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Revealed Preference and the Subjective State Space Hypothesis*

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Abstract

This paper provides, for finite sets of choice data, revealed preference characterizations for the additive representations considered in Dekel, Lipman and Rustichini (2001). For a particular class of data sets, it is shown that the characterizing conditions can be reformulated as nonlinear systems of inequalities for which the existence of solutions can be verified using numerical methods.

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1 Introduction

Kreps (1979) studies preferences over menus of deterministic alternatives. He shows that a simple set of axioms characterizes a representation that can be interpreted as if the agent is uncertain about her future tastes. This taste uncertainty is summarized by a set of possible future preferences which is referred to as the subjective state space. In Kreps' model, the subjective state space is not completely pinned down by the preference over menus. Dekel, Lipman and Rustichini (2001)(henceforth DLR) extend Kreps' analysis to menus of lotteries. This richer domain allows them to show that, under certain assumptions, the subjective state space is "essentially unique" given the preference over menus of lotteries.

Although DLR provide axiomatic foundations for several representations that have a subjective state space component, we only consider the additive ones which have the form

$$V(A) = \int_S \sup_{p \in A} U_s(p) \mu(ds),$$

where: A is a menu of lotteries, S is a non-empty set, U_s is an expected utility function for every $s \in S$, and μ is a finitely additive (signed) measure over S. As is typical of the axiomatic approach, verification of their axioms requires that the entire preference order be observable. This paper provides a corresponding revealed preference analysis assuming that only finitely many choices are observed. In particular, we assume that we are given finitely many observations, each observation consists of a *budget*, that is, a collection of menus of lotteries, and a *choice* from each budget.

Our main result shows that a suitable adaptation of the Independence axiom for menus of lotteries to finite choice data sets characterizes the general additive representation. We also provide necessary and sufficient conditions for the cases in which μ is a positive measure; and when the subjective state space S is a singleton.

In their framework, DLR show that there is a distinction between preferences that have an additive representation with an infinite state space and preferences that have a representation with a finite state space. An implication of our results is that for finite data sets there is no empirically meaningful distinction.

Our analysis builds on the revealed preference analysis for vNM utility theory carried out by Fishburn (1975), Border (1992) and Kim (1996). Fishburn (1975) considers lotteries over a finite abstract prize space; Border (1992) assumes the prize space is a compact subset of the real line and studies the case in which the vNM utility index is increasing. Kim (1996) generalizes Fishburn (1975) and Border (1992) by considering lotteries over an abstract compact metric space. As in Fishburn (1975), we consider an abstract finite prize space but we generalize the choice domain to menus of lotteries. Restricted to singleton menus, our characterizing conditions are equivalent to Fishburn's condition for vNM utility theory. To the best of our knowledge this is the first paper to provide a revealed preference analysis for any model with a *subjective* state space component.

Finally, for the case in which each menu of lotteries is either finite or finitely generated (that is, equal to the convex hull of a finite set), we show how our results can be reformulated as a nonlinear system of inequalities and discuss the numerical methods that can be used to verify them. Hence, for this case, our results provide a test that is, in principle, implementable. See Varian (1983), Chiappori (1988), Diewert (2012) and Demuynck and Seel (2018) for other instances in the revealed preference literature where characterizing conditions lead to nonlinear inequalities.

The paper proceeds as follows. Section 2 gives the general model. Section 3 states the main results. Section 4 contains a discussion of our results. Appendix A contains the proofs of our results and Appendix B describes the aforementioned systems of inequalities and applicable numerical methods.

2 The Model

Let X be a **finite** set of cardinality n. A lottery is a probability measure over X. The set of all lotteries is denoted $\Delta(X)$ and $P(\Delta(X))$ denotes the set of all its non-empty subsets.

A **choice problem** $(c, \mathcal{B}, T)$ is an index set $T = \{1, ..., T\}$, a collection of sets of menus of lotteries, called **budgets**; $\mathcal{B} = \{B_t | t \in T\}$ where $B_t \subseteq P(\Delta(X))$; and a function $c : T \to P(\Delta(X))$ such that $c_t \in B_t$ for all $t \in T$. Generic menus will be denoted A, B and generic lotteries will be denoted p, q.

Definition 2.1. *A choice problem $(c, \mathcal{B}, T)$ is* ***rationalizable*** *by an* ***additive representation*** *if there exist a non-empty set S, a state dependent utility function $U : S \times \Delta(X) \to \mathbb{R}$ such that U_s is an expected utility function for all $s \in S$, and a finitely additive (signed) measure μ on S, such that: for all $t \in T$ and $d_t \in B_t \backslash c_t$,*

$$\int_S \sup_{p \in c_t} U_s(p)\mu(ds) > \int_S \sup_{q \in d_t} U_s(q)\mu(ds).$$

If μ is positive, we say $(c, \mathcal{B}, T)$ is rationalizable by a ***monotone additive representation****. If $|S| = 1$, we say $(c, \mathcal{B}, T)$ is rationalizable by a* ***strategically rational representation****.*[1]

[1] To be more precise, we require that S be a measurable space and that U be measurable with respect to this space. Since we make no explicit use of such measurability considerations, we avoid a discussion of the details.

Our notion of rationalizability requires that the chosen menu be strictly prefered to any of the rejected menus. If we were to require only weak preference, then any choice problem would be rationalizable by a strategically rational representation with a constant expected utility function.

The case $|S| = 1$ is of particular interest because of its relation to vNM utility theory. A strategically rational agent knows what she is going to choose from the menu. If the agent chooses A over B, then there exists a lottery $p \in A$ such that she would choose $\{p\}$ over B.

The difference between a monotone additive and an additive representation is that the former does not allow smaller menus to be strictly preferable. It prescribes that given menus $B \subset A$, the agent would choose A over B because of the extra flexibility A provides. However, there are situations (temptation and costly self control, see Gul and Pesendorfer (2001)) in which flexibility is costly and commitment is valuable. The additive representation allows for both.

Given a collection $A_1, ..., A_T$ of menus of lotteries and a probability measure $\lambda \in \Delta(T)$, a mixture of these menus is defined as

$$\sum_t \lambda_t A_t = \{p \in \Delta(X) | p = \sum_t \lambda_t p_t, p_t \in A_t\}.$$

3 Main Results

In this section we state our main result for choice problems $(c, \mathcal{B}, T)$ such that each budget is binary. In the next section we discuss to what extent this binariness assumption is without loss of generality.

Theorem 3.1. *Let $(c, \mathcal{B}, T)$ be a choice problem such that for all $t \in T$, $B_t = \{c_t, d_t\}$ where c_t and d_t are closed and convex sets of lotteries and c_t is the chosen menu out of B_t. For each of the following rationalizability properties the conditions indicated are necessary and sufficient.*

Strategically Rational *There exist $p_t \in c_t$, $t = 1, ..., T$, such that* $\sum_t \lambda_t p_t \notin \sum_t \lambda_t d_t \ \forall \lambda \in \Delta(T)$.

Monotone Additive $\sum_t \lambda_t c_t \nsubseteq \sum_t \lambda_t d_t \ \forall \lambda \in \Delta(T)$.

Additive $\sum_t \lambda_t c_t \neq \sum_t \lambda_t d_t \ \forall \lambda \in \Delta(T)$.

Moreover, if $(c, \mathcal{B}, T)$ is rationalizable by an (monotone) additive representation, then it is also rationalizable by an (monotone) additive representation with a finite state space of cardinality less than or equal to T.

At first glance it may seem that the conditions characterizing the strategically rational and monotone additive cases are equivalent. This is not true: the strategically rational case requires that for some fixed $p_1, ..., p_T$ such that $p_t \in c_t$ for all

$t \in T$, $\sum_t \lambda_t p_t \notin \sum_t \lambda_t d_t \; \forall \; \lambda \in \Delta(T)$, whereas in the monotone additive case the p_t's can vary with λ. To illustrate, let $(c, \mathcal{B}, T)$ be such that

$$T = \{1,2\}, c_1 = \{p\}, d_1 = \{q\}, c_2 = \Delta(\{p,q\}) \text{ and } d_2 = \{p\}.$$

This choice problem is rationalizable by a monotone additive representation but not by a strategically rational one: fix any $r \in \Delta(\{p,q\})$, then

$$\begin{aligned} \lambda p + (1-\lambda) r &= \lambda p + (1-\lambda)(\alpha p + (1-\alpha) q) \\ &= (\lambda + (1-\lambda)\alpha) p + (1-\lambda)(1-\alpha) q, \end{aligned}$$

for some $\alpha \in [0,1]$. Thus, if $\lambda = \frac{(1-\alpha)}{2-\alpha}$, then $\lambda p + (1-\lambda) r = \lambda q + (1-\lambda) p$. Hence, $\nexists \; p_1 \in c_1, p_2 \in c_2$ such that $\lambda p_1 + (1-\lambda) p_2 \notin \lambda d_1 + (1-\lambda) d_2$ for all $\lambda \in \Delta(T)$, which is a violation of our condition for strategic rationality.[2]

The main content in Theorem 3.1 is that the indicated conditions are sufficient for rationalizability. In addition, the theorem shows that there is no meaningful empirical difference between representations with infinite state spaces and representations with finite state spaces. Moreover, the number of budgets provides an upper bound on the minimum number of states needed for rationalizability.[3]

As described next, it is straightforward to show that the indicated conditions are necessary for rationalizability. Any preference $\succeq$ over $P(\Delta(X))$ that has a strategically rational, monotone additive or additive representation satisfies the following axiom (by DLR's Theorem 4).[4]

Independence For any $\lambda \in (0,1]$, $A \succeq B$ implies $\lambda A + (1-\lambda) C \succeq \lambda B + (1-\lambda) C$.

If $(c, \mathcal{B}, T)$ is rationalizable by an additive representation, then the preference $\succeq$ over $P(\Delta(X))$ that corresponds to the representation is such that $c_t \succ d_t$ for every $t \in T$. Then, by Independence, $\sum_t \lambda_t c_t \succ \sum_t \lambda_t d_t$ for every $\lambda \in \Delta(T)$. Hence, $\sum_t \lambda_t c_t \neq \sum_t \lambda_t d_t$ for every $\lambda \in \Delta(T)$. Moreover, if the representation is monotone additive, then (by DLR's Theorem 4), $B \subseteq A$ implies $A \succeq B$. Hence, there cannot exist $\lambda \in \Delta(T)$ such that $\sum_t \lambda_t c_t \subseteq \sum_t \lambda_t d_t$. Finally, if the representation is strategically rational, then for any closed and convex menu A there exists a lottery $p \in A$ such that $\{p\} \sim A$. To see this take $p \in \arg\max_{p \in A} U(p)$, then $\{p\} \sim A$. The p_t's that appear in the condition characterizing strategic rationality are the ones prescribed by this property. Hence, Independence implies that $\sum_t \lambda_t p_t \notin \sum_t \lambda_t d_t \; \forall \lambda \in \Delta(T)$.

[2] Any monotone additive representation such that $S = \{s_1, s_2\}$, $\mu(s_1) = \mu(s_2) > 0$ and $U_{s_1}(p) > U_{s_2}(q) > U_{s_2}(p) > U_{s_1}(q)$ rationalizes $(c, \mathcal{B}, T)$.

[3] The argument for the proof of the bound on the state space was inspired by an exchange with Chris Chambers.

[4] A preference is a complete and transitive binary relation.

The conditions that appear in Theorem 3.1 cannot be verified directly as there are uncountably many λ's in $\Delta(T)$ if $|T| > 1$. Moreover, an existential quantifier appears in the condition that characterizes the strategically rational representation. Appendix B shows that for the case in which each menu is either finite or finitely generated, the conditions can be expressed as nonlinear systems of inequalities. Then it discusses numerical methods that can be used to check for existence of a solution.

Finally, the next proposition shows that whenever the number of observations T is less than or equal to the cardinality n of the set of alternatives X, then the bound given by Theorem 3.1 is tight.

Proposition 3.1. *If $n \geq T$, then there exists a rationalizable choice problem $(c, \mathcal{B}, T)$ such that every (monotone) additive representation that rationalizes $(c, \mathcal{B}, T)$ has at least T states.*

To provide intuition about the previous proposition we give a sketch of the proof: Suppose $n \geq T$ and let $(c, \mathcal{B}, T)$ be such that

$$c_t = ch(\{\delta_1, ..., \delta_T\})$$
$$d_t = ch(\{\delta_1, ..., \delta_{t-1}, \delta_{t+1}, ..., \delta_T\})$$

where $ch(.)$ denotes the convex hull and δ_i is the degenerate lottery that gives alternative x_i with probability 1. By Theorem 3.1 $(c, \mathcal{B}, T)$ is rationalizable by an additive representation. The proof of the proposition amounts to showing that every additive representation that rationalizes $(c, \mathcal{B}, T)$ has at least T states. Intuitively, the choices reveal that for each δ_i there is a state s such that $U_s(\delta_i) > U_s(q)$ for all $q \in \Delta(\{\delta_1, ..., \delta_T\}) \backslash \{\delta_i\}$. Hence, any additive representation that rationalizes $(c, \mathcal{B}, T)$ has at least T states.

4 Discussion

Above we assumed that for each $t \in T$, the agent chooses between two menus. For the characterization part of Theorem 3.1, this assumption is without loss of generality as long as each budget has finitely many menus. To see this, consider a choice problem $(c, \mathcal{B}, T)$ such that $T = \{1\}$ and $B_1 = \{c_1, d_{1,1}, d_{1,2}\}$. Let $(c', \mathcal{B}', T')$ be such that $T' = \{1, 2\}$, $B'_1 = \{c_1, d_{1,1}\}$, $B'_2 = \{c_1, d_{1,2}\}$, $c'(1) = c_1$ and $c'(2) = c_1$. Then $(c, \mathcal{B}, T)$ is rationalizable by a strategically rational, monotone additive or additive representation if and only if $(c', \mathcal{B}', T')$ is rationalizable by a strategically rational, monotone additive or additive representation respectively. This kind of "decomposition of choices" can be done for any $(c, \mathcal{B}, T)$ such that each budget contains finitely many menus. Hence, it is enough to consider budgets that contain only two menus. However, the bound on the cardinality of the state space must be modified if binariness is dropped. In general, the bound equals the total number of rejected menus (that is, $\sum_{t\in T}|B_t\backslash c_t|$). For example, if we were to observe a single choice from a budget that contains 26 menus, then the bound equals 25.

The closed and convex assumption is also without loss of generality: consider a choice problem $(c, \mathcal{B}, T)$ such that at least one of the menus is not closed or convex. We can always construct another choice problem $(c', \mathcal{B}', T)$ by replacing each menu with the closure of its convex hull. It follows from Lemma 1 and Lemma 2 in DLR that $(c, \mathcal{B}, T)$ is rationalizable by a strategically rational, monotone additive or additive representation if and only if $(c', \mathcal{B}', T)$ is rationalizable by a strategically rational, monotone additive or additive representation respectively.

Note that Theorem 3.1 implies that if we observe a single choice $c_1 \subset d_1$, then the choice cannot be rationalized by a monotone additive representation. However, we cannot conclude that the agent does not value flexibility. It could be the case that the agent does value flexibility but that the extra lotteries in d_1 do not add any flexibility. Hence, whenever a choice problem does not satisfy the characterizing condition for monotone additive representation in Theorem 3.1 we can only conclude that the choices do not *reveal* a (strict) preference for flexibility. This limitation comes from the requirement that c_t be strictly preferred to d_t.

An agent that behaves as the strategically rational representation prescribes has no subjective uncertainty about her future taste. When evaluating a menu she considers only the best lottery according to her future tastes. Our characterizing condition reflects this taste certainty as the p_t's that appear in the condition are understood to be what the agent anticipates she will choose. For an agent that admits a monotone additive representation this is no longer true. In particular, she is uncertain about her future tastes. This taste uncertainty makes "smaller" menus less desirable because the agent does not desire to commit to any particular lottery. Any behavior that would suggest that the agent would prefer to commit to a smaller menu when a bigger one (in the sense of set inclusion) is available is ruled out by our condition. Finally, an agent that admits an additive representation is also uncertain about her future tastes but considers some of her possible future tastes to be harmful ex ante. These "bad" states are the ones that are assigned a negative weight by the signed measure. Therefore in some situations bigger menus might be less desirable than smaller ones and in other situations they might be more desirable. Thus, there is no consistency requirement between the choices of the agent other than rationality in the sense of weak order and Independence which are completely captured by our condition.

We conclude the discussion of our results by relating our characterizing conditions to DLR's axioms. DLR shows that Independence is the main ingredient of all the representations we consider. Moreover, they show that the only distinction between the monotone additive and the additive representations is that the former satisfies monotonicity.[5] Our condition for the monotone additive representation captures the joint content of monotonicity and Independence, while the condition for the additive representation captures the content of Independence for finite choice data.

4.1 Relation to Fishburn (1975) and Border (1992)

The condition for strategic rationality in Theorem 3.1 is an adaptation of Fishburn's result to menus of lotteries. Fishburn's result can be expressed in our setup as follows.

Let $(c, \mathcal{B}, T)$ be a choice problem such that $B_t = \{\{p_t\}, \{q_t\}\}$ and $c(t) = \{p_t\}$ for all $t \in T$. $(c, \mathcal{B}, T)$ is rationalizable by vNM if there exists an expected utility function $u : \Delta(X) \to \mathbb{R}$ such that for all $t \in T$,

$$u(p_t) > u(q_t).$$

Theorem 4.1. *(Fishburn) Let $(c, \mathcal{B}, T)$ be a choice problem such that $B_t = \{\{p_t\}, \{q_t\}\}$ and $c(t) = \{p_t\}$ for all $t \in T$. Then, $(c, \mathcal{B}, T)$ is rationalizable by vNM if and only if $\sum_t \lambda_t p_t \neq \sum_t \lambda_t q_t \ \forall \lambda \in \Delta(T)$.*

To see the relation to our characterizing condition for strategic rationality, fix a choice problem $(c, \mathcal{B}, T)$ as in Theorem 3.1. Our condition is satisfied if and only if there exist $p_1, ..., p_T$ such that any choice problem $(c', \mathcal{B}', T)$, with

$$B'_t = \{\{p_t\}, \{q_t\}\}, c(t) = \{p_t\} \text{ and } q_t \in d_t \text{ for all } t \in T,$$

satisfies Fishburn's condition. Note that for choice problems such that every menu contains a single lottery, each of the conditions in Theorem 3.1 is equivalent to Fishburn's.

Border (1992) considers an observable choice function over sets of lotteries, where all lotteries are over monetary outcomes. He shows that if the choices are not consistent with expected utility maximization of an increasing vNM utility function, then there is a way to create a compound lottery using the rejected lotteries that stochastically dominates the compound lottery created by using the chosen lotteries. Our result for the monotone additive case is in the same spirit. In particular, Theorem 3.1 implies that if a choice problem is not rationalizable by a monotone additive representation, then we can construct a compound menu using the rejected menus that dominates, in the sense of set inclusion, the compound menu created by using the chosen menus.

[5]A preference $\succeq$ over $P(\Delta(X))$ satisfies monotonicity if $B \subseteq A$ implies $A \succeq B$

4.2 Relation to Kreps (1979)

To understand the relation between our results and Kreps (1979), we define the analogous setup for menus of alternatives.

Definition 4.1. *A* ***Kreps choice problem*** *is a tuple* $(c^K, \mathcal{B}^K, T)$ *such that* $T = \{1, ..., T\}$*,* $\mathcal{B}^K = \{B_t^K | t \in T\}$ *where* $B_t^K \subseteq P(X)$ *and* $c^K : T \to P(X)$ *is such that* $c_t^K \in B_t^K$ *for all* $t \in T$*.*

Generic menus of alternatives will be denoted by A^K and B^K and generic alternatives will be denoted a, b. We proceed to define the representation considered in Kreps (1979) in this setup.[6]

Definition 4.2. *A Kreps choice problem* $(c^K, \mathcal{B}^K, T)$ *is* ***rationalizable*** *by a* ***Kreps additive representation*** *if there exist a non-empty finite set* S*, and a state dependent utility function* $U : S \times X \to \mathbb{R}$ *such that: for all* $t \in T$ *and* $d_t^K \in B_t^K \backslash c_t^K$*,*

$$\sum_{s \in S} \max_{a \in c_t^K} U_s(a) > \sum_{s \in S} \max_{b \in d_t^K} U_s(b).$$

A Kreps choice problem $(c^K, \mathcal{B}^K, T)$ *is* ***rationalizable*** *by a* ***utility function*** *if there exists a function* $U : P(X) \to \mathbb{R}$ *such that: for all* $t \in T$ *and* $d_t^K \in B_t^K \backslash c_t^K$*,*

$$U(c_t^K) > U(d_t^K).$$

Given a Kreps choice problem $(c^K, \mathcal{B}^K, T)$, construct the analogous choice problem $(c, \mathcal{B}, T)$ such that for all $t \in T$,

$$c_t = ch(\{\delta_a | a \in c_t^K\})$$
$$d_t = ch(\{\delta_b | b \in d_t^K\})$$

where δ_a and δ_b are the degenerate lotteries that give a and b with probability one respectively and $ch(.)$ denotes the convex hull. Hence, we can map Kreps data into our setup.

The next proposition shows that Kreps choice problems are rationalizable by a Kreps additive representation if and only if the analogous choice problem in our setup is rationalizable by a monotone additive representation. Moreover, a Kreps choice problem is rationalizable by a utility function if and only if the analogous choice problem in our setup is rationalizable by an additive representation.

Proposition 4.1. *Given a Kreps choice problem* $(c^K, \mathcal{B}^K, T)$ *and its analogue* $(c, \mathcal{B}, T)$*:*

[6] Since we assumed that the set of alternatives is finite, by Theorem 1 in Kreps (1979), it is enough to consider representations with finite state spaces.

1. $(c^K, \mathcal{B}^K, T)$ *is rationalizable by a Kreps additive representation if and only if* $(c, \mathcal{B}, T)$ *is rationalizable by a monotone additive representation.*

2. $(c^K, \mathcal{B}^K, T)$ *is rationalizable by a utility function if and only if* $(c, \mathcal{B}, T)$ *is rationalizable by an additive (not necessarily monotone) representation.*

Part 1 allows us to conclude that our results also provide a characterization of Kreps' representation. Moreover, for any Kreps choice problem, all the menus of lotteries in the budgets of the analogous choice problem in our setup are finitely generated. Hence, the results in Appendix B provide an implementable test for the Kreps additive representation. Part 2 follows from Propositions 1 and 3 in Gorno (2016).

Kreps' Theorem 1 shows that a preference $\succeq^K$ over $P(X)$ has a Kreps additive representation if and only if it satisfies

K-1 $A^K \subseteq B^K$ implies $B^K \succeq^K A^K$

K-2 $A^K \sim^K A^K \cup B^K$ implies that for all D^K, $A^K \cup D^K \sim^K A^K \cup B^K \cup D^K$.

Our condition reflects the first axiom because any violation of **K-1** in $(c^K, \mathcal{B}^K, T)$ would imply a violation of our condition in $(c, \mathcal{B}, T)$. However, our condition does not reflect the second axiom. This is not surprising because, in the menu of lotteries framework, **K-2** is redundant in the presence of monotonicity and Independence. Both of these are reflected in our characterizing condition.

Appendix A: Proofs

Proof of Theorem 3.1

Necessity in each case is provided in the text. Here we prove sufficiency and the assertion about the cardinality of the subjective state space.

Strategic Rationality

Suppose that there exist $p_1, ..., p_T$ such that $p_t \in c_t$ for all $t \in T$ and $\sum_t \lambda_t p_t \notin \sum_t \lambda_t d_t$ for all $\lambda \in \Delta(T)$. Define $C = \{(p - q) | p = \sum_t \lambda_t p_t, q \in \sum_t \lambda_t d_t$ and $\lambda \in \Delta(T)\}$. Then, C is closed, convex and $0 \notin C$.

[Proof that C is convex. Take $(p-q), (p'-q') \in C$ and $\alpha \in (0,1)$. Then, $\alpha p + (1-\alpha)p' = \alpha \sum_t \lambda_t p_t + (1-\alpha)\sum_t \lambda'_t p_t = \sum_r (\alpha\lambda_t + (1-\alpha)\lambda'_t)p_t$. Hence, it suffices to show that $\alpha q + (1-\alpha)q' \in \sum_t (\alpha\lambda_t + (1-\alpha)\lambda'_t)d_t$. By definition of C, $q = \sum_t \lambda_t q_t$ and $q' = \sum_t \lambda'_t q'_t$. WLOG assume that $\nexists\ t \in T$ such that $\lambda_t = \lambda'_t = 0$. Let q''_t be such that $q''_t = \frac{\alpha\lambda_t}{\alpha\lambda_t + (1-\alpha)\lambda'_t} q_t + \frac{(1-\alpha)\lambda'_t}{\alpha\lambda_t + (1-\alpha)\lambda'_t} q'_t$. Then, $q''_t \in d_t$ for all $t \in T$ and $\sum_t (\alpha\lambda_t + (1-\alpha)\lambda'_t)q''_t = \alpha q + (1-\alpha)q'$. Hence, $\alpha q' + (1-\alpha)q' \in \sum_t (\alpha\lambda_t + (1-\alpha)\lambda'_t)d_t$]

By the Hyperplane Separation Theorem, there exists an expected utility function $u : \Delta(X) \to \mathbb{R}$ such that $u(p) > u(q)$ for all $(p - q) \in C$. Hence, for all $\lambda \in \Delta(T)$ and $q \in \sum_t \lambda_t d_t$,

$$\max_{p \in \sum_t \lambda_t c_t} u(p) \geq u(\sum_t \lambda_t p_t) > u(q).$$

Note that if λ is such that $\lambda_t = 1$ for some $t \in T$, then the previous inequality implies that

$$\max_{p \in c_t} u(p) > \max_{q \in d_t} u(q).$$

Hence, $(c, \mathcal{B}, T)$ is rationalizable by a strategically rational representation.

Monotone Additive

Step 1: Find all the candidate expected utility functions.

Suppose that for every $\lambda \in \Delta(T)$, $\sum_t \lambda_t c_t \not\subseteq \sum_t \lambda_t d_t$. Then, for every $\lambda \in \Delta(T)$ there exists $p_\lambda = \sum_t \lambda_t p_{t,\lambda}$ such that $p_{t,\lambda} \in c_t \ \forall t \in T$ and $p_\lambda \in \sum_t \lambda_t c_t \backslash \sum_t \lambda_t d_t$. Define $C_\lambda = \{(p_\lambda - q) | q \in \sum_t \lambda_t d_t\}$. Then, C_λ is closed and convex and $0 \notin C_\lambda$. Hence, by the Hyperplane Separation Theorem, there exists an expected utility function $u_\lambda : \Delta(X) \to \mathbb{R}$ and a real number κ_λ such that $u_\lambda(p_\lambda) > \kappa_\lambda > u_\lambda(q)$ for all $q \in \sum_t \lambda_t d_t$. The set $U_\Lambda = \{u_\lambda | \lambda \in \Delta(T)\}$ contains all the expected utility functions that are candidates for the representation.

Step 2: Choose a finite number of expected utility functions in U_Λ to form the representation.

Fix a $\lambda \in \Delta(T)$ and let $p_\lambda = \sum_t \lambda_t p_{t,\lambda}$ be the associated compund lottery described in step 1. Define $p_{\lambda',\lambda} = \sum_t \lambda'_t p_{t,\lambda}$, then $p_{\lambda',\lambda} \in \sum_t \lambda'_t c_t$.

We are going to show that there exists an $\epsilon_\lambda > 0$ such that for all $\lambda' \in B_{\epsilon_\lambda}(\lambda) \cap \Delta(T)$,

$$u_\lambda(p_{\lambda',\lambda}) > u_\lambda(q) \text{ for all } q \in \textstyle\sum_t \lambda'_t d_t.$$

First note that for any $\lambda' \in \Delta(T)$ close enough to λ, $u_\lambda(p_{\lambda',\lambda}) > \kappa_\lambda$. Define the correspondence $C : \Delta(T) \rightsquigarrow \Delta(X)$ as $C(\lambda') = \sum_t \lambda'_t d_t$. C is a compact-valued and continuous correspondence. By the Maximum Theorem, for any λ' close enough to λ, $\kappa_\lambda > u_\lambda(q)$ for all $q \in \sum_t \lambda'_t d_t$. Hence, there exists $\epsilon_\lambda > 0$ such that for for all $\lambda' \in B_{\epsilon_\lambda}(\lambda) \cap \Delta(T)$, $u_\lambda(p_{\lambda',\lambda}) > u_\lambda(q)$ for all $q \in \sum_t \lambda'_t d_t$.

$\{B_{\epsilon_\lambda} | \lambda \in \Delta(T)\}$ forms an open cover of $\Delta(T)$ which is compact. Hence, there exists a finite sub cover $\Lambda'_1, ..., \Lambda'_k$. Define $\Lambda_i = \Lambda'_i \cap \Delta(T)$. Note that $\cup_i \Lambda_i = \Delta(T)$.

Let u_i be the expected utility function associated with Λ_i' be the compound lottery described in step 1. By construction, for every $\lambda \in \Lambda_i$, $u_i(p_{\lambda,\lambda_i}) > u_i(q)$ for every $q \in \sum_t \lambda_t d_t$.

The u_i's will be the expected utility functions in the additive representation.

Step 3: Obtain the positive measure over $S = \{1, ..., k\}$.

Define

$$u_{t,j}^* = \max_{p \in c_t} u_j(p) \text{ and } u_{t,j} = \max_{q \in d_t} u_j(q)$$

for $t = 1, ..., T$ and $j = 1, ..., k$. According to the expected utility function u_j, $u_{t,j}^*$ is the utility of the best lottery in c_t and $u_{t,j}$ is the utility of the best lottery in d_t.

Let $A =$

$$\begin{bmatrix} u_{11}^* - u_{11} & u_{12}^* - u_{12} & u_{13}^* - u_{13} & \dots & u_{1k}^* - u_{ik} \\ u_{21}^* - u_{21} & u_{22}^* - u_{12} & u_{23}^* - u_{23} & \dots & u_{2k}^* - u_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{T1}^* - u_{T1} & u_{T2}^* - u_{T2} & u_{T3}^* - u_{T3} & \dots & u_{Tk}^* - u_{Tk} \end{bmatrix}$$

We claim that for any $\lambda \in \Delta(T)$, $A^\intercal \lambda$ has a positive entry.[7] To see this note that any $\lambda \in \Delta(T)$ must be in some Λ_i and by step 2, $u_i(p_{\lambda,\lambda_i}) > u(q)$ for all $q \in \sum_t \lambda_t d_t$. Hence,

$$\max_{p \in \sum_t \lambda_t c_t} u_i(p) > \max_{q \in \sum_t \lambda_t d_t} u_i(q),$$

which implies that the i^{th} entry of $A^\intercal \lambda$ is positive.

If there exists a $\mu > 0$ such that $A\mu > 0$, then $(c, \mathcal{B}, T)$ is rationalizable by a monotone additive representation. Ville's Theorem of the Alternative [13, p.36] characterizes the conditions under which such systems admit solutions.

Theorem A.1. *Either*

1. $Ax > 0$, $x > 0$ *has a solution* x,
2. $A^\intercal y \leq 0$, $y \geq 0$, $y \neq 0$ *has a solution* y,

but never both.

It follows from above that $A^\intercal y$ has a positive entry for any $y \geq 0$ and $y \neq 0$. Hence, $(c, \mathcal{B}, T)$ is rationalizable by a monotone additive representation.

[7] $A^\intercal$ is the transpose of A.

Additive

Suppose that for every $\lambda \in \Delta(T)$, $\sum_t \lambda_t c_t \neq \sum_t \lambda_t d_t$. Then, for every $\lambda \in \Delta(T)$, there are two cases to consider.

Case 1: $\sum_t \lambda_t c_t \not\subseteq \sum_t \lambda_t d_t$. Then, there exists $p_\lambda = \sum_t \lambda_t p_{t,\lambda}$ such that $p_{t,\lambda} \in c_t$ for $t \in T$ and $p_\lambda \in \sum_t \lambda_t c_t \backslash \sum_t \lambda_t d_t$. Define $C_\lambda = \{(p_\lambda - q) | q \in \sum_t \lambda_t d_t\}$. C_λ is closed, convex and $0 \notin C_\lambda$. By the Hyperplane Separation Theorem, there exists an expected utility function $u_\lambda : \Delta(X) \to \mathbb{R}$ and a number κ_λ such that $u_\lambda(p_\lambda) > \kappa_\lambda > u_\lambda(q)$ for all $q \in \sum_t \lambda_t d_t$.

Case 2: $\sum_t \lambda_t c_t \subseteq \sum_t \lambda_t d_t$. Then, $\sum_t \lambda_t d_t \not\subseteq \sum_t \lambda_t c_t$. Hence, there exists $q_\lambda = \sum_t \lambda_t q_{t,\lambda}$ such that $q_{t,\lambda} \in d_t$ for all $t \in T$ and $q_\lambda \in \sum_t \lambda_t d_t \backslash \sum_t \lambda_t c_t$. Define $C_\lambda = \{(q_\lambda - p) | p \in \sum_t \lambda_t c_t\}$. C_λ is closed, convex and $0 \notin C_\lambda$. By the Hyperplane Separation Theorem, there exists an expected utility function $u_\lambda : \Delta(X) \to \mathbb{R}$ and a number κ_λ such that $u_\lambda(q_\lambda) > \kappa_\lambda > u_\lambda(p)$ for all $p \in \sum_t \lambda_t c_t$.

By an analogous argument to the one used in the proof of the monotone additive case there exists $\Lambda_1, ..., \Lambda_k \subseteq \Delta(T)$ such that $\cup_i \Lambda_i = \Delta(T)$ and $\lambda \in \Lambda_i$ implies

$$\max_{p \in \sum_t \lambda_t c_t} u_i(p) > \max_{q \in \sum_t \lambda_t d_t} u_i(q)$$

or

$$\max_{q \in \sum_t \lambda_t d_t} u_i(q) > \max_{p \in \sum_t \lambda_t c_t} u_i(p).$$

Define the A matrix as above. We claim that $A^\intercal \lambda$ has at least one non zero entry for every $\lambda \in \Delta(T)$. To see this note that any $\lambda \in \Delta(T)$ must be in some Λ_i. Hence, either

$$\max_{p \in \sum_t \lambda_t c_t} u_i(p) > \max_{q \in \sum_t \lambda_t d_t} u_i(q)$$

or

$$\max_{q \in \sum_t \lambda_t d_t} u_i(q) > \max_{p \in \sum_t \lambda_t c_t} u_i(p).$$

which implies that the i^{th} entry if $A^\intercal \lambda$ is non zero.

If there exists a μ such that $A\mu > 0$, then $(c, \mathcal{B}, T)$ is rationalizable by an additive representation. Gordan's Theorem of the Alternative [13, p.31] characterizes the conditions under which such systems admits solutions.

Theorem A.2. *Either*

1. $Ax > 0$ *has a solution* x,
2. $A^\intercal y = 0$, $y \geq 0$, $y \neq 0$ *has a solution* y,

but never both.

It follows from above that $A^\intercal y$ has a non zero entry for any $y \geq 0$ and $y \neq 0$. Hence, $(c, \mathcal{B}, T)$ is rationalizable by an additive representation.

Bound on the cardinality of the state space

From above we can conclude that if a choice problem $(c, \mathcal{B}, T)$ is rationalizable by an (monotone) additive representation with an infinite state space, then it is also rationalizable by an (monotone) additive representation with a finite state space.

Monotone Additive

Suppose $(c, \mathcal{B}, T)$ is rationalizable by a monotone additive representation with a finite state space of cardinality $K > T$. WLOG assume μ is strictly positive. Define the A matrix as above. We are going to show that there exists a matrix A' with $r \leq T$ columns and a non-negative vector μ' such that each column of A' is a column of A and $A'\mu' > 0$. Note that $A\mu = \sum_{i=1}^{K} \mu_i a_i$ where $a_i \in \mathbb{R}^T$ is the i^{th} column of A. Since $\mu > 0$, then $A\mu \in \{\sum_{i=1}^{K} \lambda_i a_i | \lambda_i > 0 \text{ for every } i\}$. Then, by the cone version of Carathédory's Theorem [16, p.156], $A\mu$ can be written as a non-negative linear combination of T or fewer elements of $\{a_1, ..., a_K\}$. WLOG, assume that the first $r \leq T$ elements of $\{a_1, ..., a_K\}$ are the ones described by the theorem and let $\alpha_1, ..., \alpha_r \geq 0$ be the weights such that $\sum_{i=1}^{r} \alpha_i a_i = A\mu > 0$. To conclude the proof, let $A' = [a_1, ..., a_r]$ and μ' be such that $\mu'_i = \alpha_i$. Hence, $A'\mu' > 0$.

Additive

Suppose $(c, \mathcal{B}, T)$ is rationalizable by an additive representation with a finite state space of cardinality $K > T$. Define the A matrix as above. Then $rank(A) = r \leq T$. Hence, there are r linearly independent columns of A. WLOG assume $a_1, ..., a_r$ are the aforementioned columns. Let $A' = [a_1, ..., a_r]$, we are going to show that there exists μ' such that $A'\mu' > 0$. To see this note that $a_j = \sum_{i=1}^{r} \alpha_i^j a_i$ for every $j \in \{r+1, ..., K\}$ where $\alpha_i^j \in \mathbb{R}$ for every $i = 1, ..., r$ and $j = r+1, ..., K$. Hence,

$$
\begin{aligned}
0 &< A\mu \\
&= \sum_{j=1}^{K} \mu_j a_j \\
&= \sum_{j=1}^{r} \mu_j a_j + \sum_{j=r+1}^{K} \mu_j [\sum_{i=1}^{r} \alpha_i^j a_i] \\
&= \sum_{j=1}^{r} (\mu_j + \sum_{i=r+1}^{K} \mu_i \alpha_j^i) a_j.
\end{aligned}
$$

Let μ' be such that $\mu'_j = \mu_j + \sum_{j=r+1}^{K} \mu_j \alpha_i^j$. Thus, $A'\mu' > 0$.

Proof of Proposition 3.1

Let $(c, \mathcal{B}, T)$ be such that $c_t = ch(\{\delta_1, ..., \delta_T\})$ and $d_t = ch(\{\delta_1, ..., \delta_{t-1}, \delta_{t+1}, ..., \delta_T\})$ where δ_i is the degenerate lottery that gives alternative x_i with probability 1. Then, by Theorem 3.1, $(c, \mathcal{B}, T)$ is rationalizable by an additive representation. Assume, by way of contradiction, that there exists an additive representation that rationalizes $(c, \mathcal{B}, T)$ with a state space of cardinality $K < T$. Assume the (signed) measure associated with the representation is positive (hence the representation is monotone additive). Then, by DLR's Lemma 1, there exist $p_1, ..., p_K$ such that $p_i \in \{\delta_1, ..., \delta_T\}$ and $p_i \in \arg\max_{p \in ch(\{\delta_1,...,\delta_T\})} u_i(p)$. Thus, there exists $t \in T$ such that $p_1, ..., p_K \in d_t$ a contradiction. Hence, it must be the case the (signed) measure corresponding to the representation has at least one negative value.

Note that if the measure assigns negative values to all the states, then we get an immediate contradiction. Hence, there exists at least one s such that $\mu(s) > 0$. WLOG assume the first $r < K$ states are the ones with positive weight. By DLR's Lemma 1 there exist $p_1, ..., p_r$ such that $p_i \in \{\delta_1, ..., \delta_T\}$ and $p_i \in \arg\max_{p \in ch(\{\delta_1,...,\delta_T\})} u_i(p)$. Then, $\{p_1, ..., p_r\} \subseteq d_t$ for some $t \in T$. Hence,

$$\begin{aligned}
\sum_{s \in S} \max_{q \in \{p_1,...,p_r\}} U_s(q)\mu(s) &\geq \sum_{s \in S} \max_{q \in d_t} U_s(q)\mu(s) \\
&\geq \sum_{s \in S} \max_{q \in ch(d_t \cup \{\delta_t\})} U_s(q)\mu(s) \\
&= \sum_{s \in S} \max_{p \in c_t} U_s(p)\mu(s),
\end{aligned}$$

a contradiction.

Proof of Proposition 4.1

Part 2 follows from Propositions 1 and 3 in Gorno (2016). Here we prove Part 1.

Suppose $(c^K, \mathcal{B}^K, T)$ is rationalizable by a Kreps additive representation, then there exists a finite set S and a state dependent utility function $U : S \times X \to \mathbb{R}$ such that: for every $t \in T$ and $d_t^K \in B_t^K \backslash c_t^K$,

$$\sum_{s \in S} \max_{a \in c_t^K} U_s(a) > \sum_{s \in S} \max_{b \in d_t^K} U_s(b).$$

Let $\bar{U} : S \times \Delta(X) \to \mathbb{R}$ be such that $\bar{U}_s(p) = \sum_{x \in X} p(x)U_s(x)$. Then for every $s \in S$, $\bar{U}_s$ is an expected utility function. Let μ be a measure over S such that $\mu(s) = 1$ for every $s \in S$. Hence, $(S, \bar{U}, \mu)$ forms a monotone additive representation, we will show it rationalizes $(c, \mathcal{B}, T)$. To see this, note that any preference over $\succeq$

over $P(\Delta(X))$ that has a monotone additive representation satisfies the following axiom: $A \sim ch(A)$ (DLR's Lemma 1). Hence, for every $s \in S$ and every $t \in T$,

$$\max_{a \in c_t^K} U_s(a) = \max_{\delta_a \in \{\delta_a | a \in c_t^K\}} \bar{U}_s(\delta_a) = \max_{p \in c_t} \bar{U}_s(p)$$

and

$$\max_{b \in d_t^K} U_s(b) = \max_{\delta_b \in \{\delta_b | b \in d_t^K\}} \bar{U}_s(\delta_b) = \max_{q \in d_t} \bar{U}_s(q).$$

Which implies that for every $t \in T$

$$\sum_{s \in S} \max_{a \in c_t^K} U_s(a) > \sum_{s \in S} \max_{b \in d_t^K} U_s(b)$$

if and only if

$$\sum_{s \in S} \max_{p \in c_t} \bar{U}_s(p) > \sum_{s \in S} \max_{q \in d_t} \bar{U}_s(q).$$

Next, suppose $(c, \mathcal{B}, T)$ is rationalizable by a monotone additive representation. Then, by Theorem 3.1, there exist a finite set S, a state dependent utility function $U : S \times \Delta(X) \to \mathbb{R}$ and a positive measure μ over S such that: for every $t \in T$ and $d_t \in B_t \backslash c_t$,

$$\sum_{s \in S} \max_{p \in c_t} U_s(p)\mu(s) > \sum_{s \in S} \max_{q \in d_t} U_s(q)\mu(s).$$

Assume WLOG that $\mu(s) = 1$ for every $s \in S$. Let $\bar{U} : S \times X \to \mathbb{R}$ be such that $U_s(x) = U_s(\delta_x)$. Then, $(S, \bar{U})$ forms a Kreps additive representation, we will show that $(S, \bar{U})$ rationalizes $(c^K, \mathcal{B}^K, T)$. To see this, note that for every $s \in S$,

$$\max_{p \in c_t} U_s(p) = \max_{\delta_a \in \{\delta_a | a \in c_t^K\}} U_s(\delta_a) = \max_{a \in c_t^K} \bar{U}_s(s)$$

and

$$\max_{q \in d_t} U_s(q) = \max_{\delta_b \in \{\delta_b | b \in d_t^K\}} U_s(\delta_b) = \max_{b \in d_t^K} \bar{U}_s(b).$$

Hence, for every $t \in T$,

$$\sum_{s \in S} \max_{p \in c_t} U_s(p) > \sum_{s \in S} \max_{q \in d_t} U_s(q)$$

if and only if

$$\sum_{s \in S} \max_{a \in c_t^K} \bar{U}_s(a) > \sum_{s \in S} \max_{b \in d_t^K} \bar{U}_s(b).$$

Appendix B: Testability

This appendix shows that, for the class of choice problems such that each menu is either finite or finitely generated, the characterizing conditions in Theorem 3.1 can be reformulated as systems of nonlinear inequalities for which the existence of solutions can be verified using existing numerical methods. Hence, the results in this appendix provide a test for the representations which is implementable, at least in principle. We focus on the case in which each menu is the convex hull of finitely many lotteries; one can always replace a finite menu with its convex hull.

Fix a choice problem $(c, \mathcal{B}, T)$ such that for all $t \in T$, $B_t = \{c_t, d_t\}$ and

$$c_t = ch(\{p_{1_t}, ..., p_{m_t^c}\})$$
$$d_t = ch(\{q_{1_t}, ..., q_{m_t^d}\})$$

where $p_{i_t}, q_{j_t} \in \Delta(X)$ for every $i_t \in \{1, ..., m_t^c\}$, $j \in \{1, ..., m_t^d\}$ and $m_t^c, m_t^d \in \mathbb{N}$.

Given T lotteries $p_1, ..., p_T$, let $[p_1, ..., p_T]$ denote the matrix that has lottery p_t in column t. Define $\mathcal{C} = \{C \in \mathbb{R}_{n\times T} | C = [p_1, ..., p_T], p_t \in \{p_{1_t}, ..., p_{m_t^c}\}\ \forall t \in T\}$ and $\mathcal{D} = \{D \in \mathbb{R}_{n\times T} | D = [q_1, ..., q_T], q_t \in \{q_{1_t}, ..., q_{m_t^d}\}\ \forall t \in T\}$. Since $\{p_{1_t}, ..., p_{m_t^c}\}$ and $\{q_{1_t}, ..., q_{m_t^d}\}$ are finite for all $t \in T$, $\mathcal{C}$ and $\mathcal{D}$ are finite sets. Hence

$$\mathcal{C} = \{C_1, ..., C_{k^c}\}$$
$$\mathcal{D} = \{D_1, ..., D_{k^d}\}$$

where $k^c = \prod_{t=1}^T m_t^c$ and $k^d = \prod_{t=1}^T m_t^d$.

Before moving on to the systems of inequalities we state and prove a lemma we use in the proofs of the following results.

Lemma B.1. *For any $\lambda \in \Delta(T)$*

$$\sum_t \lambda_t c_t = ch(\{\sum_t \lambda_t p_t | p_t \in \{p_{1_t}, ..., p_{m_t^c}\} \forall t \in T\})$$
$$\sum_t \lambda_t q_t = ch(\{\sum_t \lambda_t q_t | q_t \in \{q_{1_t}, ..., q_{m_t^d}\} \forall t \in T\})$$

Proof. Follows from the following fact [17, p.95]:

Let $A = ch(\{x_1, ..., x_n\})$ and $B = ch(\{y_1, ..., y_m\})$ be two $\mathbb{R}^n$-polytopes and let $C = A + B$ where the sum is in Minkowski sense. Then

$$C = ch(\{x + y | x \in \{x_1, ..., x_n\}, y \in \{y_1, ..., y_m\}\}).$$

□

Strategically Rational

Proposition B.1. *$(c, \mathcal{B}, T)$ is rationalizable by a strategically rational representation if and only if for some $j \in \{1, ..., k^c\}$ the following system does not have a solution*

$$\sum_{i=1}^{k^d}(D_i\lambda)\alpha_i = C_j\lambda$$

$$\sum_{i=1}^{k^d}\alpha_i = 1,\ \alpha_i \geq 0$$

$$\sum_t \lambda_t = 1,\ \lambda_t \geq 0.$$

Proof. First note that by Lemma 1 in DLR and Theorem 3.1, $(c, \mathcal{B}, T)$ is rationalizable by a strategically rational representation if and only if there exist $p_1, ..., p_T$ such that $p_t \in \{p_{1_t}, ..., p_{m_t^c}\}\ \forall\ t \in T$ and $\sum_t \lambda_t p_t \notin \sum_t \lambda_t d_t\ \forall\ \lambda \in \Delta(T)$.

Hence, by Lemma B.1, $(c, \mathcal{B}, T)$ is rationalizable by a strategically rational representation if and only if there exist $p_1, ...p_T$ such that $p_t \in \{p_{1_t}, ..., p_{m_t^c}\}\ \forall\ t \in T$ and $\sum_t \lambda_t p_t \notin ch(\{\sum_t \lambda q_t | q_t \in \{q_{1_t}, ..., q_{m_t^d}\} \forall t \in T\})\ \forall\ \lambda \in \Delta(T)$.

Note that for any $p_1, ..., p_T$ such that $p_t \in \{p_{1_t}, ..., p_{m_t^c}\}\ \forall t \in T$ there exists a $j \in \{1, ..., k^c\}$ such that $C_j = [p_1, ..., p_T]$ and $\sum_t \lambda_t p_t = C_j\lambda$. Moreover, $ch(\{\sum_t \lambda_t q_t | q_t \in \{q_{1_t}, ..., q_{m_t^d}\} \forall t \in T\}) = ch(\{D_1\lambda, ..., D_{k^d}\lambda\})$. Hence, $(c, \mathcal{B}, T)$ is rationalizable by a strategically rational representation if and only if for some $j \in \{1, ..., k^c\}$, there does not exists a $\lambda \in \Delta(T)$ such that $C_j\lambda \in ch(\{D_1\lambda, ..., D_{k^d}\lambda\})$ which is equivalent to requiring the above system to not have a solution for some $j \in \{1, ..., k^c\}$. □

Monotone Additive

Proposition B.2. *$(c, \mathcal{B}, T)$ is rationalizable by a monotone additive representation if and only if the following system does not have a solution*

$$\sum_{i=1}^{k^d}(D_i\lambda)\alpha_{i,j} = C_j\lambda \text{ for } j = 1, ..., k^c$$

$$\sum_{i=1}^{k^d}\alpha_{i,j} = 1 \text{ for } j = 1, ..., k^c$$

$$\alpha_{i,j} \geq 0 \text{ for } i = 1, ..., k^d \text{ and } j = 1, ..., k^c$$

$$\sum_t \lambda_t = 1,\ \lambda_t \geq 0.$$

Proof. By Lemma B.1, there exists a $\lambda \in \Delta(T)$ such that $\sum_t \lambda_t c_t \subseteq \sum_t \lambda_t d_t$ if and only if $\{\sum_t \lambda_t p_t | p_t \in \{p_{1_t}, ..., p_{m_t^c}\} \forall t \in T\} \subseteq ch(\{\sum_t \lambda_t q_t | q_t \in \{q_{1_t}, ..., q_{m_t^d}\} \forall t \in T\})$. Moreover, $\{\sum_t \lambda_t p_t | p_t \in \{p_{1_t}, ..., p_{m_t^c}\} \forall t \in T\} = \{C_1\lambda, ..., C_{k^c}\lambda\}$ and $ch(\{\sum_t \lambda_t q_t | q_t \in \{q_{1_t}, ..., q_{m_t^d}\} \forall t \in T\}) = ch(\{D_1\lambda, ..., D_{k^d}\lambda\})$. Hence, by Theorem 3.1, $(c, \mathcal{B}, T)$ is rationalizable by a monotone additive representation if and only if there does not exists a $\lambda \in \Delta(T)$ such that $\{C_1\lambda, ..., C_{k^c}\lambda\} \subseteq ch(\{D_1\lambda, ..., D_{k^d}\lambda\})$ which is equivalent to requiring the above system to not have a solution. $\square$

Additive

Proposition B.3. $(c, \mathcal{B}, T)$ *is rationalizable by an additive representation if and only if the following system does not have a solution*

$$\begin{aligned}
\sum_{i=1}^{k^d} (D_i\lambda)\alpha_{i,j}^d &= C_j\lambda \textit{ for } j = 1, ..., k^c \\
\sum_{i=1}^{k^c} (C_i\lambda)\alpha_{i,j}^c &= D_j\lambda \textit{ for } j = 1, ..., k^d \\
\sum_{i=1}^{k^d} \alpha_{i,j}^d &= 1 \textit{ for } j = 1, ..., k^c \\
\sum_{i=1}^{k^c} \alpha_{i,j}^c &= 1 \textit{ for } j = 1, ..., k^d \\
\alpha_{i,j}^d &\geq 0 \textit{ for } i = 1, ..., k^d \textit{ and } j = 1, ..., k^c \\
\alpha_{i,j}^c &\geq 0 \textit{ for } i = 1, ..., k^c \textit{ and } j = 1, ..., k^d \\
\sum_t \lambda_t &= 1, \ \lambda_t \geq 0.
\end{aligned}$$

Proof. By Lemma B.1, there exists a $\lambda \in \Delta(T)$ such that $\sum_t \lambda_t c_t = \sum_t \lambda_t d_t$ if and only if

$$\{\sum_t \lambda_t p_t | p_t \in \{p_{1_t}, ..., p_{m_t^c}\} \forall t \in T\} \subseteq ch(\{\sum_t \lambda_t q_t | q_t \in \{q_{1_t}, ..., q_{m_t^d}\} \forall t \in T\})$$
$$\{\sum_t \lambda_t q_t | q_t \in \{q_{1_t}, ..., q_{m_t^d}\} \forall t \in T\} \subseteq ch(\{\sum_t \lambda_t p_t | p_t \in \{p_{1_t}, ..., p_{m_t^c}\} \forall t \in T\}).$$

Since $\{\sum_t \lambda_t p_t | p_t \in \{p_{1_t}, ..., p_{m_t^c}\} \forall t \in T\} = \{C_1\lambda, ..., C_{k^c}\lambda\}$ and $\{\sum_t \lambda_t q_t | q_t \in \{q_{1_t}, ..., q_{m_t^d}\} \forall t \in T\} = \{D_1\lambda, ..., D_{k^d}\lambda\}$, it is enough to check whether $\{C_1\lambda, ..., C_{k^c}\lambda\} \subseteq ch(\{D_1\lambda, ..., D_{k^d}\lambda\})$ and $\{D_1\lambda, ..., D_{k^d}\lambda\} \subseteq ch(\{C_1\lambda, ..., C_{k^c}\lambda\})$ to know if $\sum_t \lambda_t c_t =$

$\sum_t \lambda_t d_t$. Hence, by Theorem 3.1, $(c, \mathcal{B}, T)$ is rationalizable by an additive representation if and only if there does not exists a $\lambda \in \Delta(T)$ such that $\{C_1\lambda, ..., C_{k^c}\lambda\} \subseteq ch(\{D_1\lambda, ..., D_{k^d}\lambda\})$ and $\{D_1\lambda, ..., D_{k^d}\lambda\} \subseteq ch(\{C_1\lambda, ..., C_{k^c}\lambda\})$ which is equivalent to requiring the above system to not have a solution. $\square$

Solving the Inequalities

In this section we describe the numerical methods that can be used to check if the above systems have solutions. We restrict our attention to the monotone additive case; the strategically rational and additive cases are handled in an analogous way.

Consider the system of nonlinear equations given by Proposition B.2

$$\sum_{i=1}^{k^d} (D_i\lambda)\alpha_{i,j} = C_j\lambda, \quad \sum_{i=1}^{k^d} \alpha_{i,j} = 1, \quad \alpha_{i,j} \geq 0 \text{ for } i = 1, ..., k^d \text{ and } j = 1, ..., k^c$$
$$\sum_t \lambda_t = 1, \qquad \lambda_t \geq 0.$$

Let $F_j : \mathbb{R}^{T+k^c\times k^d} \to \mathbb{R}^n$ be such that $F_j(\lambda, \alpha) = \sum_{i=1}^{k^d} (D_i\lambda)\alpha_{i,j} - C_j\lambda$ for $j = 1, ..., k^c$. Then, F_j is twice continuously differentiable and has analytic derivatives. Let $F : \mathbb{R}^{T+k^c\times k^d} \to \mathbb{R}^{k^c\times n}$ be such that

$$F(\lambda, \alpha) = \begin{bmatrix} F_1(\lambda, \alpha) \\ F_2(\lambda, \alpha) \\ \vdots \\ F_{k^c}(\lambda, \alpha) \end{bmatrix}.$$

Hence, the above system is equivalent to

$$F(\lambda, \alpha) = 0, \quad \sum_{i=1}^{k^d} \alpha_{i,j} = 1, \quad \alpha_{i,j} \geq 0 \text{ for } i = 1, ..., k^d \text{ and } j = 1, ..., k^c$$
$$\sum_t \lambda_t = 1, \qquad \lambda_t \geq 0.$$

Consider the following optimization problem

$$\begin{aligned} \underset{\lambda,\alpha}{\text{minimize}} \quad & \frac{1}{2}\|F(\lambda, \alpha)\|^2 \\ \text{subject to} \quad & \sum_{i=1}^{k^d} \alpha_{i,j} = 1, \quad \alpha_{i,j} \geq 0 \qquad \text{for } i = 1, ..., k^d \text{ and } j = 1, ..., k^c \\ & \sum_t \lambda_t = 1, \qquad \lambda_t \geq 0. \end{aligned}$$

where $\|.\|$ is the euclidean norm. Then, *the system of nonlinear equations has a solution if and only if the global minimum of the optimization problem is equal to 0.* The above optimization problem can be solved using numerical methods like Newton's, augmented Lagrangian function, sequential quadratic programming (SQP), trust-region SQP or the interior point method. For a comprehensive textbook review of relevant numerical methods see Nocedal and Wright (2006). For more recent extensions of these methods, see for example Curtis, Jiang and Robinson (2015) and Osman (2016).

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