Bielliptic modular curves $X_0^*(N)$ with square-free levels

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Abstract

Let $N \geq 1$ be a square-free integer such that the modular curve $X_0^*(N)$ has genus ≥ 2. We prove that $X_0^*(N)$ is bielliptic exactly for 19 values of N, and we determine the automorphism group of these bielliptic curves. In particular, we obtain the first examples of nontrivial $\mathrm{Aut}(X_0^*(N))$ when the genus of $X_0^*(N)$ is ≥ 3. Moreover, we prove that the set of all quadratic points over $\mathbb{Q}$ for the modular curve $X_0^*(N)$ with genus ≥ 2 and N square-free is not finite exactly for 50 values of N.

1 Introduction

Let X be a smooth projective curve defined over a number field K of genus g_X at least two. In [9], Faltings proved the finiteness of the set of points of X defined over K, denoted by $X(K)$. After that, for a finite extension L/K, the natural object to consider was the set of points of X defined over all quadratic extensions of L, i.e. the set

$$\Gamma_2(X, L) := \cup_{[F:L]\leq 2} X(F).$$

In [13], Harris and Silverman proved that the above set is not finite for some number field L if, and only if, X is hyperelliptic or bielliptic, i.e. the curve $X \times_K \overline{K}$ admits a degree 2 map to the projective line or to an elliptic curve over a fixed algebraic closure $\overline{K}$ of K. Moreover, from the work of Abramovich, Harris and Silverman, in [5, Theorem 2.14] it is proved that the set $\Gamma_2(X, L)$ is infinite if, and only if, X is hyperelliptic over L, i.e. there is a morphism $\phi\colon X \to \mathbb{P}^1_L$ of degree two defined over L, or X is bielliptic over L, i.e. there exist an elliptic curve E over L and a morphism $\phi : X \to E$ of degree two defined over L, such that the L-rank of E is at least one.

This made the study of bielliptic curves a matter of deep interest for Arithmetic Geometry. This was developed for the modular world, because L-points of modular curves have a moduli interpretation on elliptic curves. The first work concerned the modular curves $X_0(N)$. The levels N for which the set $\Gamma_2(X_0(N), \mathbb{Q})$ is finite are determined in [4].

Later, different results determining the bielliptic curves among some modular curves recovering $X_0(N)$ can be found in [17] for $X_1(N)$, in [19] for $X_\Delta(N)$ and in [6] and [18] for $X(N)$.

Two important tools to obtain such results are the following. First, if there is a morphism of curves $X \twoheadrightarrow X'$ such that X is bielliptic and the genus of X' is at least 2, then X' is hyperelliptic or bielliptic [13, Proposition 1]. One can use results about hyperelliptic modular curves, whose study has been widely treated in the last decades. Second, X is bielliptic if, and only if, there exists an involution of X fixing $2g_X - 2$-many points, where g_X denotes de genus of X.

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In this paper we consider the modular curves $X_0^*(N)$. They are defined as the quotient of the modular curve $X_0(N)$ by the group of all Atkin-Lehner involutions, which is defined over $\mathbb{Q}$. Its K-points, which are not cuspidal, correspond to K-curves with additional data from the level N. See [8] for further information.

Here, we restrict our attention to the case where N is square-free. Under such assumption, the modular curve $X_0^*(N)$ corresponds to the quotient $X_0(N)/\operatorname{Aut}(X_0(N))$ and, thus, it does not have any natural automorphisms (in particular, involutions) coming from $X_0(N)$, except for $N = 37$. These curves have two properties that play an important role in the development of this article. On the one hand, all involutions of $X_0^*(N)$ are defined over $\mathbb{Q}$. On the other hand, the endomorphism algebra $\operatorname{End}(\operatorname{Jac}(X_0^*(N)) \otimes \mathbb{Q}$ is isomorphic to the product of totally real numbers fields (cf. [3, §2]). Any of these properties can fail when N is non square-free and the study of this case needs additional tools.

We point out that if $X_0^*(N)$ is bielliptic, then a bielliptic quotient (i.e. the quotient of $X_0^*(N)$ by a bielliptic involution), is an elliptic curve E defined over $\mathbb{Q}$ of conductor $M|N$ with odd analytic rank, because the attached newform is invariant under the Atkin-Lehner involution w_M. Hence, it is expected that the algebraic rank of E is odd.

In our case, the knowledge of the values of N for which $X_0(N)$ is bielliptic is not useful to obtain bielliptic curves $X_0^*(N)$, because, in this case, these curves have genus at most one or are hyperelliptic. A new approach is needed to deal with our case and, here, we use a method given in [11] to discard automorphisms of certain order in the automorphism group of a curve defined over a finite field. In particular, this method allows us to deduce that the automorphism group is trivial for such values of N, when such method works. This approach behaves well for odd square-free integers N, when N is the product of two or three primes. When it fails, we use the usual method of reducing modulo a prime p to discard some situations. For the remaining cases, using a Theorem of Petri [?], we implement a method to recognize whether a curve $X_0^*(N)$ is bielliptic and compute equations for the elliptic quotient.

The main result of this article is the following.

Theorem 1. *Let $N > 1$ be a square-free integer. Assume that the genus of the modular curve $X_0^*(N)$ is at least 2. Then, the modular curve $X_0^*(N)$ is bielliptic if, and only if, N is in the following table*

genus	N
2	$106, 122, 129, 158, 166, 215, 390$
3	$178, 183, 246, 249, 258, 290, 303, 318, 430, 455, 510$
4	370

For these values of N, the automorphism group of $X_0^*(N)$ has order 2 when its genus is greater than two, otherwise it is the Klein group.

Concerning $\operatorname{Aut}(X_0^*(N))$, with N square-free and genus ≥ 2, it is known that it is an abelian 2-group (cf. [3]). Moreover, when N is prime this group is nontrivial if, and only if, the genus of the curve is 2 and, in this case, the group has order 2 (cf. [3, Theorem 1.1]). In fact, it is expected that this group is trivial for almost all square-free N.

In this paper, we can observe that the Klein group appears naturally for genus two curves which are also bielliptic (cf. Remark 10). Moreover, we point out that a bielliptic curve could have several involutions and also more than one bielliptic involution when its genus is ≤ 5 (cf. [5, Prop.2.10]). Nevertheless, this does not happen in our case when the genus is > 2.

As a by-product of this work, we also obtain the following result.

Proposition 2. *The automorphism group of $X_0^*(N)$ is trivial for the following values of N:*

$$185, 202, 259, 262, 267, 273, 282, 301, 305, 310, 326, 354, 393, 394, 395, 399, 426, 427, 429, 445, 458, 462, 489, 505, 518, 546, 553, 570, 581, 582, 602, 627, 638, 710, 754, 786, 795, 814, 826, 854, 903, 915, 1001, 1015\,.$$

Moreover, the group $\mathrm{Aut}(X_0^*(366))$ *has order* 2*, and the quotient curve has genus* 2*.*

For many of these values (see Propositions 11 and 21), this result is obtained by using the method to discard the existence of involutions, which was mentioned above. A Magma code to be applied in our case can be found in

`http://mat.uab.cat/~francesc/programmesXoestrellaMagma.html`

(this html page, also contains different codes in Magma for computing the genus of $X_0^*(N)$ and its $\mathbb{F}_{p^k}$-points). For the remaining values of the above proposition (see Propositions 16, 17, 23 and 24), Theorem of Petri is the main tool.

As for quadratic points, by the work of Hasegawa and Hashimoto (cf. [15]), we know that $X_0^*(N)$ is hyperelliptic with N square-free if, and only if, the curve has genus 2. When $X_0^*(N)$ is bielliptic with genus > 2, the rank of the elliptic quotient turns out to be one. So we conclude

Theorem 3. *Let $N > 1$ be a square-free integer. Assume that the genus of the modular curve $X_0^*(N)$ is ≥ 2. Then, the set $\Gamma_2(X_0^*(N), \mathbb{Q})$ is infinite if, and only if, N lies in the set*

$$\{67, 73, 85, 93, 103, 106, 107, 115, 122, 129, 133, 134, 146, 154, 158, 161, 165, 166, 167, 170, 177, 178, 183, 186, 191, 205, 206, 209, 213, 215, 221, 230, 246, 249, 255, 258, 266, 285, 286, 287, 290, 299, 303, 318, 330, 357, 370, 390, 430, 455, 510\}\,.$$

Theorem 3 holds when we replace $\Gamma_2(X_0^*(N), \mathbb{Q})$ with $\Gamma_2(X_0^*(N), K)$, where K is any number field. This is due to the fact that if $X_0^*(N)$ is hyperelliptic or bielliptic over K, then it is hyperelliptic or biellitic over $\mathbb{Q}$ (cf. Lemma 4)

2 Preliminary results

Let $N > 1$ be an integer. We fix once and for all the following notation. We denote by g_N and g_N^* the genus of $X_0(N)$ and $X_0^*(N)$, respectively, and n is the number of primes dividing N. For any $1 \leq d | N$ with $(d, N/d) = 1$ we have an involution $w_d \in \mathrm{Aut}(X_0(N))$, called the Atkin-Lehner involution attached to d, and we denote by $B(N)$ the group of all Atkin-Lehner involutions. We denote by New_N the set of normalized newforms in $S_2(\Gamma_0(N))$, and New_N^* is the subset of New_N consisting of the newforms invariant under the action of the group $B(N)$. For an integer $m \geq 1$ and a newform $f \in \mathrm{New}_N$, $a_m(f)$ is the m-th Fourier coefficient of f. For an eigenform $f \in S_2(\Gamma_0(N)$, A_g denotes the abelian variety defined over $\mathbb{Q}$ attached by Shimura to g. As usual, ψ is the Dedekind psi function.

In the sequel, N is square-free. We recall the following result of Baker and Hasegawa.

Lemma 4 (Corollary 2.6 in [3])**.** *The group* $\mathrm{Aut}\, X_0^*(N)$ *is elementary* 2*-abelian and every automorphism of $X_0^*(N)$ is defined over $\mathbb{Q}$.*

From now on, we assume that $X_0^*(N)$ has a bielliptic involution u. Let us denote by E the elliptic quotient $X_0^*(N)/\langle u \rangle$ and by π the nonconstant morphism $X_0(N) \to X_0^*(N) \to E$, which has degree 2^{n+1} and is defined over $\mathbb{Q}$. Let M be the conductor of E. It is well-known that $M | N$ and there exist a morphism $\pi_M \colon X_0^*(M) \twoheadrightarrow E$ and a normalized newform $f_E \in \mathrm{New}_M^*$ such that $\pi_M^*(\Omega^1_{E/\mathbb{Q}}) = \mathbb{Q}(f_E(q)\mathrm{d}\, q/q)$. Moreover, $\pi^*(\Omega^1_{E/\mathbb{Q}}) = \mathbb{Q}(g(q)\mathrm{d}\, q/q)$, where g is the eigenform

$\sum_{1\le d|N/M} w_d^*(f) \in S_2(\Gamma_0(N))^{B(N)}$ and w_d stands for the Atkin-Lehner involution attached to d. More precisely, $g(q) = \sum_{1\le d|N/M} d\, f_E(q^d)$.

Since the gonality of $X_0^*(N)$ is ≤ 4, by applying Proposition 4.4 in [2], we obtain $g_N^* \le 35$. Nevertheless, this fact is not very useful in order to determine a finite set of possible values for N. The following lemma helps us to achieve this goal. Note that for a prime $p \nmid M$, $|E(\mathbb{F}_{p^2})| \le (p+1)^2$, but $|E(\mathbb{F}_{p^2})| = (p+1)^2 - a_p^2(f_E)$ due to the congruence of Eichler-Shimura.

Lemma 5. *The following inequalities hold:*

(i) *If $p \nmid N$, then*

$$\text{(a) } \frac{\psi(N)}{2^n} \le 12 \cdot \frac{2|E(\mathbb{F}_{p^2})|-1}{p-1}, \quad \text{(b) } g_N^* \le 2\frac{|E(\mathbb{F}_{p^2})|}{p-1}, \quad \text{(c) } g_N \le 2^{n+1}\frac{|E(\mathbb{F}_{p^2})|}{p-1}.$$

(ii) *If $p \mid N$, then $g_{N/p}^* \le 1$ or*

$$\frac{\psi(N/p)}{2^{n-1}} \le 12\frac{2\,|E(\mathbb{F}_{p^2})|-1}{p-1}, \quad g_{N/p}^* \le 2\frac{|E(\mathbb{F}_{p^2})|}{p-1}, \quad g_{N/p} \le 2^n\frac{|E(\mathbb{F}_{p^2})|}{p-1}, \quad \text{if } p \nmid M,$$
$$\frac{\psi(N/p)}{2^{n-1}} \le 12\frac{2p^2+1}{p-1}, \quad g_{N/p}^* \le 2\frac{p^2+1}{p-1}, \quad g_{N/p} \le 2^n\frac{p^2+1}{p-1}, \quad \text{if } p|M.$$

Proof. Assume $p \nmid N$. We generalize the argument used by Ogg in [20]. Indeed, $X_0(N)(\mathbb{F}_{p^2})$ contains 2^n cusps and at least $(p-1)\frac{\psi(N)}{12}$ many supersingular points (cf. [2, Lemma 3.20 and 3.21]). Since there is a nonconstant morphism defined over $\mathbb{Q}$ from $X_0(N)$ to an elliptic quotient E of $X_0^*(N)$ which has degree 2^{n+1}, $|X_0(N)(\mathbb{F}_{p^2})| \le 2^{n+1}|E(\mathbb{F}_{p^2})|$. Parts (b) and (c) in (i) are obtained applying [2, Lemma 3.25].

If $p|N$, then $X_0(N)/\mathbb{F}_p$ is the copy of two curves $X_0(N/p)/\mathbb{F}_p$, and the normalization of $X_0^*(N)/\mathbb{F}_p$ is the curve $X_0^*(N/p)/\mathbb{F}_p$ (cf. [10]). If the reduction of the involution u is the identity, then $g_{N/p}^*$ is the genus of $E/\mathbb{F}_p$. Otherwise, $|X_0^*(N/p)(\mathbb{F}_{p^2})|$ is at most $2(p^2+1)$ or $|E(\mathbb{F}_{p^2})|$, depending on whether $p \mid M$ or not. □

Remark 6. *The above conditions imply $n \le 4$, when N is odd, and $n \le 5$ in the even case. The values N for which $g_N^* \le 1$ can be found in [12, Proposition 3.1 and 3.2], and those for which $g_N^* = 2$ can be found in [14, Theorem 2].*

Keeping the above notation, we present the following lemma, which will used to discard some elliptic curves E for a value N.

Lemma 7. *Let E' be the elliptic curve in the $\mathbb{Q}$-isogeny class of E that is an optimal quotient of the jacobian of $X_0^*(M)$. If $M = N$, then the degree D of the modular parametrization $\pi_N \colon X_0(N) \to E'$ divides 2^{n+1}.*

Proof. The statement follows from the optimality of π_N and the fact that the degree of π is 2^{n+1}. □

Remark 8. *The degree D can be found in [7, Table 5].*

3 Odd case

If N is odd, applying Lemma 5 for $p = 2$, we have $\psi(N)/2^n \leq 204$. This fact implies $n \leq 3$, except for the values $N \in \{3 \cdot 5 \cdot 7 \cdot 11, 3 \cdot 5 \cdot 7 \cdot 13\}$. The case $n = 1$ can be discarded, since $\mathrm{Aut}(X_0^*(p))$ is trivial for all p except for $g_p^* = 2$ and, in this particular hyperelliptic case, the automorphism group has order two (cf. [3, Theorem 1.1]). We assume also that $g_N^* \geq 2$ and apply Lemma 5 for the values $\gcd(N, 3)$ for which $\psi(N)/2^n \leq 186$. There are exactly 146 values for odd N such that $n > 1$, $\psi(N) \leq 204 \cdot 2^n$ (or $\leq 186 \cdot 2^{n+1}$ if $6 \nmid N$) and all of these satisfy $1 < g_N \leq 9 \cdot 2^{n+1}$. More precisely, we have 100 cases for $n = 2$, 44 for $n = 3$ and 2 for $n = 4$. We can reduce this list by applying Lemma 5 for each candidate N, we consider the pairs (N, E), where E is the $\mathbb{Q}$-isogeny class of the elliptic curves of conductor $M|N$ such that its attached newform f_E lies in New_M^*. From [7, Table 5,Table 3], we obtain the degree D of π_N, when $M = N$, and $a_2(f_E)$, $a_3(f_E)$ and $a_5(f_E)$. In particular, we know $|E(\mathbb{F}_4)|$, $|E(\mathbb{F}_9)|$ when $3 \nmid M$, and $|E(\mathbb{F}_{25})|$ when $5 \nmid M$.

For $n > 1$, we can discard the pairs (N, E) that do not satisfy the conditions in Lemmas 5 and 7. When $g_N^* = 2$, if an elliptic quotient of $X_0^*(N)$ is not bielliptic, then we can discard N (cf. Remark 10). In particular, we discard $N = 285$ because the elliptic quotient of $X_0^*(285)$ with conductor 258 does not satisfy Lemma 7. In Table 1, we present the remaining possibilities, where the label of the elliptic curve E is the one in Cremona tables.

N	g_N	g_N^*	M	Label E
$129 = 3 \cdot 43$	13	2	N	a
			43	a
$183 = 3 \cdot 61$	19	3	61	a
$185 = 5 \cdot 37$	17	3	37	a
$215 = 5 \cdot 43$	21	2	N	a
			43	a
$237 = 3 \cdot 79$	25	5	79	a
$249 = 3 \cdot 83$	27	3	N	b
			83	a
$259 = 7 \cdot 37$	23	4	37	a
$267 = 3 \cdot 89$	29	4	89	a
$301 = 7 \cdot 43$	27	6	43	a
$303 = 3 \cdot 101$	33	3	101	a
$305 = 5 \cdot 61$	29	4	61	a
$393 = 3 \cdot 131$	43	5	131	a
$395 = 5 \cdot 79$	39	4	79	a
$415 = 5 \cdot 83$	41	8	83	a
$427 = 7 \cdot 61$	39	8	61	a
$445 = 5 \cdot 89$	43	7	89	a
$581 = 7 \cdot 83$	55	8	83	a
$273 = 3 \cdot 7 \cdot 13$	33	4	91	a
$385 = 5 \cdot 7 \cdot 11$	45	4	77	a
$399 = 3 \cdot 7 \cdot 19$	49	4	57	a
$429 = 3 \cdot 11 \cdot 13$	53	3	143	a
$435 = 3 \cdot 5 \cdot 29$	57	5	145	a
$455 = 5 \cdot 7 \cdot 13$	53	3	65	a
			91	a
$465 = 3 \cdot 5 \cdot 31$	61	5	155	c
$555 = 3 \cdot 5 \cdot 37$	73	5	185	c
$615 = 3 \cdot 5 \cdot 41$	81	6	123	b
$645 = 3 \cdot 5 \cdot 43$	85	5	215	a
			129	a
$705 = 3 \cdot 5 \cdot 47$	88	8	141	d
$715 = 5 \cdot 11 \cdot 13$	81	8	143	a
			65	a
$795 = 3 \cdot 5 \cdot 53$	105	10	265	a
			53	a
$861 = 3 \cdot 7 \cdot 41$	109	7	123	b
$903 = 3 \cdot 7 \cdot 43$	113	11	129	a
$987 = 3 \cdot 7 \cdot 47$	125	9	141	d
$1001 = 7 \cdot 11 \cdot 13$	109	8	143	a
$1015 = 5 \cdot 7 \cdot 29$	117	11	145	a
$1155 = 3 \cdot 5 \cdot 7 \cdot 11$	185	8	77	a
$1365 = 3 \cdot 5 \cdot 7 \cdot 13$	237	9	455	a
			65	a

Table 1

First, we examine the hyperelliptic cases in Table 1, which correspond to those such that $g_N^* = 2$ (cf. [10, Theorem 2]).

Proposition 9. *The curves of genus two $X_0^*(129)$ and $X_0^*(215)$ are bielliptic.*

Proof. For $N = 129$ and $N = 215$, the jacobian of $X_0^*(N)$ is isogenous over $\mathbb{Q}$ to the product of two elliptic curves $E_1 \times E_2$, where E_1 has conductor N and E_2 has conductor $M = 43$. Hence, there exist two normalized newforms $f_1 \in \mathrm{New}_N^*$ and $f_2 \in \mathrm{New}_M^*$ such that the elliptic curves A_{f_1} and A_{f_2} are isogenous over $\mathbb{Q}$ to E_1 and E_2 respectively. The set of the regular differentials

$$\omega_1 = f_1(q)\mathrm{d}\, q/q\,, \quad \omega_2 = (f_2(q) + \ell f_2(q^\ell))\mathrm{d}\, q/q\,, \text{ with } \ell = N/M\,,$$

is a basis of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$. The functions $x = \dfrac{\omega_2}{\omega_1}$ and $y = \dfrac{\mathrm{d}\, x}{\omega_1}$ on $X_0^*(N)$ satisfy the equations

N	equations		
129	$4\,y^2$	$=$	$x^6 - 11\,x^4 + 35\,x^2 - 9$
215	$4\,y^2$	$=$	$-x^6 - 5\,x^4 - 3\,x^2 + 25$

For $N = 129$ and 215, it is clear that the curves have two bielliptic involutions $(x, y) \mapsto (-x, \pm y)$. □

Remark 10. *Assume that $X_0^*(N)$ has genus two and has an elliptic quotient. Then, $\mathrm{Jac}(X_0^*(N))$ is isogenous over $\mathbb{Q}$ to the product of two non isogenous elliptic curves E_1 and E_2. If $X_0^*(N)$ has a bielliptic involution u, then $u^*(E_i) = E_i$ and their regular differentials ω_1 and ω_2 are eigenvectors of u. Hence, $u^*(\omega_1)$ must be $\pm\omega_1$ and $u^*(\omega_2) = \mp\omega_2$. Therefore $u^*(x) = u^*(\omega_1/\omega_2) = -x$ and $u^*(y)$ is y or $-y$ depending on whether $u^*(\omega_2)$ is $-\omega_2$ or not. In any case, $y^2 = P(x^2)$ for a degree three polynomial $P \in \mathbb{Q}[x]$, and the automorphism group of the curve is the Klein group generated by u and the hyperelliptic involution w. Moreover, $X_0^*(N)$ has two bielliptic involutions u and $u \cdot w$ and both elliptic curves are bielliptic quotients.*

Now, we will apply two sieves to discard some values of N. Both are based on the values of $|X_0^*(N)(\mathbb{F}_{p^n})|$ for a prime $p \nmid N$. The first of them uses [11, Theorem 2.1], which allows us to detect some curves $X/\mathbb{Q}$ without involutions defined over $\mathbb{Q}$ (see [?, Prop.10.3.38]). More precisely, for a prime p of good reduction for a curve $X/\mathbb{Q}$ and an integer $n \geq 1$, consider the sequence

$$P_p(n) := \mathrm{mod}\,[(\sum_{d|n} \mu(n/d)|X(\mathbb{F}_{p^n})|)/n, 2]$$

where $\mathrm{mod}\,[r, 2]$ denotes 0 or 1 depending on whether r is even or not, and μ is the Moebius function. Set $Q_p(2k+1) = \sum_{n\geq 0}^{k}(2n+1)P_p(2n+1)$. If $X_0^*(N)$ has an involution defined over $\mathbb{Q}$, then

$$Q_p(2k+1) \leq 2g_N^* + 2 \text{ for all } k \geq 0\,.$$

Proposition 11. *The curve $X_0^*(N)$ is not bielliptic and, moreover, $\mathrm{Aut}(X_0^*(N))$ is the trivial group for the following values of N:*

$$259, 267, 301, 305, 393, 395, 427, 445, 581, 795, 903, 1001, 1015\,.$$

Proof. For $f \in \mathrm{New}_M$, denote by K the number field $\mathbb{Q}(\{a_n(f)\}_{n\geq 1})$. Let p be a prime not dividing M. By the Eichler-Shimura congruence, the characteristic polynomial of Frob_p acting on the Tate module of A_f is

$$\prod_{\sigma\colon K\hookrightarrow\overline{\mathbb{Q}}} (x^2 - a_p(f^\sigma) + p)\,,$$

where σ runs over the set of all $\mathbb{Q}$-embeddings of K into a fixed algebraic closure of $\mathbb{Q}$. The jacobian of $X_0^*(N)$ is isogenous over $\mathbb{Q}$ to the product $\prod_{1\le M|N}\prod_{f\in \mathrm{New}_M^*/G_{\mathbb{Q}}} A_f$, where $G_{\mathbb{Q}}$ denotes the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

To compute $|X_0^*(N)(\mathbb{F}_{p^n})|$, we proceed as follows. By using Magma, we determine New_M^* for all $M|N$ and, then, the characteristic polynomial $R_p(x)$ of Frob_p acting on the Tate module of $\mathrm{Jac}(X_0^*(N))$ is obtained as follows

$$R_p(x) = \prod_{M|N}\prod_{f\in \mathrm{New}_M{}^*} (x^2 - a_p(f)\,x + p) = \prod_{i=1}^{2\,g_N^*}(x-\alpha_i)\,.$$

Finally,

$$|X_0^*(N)(\mathbb{F}_{p^n})| = p^n + 1 - \sum_{i=1}^{2\,g_N^*}\alpha_i^n\,.$$

The statement follows from these computations:

N	$Q_p(2k+1)$	$2g_N^*+2$
$259 = 7\cdot 37$	$Q_2(9) = 17$	10
$267 = 3\cdot 89$	$Q_2(7) = 15$	10
$301 = 7\cdot 43$	$Q_2(11) = 22$	14
$305 = 5\cdot 61$	$Q_2(7) = 15$	10
$393 = 3\cdot 131$	$Q_2(11) = 23$	12
$395 = 5\cdot 79$	$Q_2(7) = 15$	10
$427 = 7\cdot 61$	$Q_2(11) = 27$	18
$445 = 5\cdot 89$	$Q_3(9) = 22$	16
$581 = 7\cdot 83$	$Q_2(13) = 20$	18

N	$Q_p(2k+1)$	$2g_N^*+2$
$795 = 3\cdot 5\cdot 53$	$Q_2(13) = 27$	22
$903 = 3\cdot 7\cdot 43$	$Q_2(13) = 31$	24
$1001 = 7\cdot 11\cdot 13$	$Q_2(11) = 22$	18
$1015 = 5\cdot 7\cdot 29$	$Q_2(13) = 30$	24

□

The second sieve is based on the following fact. For a degree two morphism of curves $X \to Y$ defined over $\mathbb{Q}$ and a prime p of good reduction for X, one has

$$|X(\mathbb{F}_{p^k})| - 2|Y(\mathbb{F}_{p^k})| \le 0\,,\ \text{for all } k > 0\,.$$

Proposition 12. *The pairs (N, E) in the set*

$$\{(273, 91a), (385, 77a), (415, 83a), (429, 143a), (435, 145a), (455, 91a),\\ (465, 155c), (555, 185c), (615, 123b), (705, 141d), (715, 65a), (715, 143a),\\ (861, 123b), (987, 141d), (1155, 77a), (1365, 65a), (1365, 455a)\}$$

are not bielliptic. In particular, the curve $X_0^*(N)$ is not bielliptic for the following values of N:

$$273, 385, 415, 429, 435, 465, 555, 615, 705, 715, 861, 987, 1155, 1365\,.$$

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Proof.

N	p^k	E	$\lvert X_0^*(\mathbb{F}_{p^k})\rvert - 2\lvert E(\mathbb{F}_{p^k})\rvert$
273	8	$91a$	3
385	9	$77a$	3
415	9	$83a$	4
429	16	$143a$	5
455	4	$91a$	1
435	8	$145a$	2
465	7	$155c$	2
555	2	$185c$	1
615	11	$123b$	4
705	4	$141d$	1
715	9	$65a$	7
715	9	$143a$	1
861	4	$123b$	1
987	25	$141d$	10
1155	2	$77a$	1
1365	4	$65a$	1
1365	2	$455a$	3

□

After applying the two sieves, the following possibilities for the pairs (N, E), ordered by the genus, remain.

N	g_N^*	E
183	3	$61a$
185	3	$37a$
249	3	$83a$
		$249b$
303	3	$101a$
455	3	$65a$

N	g_N^*	E
399	4	$57a$
237	5	$79a$
645	5	$129a$
		$215a$

Table 2

Finally, in order to decide which values N in Table 2 correspond to bielliptic curves, we shall use equations.

We recall that, for a nonhyperelliptic curve X defined over $\mathbb{C}$ with genus $g > 2$, the image of the canonical map $X \to \mathbf{P}^{g-1}$ is the common zero locus of a set of homogeneous polynomials of degree 2 and 3, when $g > 3$, or of a homogenous polynomial of degree 4, if $g = 3$.

More precisely, assume that X is defined over $\mathbb{Q}$ and choose a basis $\omega_1, \cdots \omega_g$ of $\Omega^1_{X/\mathbb{Q}}$. For any integer $i \geq 2$, let us denote by $\mathcal{L}_i$ the $\mathbb{Q}$-vector space of homogeneous polynomials $Q \in \mathbb{Q}[x_1, \cdots, x_g]$ of degree i that satisfy $Q(\omega_1, \cdots, \omega_g) = 0$. Of course, $\dim \mathcal{L}_i \leq \dim \mathcal{L}_{i+1}$ because one has $x_j \cdot Q \in \mathcal{L}_{i+1}$ for all $Q \in \mathcal{L}_i$ and for $1 \leq j \leq g$.

If $g = 3$, then $\dim \mathcal{L}_2 = \dim \mathcal{L}_3 = 0$ and $\dim \mathcal{L}_4 = 1$. Any generator of $\mathcal{L}_4$ provides an equation for X. For $g > 3$, $\dim \mathcal{L}_2 = (g-2)(g-3)/2 > 0$ and a basis of $\mathcal{L}_2 \bigoplus \mathcal{L}_3$ provides a system of equations for X. When X is neither trigonal nor a smooth plane quintic ($g = 6$), it suffices to take a basis of $\mathcal{L}_2$.

For the curve $X_0^*(N)$ there exists a set of normalized eigenforms $g_1, \cdots, g_k \in S_2(\Gamma_0(N))^{B(N)}$ such that $\mathrm{Jac}(X_0^*(N)) \overset{\mathbb{Q}}{\sim} A_{g_1} \times \cdots \times A_{g_k}$, where the symbol $\overset{\mathbb{Q}}{\sim}$ means isogenous over $\mathbb{Q}$. These abelian varieties are simple and pairwise nonisogenous over $\mathbb{Q}$. Hence, any involution u of the

curve leaves stable A_{g_i} and acts on $\Omega^1_{A_{g_i}}$ as the product by -1 or the identity, because the endomorphism algebra $\mathrm{End}_{\mathbb{Q}} A_{g_i} \otimes \mathbb{Q}$ is isomorphic to a (totally real) number field.

We choose a basis $\{\omega_1, \cdots, \omega_{g_N^*}\}$ of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ obtained as the union of bases of all $\Omega^1_{A_{g_i}/\mathbb{Q}}$. An involution u of $X_0^*(N)$ induces a linear map $u^* : \Omega^1_{X_0^*(N)/\mathbb{Q}} \to \Omega^1_{X_0^*(N)/\mathbb{Q}}$ sending $(\omega_1, \cdots, \omega_{g_N^*})$ to $(\varepsilon_1\omega_1, \cdots, \varepsilon_n\omega_{g_N^*})$ with $\varepsilon_i = \pm 1$ for all $i \le g_N^*$ and satisfying

$$Q(\varepsilon_1 x_1, \cdots, \varepsilon_{g_N^*} x_{g_N^*}) \in \mathcal{L}_i \text{ for all } Q \in \mathcal{L}_i \text{ and for all } i\,. \tag{3.1}$$

Conversely, for a linear map u^* as above satisfying condition (3.1), only one of the two maps $\pm u^*$ comes from an involution of the curve, because we are assuming that X is nonhyperelliptic.

We particularize this fact to our case.

Lemma 13. *Assume $X_0^*(N)$ is nonhyperelliptic. Let $\omega_1, \cdots, \omega_{g_N^*}$ be a basis of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ as above, such that ω_1 is the differential attached to an elliptic curve E. Then, the pair (N, E) is bielliptic if, and only, if*

$$Q(-x_1, x_2, \cdots, x_{g_N^*-1}, x_{g_N^*}) \in \mathcal{L}_i \text{ for all } Q \in \mathcal{L}_i \text{ and for all } i\,. \tag{3.2}$$

Proof. If u is an involution of $X_0^*(N)$ such that E is $\mathbb{Q}$-isogenous to $X_0^*(N)/\langle u\rangle$, then $u^*(\omega_1) = \omega_1$ and $u^*(\omega_i) = -\omega_i$ for $i > 1$. Hence, condition (3.2) is satisfied. Conversely, since the curve is nonhyperelliptic the condition (3.2) implies that only one of the two linear maps

$$(\omega_1, \omega_2, \cdots, \omega_{g_N^*-1}, \omega_{g_N^*}) \mapsto \pm(-\omega_1, \omega_2, \cdots, \omega_{g_N^*-1}, \omega_{g_N^*})$$

comes from an involution u of the curve. The genus g_u of the curve $X_0^*(N)/\langle u\rangle$ agrees with the number of differentials ω_i invariant under the action of u. When $g_N^* > 3$, it follows that g_u must be 1 because it cannot be $g_N^* - 1$ due to Riemann-Hurwitz formula. For $g_N^* = 3$, the genus g_u must be different from 2, since otherwise the curve would be hyperelliptic (cf. [1, Lemma 5.10]). □

Remark 14. *When $g_N^* > 4$, $\dim \mathcal{L}_2 > 1$. If ω_j is the differential attached to an elliptic curve, we need to check that the vector space*

$$\mathcal{L}_{2,j} := \{Q \in \mathcal{L}_2 \colon Q(x_1, \cdots, -x_j, \cdots, x_{g_N^*}) \in \mathcal{L}_2\}$$

is $\mathcal{L}_2$. Note that

$$\mathcal{L}_{2,j} = \{Q \in \mathcal{L}_2 \colon Q(x_1, \cdots, x_j, \cdots, x_{g_N^*}) = Q(x_1, \cdots, -x_j, \cdots, x_{g_N^*})\}$$

Indeed, if $Q \in \mathcal{L}_{2,j}$, then $H := Q(x_1, \cdots, x_j, \cdots, x_{g_N^*}) - Q(x_1, \cdots, -x_j, \cdots x_{g_N^*}) \in \mathcal{L}_2$. Therefore, $H = x_j P$ for an homogenous polynomial $P \in \mathbb{Q}[x_1, \cdots, x_{g_N^*}]$ of degree at most 1. Hence, P must be 0, otherwise $P(\omega_1, \cdots, \omega_{g_N^*}) = 0$.

Remark 15. *Recall that, for each one of the normalized eigenforms $g_i \in S_2(\Gamma_0(N))^{B(N)}$, there is $f_i \in \mathrm{New}_M^*$ such that $g_i = \sum_{d|N/M} d\, f_i(q^d)$ and $A_{g_i} \overset{\mathbb{Q}}{\sim} A_{f_i}$. To get a basis of $\Omega^1_{A_{g_i}/\mathbb{Q}}$ we can proceed as follows. If $\dim A_{f_i} = 1$, we take as basis $g_i(q)\, dq/q$. When $\dim A_{f_i} = r > 1$, the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(A_{f_i}) \otimes \mathbb{Q}$ is generated by Hecke operators and is isomorphic to the totally real number field $K_i := \mathbb{Q}(\{a_n(f_i)\}_{n>0})$ of degree r. Let $\mathcal{I}$ be the set of $\mathbb{Q}$-embeddings of K_i into a fixed algebraic closure of $\mathbb{Q}$. For every $a \in K_i$ there is a Hecke operator T such that*

$T(f_i^\sigma) = a^\sigma f_i^\sigma$ for all $\sigma \in \mathcal{I}$. The two cusp-form $h = \sum_{\sigma\in\mathcal{I}} f_i^\sigma$ is nonzero because the coefficient of q is g. Hence, taking T such that a is a primitive element of K_i, the set

$$f_j' = T^j(h) = \sum_{\sigma\in\mathcal{I}} (a^j)^\sigma f_i^\sigma \in \mathbb{Q}[[q]],\ 0 \le j \le r-1\,,$$

is a basis of the vector space generated by f_i^σ. Therefore,

$$\omega_j' = \Big(\sum_{d|N/M} d\, f_i'(q^d)\Big)\, dq/q \ \ 0 \le j \le r-1$$

is a basis of $\Omega^1_{A_{g_i}/\mathbb{Q}}$. One can take a as the value provided by Magma in the q-expansion of f_i and, in this case, $f_j' \in \mathbb{Z}[[q]]$. The curve $X_0^*(N)$ is determined by the first Fourier coefficients of the chosen basis for $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ (cf [2, Proposition 2.8]). In order to get shorter equations, it is suitable to replace the basis f_j' with a basis of the $\mathbb{Z}$-module $\mathbb{Q}(f_1', \cdots, f_g') \cap (\mathbb{Z}[[q]]\, dq)$.

Proposition 16. *Among the curves of genus three $X_0^*(183)$, $X_0^*(185)$, $X_0^*(249)$, $X_0^*(303)$, and $X_0^*(455)$, only $X_0^*(183)$, $X_0^*(249)$, $X_0^*(303)$ and $X_0^*(455)$ are bielliptic. The corresponding elliptic quotients are labeled as $E61a1$, $E249b1$, $E101a1$ and $E65a1$, respectively. In all these cases, the automorphism group has order 2. The automorphism groups of the remaining curves are trivial.*

Proof. For these values of N, the splitting of the jacobian of $X_0^*(N)$, $J_0^*(N)$, is as follows:

$$\begin{array}{llllll}
J_0^*(183) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^2 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E61a\,, & f_2 \in \mathrm{New}_{183}^*\,, & \dim A_{f_2} = 2\,, & \\
J_0^*(185) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E37a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E185b\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E185c\,, & \\
J_0^*(249) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E83a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E249a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E249b\,, & \\
J_0^*(303) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^2 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E101a\,, & f_2 \in \mathrm{New}_{303}^*\,, & \dim A_{f_2} = 2\,, & \\
J_0^*(455) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E65a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E91a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E455a\,. &
\end{array}$$

We take a basis of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ following the order exhibited in the splitting of its jacobian and we obtain the following generators $Q \in \mathcal{L}_4$:

N	Q
183	$x^4 - 10x^2y^2 + 9y^4 - 24x^2yz + 24y^3z + 32y^2z^2 + 32yz^3 - 16z^4$
185	$243x^4 - 1728x^2y^2 - 768xy^3 - 2816y^4 - 324x^3z - 1728x^2yz - 1152xy^2z$ $+5888y^3z - 918x^2z^2 + 1152xyz^2 - 192y^2z^2 - 852xz^3 + 2624yz^3 + 571z^4$
249	$16x^4 - 64x^3y - 12x^2y^2 + 44xy^3 + 97y^4 - 180x^2z^2 + 36xyz^2 - 18y^2z^2 + 81z^4$
303	$x^4 + 2x^2y^2 - 3y^4 - 16y^2z^2 + 16yz^3 - 16z^4$
455	$81x^4 - 162x^2y^2 - 79y^4 - 324x^2yz + 244y^3z + 192y^2z^2 + 64yz^3 - 16z^4$

By Lemma 13, only the curves corresponding to $N = 183, 249, 303, 455$ are bielliptic with an only bielliptic involution u. The affine equations for the bielliptic quotients are

N	$X_0^*(N)/\langle u\rangle$
183	$x^2 - 10x + 9 - 24xz + 24z + 32z^2 + 32z^3 - 16z^4 = 0$
249	$97 + 44x - 12x^2 - 64x^3 + 16x^4 - 18z + 36xz - 180x^2z + 81z^2 = 0$
303	$x^2 + 2x - 3 - 16z^2 + 16z^3 - 16z^4 = 0$
455	$81x^2 - 162x - 79 - 324xz + 244z + 192z^2 + 64z^3 - 16z^4 = 0$

have genus one and their j-invariants are $-\frac{912673}{61}$, $\frac{357911}{249}$, $\frac{262144}{101}$ and $\frac{117649}{65}$. They correspond to the elliptic curves $E61a1$, $E249b1$, $E101a1$ and $E65a1$.

Taking into account the splitting of their jacobians and their equations, all their automorphism groups have order 2. The remaining curves have trivial automorphism groups. For instance, for $N = 249$, the linear map $(\omega_1, \omega_2, \omega_3) \mapsto (\omega_1, \omega_2, -\omega_3)$ is the only option to be considered and $Q(-x, y, z) \notin \mathcal{L}_4$. For $N = 185$, none of the polynomials $Q(-x, y, z)$,$Q(x, -y, z)$, $Q(x, y, -z)$ lies in $\mathcal{L}_4$. □

Proposition 17. *The curves of genus four $X_0^*(399)$ is not bielliptic and its automorphism group is trivial.*

Proof. The splitting of $J_0^*(399)$ is:

$$J_0^*(399) \overset{\mathbb{Q}}{\sim} A_{f_1} \times A_{f_2} \times A_{f_3}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E57a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E399a\,, \quad f_3 \in \mathrm{New}_{133}^*\,, \quad \dim A_{f_3} = 2\,.$$

In this case $\dim \mathcal{L}_2 = 1$. As in the previous proposition, we take a basis of $\Omega^1_{X_0^*(399)\mathbb{Q}}$ following the order exhibited in the splitting of its jacobian. Next, we show a generator $Q(x, y, z, t) \in \mathcal{L}_2$:

N	Q
399	$-99t^2 + 90tx + 125x^2 + 189ty + 80xy - 151y^2 + 306tz - 105xz + 42yz + 9z^2$

Since $Q(-x, y, z, t) \notin \mathcal{L}_2$, the curve is not bielliptic. The conditions $Q(x, -y, z, t), Q(-x, -y, z, t) \notin \mathcal{L}_2$ imply that the curve does not have any nontrivial involutions. □

Proposition 18. *The curves of genus five $X_0^*(237)$ and $X_0^*(645)$ are not bielliptic.*

Proof. The splitting of $J_0^*(N)$ is:

$$\begin{aligned} J_0^*(237) &\overset{\mathbb{Q}}{\sim} \textstyle\prod_{i=1}^{2} A_{f_i}\,, && A_{f_1} \overset{\mathbb{Q}}{\sim} E79a\,, && f_2 \in \mathrm{New}_{237}^*\,, && \dim A_{f_2} = 4\,, \\ J_0^*(645) &\overset{\mathbb{Q}}{\sim} \textstyle\prod_{i=1}^{4} A_{f_i}\,, && A_{f_1} \overset{\mathbb{Q}}{\sim} E43a\,, && A_{f_2} \overset{\mathbb{Q}}{\sim} E129a\,, && A_{f_3} \overset{\mathbb{Q}}{\sim} E215a\,, \quad f_4 \in \mathrm{New}_{645}^*\,, \quad \dim A_{f_4} = 2\,. \end{aligned}$$

Now, $\dim \mathcal{L}_2 = 3$. For $N = 237$, $\dim \mathcal{L}_{2,1} = 0$ and for $N = 645$ we also have $\dim \mathcal{L}_{2,2} = \dim \mathcal{L}_{2,3} = 0$. □

As a consequence, we obtain the statement of Theorem 1 for N odd.

Corollary 19. *For N odd, $X_0^*(N)$ is bielliptic if, and only if, $N \in \{129, 183, 215, 249, 303, 455\}$. For these values of N, the automorphism group has order 2 when $g_N^* > 2$, otherwise it is the Klein group.*

4 Even case

By applying Lemma 5, we determine a finite set of possible values of N. Then, we proceed as in the odd case and we obtain the pairs (N, E) exhibited in Table 3 together the genera of $X_0(N)$ and $X_0^*(N)$. As in the odd case, for $g_N^* = 2$ we can discard $N = 154, 286$ because both curves $X_0^*(N)$ have an elliptic quotient of conductor N that does not satisfy Lemma 7. Table 3 is divided into 4 cases: $3 \nmid N$, $6 \mid N$ and $30 \nmid N$, $30 \mid N$ and $210 \nmid N$ and, finally, $210 \mid N$.

$N, 3 \nmid N$	g_N	g_N^*	M	Label E
$106 = 2 \cdot 53$	12	2	106	b
			53	a
$122 = 2 \cdot 61$	14	2	122	a
			61	a
$158 = 2 \cdot 79$	19	2	158	b
			79	a
$166 = 2 \cdot 83$	20	2	166	a
			83	a
$178 = 2 \cdot 89$	21	3	89	a
$202 = 2 \cdot 101$	24	4	101	a
$262 = 2 \cdot 131$	32	4	131	a
$394 = 2 \cdot 197$	48	10	197	a
$458 = 2 \cdot 229$	56	10	229	a
$290 = 2 \cdot 5 \cdot 29$	41	3	58	a
			145	a
$310 = 2 \cdot 5 \cdot 31$	45	3	155	c
$370 = 2 \cdot 5 \cdot 37$	53	4	185	a
			185	c
			370	a
$410 = 2 \cdot 5 \cdot 41$	59	5	82	a
$430 = 2 \cdot 5 \cdot 43$	63	3	43	a
			215	a
$530 = 2 \cdot 5 \cdot 53$	77	7	106	b
			265	a
$574 = 2 \cdot 7 \cdot 41$	81	5	82	a
$590 = 2 \cdot 5 \cdot 59$	87	6	118	a
$602 = 2 \cdot 7 \cdot 43$	85	9	43	a
$710 = 2 \cdot 5 \cdot 71$	105	7	142	b
$742 = 2 \cdot 7 \cdot 53$	105	10	371	a
$770 = 2 \cdot 5 \cdot 7 \cdot 11$	137	5	77	a
			154	a
$910 = 2 \cdot 5 \cdot 7 \cdot 13$	161	5	65	a
			91	a
			455	a
$1190 = 2 \cdot 5 \cdot 7 \cdot 17$	209	8	238	b

$6 \mid N, 30 \nmid N$	g_N	g_N^*	M	Label E
$246 = 2 \cdot 3 \cdot 41$	39	3	82	a
			123	b
$258 = 2 \cdot 3 \cdot 43$	41	3	43	a
			129	a
$282 = 2 \cdot 3 \cdot 47$	45	3	141	d
$318 = 2 \cdot 3 \cdot 53$	51	3	53	a
			106	b
$354 = 2 \cdot 3 \cdot 59$	57	4	118	a
$366 = 2 \cdot 3 \cdot 61$	59	4	61	a
			122	a
$402 = 2 \cdot 3 \cdot 67$	65	5	201	a
			201	c
$426 = 2 \cdot 3 \cdot 71$	69	4	142	b
$438 = 2 \cdot 3 \cdot 73$	71	5	219	a
			219	c
$474 = 2 \cdot 3 \cdot 79$	77	7	79	a
			158	b
$498 = 2 \cdot 3 \cdot 83$	81	7	83	a
			166	a
			249	a
			249	b
$534 = 2 \cdot 3 \cdot 89$	87	8	89	a
$582 = 2 \cdot 3 \cdot 97$	95	7	291	c
$606 = 2 \cdot 3 \cdot 101$	99	8	101	a
$642 = 2 \cdot 3 \cdot 107$	105	9	214	b
$786 = 2 \cdot 3 \cdot 131$	129	11	131	a
$462 = 2 \cdot 3 \cdot 7 \cdot 11$	89	3	77	a
			154	a
$546 = 2 \cdot 3 \cdot 7 \cdot 13$	105	4	91	a
$714 = 2 \cdot 3 \cdot 7 \cdot 17$	137	5	102	a
			238	b
$798 = 2 \cdot 3 \cdot 7 \cdot 19$	153	5	57	a
			399	a
$858 = 2 \cdot 3 \cdot 11 \cdot 13$	161	6	143	a
			286	c
$966 = 2 \cdot 3 \cdot 7 \cdot 23$	185	8	138	a
$1122 = 2 \cdot 3 \cdot 11 \cdot 17$	209	9	374	a
$1254 = 2 \cdot 3 \cdot 11 \cdot 19$	233	12	57	a

$30 \mid N, 210 \nmid N$	g_N	g_N^*	M	Label E
$390 = 2 \cdot 3 \cdot 5 \cdot 13$	77	2	65	a
			390	a
$510 = 2 \cdot 3 \cdot 5 \cdot 17$	101	3	102	a
$570 = 2 \cdot 3 \cdot 5 \cdot 19$	113	4	57	a
			190	b
			285	b
$690 = 2 \cdot 3 \cdot 5 \cdot 23$	137	6	138	a
$870 = 2 \cdot 3 \cdot 5 \cdot 29$	173	7	58	a
			145	a
			290	a
$930 = 2 \cdot 3 \cdot 5 \cdot 31$	185	8	155	c
$1110 = 2 \cdot 3 \cdot 5 \cdot 37$	221	9	185	c

$30 \mid N, 210 \nmid N$	g_N	g_N^*	M	Label E
$1230 = 2 \cdot 3 \cdot 5 \cdot 41$	245	10	123	b
			615	a
$1290 = 2 \cdot 3 \cdot 5 \cdot 43$	257	10	129	a
			215	a
$1410 = 2 \cdot 3 \cdot 5 \cdot 47$	281	14	141	d
			705	a
$1590 = 2 \cdot 3 \cdot 5 \cdot 53$	317	14	53	a
			265	a
			795	a

$210 \mid N$	g_N	g_N^*	M	Label E
$2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$	561	12	77	a
			1155	a
$2730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$	657	14	65	a
			455	a

Table 3

As in the odd case, first we examine the hyperelliptic curves.

Proposition 20. *All the curves of genus two appearing in Table 3, i.e. $X_0^*(106)$, $X_0^*(122)$, $X_0^*(158)$, $X_0^*(166)$ and $X_0^*(390)$, are bielliptic.*

Proof. For $N \in \{106, 122, 154, 158, 166, 286, 390\}$, the jacobian of $X_0^*(N)$ is isogenous over $\mathbb{Q}$ to the product of two elliptic curves E and F of conductors N and $M < N$, respectively. We have that $M = N/6$ for $N = 390$ and $M = N/2$ otherwise. Let $f_E \in \mathrm{New}_N^*$ and $f_F \in \mathrm{New}_M^*$ be the corresponding newforms attached to these elliptic curves. The functions

$$x := \frac{f_E(q)}{\sum_{d|N/M} d\, f_F(q^d)}\,, \quad y := \frac{2\,q\,\mathrm{d}\,x}{\sum_{d|N/M} d\, f_F(q^d)}\,,$$

provide the following equations

$$\begin{aligned}
X_0^*(106)\colon \quad y^2 &= 4x^6 + 17x^4 - 6x^2 + 1\,,\\
X_0^*(122)\colon \quad y^2 &= 4x^6 + x^4 + 10x^2 + 1\,,\\
X_0^*(158)\colon \quad y^2 &= -2x^6 + 11x^4 + 8x^2 - 1\,,\\
X_0^*(166)\colon \quad y^2 &= -4x^6 + 17x^4 + 2x^2 + 1\,,\\
X_0^*(390)\colon \quad y^2 &= -(3x^2+1)(4x^4 - 7x^2 - 1)\,.
\end{aligned}$$

In all cases, one has the involutions $(x, y) \mapsto \pm(-x, \pm y)$. □

Next, we use the sieve based on [11, Theorem 2.1].

Proposition 21. *The curve $X_0^*(N)$ is not bielliptic and its automorphism group is trivial for the following values of N:*

$$394, 458, 582, 602, 710, 786, .$$

Proof. For a prime $p \nmid N$, let $Q_p(2k+1)$ be as in Proposition 11. After the following computations,

N	$Q_p(2k+1)$	$2g_N^* + 2$
$394 = 2 \cdot 197$	$Q_3(9) = 25$	22
$458 = 2 \cdot 229$	$Q_5(15) = 36$	22
$582 = 2 \cdot 3 \cdot 97$	$Q_{13}(9) = 18$	16
$602 = 2 \cdot 7 \cdot 43$	$Q_5(11) = 31$	20
$710 = 2 \cdot 5 \cdot 71$	$Q_3(9) = 25$	16
$786 = 2 \cdot 3 \cdot 131$	$Q_5(11) = 27$	24

the statement follows. □

Now, we apply the sieve based on the values of $|X_0^*(N)(\mathbb{F}_{p^n})| - 2|E(\mathbb{F}_{p^n})|$ and a modification for primes p dividing the conductor N.

Proposition 22. *The pairs (N, E) in the set*

$$\begin{array}{cccccc}
\{(290, 58a), & (370, 37a), & (370, 185a), & (402, 201a), & (410, 82a), & (438, 219c), \\
(474, 79a), & (474, 158b), & (498, 83a), & (498, 166a), & (498, 249a), & (498, 249b), \\
(530, 106b), & (530, 265a), & (534, 89a), & (574, 82a), & (590, 118a), & (606, 101a), \\
(642, 214b) & (714, 102a), & (742, 371a), & (770, 77a), & (770, 154a), & (798, 57a), \\
(870, 58a), & (910, 65a), & (930, 155c), & (966, 138a), & (1122, 374a), & (1190, 238b), \\
(1230, 123b), & (1230, 615a), & (1254, 57a), & (1290, 129a), & (1290, 215a), & (1410, 141d), \\
(1410, 705d), & (1590, 53a), & (1590, 265a), & (1590, 795a), & (2310, 77a), & (2310, 1155a), \\
(2730, 65a), & (2730, 455a)\} & & & &
\end{array}$$

are not bielliptic. In particular, the curve $X_0^*(N)$ is not bielliptic for the following values of N:

$$\begin{array}{l}
410, 474, 498, 530, 534, 574, 590, 606, 642, 742, 770, 930, 966, 1122, 1190, \\
1230, 1254, 1290, 1410, 1590, 2310, 2730\,.
\end{array}$$

Proof. We put $n(N, E, p^k) = |X_0^*(N)(\mathbb{F}_{p^k})| - 2|E(\mathbb{F}_{p^k})|$.

N	p^k	E	$n(N,E,p^k)$
290	9	$58a$	5
370	9	$37a$	6
	3	$185a$	2
402	5	$201a$	2
410	13	$82a$	1
438	25	$219c$	12
474	25	$79a$	10
	11	$158b$	7
498	11	$83a$	5
	7	$166a$	3
	7	$249a$	1
	13	$249b$	5
530	25	$106b$	35
	7	$265a$	8
534	5	$89a$	2
574	9	$82a$	4
590	9	$118a$	2

N	p^k	E	$n(N,E,p^k)$
606	5	$101a$	2
642	7	$214b$	9
714	25	$102a$	6
742	9	$371a$	4
770	9	$77a$	9
	3	$154a$	1
798	25	$57a$	3
870	13	$58a$	1
910	9	$65a$	3
930	7	$155c$	2
966	25	$138a$	7
1122	5	$374a$	5
1190	3	$238b$	1

N	p^k	E	$n(N,E,p^k)$
1230	49	$123b$	19
	11	$615a$	2
1254	5	$57a$	1
1290	49	$129a$	3
	49	$215a$	3
1410	49	$141d$	20
	7	$705d$	10
1590	49	$53a$	54
	7	$265a$	6
	7	$795a$	10
2310	17	$77a$	1
	17	$1155a$	9
2730	11	$65a$	7
	11	$455a$	3

[1]. □

After applying the two sieves, the following possibilities for the pairs (N, E), ordered by the genus, remain:

N	g_N^*	E
178	3	$89a$
246	3	$82a, 123b$
258	3	$43a, 129a$
282	3	$141d$
290	3	$145a$
310	3	$155c$
318	3	$53a, 106b$
430	3	$43a, 215a$
462	3	$77a, 154a$
510	3	$102a$

N	g_N^*	E
202	4	$101a$
262	4	$131a$
354	4	$118a$
366	4	$61a, 122a$
370	4	$185c, 370a$
426	4	$142b$
546	4	$91a$
570	4	$57a, 190b, 285b$

N	g_N^*	E
402	5	$201c$
438	5	$219a$
714	5	$238b$
798	5	$399a$
910	5	$91a, 455a$
690	6	$138a$
858	6	$143a, 286c$
870	7	$145a, 290a$
1110	9	$185c$

Table 4

Proposition 23. *Among the curves of genus three $X_0^*(178)$, $X_0^*(246)$, $X_0^*(258)$, $X_0^*(282)$, $X_0^*(290)$, $X_0^*(310)$, $X_0^*(318)$, $X_0^*(430)$, $X_0^*(462)$ and $X_0^*(510)$, only $X_0^*(178)$, $X_0^*(246)$, $X_0^*(258)$, $X_0^*(290)$, $X_0^*(318)$,$X_0^*(430)$ and $X_0^*(510)$ are bielliptic. The corresponding elliptic quotients are labeled as $E89a1$, $E82a1$, $E43a1$, $E145a1$, $E53a1$, $E43a1$ and $E102a1$, respectively. In all these cases, the automorphism group of $X_0^*(N)$ has order 2. The automorphism groups of the remaining curves are trivial.*

Proof. For these values of N, the splitting of the jacobian of $X_0^*(N)$, $J_0^*(N)$, is as follows:

$$
\begin{array}{llllll}
J_0^*(178) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{2} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E89a\,, & f_2 \in \mathrm{New}_{178}^*\,, & \dim A_{f_2} = 2\,, & & \\
J_0^*(246) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{3} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E82a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E123b\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E246d\,, & & \\
J_0^*(258) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{3} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E43a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E129a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E258a\,, & & \\
J_0^*(282) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{2} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E141d\,, & f_2 \in \mathrm{New}_{282}^*\,, & \dim A_{f_2} = 2\,, & & \\
J_0^*(290) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{3} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E145a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E58a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E290a\,, & & \\
J_0^*(310) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{2} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E155c\,, & f_2 \in \mathrm{New}_{310}^*\,, & \dim A_{f_2} = 2\,. & & \\
J_0^*(318) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{3} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E53a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E106b\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E318c\,, & & \\
J_0^*(430) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{3} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E43a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E215a\,, , & A_{f_3} \overset{\mathbb{Q}}{\sim} E430a\,, & & \\
J_0^*(462) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{3} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E77a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E154a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E462a\,, & & \\
J_0^*(510) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^{2} A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E102a\,, & f_2 \in \mathrm{New}_{85}^*\,, & \dim A_{f_2} = 2\,, & &
\end{array}
$$

[1] F:Bars, is a problem because we reduce modulo 2 a degree two morphism TO ARISE: Now, suppose that the pair (N, E) is bielliptic. If $p|N$, we know that $X_0(N)$ modulo p is the copy of two curves $X_0(N/p)/\mathbb{F}_p$, and the normalization of $X_0^*(N)/\mathbb{F}_p$ is the curve $X_0^*(N/p)/\mathbb{F}_p$ (cf. [10]). If p does not divide the conductor of E, then $E/\mathbb{F}_p$ is an elliptic curve. If $g_{N/p}^* > 1$, then this is the quotient curve of $X_0^*(N/p)/\mathbb{F}_p$ by an involution defined over $\mathbb{F}_p$. Therefore, $n(N/p, E, p^k) \leq 0$. Taking $N = 1110$, $p = 2$ and E with the label $185c$, we get $n(555, E, 2) = 1$ and, thus, the pair $(1110, E)$ can be discarded

We take a basis of $\Omega^1_{X_0^*(N)\mathbb{Q}}$ following the order exhibited in the splitting of the jacobian, and we obtain the following generators $Q \in \mathcal{L}_4$:

N	Q
178	$x^4 - 2x^2y^2 + y^4 - 12x^2yz - 4y^3z - 4x^2z^2 + 20y^2z^2 + 32yz^3$
246	$81x^4 - 16y^4 + 324x^2yz - 32y^3z - 162x^2z^2 + 48y^2z^2 - 260yz^3 + 17z^4$
258	$81x^4 - 90x^2y^2 + 41y^4 + 288x^2yz - 176y^3z - 36x^2z^2 + 84y^2z^2 - 128yz^3 - 64z^4$
282	$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 - 6x^3z - 18x^2yz - 27xy^2z - 30y^3z$ $-9x^2z^2 + 45xyz^2 - 117y^2z^2 + 54xz^3 + 108yz^3$
290	$27x^4 + 36x^2y^2 + 36x^2yz + 32y^3z - 126x^2z^2 + 12y^2z^2 - 84yz^3 + 67z^4$
310	$x^4 + 2x^3y - 3x^2y^2 - 4xy^3 + 4y^4 - 81xy^2z - 81xyz^2 - 81y^2z^2 - 54xz^3 + 54yz^3 + 81z^4$
318	$9x^4 - 10x^2y^2 + y^4 - 28x^2yz + 12y^3z + 20x^2z^2 - 4y^2z^2 - 32yz^3 + 32z^4$
430	$81x^4 + 54x^2y^2 - 7y^4 - 432x^2yz - 128y^3z - 108x^2z^2 + 444y^2z^2 + 64yz^3 + 32z^4$
462	$128x^4 - 320x^3y + 264x^2y^2 - 44xy^3 - y^4 - 448x^3z + 48x^2yz + 492xy^2z - 92y^3z$ $+840x^2z^2 + 12xyz^2 + 498y^2z^2 - 972xz^3 - 972yz^3 + 567z^4$
510	$3x^4 - 4x^2y^2 + y^4 + 18x^2yz - 10y^3z + 14x^2z^2 + 18y^2z^2 + 40yz^3 + 16z^4$

By Lemma 13, only the curves corresponding to $N = 178, 246, 258, 290, 318, 430, 510$ are bielliptic and only have a bielliptic involution u. The affine equations for the bielliptic quotients are

N	$X_0^*(N)/\langle u\rangle$
178	$-4x + x^2 + 32y - 12xy + 20y^2 - 2xy^2 - 4y^3 + y^4 = 0$
246	$17 - 162x + 81x^2 - 260y + 324xy + 48y^2 - 32y^3 - 16y^4 = 0$
258	$-64 - 36x + 81x^2 - 128y + 288xy + 84y^2 - 90xy^2 - 176y^3 + 41y^4 = 0$
290	$67 - 126x + 27x^2 - 84y + 36xy + 12y^2 + 36xy^2 + 32y^3 = 0$
318	$32 + 20x + 9x^2 - 32y - 28xy - 4y^2 - 10xy^2 + 12y^3 + y^4 = 0$
430	$32 - 108x + 81x^2 + 64y - 432xy + 444y^2 + 54xy^2 - 128y^3 - 7y^4 = 0$
510	$16 + 14x + 3x^2 + 40y + 18xy + 18y^2 - 4xy^2 - 10y^3 + y^4 = 0$

have genus one and their j-invariants are $-\frac{117649}{89}$, $\frac{389017}{16}$, $-\frac{4096}{43}$, $\frac{2146689}{145}$, $\frac{3375}{53}$, $-\frac{4096}{43}$ and $\frac{1771561}{612}$. They correspond to the elliptic curves in the statement. By the splitting of the jacobians and the equations of these curves, we obtain that all their automorphism groups have order 2. It is easy to check that the automorphism groups of the remaining curves are trivial. □

Proposition 24. *Among the curves of genus four $X_0^*(202)$, $X_0^*(262)$, $X_0^*(354)$, $X_0^*(366)$, $X_0^*(370)$, $X_0^*(426)$, $X_0^*(546)$ and $X_0^*(570)$, only $X_0^*(370)$ is bielliptic. The corresponding quotient curve is the elliptic curve labeled as $E370a1$ and the automorphism group of $X_0^*(370)$ has order 2. The automorphism groups of the curves $X_0^*(202)$, $X_0^*(262)$, $X_0^*(354)$, $X_0^*(426)$, $X_0^*(546)$ and $X_0^*(570)$ are trivial. The automorphism group of $X_0^*(366)$ has order 2 and the quotient curve has genus 2.*

Proof. The splitting of $J_0^*(N)$ is:

$$
\begin{array}{lllllll}
J_0^*(202) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^2 A_{f_i}\,, & & A_{f_1} \overset{\mathbb{Q}}{\sim} E101a\,, & f_2 \in \mathrm{New}_{202}^*\,, & \dim A_{f_2} = 3\,, & \\
J_0^*(262) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E131a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E262b\,, & f_3 \in \mathrm{New}_{262}^*\,, & \dim A_{f_3} = 2\,, & \\
J_0^*(354) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E118a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E354b\,, & f_3 \in \mathrm{New}_{177}^*\,, & \dim A_{f_3} = 2\,, & \\
J_0^*(366) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E61a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E122a\,, & f_3 \in \mathrm{New}_{183}^*\,, & \dim A_{f_3} = 2\,, & \\
J_0^*(370) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E370a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E185a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E185c\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E37a\,, & \\
J_0^*(426) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E142a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E426b\,, & f_3 \in \mathrm{New}_{243}^*\,, & \dim A_{f_3} = 2\,, & \\
J_0^*(546) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^2 A_{f_i}\,, & & A_{f_1} \overset{\mathbb{Q}}{\sim} E91a\,, & f_2 \in \mathrm{New}_{273}^*\,, & \dim A_{f_2} = 3\,, & \\
J_0^*(570) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E57a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E190b\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E285b\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E570a\,. &
\end{array}
$$

In all cases, $\dim \mathcal{L}_2 = 1$. Next, we show a generator $Q_2(x,y,z,t) \in \mathcal{L}_2$:

N	Q_2
202	$-9t^2 + x^2 + 9ty - 2xy + y^2 + 9tz - xz - 8yz - 2z^2$
262	$7t^2 - tx + x^2 - 4ty - xy - 4tz - xz + yz$
354	$-42t^2 - 5tx + 2ty - 10xy + 22y^2 + 48tz - 15xz + 6yz - 3z^2$
366	$-4t^2 + x^2 + 2xy - y^2 - 2z^2$
370	$-27t^2 + 144x^2 + 72ty + 32y^2 - 54tz - 40yz - 127z^2$
426	$-108t^2 + 165tx + 100x^2 + 78ty + 55xy - 137y^2 + 27tz - 30xz + 84yz - 72z^2$
546	$388t^2 + 36tx + x^2 - 68ty - 10xy + 9y^2 - 420tz - 34xz + 58yz + 125z^2$
570	$-327t^2 - 232tx + 176x^2 + 36ty + 64xy + 192y^2 + 70tz - 40xz + 188yz - 127z^2$

Only $X_0^*(370)$ could be bielliptic. In this case, the curve is trigonal (see [16, Proposition 1]) and $\dim \mathcal{L}_3 = 5$. By computing a polynomial $Q_3 \in \mathcal{L}_3$ that is not multiple of Q_2, we get

$$\begin{aligned} Q_3(x,y,z,t) = \; & -81t^3 - 3024tx^2 + 3456x^2y + 1056ty^2 - 2016x^2z - 4056tyz - 1408y^2z \\ & +1479tz^2 + 248yz^2 + 4346z^3\,. \end{aligned}$$

Since $Q_3(-x,y,z,t) \in \mathcal{L}_3$, the curve $X_0^*(370)$ is bielliptic by Lemma 13. Let us check this result. Set $P(X,Y) := \text{Resultant}\,(Q_2(X,Y,Z,1), Q_3(X,Y,Z,1), Z)$. More precisely,

$$\begin{aligned} P(X,Y) = \; & -592 + 12952X^2 - 109468X^4 + 45796X^6 + (10816X^2 + 60978X^4)Y \\ & +(3304 - 6112X^2 - 1763X^4)Y^2 + (3712 - 26412X^2)Y^3 \\ & +(908 + 5066X^2)Y^4 + 1210Y^5 - 395Y^6\,. \end{aligned}$$

The curve determined by the equation $P(X,Y) = 0$ has genus 4. Hence, it is a plane model for $X_0^*(370)$. The model admits the involution $u\colon (X,Y) \mapsto (-X,Y)$. Replacing X^2 with X, we obtain a genus one curve, whose j-invariant is $15438249/2960$. Checking [7, Table1], the elliptic quotient has conductor 370 and label $a1$. The polynomials Q_2 and Q_3 show that u is the only nontrivial involution of the curve. It is clear that the remaining curves, except $X_0^*(366)$, have trivial automorphism group.

Looking at the polynomial Q_2 for $N = 366$, we may ask whether one of the two linear maps $(\omega_1,\omega_2,\omega_3,\omega_4) \mapsto \pm(-\omega_1,-\omega_2,\omega_3,\omega_4)$ comes from an involution u of $X_0^*(366)$. After determining $\mathcal{L}_3$, the answer is affirmative. Hence, $\mathrm{Jac}(X_0^*(366))$ is isogenous over $\mathbb{Q}$ to $E61a \times E122a$ or A_{f_3}. After checking which of the vector subspaces $\langle\omega_1,\omega_2\rangle$ or $\langle\omega_3,\omega_4\rangle$ provides a hyperelliptic curve, the right answer is A_{f_3}, and an equation for the quotient curve $X_0^*(366)/\langle u\rangle$ is

$$Y^2 = X^6 - 6X^5 + 23X^4 - 42X^3 + 53X^2 - 24X + 4\,. \qquad \square$$

Remark 25. *It is expected that the automorphism group of $X_0^*(N)$ is trivial for a large enough N and, thus, the genera of the quotients curves by nontrivial involutions are bounded. The curve $X_0^*(366)$ shows that if this bound exists, then it is at least 2.*

Proposition 26. *The curves $X_0^*(402)$, $X_0^*(438)$, $X_0^*(714)$, $X_0^*(798)$, $X_0^*(910)$, $X_0^*(690)$, $X_0^*(858)$ and $X_0^*(870)$ are not bielliptic.*

Proof. The splitting of $J_0^*(N)$ for the curves of genus 5 in the statement is:

$$\begin{array}{lllll}
J_0^*(402) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E201c\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E201a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E402a\,, & f_4 \in \mathrm{New}_{67}^*\,, \\
 & & & & \dim A_{f_4} = 2\,, \\
J_0^*(438) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E219a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E219c\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E438a\,, & f_4 \in \mathrm{New}_{73}^*\,, \\
 & & & & \dim A_{f_4} = 2\,, \\
J_0^*(714) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E238b\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E102a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E714a\,, & f_4 \in \mathrm{New}_{357}^* \\
 & & & & \dim A_{f_4} = 2\,, \\
J_0^*(798) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E399a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E57a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E798a\,, & f_4 \in \mathrm{New}_{133}^*\,, \\
 & & & & \dim A_{f_4} = 2\,, \\
J_0^*(910) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E91a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E455a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E65a\,, & f_4 \in \mathrm{New}_{910}^*\,, \\
 & & & & \dim A_{f_4} = 2\,.
\end{array}$$

In all cases, $\dim \mathcal{L}_{2,1} = 0$. For $N = 910$, also $\dim \mathcal{L}_{2,2} = 0$.

For the curves of genus 6, the splitting of $J_0^*(N)$ is:

$$\begin{array}{llllll}
J_0^*(610) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E138a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E690a\,, & f_3 \in \mathrm{New}_{345}^*\,, & f_4 \in \mathrm{New}_{115}^*\,, \\
 & & & \dim A_{f3} = 2 & \dim A_{f_4} = 2\,, \\
J_0^*(858) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E143a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E290a\,, & f_3 \in \mathrm{New}_{429}^*\,, & f_4 \in \mathrm{New}_{898}^* \\
 & & & \dim A_{f_3} = 2 & \dim A_{f_4} = 2\,,
\end{array}$$

In all cases, $\dim \mathcal{L}_{2,1} = 1$. Hence, $\dim \mathcal{L}_{2,1} = 1 < \dim \mathcal{L}_2 = 6$. For $N = 898$, also $\dim \mathcal{L}_{2,1} = 1$.

Finally, the splitting for the curve $X_0^*(870)$ of genus seven is:

$$\begin{array}{lllll}
J_0^*(870) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E145a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E290a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E58a\,, & f_4 \in \mathrm{New}_{435}^*\,, \\
 & & & & \dim A_{f_4} = 4\,.
\end{array}$$

In this case, $\dim \mathcal{L}_{2,1} = \dim \mathcal{L}_{2,2} = 4 < \dim \mathcal{L}_2 = 10$. □

As a consequence of the previous results, we obtain the statement of Theorem 1 for N even.

Corollary 27. *For N even, the curve $X_0^*(N)$ is bielliptic exactly for the thirteen values of N in the set*

$$\{106, 122, 158, 166, 178, 246, 258, 290, 318, 370, 390, 430, 510\}\,.$$

For these values of N automorphism group has order 2 when $g_N^* > 2$, otherwise it is the Klein group.

5 Quadratic points

Let us now prove Theorem 2. We know by [15] that if N is square-free and $X_0^*(N)$ is hyperelliptic, then $g_N^* = 2$. On the other hand, a genus two curve defined over a number field K is hyperelliptic over K and, thus, all genus two curves $X_0^*(N)$ are hyperelliptic over $\mathbb{Q}$. The set of values of N in Theorem 2 are those for which $g_N^* = 2$ and those such that $X_0^*(N)$ is bielliptic and $g_N^* \geq 3$. This is due to the fact that, when $g_N^* \geq 3$, the quotient curve is always an elliptic curve with rank equal to 1 (see [7, Table1]). Hence, all these values of N are exactly the values for which $\Gamma_2(X_0^*(N), \mathbb{Q})$ is infinite (cf. [5, Theorem 2.14]).

6 Appendix

Here we list the values N such that $g_N^* \leq 2$. The table for genus 2 reproduces the one in [14]. The tables for genus 0 or 1 are taken from [12]. We note that the value $N = 141$, which does not appear in Proposition 1.1 of [12], is included in the appendix of this paper and here.

$g_N^* = 0$	2,3,5,6,7,10,11,13,14,15,17,19,21,22,23,26,29,30,31,33,34,35,38,39, 41,42,46,47,51,55,59,62,66,69,70,71,78,87,94,95,105,110,119 .
$g_N^* = 1$	37,43,53,57,58,61,65,74,77,79,82,83,86,89,91,101,102,111,114,118,123, 130,131,138,141,142,143,145,155,159,174,182,190,195,210,222,231,238 .
$g_N^* = 2$	67, 73,85,93,103,106,107,115,122, 129,133,134,146,154,158,161,165, 166,167 170, 177,186,191,205,206,209,213,215,221,230,255,266,285,286,287, 299 330,357,390 .

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