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# Limit cycles in continuous and discontinuous piecewise linear differential systems with two pieces separated by a straight line

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**Abstract.** This paper is a survey on the study of the maximum number of limit cycles of planar continuous and discontinuous piecewise linear differential systems defined in two half–planes separated by a straight line $L$. We restrict our attention to the crossing limit cycles, i.e. to the limit cycles having exactly two points on the straight line $L$. We summarize the results known by now and describe the tools for obtaining them.

**Mathematics subject classification:** 34C05, 34C07, 37G15.
**Keywords and phrases:** Discontinuous differential system, continuous differential system, limit cycle, piecewise linear differential system.

## 1 Introduction and results

Every continuous or discontinuous piecewise linear differential system with two pieces separated by a straight line in the plane $\mathbb{R}^2$ after a linear change of variables can be written into the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^- & a_{12}^- \\ a_{21}^- & a_{22}^- \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix} \text{ for } x < 0;$$

and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix} \text{ for } x > 0.$$

The goal of this paper is to summarize what is known about the following question: *How many limit cycles can such continuous or discontinuous piecewise linear differential systems have*? For continuous piecewise systems both linear differential systems coincides on the straight line $x = 0$, and for the definition of discontinuous piecewise systems we follow the rules of Filippov [9].

The study of the piecewise linear differential systems goes back to Andronov, Vitt and Khaikin [1], and nowadays such systems still continue to receive the attention of many researchers. These differential systems are widely used to model processes appearing in electronics, mechanics, economy, etc., see for instance the books of di

Bernardo [3] and Simpson [37], the survey of Makarenkov and Lamb [35], as well as hundreds of references quoted in these last three works.

The simplest possible continuous but nonsmooth piecewise linear differential systems are the ones having only two pieces separated by a straight line in the plane $\mathbb{R}^2$. In 1990 Lum and Chua conjectured in [33, 34] that such continuous piecewise linear differential systems have at most one limit cycle. In 1998 this conjecture was proved by Freire et al. [10]. In 2013 a new and shorter proof was done by Llibre, Ordóñez and E. Ponce [26], and recently another proof has been provided by Carmona, Fernández-Sánchez and Novaes [6]. But in all these papers the authors forgot to analyze the case when the two linear differential systems which form the piecewise linear differential system have no equilibrium points, this case was studied in Llibre and Teixeira [28] in 2016 where it is proved that the continuous piecewise systems have no limit cycles, but the discontinuous piecewise system can have at most one limit cycle.

The main objective of this paper is to study the problem of Lum and Chua just for the class of discontinuous piecewise linear differential systems in the plane with two pieces separated by a straight line. Several authors have tried to determine the maximum number of limit cycles for this class of discontinuous piecewise linear differential systems. Thus in one of the first papers dedicated to this problem Giannakopoulos and Pliete [13] in 2001 shown the existence of discontinuous piecewise linear differential systems with two limit cycles. Then in 2010 Han and Zhang [16] found other discontinuous piecewise linear differential systems with two limit cycles and they conjectured that the maximum number of limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line is two. But in 2012 Huan and Yang [17] provided numerical evidence of the existence of three limit cycles in this class of discontinuous piecewise linear differential systems. In 2012 Llibre and Ponce [27] inspired by the numerical example of Huan and Yang proved for the first time that there are discontinuous piecewise linear differential systems with two pieces separated by a straight line having three limit cycles. More precisely, they showed that the following discontinuous piecewise linear differential system with two zones separated by the straight line $x = 1$

$$\dot{\mathbf{X}} = \begin{cases} A^+\mathbf{X} & \text{if } x \geq 1, \\ A^-\mathbf{X} & \text{if } x < 1, \end{cases} \tag{1}$$

where $\mathbf{X} = (x, y)^T$ with

$$A^+ = \begin{pmatrix} \dfrac{19}{50} & -1 \\ 1 & \dfrac{19}{50} \end{pmatrix} \quad \text{and} \quad A^- = \begin{pmatrix} \dfrac{4}{3} & -\dfrac{20}{3} \\ \dfrac{377}{750} & -\dfrac{26}{15} \end{pmatrix}$$

has the three limit cycles shown in Figure 1.

Later on other authors obtained also three limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line, see Braga

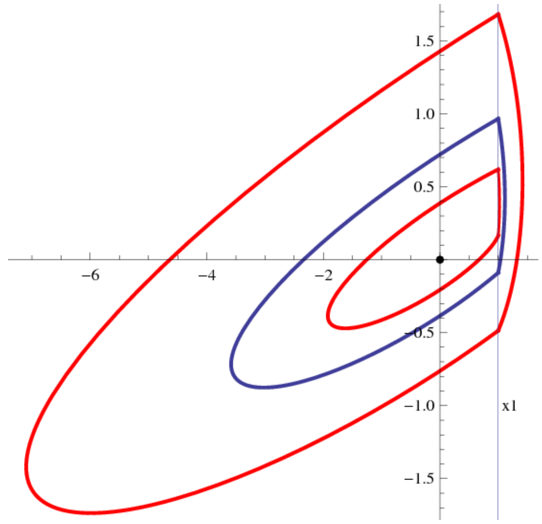


Figure 1. The first example with 3 limit cycles.

and Mello [4], 2013, Buzzi, Pessoa and Torregrosa [5], 2013, Liping Li [21], 2013, Freire, Ponce and Torres [12], 2014, and Llibre, Novaes and Teixeira [23], 2015.

The linear differential system that we consider in every half–plane extended to the full plane is either a focus (F) (include in this class of foci the centers), or a node (N) or a saddle (S). Thus in the beginning we distinguish six classes or types of planar discontinuous piecewise linear differential systems: FF, FN, FS, NN, NS and SS.

Limit cycles of discontinuous piecewise linear differential systems with two pieces separated by a straight line have been studied by many authors. In addition to the papers quoted by now we mention also some more articles.

In 2010 Shui, Zhang and Li [36] obtained some general results on the class of discontinuous piecewise linear differential systems here considered, mainly about the existence of heteroclinic and homoclinic orbits.

In 2012 Freire, Ponce and Torres [11] obtained a Liénard-like canonical form which is topologically equivalent to the original system if one restricts the attention to orbits with no points in the sliding set. Moreover they characterize some bifurcations leading to the birth of crossing periodic orbits and localize certain parameter regions where the number of crossing limit cycles is exactly two.

In 2013 Artés, Llibre, Medrado and Teixeira [2] studied the case SS and showed that this case can produce a focus on the straight line of discontinuity, and that the maximum number of limit cycles that can bifurcate from this focus is two.

The paper of Llibre, Teixeira and Torregrosa [30] was submitted at the beginning of 2012 and published in 2013. In it for the six types of systems FF, FN, FS, NN, NS, SS they provide examples with two limit cycles.

In 2013 Huan and Yang [18] showed that systems SS can exhibit two limit cycles, and in 2014 they proved in [19] that systems NN can also exhibit two limit cycles.

In 2015 Euzébio and Llibre [8] proved that the maximum number of crossing limit cycles, when one of the two linear differential systems has its equilibrium point on the straight line of discontinuity, is larger than two and smaller than four. This result was improved by Llibre, Novaes and Teixeria [24] in 2015 by proving that such a maximum is two.

In 2015 Gouveia, Llibre and Novaes [14] studied the linear differential center $(x', y') = (-y, x)$ perturbed in the class of all discontinuous piecewise linear differential systems with two zones separated by the straight line $y = 0$. Using the Bendixson transformation they provided sufficient conditions to ensure the existence of a crossing limit cycle coming from the infinity.

In 2017 Castillo, Llibre and Verduzco [7] studied the creation or destruction of a crossing limit cycle when a sliding segment changes its stability, this phenomenon is known as a pseudo–Hopf bifurcation. In that paper, under generic conditions, the authors did an unfolding for such bifurcation, and proved the existence and uniqueness of a crossing limit cycle for such class of piecewise systems.

In 2019 Shimin Li and Llibre [22] provided the phase portraits in the Poincaré disc of those piecewise systems having a unique finite singular point which is a node or a focus. Moreover they also gave sufficient and necessary conditions for the existence and uniqueness of limit cycles in that class of systems.

We recall that there are three classes of linear nodes: nodes with different eigenvalues N, nodes with equal eigenvalues whose linear part does not diagonalize N', and nodes with equal eigenvalues whose linear part diagonalizes, called star nodes. Clearly if we have a star node this prevents the existence of crossing periodic orbits. Moreover now we shall distinguish between focus F and centers C. Then we can consider 15 classes or types of planar discontinuous piecewise linear differential systems: FF, FC, FN, FN', FS, CC, CN, CN', CS, NN, NN', NS, N'N', N'S and SS.

Summarizing the results of these articles we have that the maximum number of known limit cycles which one of these systems can exhibit is given in the following table.

| | F | C | N | N' | S |
|---|---|---|---|---|---|
| F | 3 | 2 | 3 | 3 | 3 |
| C | – | 0 | 1 | 1 | 1 |
| N | – | – | 2 | 2 | 2 |
| N' | – | – | – | 2 | 2 |
| S | – | – | – | – | 2 |

But the main **open question** remains: *Is* 3 *the maximum number of crossing limit cycles which a discontinuous piecewise linear differential systems with a straight line of separation can have*?

## 2 Computation of the crossing periodic orbits

We want to recall the four different more usual techniques for computing analytically the periodic solutions of discontinuous piecewise linear differential systems based on the first integrals, or on the Poincaré map, or on the Poincaré map plus the Newton-Kantorovich Theorem, or on the averaging theory or equivalently the Melnikov integral.

### 2.1 Computation of crossing periodic orbits using first integrals

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Consider a differential system

$$\dot{x} = f(x), \qquad x \in \Omega \subset \mathbb{R}^n, \tag{2}$$

where $\Omega$ is an open subset of $\mathbb{R}^n$, and $f(x) = (f_1(x), \dots, f_n(x))$ is a $C^k$ function defined in $\Omega$ with $k \geq 1$. A *first integral* of the differential system (2) is a continuous function $H(x)$ defined in the domain of definition $\Omega$ of the differential system, which is not constant in any neighborhood, but such that $H(x)$ is constant on each orbit of the differential system (2). If $H(x)$ is $C^1$ then it is a *first integral* if and only if it satisfies

$$\frac{dH}{dt} = \frac{\partial H}{\partial x_1} f_1(x) + \ldots + \frac{\partial H}{\partial x_n} f_n(x) = 0,$$

for all the points $x \in \Omega$.

Using the first integrals of the linear differential systems, Llibre and Zhang [31] have proved that the numbers of the previous table, for the discontinuous piecewise linear differential systems in $\mathbb{R}^2$ formed by two pieces separated by a straight line when at least one of the two linear differential systems is a center, are exactly the maximum number of crossing limit cycles that such systems can exhibit. This result also was proved previously without using the first integrals by Llibre, Novaes and Teixeira in [24].

See also the paper of Llibre and Teixeira [29] for additional examples of computing periodic orbits using first integrals.

### 2.2 Computation of crossing periodic orbits using Poincaré maps

The use of the Poincaré maps for computing periodic orbits is the most often used method. In fact the methods for computing periodic solutions using the averaging theory or equivalently the Melnikov integral are also based on the Poincaré map.

The first proof that there are discontinuous piecewise linear differential systems with two zones separated by a straight line having 3 limit cycles was given for the system (2). That proof uses first the Poincaré map, and then the Newton–Kantorovich Theorem.

Let $B_r(x_0)$ be the points $x \in \mathbb{R}^n$ such that $|x - x_0| < r$, i.e. the open ball with center $x_0$ and radius $r$. We denote by $\overline{B_r(x_0)}$ the closure of $B_r(x_0)$. For a proof of the next theorem see [20].

**Theorem 1** (Newton–Kantorovich Theorem)**.** *Given a function $f : C \subset \mathbb{R}^n \to \mathbb{R}^n$ and a convex $C_0 \subset C$, assume that $f$ is $\mathcal{C}^1$ in $C_0$ and that the following assumptions hold:*

*(a)* $|Df(z) - Df(z')| \leq \gamma |z - z'|$ *for all* $z, z' \in C_0$,

*(b)* $|Df(z_0)^{-1} f(z_0)| \leq \alpha$,

*(c)* $|Df(z_0)^{-1}| \leq \beta$,

*for some $z_0 \in C_0$. Consider*

$$h = \alpha\beta\gamma, \qquad r_{1,2} = \frac{1 \pm \sqrt{1 - 2h}}{h}\alpha.$$

*If $h \leq 1/2$ and $\overline{B_{r_1}(z_0)} \subset C_0$, then the sequence $\{z_k\}$ defined by*

$$z_{k+1} = z_k - Df(z_k)^{-1} f(z_k) \quad \textit{for} \quad k = 0, 1, \ldots$$

*is contained in $B_{r_1}(z_0)$ and converges to the unique zero of $f(z)$ contained in $C_0 \cap B_{r_2}(z_0)$.*

Now we shall present the scheme of the proof that the piecewise linear differential system (2) has the three limit cycles of Figure 1. The solution $(x^+(t), y^+(t))$ of system $\dot{\mathbf{X}} = A^+\mathbf{X}$ which pass through the point $(1, Y)$ when the time $t = 0$ is

$$\begin{aligned} x^+(t) &= e^{19t/50} \left(\cos t - Y \sin t\right), \\ y^+(t) &= e^{19t/50} \left(Y \cos t + \sin t\right), \end{aligned}$$

The solution $(x^-(t), y^-(t))$ of system $\dot{\mathbf{X}} = A^-\mathbf{X}$ which passes through the point $(1, Y)$ when the time $t = 0$ is

$$\begin{aligned} x^-(t) &= \frac{1}{15} e^{-t/5} \left(15 \cos t - 100Y \sin t + 23 \sin t\right), \\ y^-(t) &= \frac{1}{750} e^{-t/5} \left(750Y \cos t - 1150Y \sin t + 377 \sin t\right). \end{aligned}$$

Assume that through the point $(1, Y)$ passes a periodic solution $(x^+(t), y^+(t)) \cup (x^-(t), y^-(t))$. Then if $t^+ > 0$ is the smallest time such that

$$x^+(-t^+) = 1,$$

and $t^- > 0$ is the smallest time such that

$$x^-(t^-) = 1,$$

we have that

$$y^+(-t^+) = y^-(t^-).$$

Hence a periodic solution of our system is characterized by a solution $(t^+, t^-, Y)$ of the system

$$\begin{aligned} f_1(t^+, t^-, Y) &= x^+(-t^+) - 1 = 0, \\ f_2(t^+, t^-, Y) &= x^-(t^-) - 1 = 0, \\ f_3(t^+, t^-, Y) &= y^+(-t^+) - y^-(t^-) = 0. \end{aligned} \tag{3}$$

So we need to find the solutions $(t^+, t^-, Y)$ of system (3). We shall prove that there are 3 isolated solutions $(t_k^+, t_k^-, Y_k)$ for $k = 1, 2, 3$ of the previous system near

$$\begin{aligned} t_1^+ &= 1.4866328036550172706875201659 5..,\\ t_1^- &= 3.45108332296097156801770033378..,\\ Y_1 &= 1.68119451051893996990946394580..,\\[1ex] t_2^+ &= 0.85668322292111353096467693501..,\\ t_2^- &= 3.78234111866076263903523393333..,\\ Y_2 &= 0.96579985584668225513544927484..,\\[1ex] t_3^+ &= 0.39178388598443280650466201997..,\\ t_3^- &= 4.66507269554569223774066305515..,\\ Y_3 &= 0.61885416518252501376067323768... \end{aligned}$$

So these 3 solutions will provide the 3 limit cycles of our discontinuous piecewise linear differential system (2). For proving the existence of these 3 isolated solutions of system (3) we shall use the Newton–Kantorovich Theorem. For the complete details of the proof see [27].

## 2.3 Computation of crossing periodic orbits using averaging theory or equivalently the Melnikov integral

The equivalence between the methods based on the averaging theory and the Melnikov integral for finding periodic orbits was proved by Han, Romanovski and Zhang [15].

The next theorem is the first-order averaging theory developed, for computing periodic orbits of discontinuous piecewise differential systems separated by the curve $h(t, x) = 0$, by Llibre, Novaes and Teixeira [25], and it can be used for studying the periodic orbits of discontinuous piecewise linear differential systems separated by a straight line.

**Theorem 2.** *Consider the following discontinuous differential system*

$$x'(t) = \varepsilon F(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{4}$$

*with*

$$\begin{aligned} F(t,x) = &\quad F_1(t,x) + \text{sign}(h(t,x))F_2(t,x), \\ R(t,x,\varepsilon) = &\quad R_1(t,x,\varepsilon) + \text{sign}(h(t,x))R_2(t,x,\varepsilon), \end{aligned}$$

*where* $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$*,* $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ *and* $h : \mathbb{R} \times D \to \mathbb{R}$ *are continuous functions,* $T$*–periodic in the variable* $t$*,* $D$ *is an open subset of* $\mathbb{R}^n$*,* $h$ *is a* $\mathcal{C}^1$ *function having* $0$ *as a regular value, and* $\Sigma_0 = \{0\} \times D \setminus h^{-1}(0)$*.*

*Define the averaged function* $f : D \to \mathbb{R}^n$ *as*

$$f(x) = \int_0^T F(t,x)dt.$$

*We assume*

*(i)* $F_1$*,* $F_2$*,* $R_1$*,* $R_2$ *and* $h$ *are locally* $L$*–Lipschitz with respect to* $x$*;*

*(ii) for* $a \in \Sigma_0$ *with* $f(a) = 0$*, there exist a neighborhood* $V$ *of* $a$ *such that* $f(z) \neq 0$ *for all* $z \in \overline{V} \setminus \{a\}$ *and* $d_B(f, V, a) \neq 0$*, i.e. the Brouwer degree of* $f$ *at* $a$ *is not zero.*

*(iii) If* $\partial h/\partial t(t_0, z_0) = 0$ *for some* $(t_0, z_0) \in \mathcal{M}$*, then* $\big(\langle \nabla_x h, F_1 \rangle^2 - \langle \nabla_x h, F_2 \rangle^2\big)(t_0, z_0) > 0$*.*

*Then for* $|\varepsilon| > 0$ *sufficiently small, there exists a* $T$*–periodic solution* $x(t, \varepsilon)$ *of system* (4) *such that* $x(0, \varepsilon) \to a$ *as* $\varepsilon \to 0$*.*

We note that if the function $f(z)$ is $\mathcal{C}^1$ and the Jacobian of $f$ at $a$ is not zero, then $d_B(f, V, a) \neq 0$, see for instance Lloyd [32].

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