], we have

$$r_{13}r_{24} = r_{12}r_{34} + r_{23}r_{14}. \tag{10}$$

From (9) we can get

$$r_{23} = r_{12} + r_{34} - r_{14}, \tag{11}$$

then substitute (11) into (10), we obtain

$$r_{34} = \frac{r_{14}^2 + r_{13}r_{24} - r_{14}}{1 + r_{14}}. \tag{12}$$

□

3. Proof of Theorem 1

Since we study classes of central configurations, without loss of generality, we can assume $m_1 = 1$, $r_{12} = 1$, and that r_{12} is the longest side of the quadrilateral.

From (3) we can obtain the masses expressed in terms of the distances r_{ij} and the areas D_k, i.e.

$$\begin{aligned} m_2 &= \frac{D_2 r_{23}^3 r_{24}^3 (r_{13}^3 - r_{14}^3)}{D_1 r_{13}^3 r_{14}^3 (r_{23}^3 - r_{24}^3)}, \\ m_3 &= \frac{D_3 r_{23}^3 r_{34}^3 (1 - r_{14}^3)}{D_1 r_{14}^3 (r_{23}^3 - r_{34}^3)}, \\ m_4 &= \frac{D_4 r_{24}^3 r_{34}^3 (1 - r_{13}^3)}{D_1 r_{13}^3 (r_{24}^3 - r_{34}^3)}. \end{aligned} \tag{13}$$

Substituting these masses into the first six Dziobek equations (3), and taking only the numerators of these six equations because the denominators do not vanish, we have

$$\begin{aligned} e_1 = \;& D_2(c_2 D_1^3 r_{12}^3 r_{13}^3 r_{14}^2 (r_{24}^3 - r_{23}^3) - r_{23}^3 r_{24}^3 (c_1 r_{12}^3 - 1)(r_{13}^3 - r_{14}^3)), \\ e_2 = \;& D_3(c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^3 (r_{34}^3 - r_{23}^3) - r_{23}^3 r_{34}^3 (c_1 r_{13}^3 - 1)(r_{12}^3 - r_{14}^3)), \\ e_3 = \;& D_2 D_3 (r_{23}^3 r_{24}^3 r_{34}^3 (c_1 r_{23}^3 - 1)(r_{12}^3 r_{14}^3)(r_{14}^3 - r_{13}^3) - \\ & c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^6 (r_{23}^3 - r_{24}^3)(r_{23}^3 - r_{34}^3)), \\ e_4 = \;& D_4(c_2 D_1^2 r_{12}^3 r_{13}^3 r_{14}^3 (r_{34}^3) - r_{24}^3 - r_{24}^3 r_{34}^3 (c_1 r_{14}^3 - 1)(r_{12}^3 - r_{13}^3)), \end{aligned} \tag{14}$$

$$
\begin{aligned}
e_5 = \ & D_2 D_4(-r_{23}^3 r_{24}^3 r_{34}^3 (c_1 r_{24}^3 - 1)(r_{12}^3 - r_{13}^3)(r_{13}^3 - r_{14}^3) - \\
& c_2 D_1^2 r_{12}^3 r_{13}^6 r_{14}^3 (r_{23}^3 - r_{24}^3)(r_{24}^3 - r_{34}^3)), \\
e_6 = \ & D_3 D_4 (c_2 D_1^2 r_{12}^6 r_{13}^3 r_{14}^3 (r_{23}^3 - r_{34}^3)(r_{34}^3 - r_{24}^3) - \\
& r_{23}^3 r_{24}^3 r_{34}^3 (c_1 r_{34}^3 - 1)(r_{12}^3 - r_{13}^3)(r_{12}^3 - r_{14}^3)).
\end{aligned}
$$

We remark that the last six equations of (2) are identically zero when we substitute the coordinates of the four bodies of Figure 1.

Notice that $D_i (i = 1, 2, 3, 4)$ is non-zero, so we can eliminate the $D_i' s$ which appear as a factor in equation (14). First, we solve the first two equations with respect to c_1 and c_2, and then we substitute c_1 and c_2 in the last four equations of (14). We obtain

$$
\begin{aligned}
e_3 &= \frac{D}{d} r_{23}^6 r_{24}^3 r_{34}^3 (r_{14}^3 - r_{12}^3)(r_{14}^3 - r_{13}^3), \\
e_4 &= 0, \\
e_5 &= \frac{D}{d} r_{23}^3 r_{24}^6 r_{34}^3 (r_{13}^3 - r_{12}^3)(r_{13}^3 - r_{14}^3), \\
e_6 &= \frac{D}{d} r_{23}^3 r_{24}^3 r_{34}^6 (r_{13}^3 - r_{12}^3)(r_{12}^3 - r_{14}^3),
\end{aligned} \tag{15}
$$

where $D = 0$ is the Dziobek equation (7), and

$$
d = r_{12}^3 (r_{13}^3 r_{23}^3 (r_{24}^3 - r_{34}^3) + r_{14}^3 r_{24}^3 (r_{34}^3 - r_{23}^3)) + r_{13}^3 r_{14}^3 r_{34}^3 (r_{23}^3 - r_{24}^3),
$$

is the denominator which comes from the denominators of c_1 and c_2.

From MacMillan and Bartky [33] we know that for every convex central configuration we have the following inequalities

$$
r_{13},\, r_{24} > r_{12},\, r_{23},\, r_{34},\, r_{14}. \tag{16}
$$

Since $D_1,\ D_3 > 0, D_2,\ D_4 < 0$, and $r_{13} > r_{12},\ r_{13} > r_{14}$, the solutions of system (15) are satisfied if and only if $D = 0$.

Solving the Dziobeck equation (8), after substituting $r_{12} = 1,\ r_{23},\ r_{34}$ from Lemma 1, we get the solution

$$
r_{13} = \frac{2 r_{14}}{r_{24}}. \tag{17}
$$

Then from (8) and (17), for a bicentric quadrilateral central configuration, we have

$$
r_{12} = r_{23} = 1, \quad r_{14} = r_{34}, \quad r_{13} = \frac{2 r_{14}}{r_{24}}. \tag{18}
$$

So the bicentric quadrilateral central configurations are also kite quadrilaterals and from (13) we obtain $m_1 = m_3$.

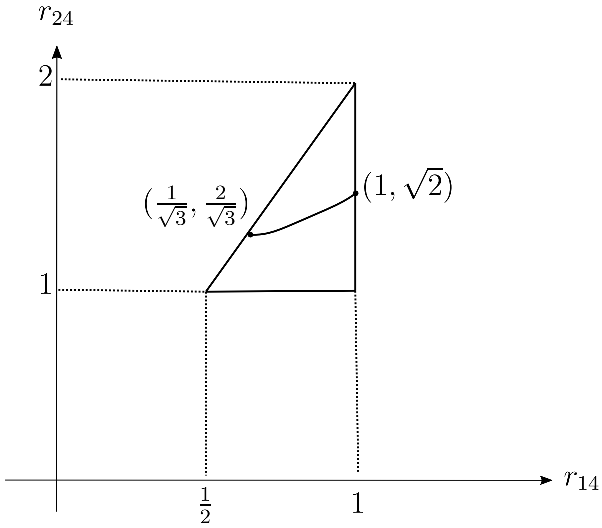

FIGURE 3.

Since the four bodies are co-circular, and a convex quadrilateral is cyclic if and only if its opposite angles are supplementary [46], we can deduce r_{14} and r_{12} are perpendicular, r_{23} and r_{34} are perpendicular.

Then we have

$$r_{12} = r_{23} = 1, \quad r_{34} = r_{14}, \quad r_{13} = \frac{2r_{14}}{\sqrt{1 + r_{14}^2}} \quad r_{24} = \sqrt{1 + r_{14}^2}. \quad (19)$$

In order to find the bicentric quadrilateral central configurations with positive masses, from (13), (16), (19) and since $r_{12} = 1$ is the longest side of the quadrilateral the following conditions hold

$$1 = r_{23} < r_{24},$$
$$r_{34},\, r_{14} < r_{13},\, r_{24},$$
$$r_{14} \leq 1,\, 1 < r_{13} = \frac{2r_{14}}{r_{24}}.$$

After some simplification, we obtain

$$r_{14} \leq 1 < r_{24} < 2r_{14} < 2. \quad (20)$$

Denote $\Lambda = \{(r_{14}, r_{24}) \,|\, r_{14} \leq 1 < r_{24} < 2r_{14} < 2\}$.

By simple computation, the intersection point of $r_{24} = 2r_{14}$ with $r_{24} = \sqrt{1 + r_{14}^2}$ is $(1/\sqrt{3}, 2/\sqrt{3})$, and the intersection point of $r_{14} = 1$ with $r_{24} = \sqrt{1 + r_{14}^2}$ is $(1, \sqrt{2})$, then we obtain $r_{14} \in [1/\sqrt{3}, 1]$. Then for each $r_{14} \in (1/\sqrt{3}, 1)$ we have a bicentric quadrilateral central configurations with positive masses satisfying (19).

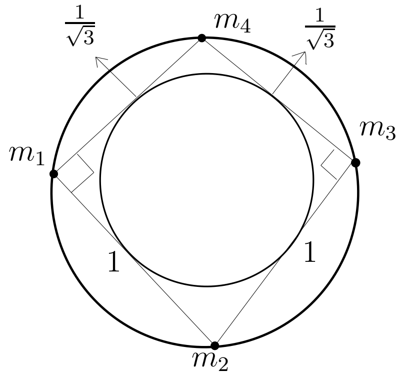

FIGURE 4.

Now we shall study if we have or not a bicentric quadrilateral central configuration when $r_{14} = 1/\sqrt{3}$ and $r_{14} = 1$.

Case for $r_{14} = 1/\sqrt{3}$. From (19), we obtain

$$r_{12} = r_{23} = 1, \quad r_{14} = r_{34} = 1/\sqrt{3}, \quad r_{13} = 1, \quad r_{24} = 2/\sqrt{3} \tag{21}$$

Next we check the masses for this central configuration. From (13) and (21), we have that $m_2 > 0, \quad m_3 = 1, \quad m_4 = 0$, and the central configuration in this case is the kite of Figure 4 but with $m_4 = 0$.

Case for $r_{14} = 1$. From (19), we obtain

$$r_{12} = r_{23} = r_{34} = r_{14} = 1, \quad r_{13} = r_{24} = \sqrt{2}, \tag{22}$$

so the configuration is a square. Thus the bicentric quadrilateral central configuration satisfying (22) corresponds to the square central configuration with $m_1 = m_2 = m_3 = m_4 = 1$.

In summary we have bicentric quadrilateral central configurations with positive masses for r_{ij} satisfying

$$r_{12} = r_{23} = 1, \quad r_{34} = r_{14}, \quad r_{24} = \sqrt{1 + r+_{14}^2}, \quad r_{13} = \frac{2r_{14}}{\sqrt{1 + r+_{14}^2}},$$

where $r_{14} \in (1/\sqrt{3} < r_{14} \leq 1]$. This proves the statement (a) of Theorem 1.

ACKNOWLEDGEMENTS

This work is supported by the Ministerio de Economa, Industria y Competitividad, Agencia Estatal de Investigacin grant MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de

Recerca grant 2017 SGR 1617, and the European project Dynamics-H2020-MSCA-RISE-2017-777911.

The second author is partially supported by Fundamental Research Funds for the Central Universities (NO.XDJK2015C139), China Scholarship Council(N0.201708505030).

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