

## The Understanding of the Derivative Concept in Higher Education

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### ABSTRACT

The aim of this work was to identify and characterize the levels of development of derivative *schema*. In order to do so, a questionnaire to 103 university students with previous instruction in Differential Calculus was applied. The questionnaire was composed of three tasks. For the identification of the levels of development of *schema* and their subsequent characterization, we consider the framework proposed by the APOS theory. In particular, this framework was operationalized through the establishment of 27 variables that allowed for the breakdown of the resolution protocols from the questionnaire into discrete elements. In this way, we obtained a vector associated with each of these variables. The identification of students assigned to each level of development of *schema* was carried out by a cluster analysis. Subsequently, we performed a statistical analysis of frequencies and implicative, with the 27 variables, which allowed to characterize the levels of development identified.

**Keywords:** derivative schema, levels of development, cluster analysis, implicative analysis, calculus

### INTRODUCTION

Calculus is one of the greatest and most important achievements of the human intellect, and is a hallmark of the development of mathematics today, whose power and flexibility may be seen in its ability to reduce complex problems to rules and simple procedures in the most diverse areas of knowledge, such as mathematics, physics, engineering, social sciences and biology, among others (Berry & Nyman, 2003; Kleiner, 2001).

In this sense, Ferrini-Mundy and Lauten (1994, p.120) describe Calculus as: "Calculus is a critical landmark in the mathematical preparation of students intending to pursue nearly all areas of science" For his part, Tall (1997, p. 289) states that Calculus "is both a climax of school mathematics and a gateway to further theoretical developments", which makes it a transitional point between elementary mathematics and advanced mathematics.

Despite the relevance of Calculus, one problem that has still remained unresolved is how to achieve understanding by university students in the fundamentals concepts of this course. Moreover, Calculus is typically considered a difficult subject for university students; it is noted that these students often can solve problems involving the proper application of rules or algorithmic procedures, but nonetheless have difficulties when they have to solve non-routine problems involving the understanding of concepts (Selden, Selden, Hauk, & Mason, 1999), or the application of that understanding to real-world problems (Tall, 1992).

In particular, this study focuses on the concept of derivatives, which is one of the central and structural elements of any Calculus course, and is also a fundamental tool in the study and understanding of phenomena that involve changing or varying magnitudes (Vrancken & Engler, 2014). Therefore, it corresponds to a basic concept applicable to many other fields in university curricula in mathematics, engineering, and other sciences.

Despite the importance of the derivative concept, the results of research related to understanding they find that this is very complex, showing a significant amount of university students who only manages to achieve a partial

**Contribution of this paper to the literature**

- This research describes the quantitative analysis of data that are generally approached by qualitative methods.
- The results of this research confirm and extend conclusions obtained through qualitative studies on the development of the derivative schema.
- The methodological design proposed in this research can serve as a model for the analysis of schemas of different mathematical concepts.

understanding of this concept (Asiala et al., 1997; Baker, Cooley & Trigueros, 2000; Cooley, Trigueros & Baker, 2007; Ferrini-Mundy & Graham, 1994; Fuentealba, Sánchez-Matamoros, Badillo & Trigueros, 2017; Orton, 1983; Sánchez-Matamoros, 2004; Sánchez-Matamoros, Garcia & Llinares, 2006, 2008; White & Mitchelmore, 1996).

An aspect that has caused difficulties in the understanding of the derivative concept, relates to the use of teaching practices that have favored the learning of algorithmic methods by students (Artigue, 1995). This predilection for teaching algorithmic procedures makes students show serious difficulties and errors when faced with solving tasks that require understanding the meaning of the derivative, either through its analytical expression, as the limit of incremental ratio, or of its geometrical interpretation as slope of the tangent line (Baker et al., 2000; Cooley et al., 2007; Sánchez-Matamoros et al., 2006; Sánchez-Matamoros et al., 2008).

The problem of understanding of the derivative concept, though not new, is still one of the biggest challenges of mathematics education at the university level, and is a constant concern for institutions of higher education as it leads to low grades, high rates of failure, and the abandonment in Calculus courses (Bressoud, Mesa, & Rasmussen, 2015; Ferrini-Mundy & Graham, 1991). Considering this situation and the social demand concerning research in mathematics education, not only analyzing the problems of the teaching and learning of the discipline, but also contributing to the solution, our focus in this research is to determine the following: how is the understanding of the derivative concept is developed in university students with prior instruction in Differential Calculus? What are the characteristics of the different levels of development of derivative *schema*? Thus, we intend to deepen the understanding of how the concept of derivatives is formed in the minds of students in order for this information to contribute in the future to solving the problems associated with learning.

## BACKGROUND AND THEORETICAL FOUNDATIONS

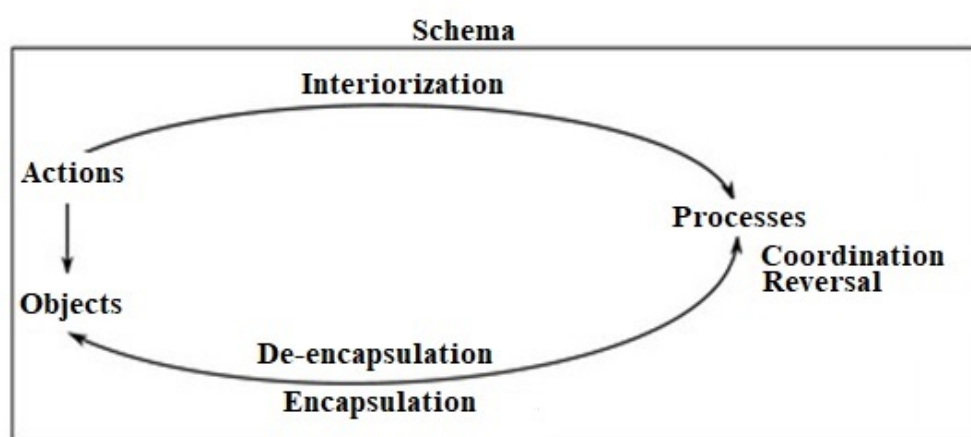
The understanding of concepts is an important goal for mathematics education, and it is generally accepted that this occurs when the pieces of knowledge that make up a concept are mentally connected together (Hiebert & Carpenter, 1992). As students face new situations and experiences, they interact with your existing knowledge (von Glasersfeld, 1983), thus allowing such connections are made. New ideas can be integrated into existing *schema* or mindsets of the students, and can make the existing knowledge reconnect in new ways (Siegler, 1986). Understanding is seen in this way as a connected network, and pieces of knowledge that are built over time through the interaction of prior knowledge with new information.

In particular, this work has been considered a cognitive approach that, like the above ideas, consider the understanding of a mathematical concept as a gradual process of building and relationship between cognitive structures. As its epistemological reference, this approach takes constructivist ideas based on the genetic psychology of Piaget, which have been considered by Dubinsky and a group of researchers known as "Research on Undergraduate Mathematics Education" (RUMEC), who have developed a theoretical framework known as APOS theory (*action-process-object-schema*). This framework is the result of the interpretation of Piagetian ideas concerning *reflective abstraction*.

To understand how the APOS theory operates, it is important to note that the principle of *reflective abstraction* was considered by Piaget as the main mechanism for all mental construction, as well as the mechanism through which all logical-mathematical structures can develop in the mind of an individual (Arnon et al., 2014). According to Piaget, this principle has two parts:

The first part involves reflection, in the sense of awareness and contemplative thought, about what Piaget called content and operations that content, and in the sense of reflecting content and operations from a lower cognitive level or stage to a higher one [...]. The second part consists of the reconstruction and reorganization of the content and operations on this higher stage, that results in the operations themselves becoming content, to which new operations can be applied (Piaget, 1973; quoted in Arnon et al., 2014, p. 6)

In the APOS theory, *actions*, *processes*, *objects* and *schema* are mental structures that, according to this framework, an individual constructs when learning a particular mathematical concept, while the passage through these stages is not necessarily sequential (Trigueros, 2005). The mechanism to move from one state of construction of mathematical knowledge to another, in this theory, is *reflective abstraction*, which is a mental tool or device used in



**Figure 1.** Structures and mental mechanisms involved in understanding a mathematical concept (based on Arnon et al., 2014, p. 18)

the process of knowledge construction, which allows the student to infer their properties on the basis of *actions* on *objects*, or to infer the relationships between *objects* of the same level of thinking. This implies, among other things, the organization of information in an intellectual framework organized higher-level (Dubinsky, 1991).

One of the hypotheses on which the APOS theory is based is that the construction of a new concept is based on the transformation of previous concepts, for which reason these concepts should initially be perceived by the individual as *objects*. Therefore, an *action* is a transformation of *objects* (previously constructed), perceived by the individual and external, in the sense that each step of the transformation requires explicit targeting and also needs an external stimulus to execute (Arnon et al., 2014). The structure of *action* is considered as the simplest within the APOS theory, but no less important, because it is essential in the construction of any mathematical concept. Also, a *process* is considered an internalized *action*, that is to say “mental”; in which the individual is aware and has control over the transformation produced by the *action*. This is characterized by the ability to imagine, skip, or reverse the steps involved in this transformation, without the need of an external stimulus. *Interiorization* is the mechanism that allows for the change of structure, from *action* to *process*, which is accomplished by repeating reflection on the *actions* (Arnon et al., 2014). A *process* can not only be generated by the interiorization of actions, but they can also be constructed from the coordination and reversal mechanisms of processes. When an individual is aware of the *process* and is able to conceive how it can be transformed as a whole by applying *actions* or *processes*, it is then said that the *process* has been *encapsulated* in a cognitive *object* (Arnon et al., 2014; Asiala et al., 1996). The mechanism associated with this change of state of the process is called *encapsulation*. Nevertheless, once a *process* is *encapsulated* in an *object*, if the individual is required to return to the process that gave rise to the *object*, this can be done by the *de-encapsulation* mechanism.

Finally, the last cognitive structure proposed by APOS theory is called a *schema*. A *schema* of a mathematical concept in particular is a coherent collection of *actions*, *processes*, *objects* and even other *schemas* and their interrelationships, grouped consciously or unconsciously in the mind of an individual, which can be used in the solution of a situation or mathematical problem involving the concept in question. The coherence of the *schema* is referred to as the individual’s ability to recognize situations in which the *schema* is applicable and which are not (Trigueros, 2005).

The *schema* is regarded as a dynamic and complex cognitive structure that is constantly developing and evolving as the individual learns. Although sometimes it is generally thought that this structure is just beginning to form once the *objects* (due to progression of *action*, *process*, *object* and *schema*) are constructed, construction may begin even at the time the individual performs *actions*.

Although the mental structures (*action*, *process*, *object* and *schema*) of an individual during an individual cannot be directly observed during the learning process, these structures can be inferred from the observation on what the individual can do, or not to face a particular situation or mathematical problem (Dubinsky, 1991). Thus, through observation and analysis, it is possible to characterize the stage of construction of a specific mathematical concept in which an individual may be found. Figure 1 shows the relationship between the mechanisms and the mental structures mentioned above.

Trigueros (2005) indicates that when a student is facing a specific problem in the field of mathematics, the student conjures up a *schema* to address the resolution. Upon evoking it, that student brings those built structures and relationships that he has at that time into play. Faced with the same task, different students can use different

structures and different relationships between them. Thus, when considering the relationships established between the built structures, different levels of development of the *schema* can be identified in the responses of students who meet the same task or set of tasks. To address and characterize these differences in further developing the dynamism of the *schema*, APOS theory proposes the study of the development of a *schema* through the use of the triad of Intra, Inter and Trans, proposed by Piaget and García (1983), which classifies it in one of these stages, depending on the level of relations that an individual can establish between the *schema* components and other cognitive structures. Following this idea, Piaget and García (1983) define the levels of development of a *schema* as follows:

- Intra: this level is characterized by the discovery of any operative action, and the pursuit of analyzing its various internal properties or its immediate consequences, but with a two-part limitation. First, there is no coordination of this pre-operation with others in an organized grouping; but also, the internal analysis of the operation involved is accompanied by progressively corrected errors and gaps in the inference that can be derived from it (p. 163).
- Inter: once an initial step is comprised, it is possible to deduce the operations that are involved, or coordinate with other more or less similar ones for the creation of systems that involve certain transformations. While there is a new situation here, there are nevertheless limitations that arise from the fact that the compositions are restricted because they can only proceed with contiguous elements (p.165).
- Trans: This level is easily defined in terms of the above as involving, in addition to the transformations, a synthesis between them. This synthesis arrives at the construction of “structures” (p.167).

From this definition of the triad of the level of development of a *schema*, in recent decades, various studies have been made concerning the development of *schemas*. For example, research by Baker et al. (2000) analyzed the calculus graphing *schema* through the solutions given by students of a non-routine a graphic problem, in which it was not the expression of the function but rather a set of analytical conditions of that function that was presented. In their study, they posited that the Calculus graphic *schema* was formed by the interaction of two *schemas* that they called “property *schema*” and “interval *schema*”. For his part, Badillo, Azcárate and Font (2011) uses an idea similar to that posed by Baker et al. (2000) for an analysis of the understanding of the concept of derivatives in a group of five mathematics and physics teachers from Colombia. The idea of coordination of *schemas* used in their research, considering that the derivative *schema* was formed by coordinating the “algebraic *schema*” and “graphic *schema*”.

For the specific case of this study, reference was made to work by Sánchez-Matamoros (2004), in which an analysis of derivative *schema* in terms of logical relationships (conjunction, contrapositive, and equivalence) that students establish between different mathematical elements when solving problems. Therefore, consistent with the study by Sánchez-Matamoros (2004), it is understood that a mathematical element is “the product of the dissociation or segregation of the concept relating to the concept and its properties” (Piaget, 1973, p. 72). From this definition, it may be indicated that the derivative concept possesses structural elements of a different nature, characterized by the modes of representation and the character of these elements. With regard to the modes of representation, this paper considers that the concept of the derivative consists of two types of elements: analytical and graphic elements. Also, regarding their character or nature, it considers elements of a specific nature, if these elements refer to a specific property at a point, or apply overall, as appropriate, to a range corresponding to a property. Thus, it is understood that a *schema* corresponds to the mathematical structure formed by the mathematical elements and logical relationships established between them, and that can be evoked for solving a problem (Sánchez-Matamoros, 2004). From the identification of mathematical elements and logical relationships established between them, Sánchez-Matamoros (2004) defined, through a qualitative analysis:

- Intra-derivative level. No logical relationships are established between the mathematical elements (either graphical or analytical, specific or overarching), and the possible outlines of the relationship (the logical conjunction type) between them was made erroneously. Use of mathematical elements in isolation, and sometimes incorrectly.
- Inter-derivative level. Logical relationships are established between the mathematical elements used, but with limitations, the predominating using logical conjunction and relate only specific and/or overarching mathematical elements that are in the same mode of representation, analytical or graphic. More mathematical elements are used correctly than in the previous level.
- Trans-derivative level. Increases the repertoire of the use of the logical relationships (and logic, contrapositive, logical equivalence) between mathematical elements. At this level, the “synthesis” of modes of representation occurs. This leads to the construction of the mathematical structure. (p. 73-74)

Regarding the synthesis portion of the description of the level of development, this applies to situations in which it is necessary to relate (make logical connections) with graphic and analytical information, that is, using information from both imaging systems to consider together and arrive at a “thing” that was not known. “Considering the information together” means establishing some sort of logical relationship between mathematical

elements to make a decision concerning the situation that currently exists (Sánchez-Matamoros, 2004). In this paper, the descriptors for each level of development based on mathematical elements and logical relationships for defining the study variables presented in the study design were considered.

Also, another important aspect in the development of *schemas* related to the issues raised by Piaget and Garcia (1983) on the gradual growth of the *schema* and the nature of the triad. In particular, these authors point out that:

The nature of the elements of the triad is functional rather than structural. Therefore, they follow a necessary order, since the development of Trans, as a system of transformations together in entirely new properties, involves the formation of some of these transformations at the Inter level, and that this latter group involves the knowledge of characters analyzed in the Intra (p. 171).

This last idea has been adapted by the APOS theory to analyze how a schema of a specific mathematical concept is developed, which for the case of this study corresponds to the derivative, in addition, allows to characterize the different levels of schema development, and how these relate to each other, which correspond to the goals of this research.

## METHODOLOGICAL DESIGN

This work is part of a wider investigation; whose purpose is to analyze the derivative *schema* in college students with prior instruction in Differential Calculus. The methodological approach adopted is of mixed type with exploratory and confirmatory character, we worked with qualitative data and statistical analysis (Rocco, Bliss, Gallagher & Perez-Prado, 2003). In this part of the work, we have focused on quantitative analysis.

### Participants

The participants in this study were 103 college students from the academic years 2014/2015 and 2015/2016, with double majors in Mathematics and Physics at a public university in the province of Barcelona. All students had taken and passed at least one subject which included the topics of Differential Calculus. The option of choosing students who had already completed one or more courses in Differential Calculus is intentional and is based on two aspects: (1) our interest in characterizing the levels of development of the derivative *schema*, reached by college students after the instructional process; and, (2) the difficulty associated with the characterization of the encapsulation mechanism of processes, demonstrated in previous studies (Cooley et al., 2007; García et al., 2011; Font, Trigueros, Badillo & Rubio, 2016; Fuentealba et al., 2017; Sánchez-Matamoros, 2004).

With regard to the characteristics of the participants, it may be mentioned that there is great variability in their age range, prior training, and academic level.

### Instrument

The instrument used (see [Figure 2](#)) and discussed in this article is a questionnaire that was constructed by adapting three tasks used in previous research on the derivative concept (Baker et al., 2000; Cooley et al., 2007; Fuentealba, Sánchez-Matamoros & Badillo, 2015; García et al., 2011; Sánchez-Matamoros, 2004; Sánchez-Matamoros et al., 2006). For it to be solved, it was necessary to use distinct mathematical elements that make up the derivative concept. This questionnaire was administered to 103 participants in this study and lasted approximately 90 minutes.

### Task 1

#### Description of aspects associated with its solution

#### Problem statement

Sketch the graph of a function  $f$  satisfying the following conditions

- $f$  is continuous on its domain
- $f(2) = 0$
- $f'(2) = f'(5) = 0$
- $\lim_{x \rightarrow -\infty} f(x) = -4$
- $\lim_{x \rightarrow -8} f(x) = -\infty$
- $f'(x) < 0$  when  $5 < x < 8$
- $f'(x) \geq 0$  when  $x < 5$
- $f''(x) < 0$  when  $3 < x < 8$
- $f''(x) > 0$  when  $x < 3$

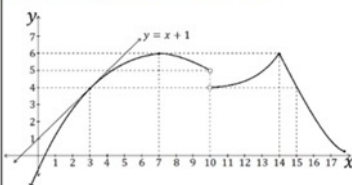
The first task provides analytical information for the function  $f$  in terms of  $f$ ,  $f'$  and  $f''$ . With these conditions students are asked to sketch the graph of the function  $f$ . The objectives of this task are, on the one hand, to observe if students are able to establish both local and global relations between *actions* and *processes* in their *schema*, that associate the sign of  $f'$  in an interval with the *process* which allows to infer from the behaviour of the rate of change  $f$  in that interval; the *process* associated with the sign of  $f''$  in an interval with the *process* which allows the concavity of  $f$  to be determined; and the *process* that associates the change of sign of  $f'$  with the possible existence of extreme values or inflection points. On the other hand, to observe if the student is able to identify the contradictions in the analytical conditions posed in the problem that would provide evidence of the construction of stronger relations between the elements that constitute the *schema*.

### Task 2

#### Description of aspects associated with its solution

#### Problem statement

Consider the graph of the function  $f$ , formed by parabolas' sections



- Get the values of  $f'(3)$ ,  $f'(7)$ ,  $f'(10)$ ,  $f'(14)$  and  $f'(15)$ . Explaining how you got them
- Make a sketch of the graph of  $f'$ . Explain how you made your sketch

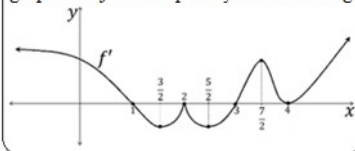
The second task, presented in graphical mode, has two parts, the first focuses on the local behaviour of the function and students are asked to calculate the derivative of  $f$  at specific points. The objective of this part is to observe if the students are able to carry out the *actions* or *processes* required to calculate the derivative at the points where the function has a different behaviour. In the second part, they are requested to sketch the graph of  $f'$  knowing the graph of  $f$ . We wanted to determine if students had constructed the derivative function as an *object*, and if they were able to *coordinate* the *processes* involved in the management of all the information, local and global, in an analytical context, with those corresponding to the graphical context. Notice that in task 1 a logical implication is involved, while in this task the reciprocal implication needs to be used.

### Task 3

#### Description of aspects associated with its solution

#### Problem statement

The following graph corresponds to the derivative of  $f$ . Sketch the possible graphs for  $f$  and explain your reasoning



In the third task, students were given the graph of the function  $f'$ , which includes several sign changes; increasing and decreasing slopes, zeros, extreme values and a corner point. The task asks to sketch a graph for the function  $f$  and its objective is to assess whether students could *coordinate* the *processes* related to the extreme values of  $f'$  with the *processes* associated with the extreme values and inflection points in  $f$ . Similarly, to *coordinate* the *processes* associated with the sign of  $f'$  with those related to the monotony of  $f$  and finally, to *coordinate* the *processes* related to the increase of  $f'$  with those related to the convexity of  $f$ .

Figure 2. Tasks proposed in the questionnaire and description of aspects associated with its solution

## Analysis Methods

As an initial step before performing the analysis, the resolutions protocols were discretized. For the discretize of these protocols, and to obtain a vector associated with each of them, 27 variables were defined (see Figure 3). These variables are the result of the breakdown of mathematical elements in both modes of representation

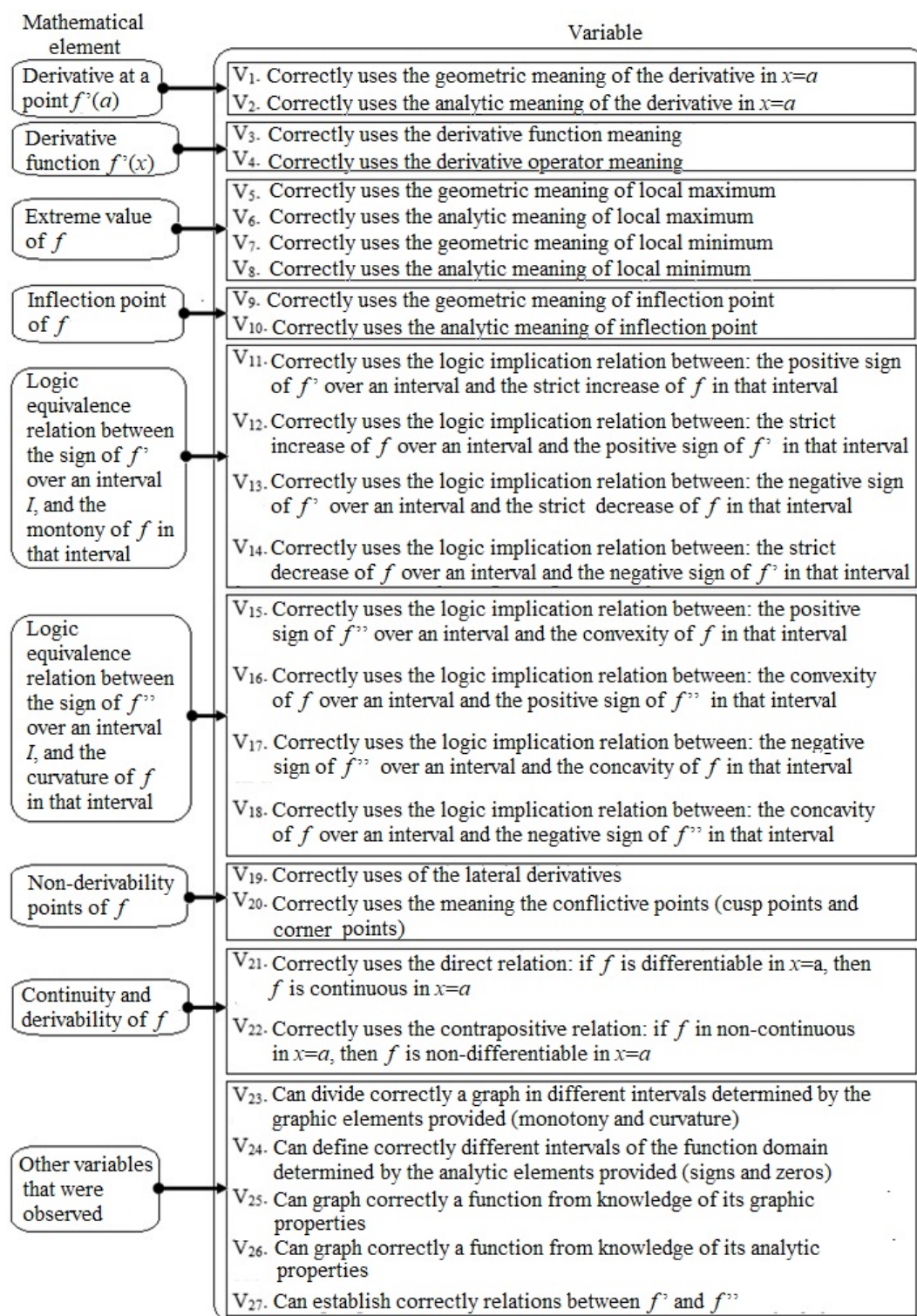


Figure 3. Variables used for the discretization of the resolution protocols of each of the questionnaires

(analytical and graphical), the use of logical relationships and previous studies using the APOS theory (Fuentealba et al., 2017; Fuentealba, Badillo & Sánchez-Matamoros, in press; Trigueros & Escandón, 2008). For example, the decomposition of the logic relationship of double implication between the positive sign of the first derivative in an interval and the strict growth of the function in said interval, allowed us to generate the variables V11 and V12. Also, by decomposing of some mathematical elements in both modes of representation, we generated other variables, for example, the variables V9 and V10, associated with correct use of the meaning of inflection point.

**Table 1.** Hierarchical Cluster obtained by varying the distance and the type of grouping selected

Distance	Cluster grouping type	Cophenetic correlation
Squared Euclidean	Complete linkage	0,859
Euclidean	Complete linkage	0,747
Manhattan	Complete linkage	0,660
Squared Euclidean	Single linkage	0,723
Euclidean	Single linkage	0,775
Manhattan	Single linkage	0,723

The purpose of establishing these variables was to conduct a cluster analysis that would identify and characterize the levels of development of the derivative *schema* (groups delivered by conglomerates). However, to quantify the presence or absence of each of the variables in the resolution protocols used by the students, it was necessary to use a measurement scale to assign a score to each of them. For the specific case of this study, a scale of a binary type, 1 or 0 (1 in the case that the correct use of variables was observed, and 0 in another case). From these two tools (variables and scale) we obtained for each resolution protocol a vector of the type ( $V_1, V_2, V_3, \dots, V_{27}$ ), wherein each variable has a value of 0 or 1. Each of these vectors was labeled with the letter E and a subscript that indicates the student belonging to it, thus 103 vectors ( $E_1, E_2, E_3, \dots, E_{103}$ ) were obtained.

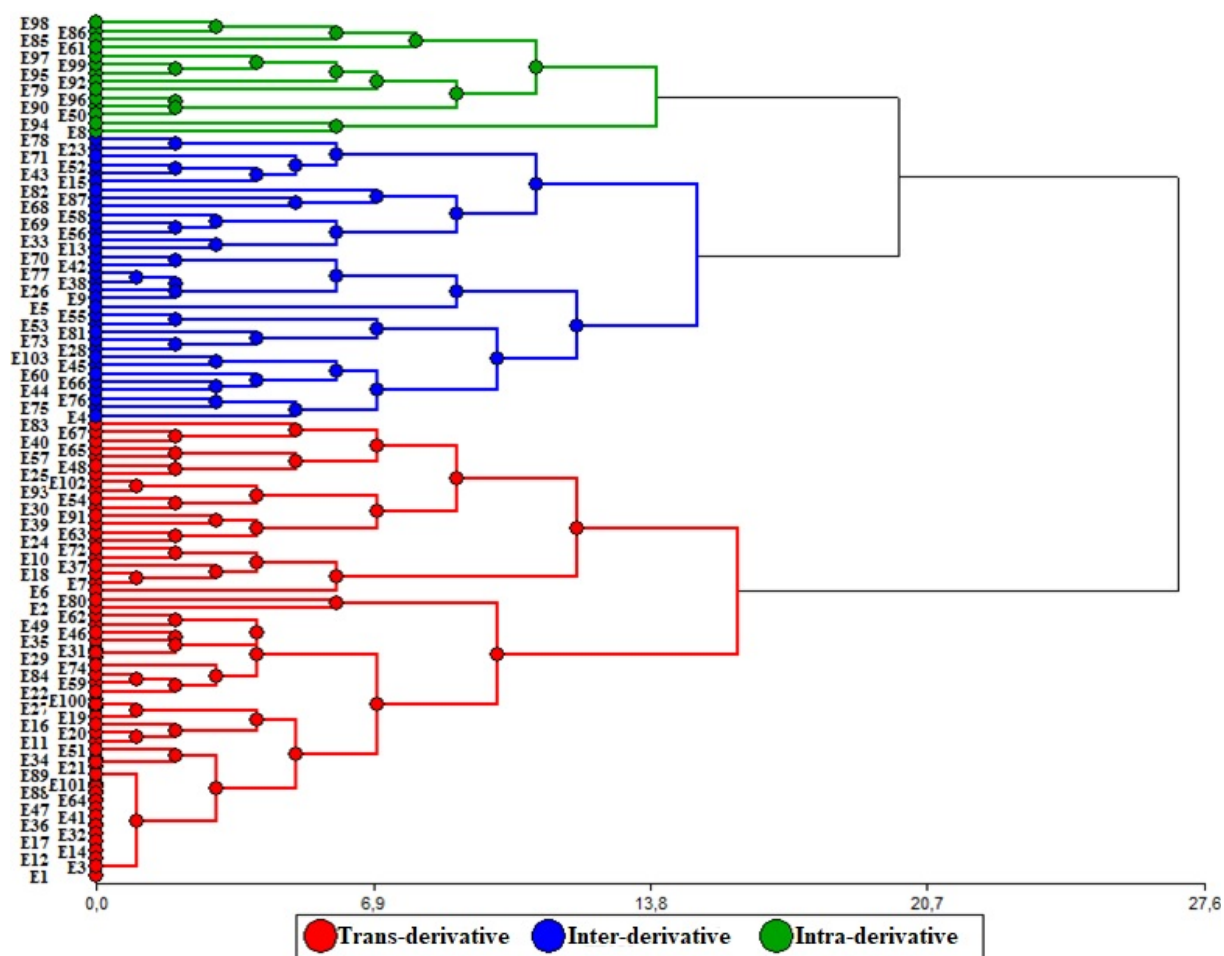
Later, after three sub-matrices of certain data (Intra, Inter and Trans) were obtained through a cluster analysis, a frequency analysis was conducted with percentage frequencies in terms of the proper or improper use of the variables. However, this type of analysis does not indicate what the most important variables are within each level of development, and also allows for the relationships to be viewed between them. Therefore, to observe the underlying structures of the group of variables, an implicative statistical analysis (*Analyse Statistique Implicative*, or ASI, in French) is carried out on each of the levels, corresponding to a method of analysis which allows, from a set of data which interrelates with a population of subjects or objects with a set of variables, for the extracting and structuring of knowledge in the form of rules and generalized rules (Zamora, Gregori & Orús, 2009). In particular, an implicative statistical analysis is a method of data mining, which is not symmetrical, and which allows to statistically model the quasi-implication  $a \Rightarrow b$ , that is to say, it attempts to quantify how likely it is to happen if the variable  $b$  has been observed in the population variable  $E$  (Lerman, Gras & Rostam, 1981). According to Trigueros and Escandón (2008), in this methodology, "the implication  $a \Rightarrow b$  shall be admissible if the number of individuals  $E$  that contradict it is very small, in probabilistic terms, in relation to the number of individuals expected under the hypothesis of having no relationship. If this happens, you can say that  $A$ , the set of observations to satisfy the feature, is "almost" contained in  $B$ , the set of observations that satisfy the characteristic  $b$ " (p. 67). Unlike data based symmetrical analysis methods, for example, on a distance or a correlation, the sets of rules sets obtained by the analysis can lead to implicative causal hypotheses (Zamora et al., 2009).

## ANALYSIS AND RESULTS

As mentioned, the first step corresponded to the embodiment of the cluster analysis with 103 vectors obtained by the discretization of the questionnaires. It is important to note that cluster analysis does not present a single solution, but rather the result of this depends on the characteristics of the selected process, i.e., the distance used and agglomerative or clustering method. While this study is intended to characterize the levels of development of the derivative *schema*, it was assumed, indicating the APOS theory as to the levels of development of an outline of any mathematical concept are three and that under this framework are the triad Intra, Inter and Trans (Arnon et al., 2014; Piaget & Garcia, 1983). Given the foregoing, it was built with the Infostat 2016 application six different conglomerates combining distances (Euclidean, squared Euclidean and Mahattan) and clustering methods (simple and complete linkage). Values of the cophenetic correlations for each of the six hierarchical clusters obtained are presented in Table 1.

Considering the results of the six conglomerates that were constructed and their cophenetic correlations we choose those whose cophenetic correlation coefficient was highest, which in this case corresponded to 0.859, which was selected. The rationale for this decision was based on the issues raised by Sokal and Rohlf (1962), who indicate that this ratio ensures a correct rating when its value is close to one.

As a result of the cluster analysis with the squared Euclidean distance and the complete agglomerative linkage method, the dendrogram shown in Figure 4 was obtained.



**Figure 4.** Complete dendrogram obtained with squared Euclidean distance and complete linkage

**Table 2.** Students assigned to the levels of development of the derivative *schema*

Level	Trans-derivative	Inter-derivative	Intra-derivative
Students	E1, E2, E3, E6, E7, E10, E11, E12, E14, E16, E17, E18, E19, E20, E21, E22, E24, E25, E27, E29, E30, E31, E32, E34, E35, E36, E37, E39, E40, E41, E46, E47, E48, E49, E51, E54, E57, E59, E62, E63, E64, E65, E67, E72, E74, E80, E83, E84, E88, E89, E91, E93, E100, E101, E102.	E4, E5, E9, E13, E15, E23, E26, E28, E33, E38, E42, E43, E44, E45, E52, E53, E55, E56, E58, E60, E66, E68, E69, E70, E71, E73, E75, E76, E77, E78, E81, E82, E87, E103.	E8, E50, E61, E79, E85, E86, E90, E92, E94, E95, E96, E97, E98, E99.
Total	55	34	14

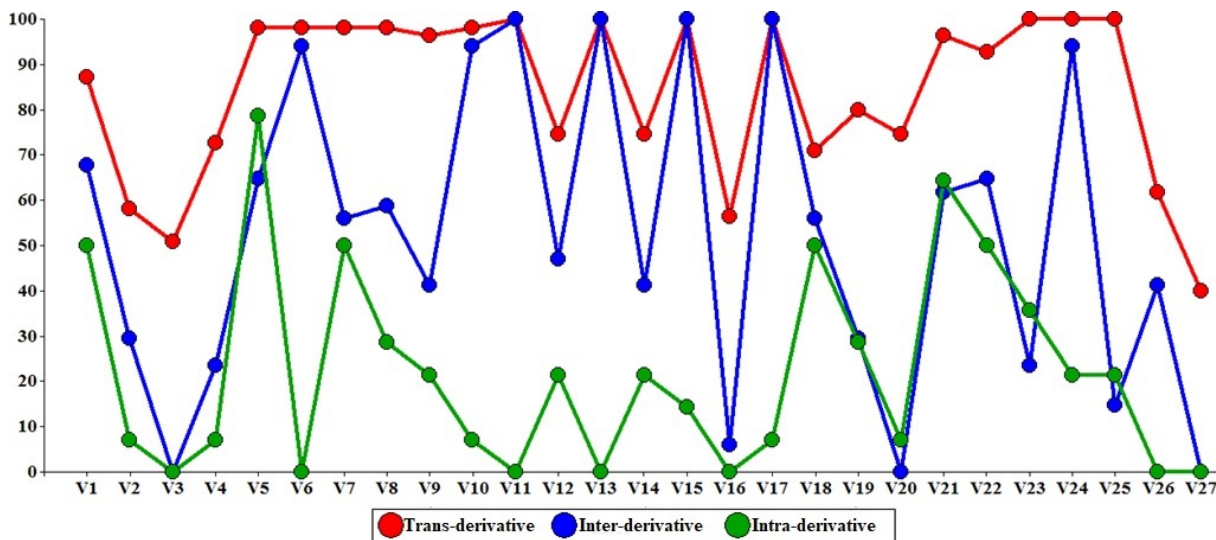
The dendrogram obtained through the cluster analysis allowed for the data matrix to be divided into three sub-matrices, one for each level of development of the derivative *schema*. In particular, in **Figure 4**, the red portion of the dendrogram corresponds to students assigned by the cluster analysis at the Trans-derivative level of development, the blue portion corresponds to students assigned to the Inter-derivative level, and the green portion are students assigned to the Intra-derivative level.

The students classified in each of the levels of development of the derivative *schema* are presented in **Table 2**.

The division of the matrix allowed for a descriptive statistical analysis of the percentages of the correct use of the 27 variables for each of the levels of development of the *schema*. The percentages of correct use of variables for each level of development are presented in **Table 3**.

**Table 3.** Frequency of correct use of the variables by mathematical criterion defining element and each variable

Elements	Criterion	Percentage		
		Trans	Inter	Intra
1. Derivative at a point $f'(a)$	V1: Geometric meaning	87	68	50
	V2: Analytic meaning	58	29	7
2. Derivative function $f'(x)$	V3: Derivative as a function	51	0	0
	V4: Derivative as an operator	73	24	24
	V5: Local maximum geometrically	98	65	79
3. Extreme value of $f$	V6: Local maximum analytically	98	94	94
	V7: Local minimum geometrically	98	56	50
	V8: Local minimum analytically	98	59	29
4. Inflection point of $f$	V9: Inflection point geometrically	96	41	21
	V10: Inflection point analytically	98	94	7
5. Logic equivalence relation between: the sign of $f'$ over an interval $I$ , and the monotony of $f$ in that interval	V11: $f' > 0 \rightarrow f$ increase strict	100	100	0
	V12: $f$ increase strict $\rightarrow f' > 0$	75	47	21
	V13: $f' < 0 \rightarrow f$ decrease strict	100	100	0
	V14: $f$ decrease strict $\rightarrow f' < 0$	75	41	21
6. Logic equivalence relation between: the sign of $f''$ over an interval $I$ , and the curvature of $f$ in that interval	V15: $f'' > 0 \rightarrow f$ convex	100	100	14
	V16: $f$ convex $\rightarrow f'' > 0$	56	6	0
	V17: $f'' < 0 \rightarrow f$ concave	100	100	7
	V18: $f$ concave $\rightarrow f'' < 0$	71	56	50
7. Non- differentiability points of $f$	V19: Lateral derivatives	80	29	29
	V20: Conflicitive points	75	0	7
8. Continuity and derivability of $f$	V21: If $f$ is differentiable in $x = a \rightarrow f$ is continous in $x = a$	96	62	64
	V22: $f$ is non-continous in $x = a \rightarrow f$ is non-differentiable in $x = a$	93	65	50
	V23: Determine intervals to sketch a function from graphical information	100	24	36
	V24: Determine intervals to sketch a function from analytical information	100	94	21
	V25: Sketch a function from its graphic properties	100	15	21
Other variables	V26: Sketch a function from its analytic properties	62	41	0
	V27: Relations between $f'$ and $f''$	40	0	0

**Figure 5.** Graph of multivariate profiles levels development derivative schema

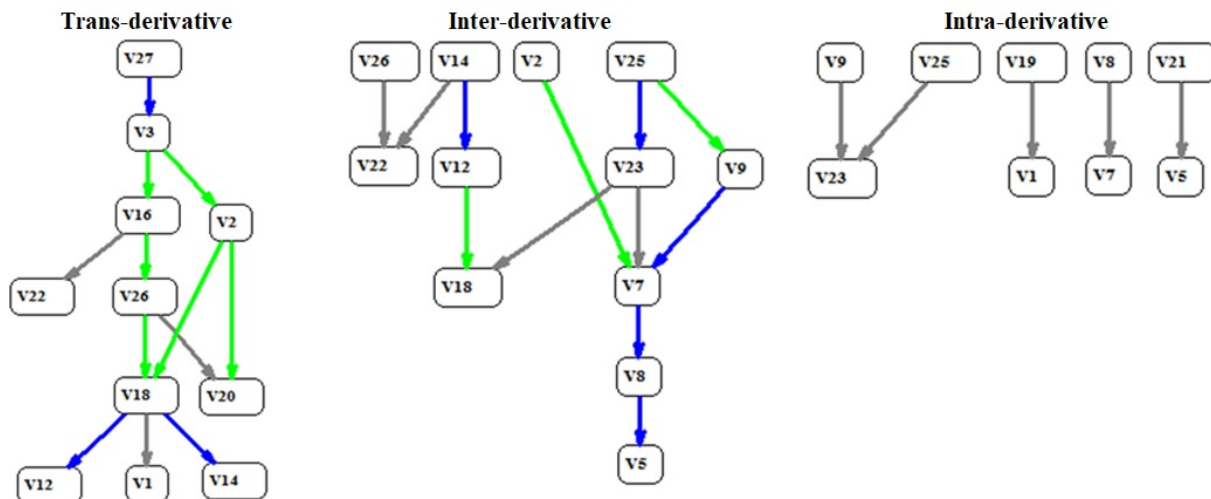
In order to more clearly visualize the differences and similarities between the three levels of development schema graph multivariate profiles are constructed from information percentages of correct use of the variables presented in Table 3. The multivariate graph profiles are presented in Figure 5.

Among the most important characteristics that can be inferred from Figure 5 are:

- Students at the levels of Intra-derivative (7%) and Inter-derivative (29%) have difficulties developing the interpretation of the derivative as the limit of the incremental ratio ( $V_2$ ). Meanwhile, students at the Trans-derivative level use it correctly in 58% of cases.

- Students at the Intra-derivative (0%) and Inter-derivative (0%) levels of development do not consider the derivative as a function ( $V_3$ ). Conversely, more than 50% of students at the Trans-derivative level of development do.
- Students at the Intra-derivative (7%) and Inter-derivative (24%) levels have difficulty with the use of the derivative as linear operator ( $V_4$ ). Meanwhile, 73% of students at the Trans-derivative level of development used this variable correctly.
- Students at the Trans-derivative level of development have no difficulty determining extreme values and inflection points ( $V_5, V_6, V_7, V_8, V_9, V_{10}$ ), regardless of the representation mode in which the task is provided. Meanwhile, students at the Inter-derivative level do have some difficulties in determining extreme values and inflection points, especially when the information on the task is provided in graphing mode ( $V_5, V_7, V_9$ ). Similarly, students at the Intra-derivative level also present difficulties in determining extremes and inflection point values, but these difficulties are even greater than the level of Inter-derivative development, moreover, presented in both modes of representation.
- Students at the Trans-derivative level of development generally have no difficulty establishing direct and contrapositive relationships, between the sign of the first derivative and the monotony of the function ( $V_{11}, V_{12}, V_{13}, V_{14}$ ). Instead, students at the Inter-derivative level have difficulties in establishing contrary relationships, which is to say, between the monotony of the function and the sign of the first derivative ( $V_{12}, V_{14}$ ). Meanwhile, students at the Intra-derivative level have difficulty in establishing both relationships (direct and contrapositive).
- Students at the Trans-derivative level have little difficulty in establishing direct and contrary relationships, between the sign of the second derivative and the curvature of the function ( $V_{15}, V_{16}, V_{17}, V_{18}$ ). Instead, students at the Inter-derivative level have difficulty establishing contrary relationships, which is to say, between the curvature of the function and the sign of the second derivative ( $V_{16}, V_{18}$ ). Meanwhile, students at the Intra-derivative level of development have serious difficulties in establishing these relationships (direct and contrapositive)
- Regarding the use of the lateral derivatives ( $V_{19}$ ) and the processing of conflict points ( $V_{20}$ ), it is noted that students at the Trans-derivative level generally do not have difficulties in their proper use. However, students of the Intra-derivative and Inter-derivative levels do have serious difficulties in using both variables.
- Students at the Trans-derivative level do not show great difficulties in the use of direct and contrapositive relations between differentiability and continuity ( $V_{21}, V_{22}$ ). Instead, while students at the Intra-derivative and Inter-derivative levels use them, they still have some difficulties in doing.
- Students at the Trans-derivative level can determine ranges from graphic information ( $V_{23}$ ) and correctly create a function from this information ( $V_{25}$ ) without difficulty. On the other hand, students at the Intra-derivative and Inter-derivative levels have difficulty doing so. Similarly, the students at the Trans-derivative level intervals can also determine from analytical information provided ( $V_{24}$ ) and can correctly create a function from this information ( $V_{26}$ ) without difficulty. Students at the Inter-derivative level can also determine the ranges from analytical information, however, they show great difficulties in outlining a function with such information.
- Only some of the students at the Trans-derivative level of development are able to establish relationships between the first and second derivative.

While all information provided by the percentages regarding the correct use of variables and multivariate profiles is valuable to characterize the levels of development of the derivative *schema*, this does not provide information regarding which variables are most important and determine the underlying structure of each level of development, and thus, diagrams were constructed to obtain such implicative information for each of the levels of development. In [Figure 6](#) shows the trees associated with each level.



**Figure 6.** Implicative trees for each level of development of derivative schema

For each level of development of the *schema*, the corresponding implicative tree diagram provides information about the most important variables of the structure. In particular, the variables that begin implicative chains are those that cause more difficulties in each level, and the ending implicative variables correspond to chains that were easier to use. Therefore, if we consider the implicative tree for the Trans-derivative level of development, we can indicate that the most important variable is related to establishment of relations between the first and second derivative ( $V_{27}$ ); also this indicates that the students who are able to establish such relationships can also successfully use other variables. In addition, it is important to note that the different colors of the arrows indicate the strength of involvement. In particular, blue corresponds to implications with 95% significance, green with 90%, and gray with 85%. In addition, as we can see, not all variables are displayed on the implicative tree, which is due to the fact that the implications of these variables with the present the diagram are below 85% of significance.

Another important aspect of note is that the implicative tree of the Trans-derivative level, as already mentioned, only has an underlying variable that determines the structure of the relations between them. In addition, these relationships are significant, with a value that is 85% or higher. Meanwhile, at the Inter-derivative level of development, there are four variables that determine the structure of relations, and in addition, their implications values are greater than or equal to 85%. In addition, differences may be noted in the number of levels of implications for each level of development. In particular, the Trans-derivative level has five levels of implication, the Inter-derivative has four levels of implication, and the Intra-derivative only has one level of implication. In addition, in this last level, no chain of implications is formed.

## DISCUSSION AND CONCLUSIONS

The results of this work have allowed us to identify levels associated with the development of the derivative *schema*. Each level has the characteristics associated with the previous level and new ones that differentiate it, which shows the progressive nature of the understanding of the concept. This characterization of the different levels of development of the *schema*, conducted through a cluster analysis and the frequency and implicative analysis confirms several conclusions from previous qualitative studies done beforehand (Baker et al., 2000; Cooley et al., 2007; Sánchez-Matamoros, 2004; Sánchez-Matamoros et al., 2006), regarding the essential role they play; the modes of representation, the ends and inflection points values, as well as the logical relationships that may be established between mathematical elements, which are those that determine and differentiate the distinct levels of the development of the *schema*.

Moreover, the results of implicative analysis carried out in each of these levels of development of the *schema* also corroborate and extend the results of previous research. Thus, for example, in research Sánchez-Matamoros (2004), Sánchez-Matamoros et al. (2006) and Garcia et al. (2011) a student that is considered to be in the Trans level of development of the derivative *schema* is able to transfer all relations  $f$  and  $f'$  to the pair  $f'$  and  $f''$ . This result is similar to that obtained in the implicative tree of the development level Trans-derivative of this work, which indicates that the most important variable is related to establishment of relations between the first and second derivatives ( $V_{27}$ ), this It shows that the student is able to establish these relationships can also successfully use the other variables ( $V_1, V_2, \dots, V_{26}$ ). In addition, the implicative analysis using different levels of statistical significance (95% blue, green 90%, and 85% gray) and the quasi-implications provide certain information regarding the consistency of the construction of the *schema*. In particular, this coherence is observed in the number of variables

that determine the implications of chains and the statistical significance of these, with the observation of a number of variables that underlie the structure of each level of development that increase from the Trans to the Intra level (one for the Trans, four for the Inter, and five for the Intra).

Also, the number of levels of involvement observed at each level of development also reinforces the conclusion that the coherence of the *schema* is only achieved at the Trans-derivative level of development, which in this case has five levels of implications, unlike what happens to the level of Inter-derivative development that has four levels of involvement, and the Intra-derivative level that only has one level of involvement.

Finally, this design and characterization of the levels of development of the derivative schema can be enhanced with a holistic view that incorporates the different meanings of the concept, such as those proposed by various studies that use the ontosemiotic approach (Font et al., 2016; Pino-Fan, Godino & Font, 2011; Pino-Fan, Godino, & Font, 2018). Likewise, we note that the methodological design used can be applied to identify and characterize levels and/or sublevels of the development of schemas of other mathematical concepts.

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