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Research Article

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Besov regularity for solutions of p-harmonic equations

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Abstract: We establish the higher fractional differentiability of the solutions to nonlinear elliptic equations in divergence form, i.e., $\operatorname{div} \mathcal{A}(x, Du) = \operatorname{div} F$, when $\mathcal{A}$ is a p-harmonic type operator, and under the assumption that $x \mapsto \mathcal{A}(x, \xi)$ belongs to the critical Besov–Lipschitz space $B^{\alpha}_{n/\alpha,q}$. We prove that some fractional differentiability assumptions on F transfer to Du with no losses in the natural exponent of integrability. When $\operatorname{div} F = 0$, we show that an analogous extra differentiability property for Du holds true under a Triebel–Lizorkin assumption on the partial map $x \mapsto \mathcal{A}(x, \xi)$.

Keywords: Nonlinear elliptic equations, p-harmonic operators, higher order fractional differentiability, Besov spaces, Triebel–Lizorkin spaces

MSC 2010: 35B65, 35J60, 42B37, 49N60

1 Introduction

In this paper we study the extra fractional differentiability of weak solutions of the following nonlinear elliptic equations in divergence form:

$$\operatorname{div} \mathcal{A}(x, Du) = \operatorname{div} F \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a domain, $u \colon \Omega \to \mathbb{R}$, $F \colon \Omega \to \mathbb{R}^n$, and $\mathcal{A} \colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function with $p - 1$ growth. This means that there exist an exponent $p \geq 2$ and constants $\ell, L, \nu > 0$ and $0 \leq \mu \leq 1$ such that

(A1) $\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle \geq \nu(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2$,
(A2) $|\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leq L(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|$,
(A3) $|\mathcal{A}(x, \xi)| \leq \ell(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}$

for every $\xi, \eta \in \mathbb{R}^n$ and for a.e. $x \in \Omega$.

When dealing with p-harmonic equations, regularity results usually refer to the auxiliary function

$$V_p(Du) = (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du,$$

which takes into account the p-growth of the operator. Obviously, $V_p(Du)$ reduces to the gradient of the solution for $p = 2$. Heuristically, thinking of the classical p-Laplace equation

$$\operatorname{div}(|Du|^{p-2} Du) = 0$$

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and setting $w = |V_p(Du)|^2$, we have that Dw is a subsolution of a linear elliptic equation (for more details, we refer to [10, p. 272]). Therefore, the function $V_p(Du)$, that takes into account the nonlinearity of the equation, is the natural substitute of the gradient of the solution when passing from the linear to the p-harmonic setting.

It is well known that the Lipschitz continuity of the partial map $x \to \mathcal{A}(x, \cdot)$ is a sufficient condition for the higher differentiability of the solutions when the right-hand side of the equation is sufficiently regular (we refer again to [10] for an exhaustive treatment). Also, it is clear that no extra differentiability can be expected for solutions, even if F is smooth, unless some differentiability is assumed on the x-dependence of $\mathcal{A}$.

Recent developments show that the Lipschitz regularity of the partial map $x \to \mathcal{A}(x, \cdot)$ can be weakened in a $W^{1,n}$ assumption on the coefficients, both in the linear and in the nonlinear setting, in order to get higher differentiability of the solution of integer order. In this direction, in [9, 19, 20], the higher differentiability of the function $V_p(Du)$ is obtained from a pointwise condition on $\mathcal{A}$ that is equivalent to the $W^{1,n}$ regularity of the map $x \to \mathcal{A}(x, \cdot)$. More precisely, it is assumed that there exists a non negative function $g \in L^n_{\mathrm{loc}}(\Omega)$ such that

$$|\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)| \le |x - y|(g(x) + g(y))(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} \tag{1.2}$$

for almost every $x, y \in \Omega$ and every $\xi \in \mathbb{R}^n$. Related results concerning the planar Beltrami equation [3] and minimizers of non uniformly convex functionals [6–8] can also be found.

It turns out that the higher differentiability of the solutions can also be analyzed in the case of fractional Sobolev regularity of the coefficients. We mention previous contributions made in [2, 4, 5] for the case of planar Beltrami systems, and [16, 17] for higher dimensional results with not necessarily linear growth. Closer to the subject of the present paper, and assuming that $\mathcal{A}(x, \xi)$ has linear growth with respect to the gradient variable and enjoys either a Triebel–Lizorkin or a Besov–Lipschitz smoothness (roughly speaking enjoys a fractional differentiability property) with respect to the x-variable, it is proven in [1] that the fractional differentiability of $\mathcal{A}(x, \cdot)$ transfers to the gradient of the solution with no losses in the order of differentiation.

The aim of this paper is to extend the results of [1] to the case of p-harmonic type operators with $p \ge 2$. More precisely, we will show that a fractional differentiability assumption for the operator $\mathcal{A}$ with respect to the x-variable yields a fractional differentiability for the solutions. In this case, the fractional differentiability of $\mathcal{A}(x, \cdot)$ transfers to $V_p(Du)$.

Our first result concerns the case of Triebel–Lizorkin coefficients, i.e., we assume that there exists a function $g \in L^{\frac{n}{\alpha}}_{\mathrm{loc}}(\Omega)$ such that

$$|\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)| \le |x - y|^\alpha (g(x) + g(y))(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} \tag{1.3}$$

for almost every $x, y \in \Omega$, and every $\xi \in \mathbb{R}^n$.

Theorem 1.1. *Let $0 < \alpha < 1$. Assume that $\mathcal{A}$ satisfies (A1)–(A3), and that (1.3) holds. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of*

$$\operatorname{div} \mathcal{A}(x, Du) = 0, \tag{1.4}$$

then $V_p(Du) \in B^\alpha_{2,\infty}$, locally, and as a consequence $Du \in B^{\frac{2\alpha}{p}}_{p,\infty}$, locally.

See Section 2 for the definition of $B^\alpha_{p,q}$ and the meaning of *locally*. It is worth mentioning that there is a jump between (1.2) and (1.3) when stated in terms of the Triebel–Lizorkin scale $F^\alpha_{p,q}$. The jump appears in the q index, when the order of differentiation becomes integer. Indeed, condition (1.2) fully describes equations with coefficients in the Sobolev space $W^{1,n}$, that is, the Triebel–Lizorkin space $F^1_{n,2}$. In contrast, condition (1.3), for $0 < \alpha < 1$, says that $\mathcal{A}(x, \cdot)$ belongs to $F^\alpha_{n/\alpha,\infty}$. More explanations about this can be found in [15, Remark 3.3] and the references therein.

The proof of Theorem 1.1 does not seem to work when the coefficients are assumed to belong to $F^\alpha_{n/\alpha,q}$ for finite values of q. As in the linear case (see [1]), the Besov setting fits better in this context. To be precise, given $0 < \alpha < 1$ and $1 \le q \le \infty$, we assume that there exists a sequence of measurable non-negative functions $g_k \in L^{\frac{n}{\alpha}}(\Omega)$ such that

$$\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(\Omega)} < \infty,$$

and at the same time the following holds:

(A4) $|\mathcal{A}(x, \xi) - \mathcal{A}(y, \xi)| \le |x - y|^\alpha (g_k(x) + g_k(y))(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}$

for each $\xi \in \mathbb{R}^n$, and almost every $x, y \in \Omega$ such that $2^{-k} \le |x-y| < 2^{-k+1}$. We will shortly write then that $(g_k)_k \in \ell^q(L^{\frac{n}{\alpha}})$. If $\mathcal{A}(x,\xi) = a(x)|\xi|^{p-2}\xi$ and $\Omega = \mathbb{R}^n$, then (A4) says that a belongs to $B^\alpha_{n/\alpha,q}$, see [15, Theorem 1.2].

Under (A4), we are able to deal with non-homogeneous equations and we prove that the extra differentiability of the solutions is related to the regularity of the datum and of the coefficients, both measured in the Besov scale. More precisely, we have the following result.

Theorem 1.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of the equation*

$$\operatorname{div}\mathcal{A}(x, Du) = \operatorname{div} F, \tag{1.5}$$

under assumptions (A1)–(A4), *with $\mu > 0$. Then the implication*

$$F \in B^\beta_{2,q} \;\Rightarrow\; V_p(Du) \in B^{\min\{\alpha,\beta\}}_{2,q}$$

holds locally, provided that $0 < \beta < 1$ and $1 \le q \le \frac{2n}{n-2\beta}$.

The parameter μ in assumption (A1) plays a very important role. When $\mu > 0$, the equation is non-degenerate elliptic while the case $\mu = 0$ corresponds to degenerate cases. For instance, a model case for $\mu > 0$ is given by

$$\mathcal{A}(x,\xi) = a(x)(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}\xi,$$

while a typical degenerate problem is the weighted p-Laplace equation

$$\mathcal{A}(x,\xi) = a(x)|\xi|^{p-2}\xi$$

for some coefficient $\nu \le a(x) \le \ell$. In the degenerate case, the ellipticity assumption (A1) is lost when $|\xi|$ approaches zero, and the estimates worsen even in the classical theory (see [22]). Actually, in this case we are not able to prove an extra fractional differentiability of the function $V_p(Du)$ completely analogous to our previous theorem. Instead, due to the degeneracy $\mu = 0$, we have the following weaker result in the sense that the differentiability of the datum F still transfers to the function $V_p(Du)$, but with a loss in the order of differentiation, even assuming the datum in a Besov space slightly smaller than $B^\beta_{2,q}$. More precisely we have the following theorem.

Theorem 1.3. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of the equation*

$$\operatorname{div}\mathcal{A}(x, Du) = \operatorname{div} F,$$

under assumptions (A1)–(A4), *with $\mu = 0$. Let $0 < \alpha, \beta < 1$ and $p' = \frac{p}{p-1}$. Then the implication*

$$F \in B^\beta_{2,qp'/2} \;\Rightarrow\; V_p(Du) \in B^{\min\{\alpha,\frac{\beta p'}{2}\}}_{2,q}$$

holds locally, provided that $1 \le \frac{qp'}{2} \le \frac{2n}{n-2\beta}$.

Note that, for $p = 2$, Theorems 1.2 and 1.3 both recover [1, Theorem 3] at the energy space and in the case $\alpha = \beta$. Actually, when dealing with equations with linear growth, the natural degree of integrability of the gradient of the solutions as well as of their extra α fractional Hajlasz gradients is 2. Therefore, the higher fractional differentiability results at the energy space are those proving that Du belongs to $B^\alpha_{2,q}$ or $F^\alpha_{2,q}$. In [1], extra fractional differentiability results for equations with linear growth have been established also in spaces different from the natural ones, i.e., it has been proven that Du belongs to $B^\alpha_{s,q}$ and $F^\alpha_{s,q}$ for some $s \neq 2$ sufficiently close to 2.

All our theorems rely on the basic fact that the Besov spaces $B^\alpha_{n/\alpha,q}$ and the Triebel–Lizorkin space $F^\alpha_{n/\alpha,\infty}$ continuously embed into the VMO space of Sarason (e.g., [11, Proposition 7.12]). Linear equations with VMO coefficients are known to have a nice L^p theory (see [12] for $n = 2$ or [13] for $n \ge 2$). A first nonlinear growth counterpart was found in [14] for $\mathcal{A}(x,\xi) = \langle A(x)\xi, \xi\rangle^{\frac{p-2}{2}} A(x)\xi$, $2 \le p \le n$, see also [17, 18]. For proving Theorems 1.1, 1.2 and 1.3, we shall use a result proved in [1] (see Theorem 2.5 in Section 2.2 below), and combine it with the Sobolev type embedding for Besov Lipschitz spaces to obtain the higher integrability of the gradient of the solutions of equation (1.1). Such higher integrability allows us to estimate the difference quotient of order α of the gradient of the solutions that yields their Besov type regularity.

The paper is structured as follows. In Section 2 we give some preliminaries on Harmonic Analysis. In Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorems 1.2 and 1.3.

2 Notations and preliminary results

In this paper we follow the usual convention and denote by c a general positive constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^n$ will be the standard euclidean one and it will be denoted by $|\cdot|$. In particular, for the vectors $\xi, \eta \in \mathbb{R}^n$, we write $\langle \xi, \eta \rangle$ for the usual inner product and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding euclidean norm.

In what follows, $B(x, r) = B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ will denote the ball centered at x of radius r. We shall omit the dependence on the center and on the radius when no confusion arises.

For the auxiliary function V_p, defined for all $\xi \in \mathbb{R}^n$ as

$$V_p(\xi) := (\mu^2 + |\xi|^2)^{\frac{p-2}{4}} \xi,$$

where $\mu \geq 0$ and $p \geq 1$ are parameters, we record the following estimate (see the proof of [10, Lemma 8.3]).

Lemma 2.1. *Let $1 < p < \infty$ and $0 \leq \mu \leq 1$. There exists a constant $c > 0$, depending only on n, p but not on $\mu > 0$, such that*

$$c^{-1}(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \frac{|V_p(\xi) - V_p(\eta)|^2}{|\xi - \eta|^2} \leq c(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}$$

for any $\xi, \eta \in \mathbb{R}^n$ such that $|\xi - \eta| \neq 0$.

Noticing now that for $p \geq 2$, one has

$$|\xi - \eta|^p = |\xi - \eta|^2 |\xi - \eta|^{p-2} \leq |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} \leq c|\xi - \eta|^2 (\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}},$$

and combining this with Lemma 2.1, we find that there exists a constant $c > 0$ such that

$$|\xi - \eta|^p \leq c|V_p(\xi) - V_p(\eta)|^2 \tag{2.1}$$

for every $\xi, \eta \in \mathbb{R}^n$.

2.1 Besov–Lipschitz spaces

Given $h \in \mathbb{R}^n$ and $v \colon \mathbb{R}^n \to \mathbb{R}$, let $\tau_h v(x) = v(x + h)$ and $\Delta_h v(x) = v(x + h) - v(x)$. As in [21, Section 2.5.12], given $0 < \alpha < 1$ and $1 \leq p, q < \infty$, we say that v belongs to the Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$\|v\|_{B^\alpha_{p,q}(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} < \infty,$$

where

$$[v]_{\dot{B}^\alpha_{p,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|v(x+h) - v(x)|^p}{|h|^{\alpha p}} \, dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}} < \infty.$$

Equivalently, we could simply say that $v \in L^p(\mathbb{R}^n)$ and $\frac{\Delta_h v}{|h|^\alpha} \in L^q(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n))$. As usually, if one simply integrates for $h \in B(0, \delta)$ for a fixed $\delta > 0$, then an equivalent norm is obtained because

$$\left(\int_{\{|h| \geq \delta\}} \left(\int_{\mathbb{R}^n} \frac{|v(x+h) - v(x)|^p}{|h|^{\alpha p}} \, dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}} \leq c(n, \alpha, p, q, \delta) \|v\|_{L^p(\mathbb{R}^n)}.$$

Similarly, we say that $v \in B^\alpha_{p,\infty}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$[v]_{\dot{B}^\alpha_{p,\infty}(\mathbb{R}^n)} = \sup_{h\in\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} \frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}\,dx\Big)^{\frac{1}{p}} < \infty.$$

Again, one can simply take the supremum over $|h| \le \delta$ and obtain an equivalent norm. By construction, $B^\alpha_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. One also has the following version of the Sobolev embeddings (a proof can be found in [11, Proposition 7.12], taking into account that $L^r = F^0_{r,2}$, with $1 < r < +\infty$).

Lemma 2.2. *Suppose that* $0 < \alpha < 1$.
(a) *If* $1 < p < \frac{n}{\alpha}$ *and* $1 \le q \le p^*_\alpha =: \frac{np}{n-\alpha p}$, *then there exists a continuous embedding* $B^\alpha_{p,q}(\mathbb{R}^n) \subset L^{p^*_\alpha}(\mathbb{R}^n)$.
(b) *If* $p = \frac{n}{\alpha}$ *and* $1 \le q \le \infty$, *then there exists a continuous embedding* $B^\alpha_{p,q}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$.

Given a domain $\Omega \subset \mathbb{R}^n$, we say that v belongs to the local Besov space $B^\alpha_{p,q,\mathrm{loc}}$ if φv belongs to the global Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ whenever φ belongs to the class $\mathcal{C}^\infty_c(\Omega)$ of smooth functions with compact support contained in Ω. The following lemma is an easy exercise.

Lemma 2.3. *A function* $v \in L^p_{\mathrm{loc}}(\Omega)$ *belongs to the local Besov space* $B^\alpha_{p,q,\mathrm{loc}}$ *if and only if*

$$\Big\|\frac{\Delta_h v}{|h|^\alpha}\Big\|_{L^q(\frac{dh}{|h|^n};L^p(B))} < \infty$$

for any ball $B \subset 2B \subset \Omega$ *with radius* r_B. *Here the measure* $\frac{dh}{|h|^n}$ *is restricted to the ball* $B(0, r_B)$ *on the h-space.*

Proof. Let us fix a smooth and compactly supported test function φ. We have the pointwise identity

$$\frac{\Delta_h(\varphi v)(x)}{|h|^\alpha} = v(x+h)\frac{\Delta_h\varphi(x)}{|h|^\alpha} + \frac{\Delta_h v(x)}{|h|^\alpha}\varphi(x).$$

It is clear that

$$\Big|v(x+h)\frac{\Delta_h\varphi(x)}{|h|^\alpha}\Big| \le |v(x+h)|\|\nabla\varphi\|_\infty|h|^{1-\alpha},$$

and therefore one always has $\frac{\Delta_h\varphi}{|h|^\alpha} \in L^q(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n))$. As a consequence, we have the equivalence

$$\varphi v \in B^\alpha_{p,q}(\mathbb{R}^n) \;\Leftrightarrow\; \frac{\Delta_h v}{|h|^\alpha}\varphi \in L^q\Big(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n)\Big).$$

However, it is clear that $\frac{\Delta_h v}{|h|^\alpha}\varphi \in L^q(\frac{dh}{|h|^n}; L^p(\mathbb{R}^n))$ for each $\varphi \in C^\infty_c(\Omega)$ if and only if the same happens for every $\varphi = \chi_B$ and every ball $B \subset 2B \subset \Omega$. The claim follows. ☐

As in [21, Section 2.5.10], we say that a function $v\colon \mathbb{R}^n \to \mathbb{R}$ belongs to the Triebel–Lizorkin space $F^\alpha_{p,q}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$\|v\|_{F^\alpha_{p,q}(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)} < \infty,$$

where

$$[v]_{\dot{F}^{p,q}_\alpha(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} \frac{|v(x+h)-v(x)|^q}{|h|^{n+\alpha q}}\,dh\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}.$$

Equivalently, we could simply say that $v \in L^p(\mathbb{R}^n)$ and $\frac{\Delta_h v}{|h|^\alpha} \in L^p(dx; L^q(\frac{dh}{|h|^n}))$.

It turns out that Besov–Lipschitz and Triebel–Lizorkin spaces of fractional order $\alpha \in (0, 1)$ can be characterized in pointwise terms. Given a measurable function $v\colon \mathbb{R}^n \to \mathbb{R}$, a *fractional α-Hajlasz gradient for* v is a sequence $(g_k)_k$ of measurable non-negative functions $g_k\colon \mathbb{R}^n \to \mathbb{R}$, together with a null set $N \subset \mathbb{R}^n$, such that the inequality

$$|v(x) - v(y)| \le |x-y|^\alpha(g_k(x) + g_k(y))$$

holds whenever $k \in \mathbb{Z}$, and $x, y \in \mathbb{R}^n \setminus N$ are such that $2^{-k} \le |x-y| < 2^{-k+1}$. We say that $(g_k) \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ if

$$\|(g_k)_k\|_{\ell^q(L^p)} = \Big(\sum_{k\in\mathbb{Z}} \|g_k\|^q_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{q}} < \infty.$$

Similarly, we write $(g_k) \in L^p(\mathbb{R}^n; \ell^q(\mathbb{Z}))$ if

$$\|(g_k)_k\|_{L^p(\ell^q)} = \Big(\int_{\mathbb{R}^n} \|(g_k(x))_k\|^p_{\ell^q(\mathbb{Z})}\, dx \Big)^{\frac{1}{p}} < \infty.$$

The following result was proven in [15].

Theorem 2.4. *Let $0 < \alpha < 1$, $1 \le p < \infty$ and $1 \le q \le \infty$. Let $v \in L^p(\mathbb{R}^n)$.*

(i) *One has $v \in B^\alpha_{p,q}(\mathbb{R}^n)$ if and only if there exists a fractional α-Hajlasz gradient $(g_k)_k \in \ell^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ for v. Moreover,*

$$\|v\|_{B^\alpha_{p,q}(\mathbb{R}^n)} \simeq \inf \|(g_k)_k\|_{\ell^q(L^p)},$$

where the infimum runs over all possible fractional α-Hajlasz gradients for v.

(ii) *One has $v \in F^\alpha_{p,q}(\mathbb{R}^n)$ if and only if there exists a fractional α-Hajlasz gradient $(g_k)_k \in L^p(\mathbb{R}^n; \ell^q(\mathbb{Z}))$ for v. Moreover,*

$$\|v\|_{F^\alpha_{p,q}(\mathbb{R}^n)} \simeq \inf \|(g_k)_k\|_{L^p(\ell^q)},$$

where the infimum runs over all possible fractional α-Hajlasz gradients for v.

2.2 VMO coefficients in $\mathbb{R}^n$

In this section, we recall a regularity result, proven in [1], that will be crucial in our proofs. Let $n \ge 2$ and let $\mathcal{A}\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function such that assumptions (A1)–(A3) hold. We also require a control on the oscillations, which is described as follows. Given a ball $B \subset \Omega$, let us denote

$$\mathcal{A}_B(\xi) = \fint_B \mathcal{A}(x, \xi)\, dx.$$

One can easily check that the operator $\mathcal{A}_B(\xi)$ also satisfies assumptions (A1)–(A3). Now set

$$V(x, B) = \sup_{\xi \neq 0} \frac{|\mathcal{A}(x, \xi) - \mathcal{A}_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}}$$

for $x \in \Omega$ and $B \subset \Omega$. If $\mathcal{A}$ is given by the weighted p-laplacian, that is, $\mathcal{A}(x, \xi) = a(x)|\xi|^{p-2}\xi$, one obtains

$$V(x, B) = |a(x) - a_B|, \quad \text{where } a_B = \fint_B a(y)\, dy,$$

and so any reasonable VMO condition on $a(x)$ requires that the mean value of $V(x, B)$ on B goes to 0 as $|B| \to 0$. Our VMO assumption on general Carathéodory functions $\mathcal{A}$ consists of a uniform version of this fact. Namely, we will say that $x \mapsto \mathcal{A}(x, \xi)$ is *locally uniformly in* VMO if for each compact set $K \subset \Omega$, we have that

$$\lim_{R \to 0} \sup_{r(B) < R} \sup_{c(B) \in K} \fint_B V(x, B)\, dx = 0. \tag{2.2}$$

Here $c(B)$ denotes the center of the ball B and $r(B)$ its radius.

The following theorem, proved in [1], is a regularity result for weak solutions of p-harmonic equations with VMO coefficients.

Theorem 2.5. *Let $2 \le p \le n$ and $q > p$. Assume that* (A1)–(A3) *hold, and that $x \mapsto \mathcal{A}(x, \xi)$ is locally uniformly in* VMO. *If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of*

$$\operatorname{div} \mathcal{A}(x, Du) = \operatorname{div} F,$$

with $F \in L^{\frac{q}{p-1}}_{\mathrm{loc}}$, then $Du \in L^q_{\mathrm{loc}}$. Moreover, there exists a constant $\lambda > 1$ such that the Caccioppoli inequality

$$\fint_B |Du|^q \le C(n, \lambda, \nu, \ell, L, p, q)\Big(1 + \frac{1}{|B|^{\frac{q}{n}}} \fint_{\lambda B} |u|^q + \fint_{\lambda B} |F|^{\frac{q}{p-1}}\Big)$$

holds for any ball B such that $\lambda B \subset \Omega$.

2.3 Difference quotient

We recall some properties of the finite difference operator that will be needed in the sequel. We start with the description of some elementary properties that can be found, for example, in [10].

Proposition 2.6. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \geq 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > |h|\}.$$

Then the following hold:
(1) $\Delta_h F \in W^{1,p}(\Omega_{|h|})$ *and* $D_i(\Delta_h F) = \Delta_h(D_i F)$.
(2) *If at least one of the functions F and G has support contained in $\Omega_{|h|}$, then*

$$\int_\Omega F \Delta_h G\, dx = -\int_\Omega G \Delta_{-h} F\, dx.$$

(3) *We have* $\Delta_h(FG)(x) = F(x+h)\Delta_h G(x) + G(x)\Delta_h F(x)$.

The next result about the finite difference operator is a kind of an integral version of the Lagrange theorem.

Lemma 2.7. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$ and $F, DF \in L^p(B_R)$, then*

$$\int_{B_\rho} |\Delta_h F(x)|^p\, dx \leq c(n,p)|h|^p \int_{B_R} |DF(x)|^p\, dx.$$

Moreover,

$$\int_{B_\rho} |F(x+h)|^p\, dx \leq \int_{B_R} |F(x)|^p\, dx.$$

3 Proof of Theorem 1.1

We first prove that if (1.3) is satisfied, then $\mathcal{A}$ has the locally uniform VMO property (2.2). The proof goes exactly as that of [1, Lemma 17], concerning the case of an operator with linear growth. We report it here for the sake of completeness.

Lemma 3.1. *Let $\mathcal{A}\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map such that* (A1)–(A3) *hold. Assume that* (1.3) *is satisfied. Then $\mathcal{A}$ is locally uniformly in* VMO*, that is,* (2.2) *holds.*

Proof. We have

$$\begin{aligned}
⨍_B V(x,B)\, dx &= ⨍_B \sup_{\xi \neq 0} \frac{|\mathcal{A}(x,\xi) - \mathcal{A}_B(\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}}\, dx \\
&\leq ⨍_B \sup_{\xi \neq 0} ⨍_B \frac{|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)|}{(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}}\, dy\, dx \\
&\leq ⨍_B \sup_{\xi \neq 0} ⨍_B (g(x) + g(y))|x-y|^\alpha\, dy\, dx \\
&= ⨍_B ⨍_B (g(x) + g(y))|x-y|^\alpha\, dy\, dx \\
&\leq \Big(⨍_B ⨍_B (g(x)+g(y))^{\frac{n}{\alpha}}\, dy\, dx \Big)^{\frac{\alpha}{n}} \Big(⨍_B ⨍_B |x-y|^{\frac{n\alpha}{n-\alpha}}\, dy\, dx \Big)^{\frac{n-\alpha}{n}} \\
&\leq \Big(\frac{1}{|B|} \int_B g^{\frac{n}{\alpha}} \Big)^{\frac{\alpha}{n}} C(\alpha,n)|B|^{\frac{\alpha}{n}} = C(n,\alpha) \Big(\int_B g^{\frac{n}{\alpha}} \Big)^{\frac{\alpha}{\nu}},
\end{aligned}$$

and thus (2.2) holds. □

Proof of Theorem 1.1. Let us fix a ball B_R such that $B_{2R} \Subset \Omega$, and consider a cut off function $\eta \in C_0^\infty(B_R)$, with $\eta \equiv 1$ on $B_{R/2}$, such that $|\nabla \eta| \le \frac{c}{R}$. For small enough $|h|$, we set $\varphi = \Delta_{-h}(\eta^2 \Delta_h u)$ as a test function in equation (1.4). Using Proposition 2.6 (1), we obtain

$$\int \langle \mathcal{A}(x, Du), \Delta_{-h} D(\eta^2 \Delta_h u) \rangle \, dx = 0,$$

which is equivalent, by Proposition 2.6 (2), to the following equality:

$$\int \langle \Delta_h(\mathcal{A}(x, Du)), D(\eta^2 \Delta_h u) \rangle \, dx = 0. \tag{3.1}$$

We can write (3.1) as follows:

$$\begin{aligned} &\int \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x+h, Du(x)), D(\eta^2 \Delta_h u) \rangle \, dx \\ &\qquad = \int \langle \mathcal{A}(x, Du(x)) - \mathcal{A}(x+h, Du(x)), D(\eta^2 \Delta_h u) \rangle \, dx, \end{aligned}$$

and therefore

$$\begin{aligned} &\int \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x+h, Du(x)), \eta^2 D(\Delta_h u) \rangle \, dx \\ &\quad = - \int \langle \mathcal{A}(x+h, Du(x+h)) - \mathcal{A}(x+h, Du(x)), 2\eta \nabla \eta \Delta_h u \rangle \, dx \\ &\qquad + \int \langle \mathcal{A}(x, Du(x)) - \mathcal{A}(x+h, Du(x)), \eta^2 D(\Delta_h u) \rangle \, dx \\ &\qquad + \int \langle \mathcal{A}(x, Du(x)) - \mathcal{A}(x+h, Du(x)), 2\eta \nabla \eta \Delta_h u \rangle \, dx \\ &\quad =: I_1 + I_2 + I_3. \end{aligned}$$

Using the ellipticity assumption (A1) in the left-hand side and Proposition 2.6 (1), the previous equality yields

$$\nu \int (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\Delta_h Du|^2 \eta^2 \, dx \le |I_1| + |I_2| + |I_3|.$$

By virtue of assumption (A2) and Young's inequality, we have

$$\begin{aligned} |I_1| &\le 2L \int (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\Delta_h Du| |\eta| |\nabla \eta| |\Delta_h u| \, dx \\ &\le \varepsilon \int (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\Delta_h Du|^2 \eta^2 \, dx \\ &\quad + C(\varepsilon, L) \int (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\nabla \eta|^2 |\Delta_h u|^2 \, dx. \end{aligned}$$

To estimate the integrals I_2 and I_3, we write $\frac{p-1}{2} = \frac{p}{4} + \frac{p-2}{4}$, and then we use assumption (1.3) and Young's inequality as follows:

$$\begin{aligned} |I_2| &\le |h|^\alpha \int (g(x+h) + g(x))(\mu^2 + |Du(x)|^2)^{\frac{p-1}{2}} |D\Delta_h u| \eta^2 \, dx \\ &= |h|^\alpha \int (g(x+h) + g(x))(\mu^2 + |Du(x)|^2)^{\frac{p}{4} + \frac{p-2}{4}} |D\Delta_h u| \eta^2 \, dx \\ &\le \varepsilon \int (\mu^2 + |Du(x)|^2)^{\frac{p-2}{2}} |D\Delta_h u|^2 \eta^2 \, dx + c_\varepsilon |h|^{2\alpha} \int (g(x+h) + g(x))^2 (\mu^2 + |Du(x)|^2)^{\frac{p}{2}} \eta^2 \, dx \end{aligned}$$

and

$$\begin{aligned} |I_3| &\le 2|h|^\alpha \int (g(x+h) + g(x))(\mu^2 + |Du(x)|^2)^{\frac{p-1}{2}} |\eta| |\nabla \eta| |\Delta_h u| \, dx \\ &= 2|h|^\alpha \int (g(x+h) + g(x))(\mu^2 + |Du(x)|^2)^{\frac{p}{4} + \frac{p-2}{4}} |\eta| |\nabla \eta| |\Delta_h u| \, dx \\ &\le c \int (\mu^2 + |Du(x)|^2)^{\frac{p-2}{2}} |\nabla \eta|^2 |\Delta_h u|^2 \, dx + c|h|^{2\alpha} \int (g(x+h) + g(x))^2 (\mu^2 + |Du(x)|^2)^{\frac{p}{2}} \eta^2 \, dx. \end{aligned}$$

Collecting the estimates of I_1, I_2 and I_3, we obtain

$$\begin{aligned}\nu\int(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\Delta_h Du|^2\eta^2\,dx &\le 2\varepsilon\int(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\Delta_h Du|^2\eta^2\,dx\\ &\quad+c\int(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\nabla\eta|^2|\Delta_h u|^2\,dx\\ &\quad+c|h|^{2\alpha}\int(g(x+h)+g(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\eta^2\,dx.\end{aligned}$$

Choosing $\varepsilon=\frac{\nu}{8}$ and reabsorbing the first integral in the right-hand side by the left-hand side, we obtain

$$\begin{aligned}\frac{3\nu}{4}\int(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\Delta_h Du|^2\eta^2\,dx &\le c\int(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\nabla\eta|^2|\Delta_h u|^2\,dx\\ &\quad+c|h|^{2\alpha}\int(g(x+h)+g(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\eta^2\,dx.\end{aligned}\tag{3.2}$$

Using Hölder's inequality and the first estimate of Lemma 2.7 in the first integral on the right-hand side, and the fact that $\operatorname{supp}\eta\subset B_R$, we get

$$\begin{aligned}\int_{B_R}(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\nabla\eta|^2|\Delta_h u|^2\,dx &\le\frac{c}{R^2}\Big(\int_{B_R}|\Delta_h u|^p\,dx\Big)^{\frac{2}{p}}\Big(\int_{B_R}(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p}{2}}\Big)^{\frac{p-2}{p}}\,dx\\ &\le\frac{c|h|^2}{R^2}\int_{B_{R+|h|}}(\mu^2+|Du(x)|^2)^{\frac{p}{2}}.\end{aligned}$$

Inserting the previous estimate in (3.2), we obtain

$$\begin{aligned}\frac{3\nu}{4}\int_{B_R}(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\Delta_h Du|^2\eta^2\,dx\\ \le\frac{c|h|^2}{R^2}\int_{B_{R+|h|}}(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx+c|h|^{2\alpha}\int_{B_R}(g(x+h)+g(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx.\end{aligned}$$

Using Lemma 2.1 in the left-hand side of the previous estimate yields

$$\int_{B_R}|\Delta_h(V_p(Du))|^2\eta^2\,dx\le c\frac{|h|^2}{R^2}\int_{B_{R+|h|}}(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx+c|h|^{2\alpha}\int_{B_R}(g(x+h)+g(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx.\tag{3.3}$$

We now divide both sides of inequality (3.3) by $|h|^{2\alpha}$, and use the fact that $\eta\ge\chi_{B_{R/2}}$ to obtain

$$\int_{B_{R/2}}\Big|\frac{\Delta_h(V_p(Du))}{|h|^\alpha}\Big|^2\,dx\le c\int_{B_R}(g(x+h)+g(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx+\frac{c|h|^{2-2\alpha}}{R^2}\int_{B_{R+|h|}}(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx,$$

where $c=c(\nu,L,p,n)$. The homogeneity of the equation together with Theorem 2.5 yields that $Du\in L^s_{\mathrm{loc}}(\Omega)$ for every finite $s>1$, and so, in particular, $Du\in L^{\frac{np}{n-2\alpha}}(B_R)$. Therefore, by Hölder's inequality,

$$\begin{aligned}\int_{B_R}(g(x+h)+g(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx&\le\Big(\int_{B_R}(g(x+h)+g(x))^{\frac{n}{\alpha}}\Big)^{\frac{2\alpha}{n}}\Big(\int_{B_R}(\mu^2+|Du(x)|^2)^{\frac{np}{2(n-2\alpha)}}\,dx\Big)^{\frac{n-2\alpha}{n}}\\ &\le c\Big(\int_{B_{R+|h|}}g^{\frac{n}{\alpha}}\Big)^{\frac{2\alpha}{n}}\Big(\int_{B_R}(\mu^2+|Du(x)|^2)^{\frac{np}{2(n-2\alpha)}}\,dx\Big)^{\frac{n-2\alpha}{n}},\end{aligned}$$

and so we conclude

$$\int_{B_{R/2}} \left|\frac{\Delta_h(V_p(Du))}{|h|^\alpha}\right|^2 dx \le c\Big(\int_{B_{R+|h|}} g^{\frac{n}{\alpha}}\Big)^{\frac{2\alpha}{n}}\Big(\int_{B_R}(\mu^2+|Du|^2)^{\frac{np}{2(n-2\alpha)}}\Big)^{\frac{n-2\alpha}{n}} + \frac{c|h|^{2-2\alpha}}{R^2}\int_{B_{R+|h|}}(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx.$$

Since the above inequality holds for every h, we can take suprema over $h \in B(0,\delta)$ for some $\delta < R$ and obtain

$$\sup_{|h|<\delta}\int_{B_{R/2}}\left|\frac{\Delta_h(V_p(Du))}{|h|^\alpha}\right|^2 dx \le c\Big(1+\Big(\int_{B_{2R}} g^{\frac{n}{\alpha}}\Big)^{\frac{2\alpha}{n}}\Big)\Big(\int_{B_{2R}}(\mu^2+|Du|^2)^{\frac{np}{2(n-2\alpha)}}\Big)^{\frac{n-2\alpha}{n}}.$$

In particular, this tells us that $V_p(Du) \in B^\alpha_{2,\infty}$, locally. □

4 Proof of Theorems 1.2 and 1.3

We first prove that if $\mathcal{A}$ satisfies (A1)–(A4), then it is locally uniformly in VMO. This result is a straightforward extension to the case of operators $\mathcal{A}$ with $(p-1)$-growth of [1, Lemma 18], which refers to operator with linear growth. We report it here for the sake of completeness.

Lemma 4.1. *Let $\mathcal{A}$ be such that* (A1)–(A4) *hold. Then $\mathcal{A}$ is locally uniformly in* VMO, *that is,* (2.2) *holds.*

Proof. Given a point $x \in \Omega$, let us write $A_k(x) = \{y \in \Omega : 2^{-k} \le |x-y| < 2^{-k+1}\}$. We have

$$\begin{aligned}
\fint_B V(x,B)\,dx &= \fint_B \sup_{\xi\neq 0}\frac{|\mathcal{A}(x,\xi)-\mathcal{A}_B(\xi)|}{(\mu^2+|\xi|^2)^{\frac{p-1}{2}}}\,dx\\
&\le \fint_B \sup_{\xi\neq 0}\fint_B \frac{|\mathcal{A}(x,\xi)-\mathcal{A}(y,\xi)|}{(\mu^2+|\xi|^2)^{\frac{p-1}{2}}}\,dy\,dx\\
&= \fint_B \sup_{\xi\neq 0}\frac{1}{|B|}\sum_k\int_{B\cap A_k(x)}\frac{|\mathcal{A}(x,\xi)-\mathcal{A}(y,\xi)|}{(\mu^2+|\xi|^2)^{\frac{p-1}{2}}}\,dy\,dx\\
&\le \frac{1}{|B|^2}\sum_k\int_B\int_{B\cap A_k(x)}|x-y|^\alpha(g_k(x)+g_k(y))\,dy\,dx.
\end{aligned}$$

The last term above is bounded by

$$\Big(\frac{1}{|B|^2}\sum_k\int_B\int_{B\cap A_k(x)}|x-y|^{\frac{n\alpha}{n-\alpha}}\,dy\,dx\Big)^{\frac{n-\alpha}{n}}\Big(\frac{1}{|B|^2}\sum_k\int_B\int_{B\cap A_k(x)}(g_k(x)+g_k(y))^{\frac{n}{\alpha}}\,dy\,dx\Big)^{\frac{\alpha}{n}} = \mathrm{I}\cdot\mathrm{II}$$

The first sum is very easy to handle, since

$$\mathrm{I} = \Big(\frac{1}{|B|^2}\sum_k\int_B\int_{B\cap A_k(x)}|x-y|^{\frac{n\alpha}{n-\alpha}}\,dy\,dx\Big)^{\frac{n-\alpha}{n}} \le C(n,\alpha)|B|^{\frac{\alpha}{n}}.$$

Concerning the second, we see that

$$\begin{aligned}
\mathrm{II} &\le c\Big(\frac{1}{|B|^2}\sum_k|B\cap A_k(x)|\int_B g_k(x)^{\frac{n}{\alpha}}\,dx\Big)^{\frac{\alpha}{n}}\\
&\le c\Big(\frac{1}{|B|^2}\sum_k\Big(\Big(\int_B g_k(x)^{\frac{n}{\alpha}}\,dx\Big)^{\frac{\alpha q}{n}}\Big)^{\frac{\alpha}{n}\frac{n}{\alpha q}}\Big(\frac{1}{|B|^2}\sum_k|B\cap A_k(x)|^{\frac{\alpha q}{\alpha q-n}}\Big)^{\frac{\alpha}{n}\frac{\alpha q-n}{\alpha q}}\\
&= c\frac{1}{|B|^{\frac{2}{q}}}\Big(\sum_k\|g_k\|^q_{L^{\frac{n}{\alpha}}(B)}\Big)^{\frac{1}{q}}\frac{1}{|B|^{2(\frac{\alpha}{n}-\frac{1}{q})}}\Big(\sum_k|B\cap A_k(x)|^{\frac{\alpha q}{\alpha q-n}}\Big)^{\frac{\alpha}{n}\frac{\alpha q-n}{\alpha q}}\\
&\le \frac{1}{|B|^{\frac{2}{q}}}\Big(\sum_k\|g_k\|^q_{L^{\frac{n}{\alpha}}(B)}\Big)^{\frac{1}{q}}\frac{1}{|B|^{2(\frac{\alpha}{n}-\frac{1}{q})}}C(n,\alpha,q)|B|^{\frac{\alpha}{n}}\\
&= C(n,\alpha,q)|B|^{-\frac{\alpha}{n}}\Big(\sum_k\|g_k\|^q_{L^{\frac{n}{\alpha}}(B)}\Big)^{\frac{1}{q}},
\end{aligned}$$

thus

$$\fint_B V(x,B)\,dx \le \mathrm{I}\cdot\mathrm{II} \le C(n,\alpha,q)\Big(\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(B)}\Big)^{\frac{1}{q}}.$$

In order to get the VMO condition, it just remains to prove that

$$\lim_{r\to 0}\sup_{x\in K}\Big(\sum_k \|g_k\|^q_{L^{\frac{n}{\alpha}}(B(x,r))}\Big)^{\frac{1}{q}} = 0$$

on every compact set $K \subset \Omega$. To this end, we can fix $r > 0$ small enough and observe that the function $x \mapsto \|g_k\|_{\ell^q(L^{\frac{n}{\alpha}}(B(x,r)))}$ is continuous on the set $\{x \in \Omega : d(x, \partial\Omega) > r\}$, as a uniformly converging series of continuous functions. As a consequence, there exists a point $x_r \in K$ (at least for small enough $r > 0$) such that

$$\sup_{x\in K}\|g_k\|_{\ell^q(L^{\frac{n}{\alpha}}(B(x,r)))} = \|g_k\|_{\ell^q(L^{\frac{n}{\alpha}}(B(x_r,r)))}.$$

Now, from $\|g_k\|_{L^{\frac{n}{\alpha}}(B(x,r))} \le \|g_k\|_{L^{\frac{n}{\alpha}}(B(x_r,r))}$ and since this belongs to ℓ^q, we can use dominated convergence to say that

$$\lim_{r\to 0}\|g_k\|_{\ell^q(L^{\frac{n}{\alpha}}(B(x_r,r)))} = \Big(\sum_k \lim_{r\to 0}\Big(\int_{B(x_r,r)} g_k^{\frac{n}{\alpha}}\Big)^{\frac{q\alpha}{n}}\Big)^{\frac{1}{q}}.$$

Each of the limits on the term on the right-hand side are equal to 0, since the points x_r cannot escape from the compact set K as $r \to 0$. This completes the proof. □

Proof of Theorem 1.2. We first assume that $\alpha \le \beta$. Let us fix a ball B_R such that $B_{2R} \Subset \Omega$ and a cut off function $\eta \in C_0^\infty(B_R)$, with $\eta \equiv 1$ on $B_{R/2}$, such that $|\nabla\eta| \le \frac{c}{R}$. For small enough h, we set $\varphi = \Delta_{-h}(\eta^2\Delta_h u)$ as a test function in equation (1.5). Using Proposition 2.6 (1), we obtain

$$\int_{B_R}\langle \mathcal{A}(x,Du), \Delta_{-h}D(\eta^2\Delta_h u)\rangle\,dx = \int_{B_R}\langle F, \Delta_{-h}D(\eta^2\Delta_h u)\rangle\,dx,$$

which, by Proposition 2.6 (2), is equivalent to

$$\int\langle \Delta_h(\mathcal{A}(x,Du)), D(\eta^2\Delta_h u)\rangle\,dx = \int\langle \Delta_h(F), D(\eta^2\Delta_h u)\rangle\,dx. \tag{4.1}$$

We can write (4.1) as follows:

$$\begin{aligned}\int\langle \mathcal{A}(x+h,Du(x+h)) &- \mathcal{A}(x+h,Du(x)), D(\eta^2\Delta_h u)\rangle\,dx\\ &= \int\langle \mathcal{A}(x,Du(x)) - \mathcal{A}(x+h,Du(x)), D(\eta^2\Delta_h u)\rangle\,dx + \int\langle\Delta_h(F), D(\eta^2\Delta_h u)\,dx\rangle,\end{aligned}$$

and therefore

$$\begin{aligned}\int\langle \mathcal{A}(x+h,Du(x+h)) &- \mathcal{A}(x+h,Du(x)), \eta^2 D(\Delta_h u)\rangle\,dx\\ &= -\int\langle \mathcal{A}(x+h,Du(x+h)) - \mathcal{A}(x+h,Du(x)), 2\eta\nabla\eta\Delta_h u\rangle\,dx\\ &\quad + \int\langle \mathcal{A}(x,Du(x)) - \mathcal{A}(x+h,Du(x)), \eta^2 D(\Delta_h u)\rangle\,dx\\ &\quad + \int\langle \mathcal{A}(x,Du(x)) - \mathcal{A}(x+h,Du(x)), 2\eta\nabla\eta\Delta_h u\rangle\,dx\\ &\quad + \int\langle\Delta_h(F), \eta^2 D(\Delta_h u)\rangle\,dx + \int\langle\Delta_h(F), 2\eta\nabla\eta\Delta_h u\rangle\,dx\\ &=: I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

Now, using assumption (A1) in the left-hand side and Proposition 2.6 (1), the previous equality yields

$$\nu\int_{B_R}(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}}|\Delta_h(Du)|^2\eta^2\,dx \le |I_1| + |I_2| + |I_3| + |I_4| + |I_5|.$$

The integrals I_1, I_2 and I_3 can be estimated exactly as we did in the proof of Theorem 1.1. After doing this for $2^{-k} \le |h| < 2^{-k+1}$, the above inequality reads as

$$\frac{3\nu}{4}\int(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\Delta_h(Du)|^2\eta^2\,dx \le c\frac{|h|^2}{R^2}\int_{B_{2R}}(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx + c|h|^{2\alpha}\int_{B_R}(g_k(x+h)+g_k(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx + |I_4| + |I_5|. \quad (4.2)$$

Now we estimate I_4 and I_5. By using Young's inequality, we get

$$\begin{aligned}|I_4| &\le \int_{B_R}\eta^2|\Delta_h(F)||\Delta_h(Du)|\,dx \\ &\le \frac{c}{\varepsilon}\int_{B_R}|\Delta_h(F)|^2\,dx + \varepsilon\int_{B_R}|\Delta_h(Du)|^2\eta^2\,dx \\ &= \frac{c}{\varepsilon}|h|^{2\beta}\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx + \frac{\varepsilon}{\mu^{p-2}}\int_{B_R}|\Delta_h(Du)|^2\mu^{p-2}\eta^2\,dx \\ &\le \frac{c}{\varepsilon}|h|^{2\beta}\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx + \frac{\varepsilon}{\mu^{p-2}}\int_{B_R}\eta^2|\Delta_h Du|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\,dx,\end{aligned}$$

since $\mu > 0$, where $\varepsilon > 0$ will be chosen later. Similarly,

$$|I_5| \le \frac{c}{R}\int_{B_R}|\Delta_h(F)||\Delta_h u|\,dx \le c\frac{|h|^2}{R^2}\int_{B_{2R}}|Du|^2 + c|h|^{2\beta}\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx,$$

where we used the first estimate in Lemma 2.7. Inserting the estimates of I_4 and I_5 in (4.2), we have

$$\begin{aligned}&\frac{3\nu}{4}\int_{B_R}(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\Delta_h(Du)|^2\eta^2\,dx \\ &\quad\le c\frac{|h|^2}{R^2}\int_{B_{2R}}(\mu^p+|Du(x)|^2+|Du(x)|^p)\,dx + c|h|^{2\alpha}\int_{B_R}(g_k(x+h)+g_k(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx \\ &\qquad + \frac{c}{\varepsilon}|h|^{2\beta}\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx + \frac{\varepsilon}{\mu^{p-2}}\int_{B_R}\eta^2|\Delta_h(Du)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\,dx.\end{aligned}$$

Choosing $\varepsilon = \frac{3\mu^{p-2}\nu}{8}$, reabsorbing the last integral in the right-hand side of the previous estimate by the left-hand side, using Lemma 2.1, the fact that $\eta \equiv 1$ on $B_{R/2}$, and dividing both side by $|h|^{2\gamma}$, $\gamma = \min\{\alpha,\beta\} = \alpha$, we conclude that

$$\begin{aligned}\int_{B_{R/2}}\Big|\frac{\Delta_h(V_p(Du))}{|h|^\gamma}\Big|^2\,dx &\le c\frac{|h|^{2-2\gamma}}{R^2}\int_{B_{2R}}(\mu^p+|Du(x)|^2+|Du(x)|^p)\,dx \\ &\quad + c|h|^{2(\alpha-\gamma)}\int_{B_R}(g_k(x+h)+g_k(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx + \frac{c|h|^{2(\beta-\gamma)}}{\mu^{p-2}}\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx,\end{aligned}$$

and so

$$\begin{aligned}\Big(\int_{B_{R/2}}\Big|\frac{\Delta_h(V_p(Du))}{|h|^\gamma}\Big|^2\,dx\Big)^{\frac12} &\le c\frac{|h|^{1-\gamma}}{R}\Big(\int_{B_{2R}}(\mu^p+|Du(x)|^2+|Du|^p)\,dx\Big)^{\frac12} + \frac{c|h|^{\beta-\gamma}}{\mu^{\frac{p-2}{2}}}\Big(\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx\Big)^{\frac12} \\ &\quad + c|h|^{\alpha-\gamma}\Big(\int_{B_R}(g_k(x+h)+g_k(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx\Big)^{\frac12},\end{aligned}$$

where $c = c(\nu, L, p, n)$. Taking the L^q norm with the measure $\frac{dh}{|h|^n}$ restricted to the ball $B(0,\delta)$ on the h-space, we obtain that

$$\begin{aligned}\Big(\int_{B_\delta}\Big(\int_{B_{R/2}}\Big|\frac{\Delta_h(V_p(Du))}{|h|^\gamma}\Big|^2\,dx\Big)^{\frac{q}{2}}\frac{dh}{|h|^n}\Big)^{\frac{1}{q}} &\le c\Big(\int_{B_\delta}|h|^{q(1-\gamma)}\Big(\int_{B_{2R}}(\mu^p+|Du(x)|^2+|Du(x)|^p)\,dx\Big)^{\frac{q}{2}}\frac{dh}{|h|^n}\Big)^{\frac{1}{q}}\\ &\quad+\frac{c}{\mu^{\frac{p-2}{2}}}\Big(\int_{B_\delta}\Big(\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx\Big)^{\frac{q}{2}}\frac{dh}{|h|^n}\Big)^{\frac{1}{q}}\\ &\quad+c\Big(\int_{B_\delta}\Big(\int_{B_R}(g_k(x+h)+g_k(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx\Big)^{\frac{q}{2}}\frac{dh}{|h|^n}\Big)^{\frac{1}{q}}\\ &=: J_1+J_2+J_3,\end{aligned}$$

where $c = c(\nu, L, R, p, n, \delta)$. For the estimate of J_1, one can easily check that

$$\begin{aligned}J_1 &= c\Big(\int_{B_{2R}}(\mu^p+|Du(x)|^2+|Du(x)|^p)\,dx\Big)^{\frac{1}{2}}\cdot\Big(\int_{B_\delta}|h|^{(1-\gamma)q-n}\,dh\Big)^{\frac{1}{q}}\\ &\le c(n)\Big(\int_{B_{2R}}(\mu^p+|Du(x)|^2+|Du(x)|^p)\,dx\Big)^{\frac{1}{2}}\cdot\Big(\int_0^\delta\rho^{(1-\gamma)q-1}\,d\rho\Big)^{\frac{1}{q}}\\ &= c(\gamma,n,q,\delta)\Big(\int_{B_{2R}}(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx\Big)^{\frac{1}{2}},\end{aligned}$$

since $\gamma < 1$. The term J_2 can be controlled by the $B^\beta_{2,q}$-seminorm of F, which is finite thanks to our assumption. Before estimating J_3, recall that, by virtue of assumption (A4), for every k, one has $g_k^2 \in L^{\frac{n}{2\alpha}}$. Also, since $q \le 2^*_\beta$, by Lemma 2.2, we have that $F \in L^{\frac{2n}{n-2\beta}}_{\mathrm{loc}}$, and so, by Theorem 2.5, we have that $Du \in L^{\frac{2n(p-1)}{n-2\beta}}_{\mathrm{loc}}$. Now, from $p \ge 2$ and $\alpha \le \beta$, we easily see that

$$\frac{2n(p-1)}{n-2\beta} \ge \frac{np}{n-2\alpha}, \tag{4.3}$$

and so we can proceed as follows. We write the L^q norm in the integral J_3 in polar coordinates, assuming without loss of generality that $\delta = 1$, so $h \in B(0,1)$ if and only if $h = r\xi$ for some $0 \le r < 1$ and some ξ in the unit sphere S^{n-1} on $\mathbb{R}^n$. We denote by $d\sigma(\xi)$ the surface measure on S^{n-1}. We bound the term J_3 by

$$\begin{aligned}&\int_0^1\int_{S^{n-1}}\Big(\int_{B_R}(g_k(x+r\xi)+g_k(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx\Big)^{\frac{q}{2}}d\sigma(\xi)\frac{dr}{r}\\ &\quad=\sum_{k=0}^\infty\int_{r_{k+1}}^{r_k}\int_{S^{n-1}}\Big(\int_{B_R}(g_k(x+r\xi)+g_k(x))^2(\mu^2+|Du(x)|^2)^{\frac{p}{2}}\,dx\Big)^{\frac{q}{2}}d\sigma(\xi)\frac{dr}{r}\\ &\quad\le\sum_{k=0}^\infty\int_{r_{k+1}}^{r_k}\int_{S^{n-1}}\|(\tau_{r\xi}g_k+g_k)(\mu^2+|Du|^2)^{\frac{p}{4}}\|^q_{L^2(B_R)}\,d\sigma(\xi)\frac{dr}{r},\end{aligned}$$

where we set $r_k = \frac{1}{2^k}$. Now, from (4.3), we see that $Du \in L^{\frac{np}{n-2\alpha}}_{\mathrm{loc}}$. This, together with the assumption $g_k \in L^{\frac{n}{\alpha}}$, gives us that

$$\|(\tau_{r\xi}g_k+g_k)(\mu^2+|Du|^2)^{\frac{p}{4}}\|_{L^2(B_R)} \le \|(\mu^2+|Du|^2)^{\frac{1}{2}}\|^{\frac{p}{2}}_{L^{\frac{np}{n-2\alpha}}(B_R)}\|(\tau_{r\xi}g_k+g_k)\|_{L^{\frac{n}{\alpha}}(B_R)}.$$

On the other hand, we note that for each $\xi \in S^{n-1}$ and $r_{k+1} \le r \le r_k$,

$$\|(\tau_{r\xi}g_k+g_k)\|_{L^{\frac{n}{\alpha}}(B_R)} \le \|g_k\|_{L^{\frac{n}{\alpha}}(B_R-r_k\xi)} + \|g_k\|_{L^{\frac{n}{\alpha}}(B_R)} \le 2\|g_k\|_{L^{\frac{n}{\alpha}}(\lambda B)},$$

where $\lambda = 2 + \frac{1}{R}$. Hence,

$$J_3 \le C(n,\alpha,q)\|(\mu^2+|Du|^2)^{\frac{1}{2}}\|^{\frac{p}{2}}_{L^{\frac{np}{n-2\alpha}}(B_R)}\|\{g_k\}_k\|_{\ell^q(L^{\frac{n}{\alpha}}(\lambda B))}.$$

Summarizing, we have

$$\Big\|\frac{\Delta_h(V_p(Du))}{|h|^\gamma}\Big\|_{L^q(\frac{dh}{|h|^n};L^2(B_{R/2}))} \le C\big(1+\|Du\|_{L^2(B_R)}+\|Du\|^{\frac{p}{2}}_{L^p(B_R)}\big) + \frac{C}{\mu^{p-2}}\Big\|\frac{\Delta_h(F)}{|h|^\beta}\Big\|_{L^q(\frac{dh}{|h|^n};L^2(2B))} + C\|Du\|^{\frac{p}{2}}_{L^{\frac{np}{n-2\alpha}}(2B)}\|\{g_k\}_k\|_{\ell^q(L^{\frac{n}{\alpha}}(\lambda B))},$$

with $C = C(\mu,\alpha,\beta,p,q,n,\nu,L)$. Lemma 2.3 now guarantees that $V_{p,\mu}(Du)\in B^\gamma_{2,q}$, locally, and this concludes the proof.

When $\alpha>\beta$, we have the embedding $B^\alpha_{n/\alpha,q}\subset B^\beta_{n/\beta,q}\subset \mathrm{VMO}$. Thus, we can assume that (A4) holds with α replaced by β, and then repeat the previous proof. The claim follows. □

Proof of Theorem 1.3. The proof goes exactly as that of Theorem 1.2 until estimate (4.2). We proceed now with the estimates of the integrals I_4 and I_5. Assume first that $\alpha\le\frac{\beta p'}{2}$. By using Hölder's and Young's inequalities, we get

$$\begin{aligned}|I_4| &\le \int_{B_R}\eta^2|\Delta_h(F)||\Delta_h(Du)|\\
&\le c(R)\Big(\int_{B_R}|\Delta_h(F)|^2\Big)^{\frac12}\Big(\int_{B_R}\eta^2|\Delta_h(Du)|^p\Big)^{\frac1p}\\
&\le c_\varepsilon\Big(\int_{B_R}|\Delta_h(F)|^2\,dx\Big)^{\frac{p'}{2}}+\varepsilon\int_{B_R}\eta^2|\Delta_h(Du)|^p\,dx\\
&\le c_\varepsilon|h|^{\beta p'}\Big(\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx\Big)^{\frac{p'}{2}}+\varepsilon\int_{B_R}\eta^2|\Delta_h(Du)|^p\,dx,\end{aligned}$$

where $\varepsilon>0$ will be chosen later. Similarly, using also Lemma 2.7,

$$|I_5|\le\frac{c}{R}\int_{B_R}|\Delta_h(F)|\eta|\Delta_h u|\,dx\le c\frac{|h|^p}{R^p}\int_{B_R}|Du|^p+c|h|^{\beta p'}\Big(\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx\Big)^{\frac{p'}{2}}.$$

Inserting the estimates of I_4 and I_5 in (4.2) and recalling that $\mu=0$, we have

$$\begin{aligned}&\frac{3\nu}{4}\int_{B_R}(|Du(x)|^{p-2}+|Du(x+h)|^{p-2})|\Delta_h(Du)|^2\eta^2\,dx\\
&\quad\le c\Big(\frac{|h|^2}{R^2}+\frac{|h|^p}{R^p}\Big)\int_{B_R}|Du|^p\,dx+c|h|^{2\alpha}\int_{B_R}(g_k(x+h)+g_k(x))^2|Du(x)|^p\,dx\\
&\qquad+c|h|^{\beta p'}\Big(\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx\Big)^{\frac{p'}{2}}+\varepsilon\int_{B_R}\eta^2|\Delta_h(Du)|^p\,dx\end{aligned}$$

and, by the elementary inequality (2.1), we get

$$\begin{aligned}&\int_{B_R}(|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}|\Delta_h(Du)|^2\eta^2\,dx\\
&\quad\le c\Big(\frac{|h|^2}{R^2}+\frac{|h|^p}{R^p}\Big)\int_{B_R}|Du(x)|^p\,dx+c|h|^{2\alpha}\int_{B_R}(g_k(x+h)+g_k(x))^2|Du(x)|^p\,dx\\
&\qquad+c|h|^{\beta p'}\Big(\int_{B_R}\Big|\frac{\Delta_h(F)}{|h|^\beta}\Big|^2\,dx\Big)^{\frac{p'}{2}}+c(p,\nu)\varepsilon\int_{B_R}(|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\Delta_h(Du)|^2\eta^2\,dx.\end{aligned}$$

We now choose $\varepsilon = \frac{1}{2c(p,\nu)}$, use Lemma 2.1, reabsorb the last integral in the right-hand side of the previous estimate by the left-hand side, and divide both side by $|h|^{2\alpha}$. We conclude that

$$\int_{B_{R/2}} \left|\frac{\Delta_h(V_p(Du))}{|h|^\alpha}\right|^2 dx \le c\left(\frac{|h|^{2-2\alpha}}{R^2} + \frac{|h|^{p-2\alpha}}{R^p}\right) \int_{B_R} |Du(x)|^p \, dx + c|h|^{\beta p'-2\alpha}\left(\int_{B_R} \left|\frac{\Delta_h(F)}{|h|^\beta}\right|^2 dx\right)^{\frac{p'}{2}} + c\int_{B_R} (g_k(x+h) + g_k(x))^2 |Du(x)|^p \, dx,$$

and so

$$\begin{aligned}\left(\int_{B_{R/2}} \left|\frac{\Delta_h(V_p(Du))}{|h|^\alpha}\right|^2 dx\right)^{\frac{1}{2}} &\le c\left(\frac{|h|^{1-\alpha}}{R} + \frac{|h|^{\frac{p}{2}-\alpha}}{R^{\frac{p}{2}}}\right)\left(\int_{B_R} |Du(x)|^p \, dx\right)^{\frac{1}{2}} + c|h|^{\frac{\beta p'}{2}-\alpha}\left(\int_{B_R} \left|\frac{\Delta_h(F)}{|h|^\beta}\right|^2 dx\right)^{\frac{p'}{4}} \\ &\quad + c\left(\int_{B_r} (g_k(x+h) + g_k(x))^2 |Du(x)|^p \, dx\right)^{\frac{1}{2}} \\ &\le c|h|^{1-\alpha}\left(\int_{B_R} |Du(x)|^p \, dx\right)^{\frac{1}{2}} + c\left(\int_{B_R} \left|\frac{\Delta_h(F)}{|h|^\beta}\right|^2 dx\right)^{\frac{p'}{4}} \\ &\quad + c\left(\int_{B_r} (g_k(x+h) + g_k(x))^2 |Du(x)|^p \, dx\right)^{\frac{1}{2}},\end{aligned}$$

for a constant c that depends also on R and δ, and where we used that $p \ge 2$. Taking the L^q norm with the measure $\frac{dh}{|h|^n}$ restricted to the ball $B(0,\delta)$ on the h-space, we obtain that

$$\begin{aligned}\left(\int_{B_\delta}\left(\int_{B_{R/2}} \left|\frac{\Delta_h(V_p(Du))}{|h|^\alpha}\right|^2 dx\right)^{\frac{q}{2}} \frac{dh}{|h|^n}\right)^{\frac{1}{q}} &\le c\left(\int_{B_\delta} |h|^{(1-\alpha)q}\left(\int_{B_R} |Du(x)|^p \, dx\right)^{\frac{q}{2}} \frac{dh}{|h|^n}\right)^{\frac{1}{q}} \\ &\quad + c\left(\int_{B_\delta}\left(\int_{B_R} \left|\frac{\Delta_h(F)}{|h|^\beta}\right|^2 dx\right)^{\frac{p'q}{4}} \frac{dh}{|h|^n}\right)^{\frac{1}{q}} \\ &\quad + c\left(\int_{B_\delta}\left(\int_{B_R} (g_k(x+h) + g_k(x))^2 |Du(x)|^p \, dx\right)^{\frac{q}{2}} \frac{dh}{|h|^n}\right)^{\frac{1}{q}} \\ &=: J_1 + J_2 + J_3,\end{aligned}$$

where $c = c(\nu, L, R, p)$. For the estimate of J_1, one can easily check that

$$\begin{aligned}J_1 &= c\left(\int_{B_R} |Du(x)|^p \, dx\right)^{\frac{1}{2}} \cdot \left(\int_{B_\delta} |h|^{(1-\alpha)q-n} \, dh\right)^{\frac{1}{q}} \\ &\le c(n)\left(\int_{B_R} |Du(x)|^p \, dx\right)^{\frac{1}{2}} \cdot \left(\int_0^\delta \rho^{(1-\alpha)q-1} \, d\rho\right)^{\frac{1}{q}} \\ &= c(\alpha, n, p, q, \delta)\left(\int_{B_R} |Du(x)|^p \, dx\right)^{\frac{1}{2}},\end{aligned}$$

since $\alpha < 1$. The term J_2 can be controlled by the $B^\beta_{2,qp'/2}$-seminorm of F, which is finite thanks to our assumption. In order to estimate J_3, we use that $g_k^2 \in L^{\frac{n}{2\alpha}}$, by our assumption. Also, we have that $|Du(x)|^p \in L^{\frac{n}{n-2\alpha}}_{\mathrm{loc}}$. To see this, use Lemma 2.2, with $\frac{qp'}{2} \le 2^*_\beta = \frac{2n}{n-2\beta}$, to deduce that $F \in L^{2^*_\beta}$. Since

$$\frac{2n(p-1)}{n-2\beta} \ge \frac{np}{n-2\alpha},$$

Theorem 2.5 implies that $Du \in L^{\frac{np}{n-2\alpha}}_{\mathrm{loc}}$.

We now write the L^q norm in the integral J_3 in polar coordinates, assuming without loss of generality that $\delta = 1$, so $h \in B(0,1)$ if and only if $h = r\xi$ for some $0 \le r < 1$ and some ξ in the unit sphere S^{n-1} on $\mathbb{R}^n$. We denote by $d\sigma(\xi)$ the surface measure on S^{n-1}. We bound the term J_3 by

$$\begin{aligned}
&\int_0^1 \int_{S^{n-1}} \Big(\int_{B_R} (g_k(x+r\xi)+g_k(x))^2 |Du(x)|^p \, dx \Big)^{\frac{q}{2}} d\sigma(\xi) \frac{dr}{r} \\
&\quad = \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \int_{S^{n-1}} \Big(\int_{B_R} (g_k(x+r\xi)+g_k(x))^2 |Du(x)|^p \, dx \Big)^{\frac{q}{2}} d\sigma(\xi) \frac{dr}{r} \\
&\quad = \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \int_{S^{n-1}} \|(\tau_{r\xi} g_k + g_k)^{\frac{2}{p}} Du\|_{L^p(B_R)}^{\frac{qp}{2}} d\sigma(\xi) \frac{dr}{r},
\end{aligned}$$

where we set $r_k = \frac{1}{2^k}$. Now, since $Du \in L_{\mathrm{loc}}^{\frac{np}{n-2\alpha}}$ and $g_k \in L^{\frac{n}{\alpha}}$, Hölder's inequality implies

$$\|(\tau_{r\xi} g_k + g_k)^{\frac{2}{p}} Du\|_{L^p(B_R)} \le \|Du\|_{L^{\frac{np}{n-2\alpha}}(B_R)} \|(\tau_{r\xi} g_k + g_k)\|_{L^{\frac{n}{\alpha}}(B_R)}^{\frac{2}{p}}.$$

On the other hand, we note that for each $\xi \in S^{n-1}$ and $r_{k+1} \le r \le r_k$,

$$\|(\tau_{r\xi} g_k + g_k)\|_{L^{\frac{n}{\alpha}}(B_R)} \le \|g_k\|_{L^{\frac{n}{\alpha}}(B_R - r_k\xi)} + \|g_k\|_{L^{\frac{n}{\alpha}}(B_R)} \le 2\|g_k\|_{L^{\frac{n}{\alpha}}(\lambda B)},$$

where $\lambda = 2 + \frac{1}{R}$. Hence,

$$J_3 \le C(n,\alpha,q) \|Du\|_{L^{\frac{np}{n-2\alpha}}(B_R)}^{\frac{p}{2}} \|\{g_k\}_k\|_{\ell^q(L^{\frac{n}{\alpha}}(\lambda B))},$$

where $C(n,\alpha,q) = 2^{1-\alpha} \log 2 \sigma(S^{n-1})^{\frac{1}{q}}$. Summarizing,

$$\Big\| \frac{\Delta_h(V_p(Du))}{|h|^\alpha} \Big\|_{L^q(\frac{dh}{|h|^n}; L^2(B_{R/2}))} \le C\|Du\|_{L^p(B_R)} + \Big\| \frac{\Delta_h F}{|h|^\beta} \Big\|_{L^q(\frac{dh}{|h|^n}; L^2(B_{2R}))} + C(n,\alpha,q)\|Du\|_{L^{\frac{np}{n-2\alpha}}(B_R)}^{\frac{p}{2}} \|\{g_k\}_k\|_{\ell^q(L^{\frac{n}{\alpha}}(\lambda B))}.$$

Lemma 2.3 now yields that $V_p(Du) \in B_{2,q}^{\alpha}$, locally.

When $\alpha > \frac{\beta p'}{2}$, we have the embedding $B_{n/\alpha,q}^{\alpha} \subset B_{2n/\beta p',q}^{\frac{\beta p'}{2}} \subset \mathrm{VMO}$. Thus, we can assume that (A4) holds with α replaced by $\frac{\beta p'}{2}$, and then repeat the previous proof. The claim follows. □

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