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SYSTOLIC GEOMETRY AND SIMPLICIAL COMPLEXITY FOR GROUPS

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ABSTRACT. Twenty years ago Gromov asked about how large is the set of isomorphism classes of groups whose systolic area is bounded from above. This article introduces a new combinatorial invariant for finitely presentable groups called *simplicial complexity* that allows to obtain a quite satisfactory answer to his question. Using this new complexity, we also derive new results on systolic area for groups that specify its topological behaviour.

1. INTRODUCTION

We focus on groups which can be presented as the fundamental group of a finite 2-dimensional simplicial complex. Thus, throughout this article, by *group* we mean *finitely presentable group*, and by *simplicial complex* we mean *finite simplicial complex*.

First recall the definition of systolic area for 2-dimensional simplicial complexes and groups. Let X be a simplicial complex of dimension 2 and suppose that its fundamental group is not trivial. Given a piecewise smooth Riemannian metric h on X the systole denoted by $\text{sys}(X, h)$ is defined as the shortest length of a non-contractible closed curve in X . We call *systolic area* the number

$$\sigma(X) := \inf_h \frac{\text{area}(X, h)}{\text{sys}(X, h)^2}$$

where the infimum is taken over all piecewise smooth Riemannian metrics on X . Following [Gro96, p.337] the *systolic area* of a group G is the number

$$\sigma(G) := \inf_X \sigma(X)$$

where the infimum is taken over all 2-dimensional simplicial complexes X with fundamental group G .

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It is straightforward that systolic area for free groups is zero. For a non-free group G , we have the following universal lower bound:

$$\sigma(G) \geq \frac{\pi}{16}.$$

This bound was proved by Rudyak & Sabourau in [RS08] where the authors also posed the following two fundamental questions: given a non-free group G , is it true that

- (1) $\sigma(G) \geq 2/\pi$?
- (2) $\sigma(G * \mathbb{Z}) = \sigma(G)$?

The first question is motivated by the fact that systolic area for surfaces is known to be minimal for $\mathbb{R}P^2$ whose corresponding fundamental group is \mathbb{Z}_2 and that its value is precisely $2/\pi$ (see [Gro83, 5.2.B] and [Pu52]).

The second question is related to finitude problems for systolic area. It is easy to check that systolic area is subadditive for free products, and that in particular

$$(1.1) \quad \sigma(G * F_n) \leq \sigma(G)$$

where F_n denotes the free group of rank n . In [Gro96, p.337] Gromov raised the following question: given a positive number T how large is the set of isomorphism classes of groups with systolic area at most T ? Because of inequality (1.1) we may hope a finitude result only if we consider groups without \mathbb{Z} as a free factor (or at least with an uniformly bounded number of such factors). In [RS08] the authors proved such a finitude result and give a (non-sharp) upper bound for the cardinality. We give the exact statement of their result in the next paragraph, but first observe that an affirmative answer to the second question above would thus imply that the set of values of the systolic area function lying in a compact interval is always finite.

Let us now make precise the notion of groups without \mathbb{Z} as a free factor. According to [Kur60, §35]—see also [Mas67] for a topological version— for any group G there exist an unique integer n and an unique subgroup H (up to conjugation) such that G decomposes into a free product

$$(1.2) \quad G = H * F_n$$

where H can not be decomposed in its turn like in (1.2) with a positive n . We call this number n the *free index* of G . A group without \mathbb{Z} as a free factor is thus a group of free index zero.

Denote by $\mathcal{G}_\sigma(T)$ the set of isomorphism classes of groups G with free index zero such that $\sigma(G) \leq T$. According to [RS08] it is a finite set whose cardinality

satisfies the upper bound

$$(1.3) \quad |\mathcal{G}_\sigma(T)| \leq A^{T^3}$$

for some explicit constant $A > 1$ and the lower bound

$$(1.4) \quad 2^T \leq |\mathcal{G}_\sigma(T)|$$

for T large enough. In this article we propose an alternative proof of the finitude of $\mathcal{G}_\sigma(T)$ which leads to an improvement of the upper bound (1.3). We also give a lower bound for the cardinality of the subset $\mathcal{A}_\sigma(T) \subset \mathcal{G}_\sigma(T)$ consisting of isomorphism classes of *finite abelian* groups.

Theorem 1.1. *There exist constants B, B' and B'' such that the following inequality holds for every $T \geq 2$*

$$\left[2^{\frac{\pi}{1+2\sqrt{3}}T} \right] \leq |\mathcal{A}_\sigma(T)| \leq |\mathcal{G}_\sigma(T)| \leq B^{T^{1+\frac{B'}{\sqrt{\ln(B''T)}}}}.$$

Here $[x]$ denotes the integral part of a number x .

We thus derive the following answer to Gromov's original question.

Corollary 1.1. *We have*

$$|\mathcal{G}_\sigma(T)| \leq B^{T^{1+\varepsilon}}$$

for any positive ε provided T is large enough. In particular,

$$\lim_{T \rightarrow \infty} \frac{\ln \ln |\mathcal{G}_\sigma(T)|}{\ln T} = 1.$$

Observe that our lower bound in Theorem 1.1 does not improve inequality (1.4), as $\pi/(1+2\sqrt{3}) \simeq 0.7$. But it shows that an exponential asymptotic growth is already realized on the class of finite abelian groups.

The main tool to improve the upper bound of Rudyak & Sabourau is a new combinatorial invariant for groups introduced in section 2. Namely, given a group G , this new invariant called *simplicial complexity* and denoted by $\kappa(G)$ is the minimal number of 2-simplices of a 2-dimensional simplicial complex with fundamental group G (see Definition 2.1). This invariant might be thought as a discrete version of area for groups.

For groups with free index zero the two invariants $\sigma(G)$ and $\kappa(G)$ are closely related. The central result of this article is the following comparison theorem.

Theorem 1.2. *Let G be a group with free index zero. Then*

$$2\pi\sigma(G) \leq \kappa(G) \leq 625(50^2 \cdot \sigma(G))^{1+\frac{2(1+\ln 5)}{\sqrt{\ln(50^2 \cdot \sigma(G))}}}.$$

This theorem is the link between the different results in this paper and will be proven in section 3. It shows that for large values the systolic area σ is a quasi-linear function of the simplicial complexity κ . More precisely, for any positive $\varepsilon > 0$

$$2\pi\sigma(G) \leq \kappa(G) \leq (\sigma(G))^{1+\varepsilon}$$

provided $\kappa(G)$ is large enough. Observe that there is no hope for a linear upper bound as for surface groups it is known that systolic area is strictly sublinear (see [BS94, BPS12]) while the simplicial complexity is linear in terms of the genus (see Example 2 in section 2.1).

Using Theorem 1.2 the problem of estimating $|\mathcal{G}_\sigma(T)|$ transforms into a purely combinatorial problem. For a positive integer T we denote by $\mathcal{G}_\kappa(T)$ the set of isomorphism classes of groups G with free index zero such that $\kappa(G) \leq T$. We also denote by $\mathcal{A}_\kappa(T)$ the subset corresponding to finite abelian groups.

Theorem 1.3. *For any $T \geq 2$*

$$\left[2^{\frac{T-3}{14}}\right] \leq |\mathcal{A}_\kappa(T)| \leq |\mathcal{G}_\kappa(T)| \leq 2^{6T \log_2 T}.$$

The lower bound is proved by estimating the simplicial complexity of \mathbb{Z}_m . For the upper bound we code simplicial complexes with a minimal number of 2-simplices by some special colored graphs and estimate their number, see section 4.

Here is another application of simplicial complexity. In [BPS12] is proved that for any group G

$$\sigma(G) \geq C \frac{b_1(G) + 1}{(\ln(b_1(G) + 2))^2}$$

for some universal constant C where $b_1(G)$ denotes the first real Betti number of G . But this lower bound is inefficient for groups whose first integral homology group has large torsion, such as \mathbb{Z}_m when m is large. In converse simplicial complexity is quite sensitive to torsion elements (see Proposition 2.2), and using correspondence of Theorem 1.2 we are able to prove that for any positive ε

$$\sigma(G) \geq (\ln |\text{Tors } H_1(G, \mathbb{Z})|)^{1-\varepsilon}$$

for groups with large torsion in homology. In Theorems 5.1 and 5.2 we complete the study of $\sigma(G)$ for abelian groups in terms of two parameters: the number of elements in G and the number of its invariant factors (which coincides with its minimal number of generators). Using this estimate we conclude that

$$(1.5) \quad (\log_2 m)^{1-\varepsilon} \leq \sigma(\mathbb{Z}_m) \leq 1.43 \log_2 m$$

for any positive ε provided m is large enough. In comparison, inequality (1.3) implies that $\sigma(\mathbb{Z}_m) \rightarrow \infty$ for large values of m but gives no information about the asymptotic behaviour of this sequence.

In literature can be found two other invariants that measure the complexity of finitely presented groups: the T -invariant of Delzant [Del96] and the c -complexity of Matveev & Pervova [MP01, PP08]. We compare in subsection 2.2 simplicial complexity with both T -invariant and c -complexity.

Delzant's T -invariant is additive for free product, this nice property being the main reason of its introduction. Nevertheless it is not sensitive to 2-torsion which makes it not pertinent for systolic considerations on groups.

The c -complexity was introduced to measure the complexity of 3-manifolds using their fundamental group. Despite the fact that simplicial complexity and c -complexity agree up to some universal constants (see Proposition 2.1), they are of a different kind: simplicial complexity is topological, while c -complexity is algebraical which makes it not suitable for systolic geometry. Indeed our central result is the comparison Theorem 1.2 and its proof reveals a natural connection between systolic area and simplicial complexity. Furthermore we reprove for simplicial complexity some analogs of classical results on c -complexity, like the lower bound in terms of the torsion (Proposition 2.2) and the estimate for abelian groups (Theorem 4.1). Besides the fact that these results give an alternative proof of the corresponding results for c -complexity (compare with [PP08]), they are always more precise than if we simply have used the corresponding results for c -complexity and the linear equivalence with simplicial complexity.

So, besides its own interest as a simple and concrete invariant associated to a group, simplicial complexity is shown here to have natural and strong geometric applications.

In the last section we present some applications of simplicial complexity to systolic geometry of higher dimensional spaces. One key result is the following estimate for the systolic volume \mathfrak{S} of any $(2n + 1)$ -dimensional lens space L_m^{2n+1} with fundamental group \mathbb{Z}_m (see section 6 for definitions).

Theorem 1.4. *There exists positive constants C_n, C'_n and D_n depending only on the dimension n such that for any integer $m \geq 2$*

$$(1.6) \quad C_n (\ln m)^{1 - \frac{C'_n}{\sqrt{\ln \ln m}}} \leq \mathfrak{S}(L_m^{2n+1}) \leq D_n m^n.$$

While the lower bound is of the same kind as that for $\sigma(\mathbb{Z}_m)$, the best known upper bound is thus polynomial of degree n in m . Observe that this degree is half of the degree in the trivial upper bound $\approx m^{2n}$ given by the round metric. In particular round metrics on lens spaces are not systolically extremal for large m . Determining the asymptotic behaviour of $\mathfrak{S}(L_m^{2n+1})$ in terms of both m and n is still an open question.

2. SIMPLICIAL COMPLEXITY

In this section we introduce the definition of simplicial complexity and give some of its basic properties. We then compare this new complexity with the two other standard complexities, namely the T -invariant of Delzant [Del96] and the c -complexity of Matveev & Pervova [MP01]. Next we show a central lower bound for the simplicial complexity in terms of the 1-torsion of the group.

2.1. Definition and examples. Given a finite simplicial complex P we denote by $s_k(P)$ the number of its k -simplices.

Definition 2.1. Let G be a group. We define its *simplicial complexity* $\kappa(G)$ by the following formula:

$$\kappa(G) := \inf_{\pi_1(X)=G} s_2(X),$$

the infimum being taken over all simplicial 2-complexes X with fundamental group G . A 2-complex X is then said *minimal for G* if $\pi_1(X) = G$, $s_2(X) = \kappa(G)$ and each vertex is incident to at least 2 edges. If G is of free index zero the last condition is equivalent to the one that each vertex is incident to a face.

It is important to note that a notion of complexity has already appeared in the context of topology and geometry of 3-manifolds. This notion originally due to S. Matveev relies on a special class of subpolyhedron called spine, and is in general very hard to compute. We refer the reader to the book [Mat07] for a good introduction as well as several applications for this complexity. Also remark that there exist variations of this notion that are defined using pseudo-triangulations or triangulations of 3-manifolds instead of spines (see the recent articles [MPV09, JRT09, JRT11, JRT13]). The articles [Cha15, Cha16] contain several comparison results between these various complexities.

Going back to our definition 2.1, we see that simplicial complexity satisfies the following properties:

1. $\kappa(G) = 0$ if and only if G is a free group.
2. The free product of two groups G_1 and G_2 satisfies

$$(2.1) \quad \kappa(G_1 * G_2) \leq \kappa(G_1) + \kappa(G_2).$$

But simplicial complexity is not additive with respect to free product: if $\kappa(G_1)$ and $\kappa(G_2)$ are both positive, inequality (2.1) can be strengthened by

$$\kappa(G_1 * G_2) < \kappa(G_1) + \kappa(G_2).$$

For this fix two 2-complexes X_1 and X_2 which are minimal for G_1 and G_2 respectively and glue them together by identifying one 2-simplex of X_1 with another 2-simplex of X_2 (the choice of these two 2-simplices being not relevant).

3. For a simplicial complex P , its *simplicial height* $h(P)$ is the total number of its simplices of any dimension. This invariant was introduced in [Gro96] and satisfies

$$h(P) \geq \kappa(\pi_1(P)).$$

Example 1. Even for groups whose structure is simple, the exact value of κ seems hard to compute. For small values up to 17 the following table describes the situation, see [Bul14]. Here K_2 denotes the fundamental group of the Klein bottle,

while the annotation (*) means that the corresponding minimal complex is unique and ? that there might be some others groups with the same simplicial complexity.

$\kappa(G)$	10	14	16	17
G	$\mathbb{Z}_2(*)$	$\mathbb{Z} \oplus \mathbb{Z}(*)$	K_2	$\mathbb{Z}_3, ?$

For instance there might be several groups with simplicial complexity 17 as $17 \leq \kappa(\mathbb{Z}_2 * \mathbb{Z}_2) \leq 18$ according to [Bul14]. The unique minimal complex for \mathbb{Z}_2 is the quotient of the icosahedron by the central symmetry, see Figure 1.

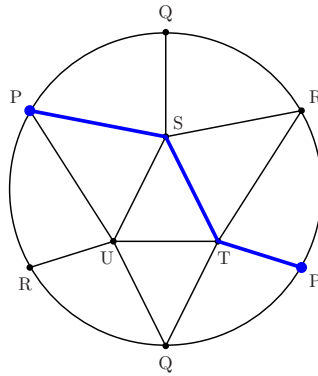


FIGURE 1. Minimal complex for \mathbb{Z}_2 .

For $\mathbb{Z} \oplus \mathbb{Z}$ the minimal complex is also unique and is given by the minimal triangulation of the 2-torus whose fundamental domain is depicted in Figure 2.

These two minimal complexes will be used in the sequel.

Example 2. For surface groups with large genus, the exact computation of their complexity remains an open problem. We can nevertheless give some bounds in terms of their genus.

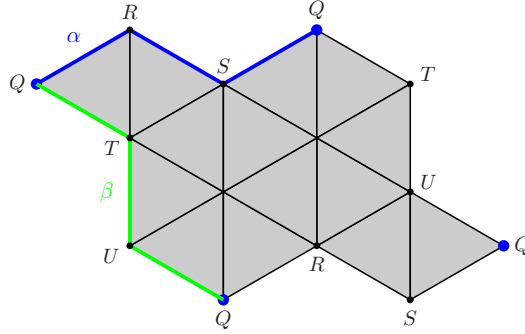
Let $\pi_1(S_l)$ be the fundamental group of an orientable surface of genus $l \geq 1$. By elementary algebraic and combinatorial considerations

$$(2.2) \quad \kappa(\pi_1(S_l)) \geq \frac{4}{3}l.$$

Besides

$$(2.3) \quad \kappa(\pi_1(S_l)) \leq 4(l-1) + 2 \left\lceil \frac{7 + \sqrt{1 + 48l}}{2} \right\rceil$$

by a result of Jungerman & Ringel [JR80]. Here $\lceil a \rceil$ denotes the integer part of $a + 1$ if a is not an integer and a for integers. Strictly speaking, the upper bound

FIGURE 2. Minimal complex for $\mathbb{Z} \oplus \mathbb{Z}$.

is only available for $l \neq 2$. For $l = 2$ we have to replace the upper bound by 24, see [JR80]. Observe that the upper bound (2.3) is sharp for $\pi_1(S_1) = \mathbb{Z} \oplus \mathbb{Z}$. Because $\kappa(\mathbb{Z} \oplus \mathbb{Z}) = 14$ we can easily derive that the free abelian group A_n of rank n satisfies

$$\frac{1}{2}n(n-1) \leq \kappa(A_n) \leq 7n(n-1).$$

Here the lower bound is given by the second Betti number. For the upper bound consider the n -dimensional torus \mathbb{T}^n and the 2-dimensional skeleton of its standard cell decomposition. This 2-skeleton consists of $\frac{n(n-1)}{2}$ tori of dimension 2, all of which we endow with the minimal triangulation shown in Figure 2. These individual triangulations can be arranged in a global triangulation of the 2-skeleton by choosing the blue and green lines in Figure 2 to lie in the 1-skeleton of the standard cell decomposition of \mathbb{T}^n . This gives a 2-complex with fundamental group A_n and a triangulation with exactly $7n(n-1)$ triangles.

Remark that the precise computation for $\kappa(A_n)$ remains open.

Subadditive property (2.1) implies that for any group G ,

$$(2.4) \quad \kappa(G * \mathbb{Z}) \leq \kappa(G).$$

As for systolic area the question to know whether or not this inequality is actually an equality is open and fundamental. Because of (2.4) we will consider in the sequel only groups with free index zero.

If G is a free index zero group and X is a minimal complex for G , it is straightforward to check that

(M_1) any edge of X is adjacent to at least two 2-simplices,

(M_2) any vertex of X is adjacent to at least four 2-simplices.

This properties of minimal complexes will be usefull in the sequel.

2.2. Comparison with other complexities. There exist two other numerical invariants which measure the complexity of groups. First recall that given a presentation

$$\mathcal{P} = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$$

of a group G , its length is the number

$$\ell(\mathcal{P}) = \sum_{i=1}^m |r_i|$$

where $|\cdot|$ denotes the word length associated to the system of generators $\{a_1, \dots, a_n\}$.

The two other types of complexity for groups are the following:

- the c -complexity introduced by Matveev & Pervova [MP01] and defined as the minimal length $c(G)$ of a finite presentation,
- the T -invariant introduced by Delzant [Del96] and defined by

$$T(G) := \min_{\mathcal{P}=\langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle} \sum_{i=1}^m \max\{|r_i| - 2, 0\}.$$

Any group can be presented with relations of length either 2 or 3, such a representation being called *triangular*. The T -invariant is nothing else than the minimal number of relations of length 3 for a triangular representation.

One fundamental property of the T -invariant is its linearity with respect to free products (see [Del96]):

$$T(G_1 * G_2) = T(G_1) + T(G_2).$$

But of course this invariant is not sensitive to torsion elements of order 2, for instance

$$T(\mathbb{Z}_2 * \dots * \mathbb{Z}_2) = 0.$$

In [PP08](see also [KS05]) it is shown that T -invariant and c -complexity satisfy the following relations:

- $T(G) \leq c(G)$,
- $c(G) \leq 9T(G)$ if G does not admit free factor isomorphic to \mathbb{Z}_2 ,
- $c(G) \leq 3T(G)$ if G has no torsion element of order 2.

On the other hand c -complexity and simplicial complexity coincide up to some universal constants:

Proposition 2.1. *For any group G we have*

$$(2.5) \quad \frac{1}{6} (\kappa(G) + h) \leq c(G) \leq 3(\kappa(G) - 4),$$

where h is the minimal number of relations over finite presentations of G .

Proof. To prove the left-hand side we start with an arbitrary finite presentation

$$\mathcal{P} = \langle a_1, \dots, a_n | r_1, \dots, r_m \rangle$$

of G . We can suppose this presentation contains no relation of length 1 and that every relation of length 2 is of the form $r_j = a_s^{\pm 2}$. Associated to \mathcal{P} we construct a simplicial 2-complex X with $\pi_1(X) = G$ and $s_2(X) \leq 6\ell(\mathcal{P}) - m$. Start with $Y = \bigvee_{i=1}^n S_i^1$ the wedge sum of n circles whose base point is denoted by P , each circle corresponding to a generator of the presentation \mathcal{P} . For each $j = 1, \dots, m$ we glue a 2-disk D_j^2 by identifying its boundary with the loop described by the relation r_j . The topological space thus obtained is denoted by X and can be triangulated as follows. First divide each circle into three edges, the base point P corresponding to one of the vertices. If $|r_j| \geq 3$ we triangulate the disk D_j^2 according to the parity of $|r_j|$. If $|r_j| = 2k$, we triangulate D_j^2 as in Figure 3 and get a contribution of $6|r_j| - 2$ triangles (remark the blocks made of nine triangles). If $|r_j| = 2k + 1$, we triangulate D_j^2 as in Figure 4 and get a contribution of $6|r_j| - 1$ triangles.

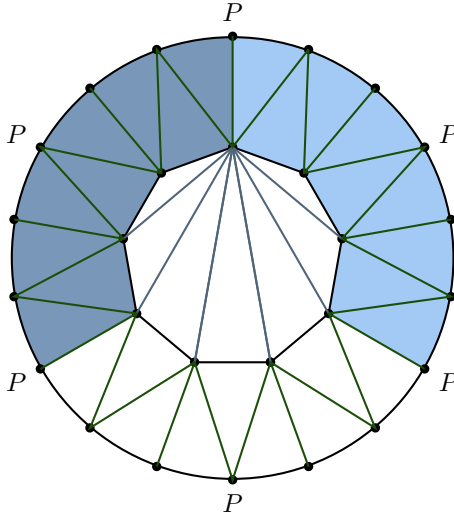


FIGURE 3. Triangulation of D_j^2 for $|r_j|$ even ($|r_j| = 6$).

If $|r_j| = 2$ we triangulate the corresponding projective plane like in Figure 1 and get a contribution of $10 = 5|r_j|$ simplices of dimension 2.

Finally the triangulation of X thus defined satisfies

$$s_2(X) \leq 6 \sum_{j=1}^m |r_j| - m \leq 6\ell(\mathcal{P}) - m.$$

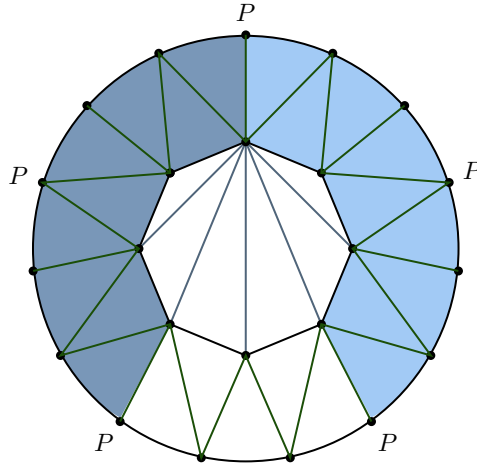


FIGURE 4. Triangulation of D_j^2 for $|r_j|$ odd ($|r_j| = 5$).

Because we start with any finite presentation \mathcal{P} we conclude that $\kappa(G) + h \leq 6c(G)$.

The right-hand side of the inequality is proved as follows. Consider a simplicial complex X minimal for G and having a minimal number of edges. Fix a maximal tree of the 1-dimensional skeleton of X with root p such that each edge incident to p belongs to the tree. We first contract every 2-simplex incident to p and then the remaining part of the maximal tree into a point. This gives a finite cell complex \tilde{X} of dimension 2 whose 2-cells are glued along at most 3 cells of dimension 1. Because X is minimal, the root p was incident to at least four 2-simplices, so the number of 2-cells of \tilde{X} is at most $s_2(X) - 4$. This topological space being homotopy equivalent to X we get a presentation of $\pi_1(\tilde{X}) = G$ whose length is at most $3(s_2(X) - 4)$. \square

Remark 2.1. If G has no torsion element of order 2, the proof implies the following inequality:

$$\kappa(G) \leq 5c(G) + T(G) \leq 13T(G).$$

2.3. Lower bound in terms of 1-torsion. The simplicial complexity $\kappa(G)$ is quite sensitive to the number of torsion elements in $H_1(G, \mathbb{Z})$. The next proposition will be used several times in the sequel.

Proposition 2.2. *Let X be a simplicial complex of dimension 2. Then*

$$s_2(X) \geq 2 \log_3 |\text{Tors } H_1(X, \mathbb{Z})|.$$

In particular, every group G satisfies the inequality

$$\kappa(G) \geq 2 \log_3 |\text{Tors } H_1(G, \mathbb{Z})|.$$

Proof. Consider the complex of simplicial cochains

$$(2.6) \quad C^1(X, \mathbb{Z}) \xrightarrow{d^1} C^2(X, \mathbb{Z}) \xrightarrow{d^2} 0.$$

The universal coefficient theorem implies a duality between homology torsion and cohomology torsion, and we have (see [Hat02, Corollary 3.3] for instance)

$$(2.7) \quad \text{Tors } H_1(X, \mathbb{Z}) \simeq \text{Tors } H^2(X, \mathbb{Z}).$$

This implies that $|\text{Tors } H_1(X, \mathbb{Z})| = |\text{Tors}(C^2(X, \mathbb{Z})/\text{Im } d^1)|$.

We endow $C^i(X, \mathbb{Z})$ with the basis dual to the simplicial basis of $C_i(X, \mathbb{Z})$. Let D denote the matrix of d^1 with respect to these bases. The matrix D has $s_2(X)$ rows, and each row has exactly three non zero elements whose value is either 1 or -1 . It follows that every row vector of D has Euclidean length $\sqrt{3}$. If we interpret the determinant of a square matrix V of order k as the volume of the parallelotope generated by its row vectors, we see that for any square submatrix V of D of order k

$$|\det V| \leq (\sqrt{3})^k.$$

Now if we denote by $t(D)$ the greatest common divisor of all minors of D of order $\text{rank}(D)$, we deduce that

$$t(D) \leq (\sqrt{3})^{\text{rank}(D)} \leq (\sqrt{3})^{s_2(X)}.$$

To conclude just remember that

$$t(D) = |\text{Tors}(\text{Coker } d^1)| = |\text{Tors}(C^2(X, \mathbb{Z})/\text{Im } d^1)|.$$

□

Remark 2.2. It is proved in [PP08] that for any group G

$$c(G) \geq 3 \log_3 |\text{Tors } H_1(G, \mathbb{Z})|.$$

This combined with the right-hand side of inequality (2.5) leads to a weaker estimate of $\kappa(G)$ than the one obtained in Proposition 2.2.

2.4. Stabilization for free products. Given a group G we denote by

$$G_{(n)} = \underbrace{G * \dots * G}_n$$

the free product of n copies of G . As the function $\kappa(G_{(n)})$ is sublinear in n according to inequality (2.1), we can define the *stable simplicial complexity* by

$$\kappa_\infty(G) := \lim_{n \rightarrow \infty} \frac{\kappa(G_{(n)})}{n}.$$

In this section we show that $\kappa_\infty(G) > 0$ for any group G which is not free. That is, albeit simplicial complexity is not additive, its asymptotic behaviour for free products of the same group is essentially linear. The analogous question for systolic area is completely open.

Proposition 2.3. *Any group G which is not free satisfies*

$$\kappa(G) - 1 \geq \kappa_\infty(G) \geq \begin{cases} 2 \log_3 2 & \text{if } G \text{ decomposes as } G = G' * \mathbb{Z}_2, \\ \frac{T(G)}{3} & \text{if } G \text{ does not admit such a decomposition.} \end{cases}$$

Proof. We start with a minimal complex X for G . Choose a triangle $\Delta \subset X$ and consider the complex

$$Y_n = \underbrace{X \cup_{\Delta} \dots \cup_{\Delta} X}_n.$$

We get

$$\kappa(G_{(n)}) \leq s_2(Y_n) = n(s_2(X) - 1) + 1 = n(\kappa(G) - 1) + 1$$

which implies the upper bound. Now if $G = G' * \mathbb{Z}_2$ then

$$|\text{Tors}H_1(G_{(n)}, \mathbb{Z})| \geq |\text{Tors}H_1((\mathbb{Z}_2)_{(n)}, \mathbb{Z})| = 2^n.$$

Using Proposition 2.2 we derive

$$\kappa_\infty(G) \geq 2 \log_3 2.$$

If G does not decompose as $G' * \mathbb{Z}_2$, because $T(G) \leq c(G) \leq 3(\kappa(G) - 4)$ we get the result as T -invariant is additive for free products. \square

3. SIMPLICIAL COMPLEXITY AND SYSTOLIC AREA

This section is completely devoted to the proof of Theorem 1.2 given in the introduction.

For the left-hand side inequality we consider a minimal simplicial complex X of dimension 2 with fundamental group G . Endow X with the metric h such that any edge is of length $\frac{2\pi}{3}$ and any face is the round hemisphere of radius 1. Because X is minimal, $s_2(X) = \kappa(G)$ and so

$$\text{area}(X, h) = 2\pi\kappa(G).$$

The definition of the metric h implies that any systolic geodesic can be homotoped to the 1-skeleton without increasing its length. Such a curve passes through at least three edges and thus $\text{sys}(X, h) \geq 2\pi$. This implies that

$$\sigma(G) \leq \sigma(X, h) \leq \frac{\kappa(G)}{2\pi}.$$

To prove the right-hand side inequality we use several ideas introduced in [Gro83, 5.3.B]. Set

$$(3.1) \quad \alpha = 25 \exp\left(\sqrt{\ln(50^2 \cdot \sigma(G))}\right).$$

By [RS08, Theorem 3.5],

$$\sigma(G) \geq \frac{1}{4}$$

as G has free index zero. The real number α is thus well defined and satisfies $\alpha > 25$. Fix some positive $\varepsilon < \frac{1}{25}$ small enough such that

$$(3.2) \quad \log_5 \frac{1}{25\varepsilon} \cdot \ln \frac{\alpha}{25} \geq \ln(50^2 \cdot \kappa(G)).$$

According to [RS08, Theorem 3.5 and Lemma 4.2] there exists a simplicial complex P of dimension 2 endowed with a metric g such that

- $\pi_1(P) = G$;
- $\text{sys}(P, g) = 1$;
- $\sigma(P, g) = \text{area}(P, g) < \sigma(G) + \varepsilon$;
- any ball $B(p, R) \subset P$ of radius $R \in [\varepsilon, \frac{1}{2}]$ centered at any point $p \in P$ satisfies the inequality

$$|B(p, R)| \geq \frac{1}{4}R^2.$$

Here $|B(p, R)|$ denote the area of the metric ball $B(p, R)$ for the metric g .

Suppose first that there exists some point p in P such that

$$|B(p, 5R)| \geq \alpha |B(p, R)|$$

for all $R \in [\varepsilon, \frac{1}{25}]$ (the reasons for considering those special values will become clear in the proof of Lemma 3.2 and the final calculation). Then, if r denotes the unique integer such that

$$\frac{1}{5^{r+2}} \leq \varepsilon < \frac{1}{5^{r+1}},$$

we can compute that

$$|B(p, 1/5)| \geq \alpha^r |B(p, 1/5^{r+1})| \geq \frac{1}{4} \alpha^r \varepsilon^2 \geq \frac{1}{50^2} \left(\frac{\alpha}{25}\right)^r \geq \frac{1}{50^2} \left(\frac{\alpha}{25}\right)^{\log_5 \frac{1}{25\varepsilon}}.$$

Thus

$$\sigma(P, g) = \text{area}(P, g) \geq |B(p, 1/5)| \geq \kappa(G)$$

according to the inequality (3.2) and the result is proved in this case.

Now suppose that for all p in P

$$|B(p, 5R)| < \alpha |B(p, R)|$$

for some $R \in [\varepsilon, \frac{1}{25}]$ and let R_p denote the supremum of such radii. In particular

$$|B(p, 5R_p)| \leq \alpha \cdot |B(p, R_p)|,$$

and for all $R \in]R_p, \frac{1}{25}]$

$$|B(p, 5R)| \geq \alpha \cdot |B(p, R)|.$$

Remark that if $R_p < 1/25$ then

$$|B(p, 5R_p)| = \alpha \cdot |B(p, R_p)|.$$

Such a ball $B(p, R_p)$ is said to be α -admissible according to Gromov's terminology (see [Gro83, Theorem 5.3.B]) and has area bounded from below as follows.

Lemma 3.1.

$$|B(p, R_p)| \geq A := \frac{1}{50^2} \left(\frac{1}{25} \right)^{\frac{\ln 50^2 (\sigma(G) + \varepsilon)}{\ln(\alpha/25)}}.$$

Proof of the lemma. Let r be the unique integer such that

$$\frac{1}{5^{r+2}} < R_p \leq \frac{1}{5^{r+1}}.$$

We have

$$\sigma(P, g) = \text{area}(P, g) \geq |B(p, 1/5)| \geq \alpha^r |B(p, R_p)| \geq \frac{1}{4} \alpha^r R_p^2 \geq \frac{1}{50^2} \left(\frac{\alpha}{25} \right)^r,$$

and so

$$r \leq \frac{\ln 50^2 (\sigma(G) + \varepsilon)}{\ln \frac{\alpha}{25}}.$$

This implies

$$|B(p, R_p)| \geq \frac{1}{4} R_p^2 \geq \frac{1}{50^2} \left(\frac{1}{25} \right)^r \geq \frac{1}{50^2} \left(\frac{1}{25} \right)^{\frac{\ln 50^2 (\sigma(G) + \varepsilon)}{\ln(\alpha/25)}}.$$

□

Using this family of α -admissible balls, we now construct a cover of (P, g) whose nerve has G as fundamental group and whose number of 2-simplices is bounded from above by the systolic area of (P, g) . This will lead to the desired upper bound on the simplicial complexity of G in terms of its systolic area.

If $B := B(p, R)$ denotes some metric ball centered at p and of radius R , we denote by nB the concentric ball $B(p, nR)$ for any positive integer n . First choose an α -admissible ball $B_1 := B(p_1, R_1)$ with $R_1 := \max\{R_p \mid p \in P\}$. At each step $i \geq 2$, we construct B_i using the data of $\{B_j\}_{j < i}$ as follows. Let R_i be the maximal radius of an α -admissible ball centered at a point in the complement of the union of the balls $\{2B_j\}_{j < i}$ and let $B_i := B(p_i, R_i)$ be such an α -admissible ball. By construction, B_i is disjoint from the other balls B_j as $R_i \leq R_j$. The process ends in a finite N number of steps when the balls $\{2B_i\}_{i=1}^N$ cover P , as $R_i \geq \varepsilon$ for every $i = 1, \dots, N$.

Consider the corresponding nerve \mathcal{N} of this cover. In general, if X is a paracompact topological space and \mathcal{U} a locally finite cover of X , there exists a canonical

map Φ from X to the nerve $\mathcal{N}(\mathcal{U})$ of the cover \mathcal{U} defined as follows. If $\{\phi_V\}_{V \in \mathcal{U}}$ denotes a partition of unity associated with \mathcal{U} ,

$$\begin{aligned} \Phi : X &\rightarrow \mathcal{N}(\mathcal{U}) \\ x &\mapsto \sum_{V \in \mathcal{U}} \phi_V(x)V. \end{aligned}$$

This map is uniquely defined up to homotopy. In our case, $\mathcal{U} = \{2B_i\}_{i=1}^N$ and Φ associates the center of such a ball to the corresponding vertex of \mathcal{N} .

Lemma 3.2. *The map $\Phi : P \rightarrow \mathcal{N}$ induces an isomorphism of fundamental groups.*

Proof of the lemma. Denote by $\mathcal{N}^{(k)}$ the k -skeleton of \mathcal{N} . We will construct a map $\Psi : \mathcal{N}^{(2)} \rightarrow X$ such that the induced map

$$\Psi_{\#} : \pi_1(\mathcal{N}) \simeq \pi_1(\mathcal{N}^{(2)}) \rightarrow \pi_1(P)$$

is the inverse of $\Phi_{\#} : \pi_1(P) \rightarrow \pi_1(\mathcal{N})$.

Let define $R_0 := 1/25$ and recall that for all $i = 1, \dots, N$ we have $R_i \leq R_0$. We have denoted the set of centers of the balls of the covering \mathcal{U} by $\{p_i\}_{i=1}^N$ and set $v_i = \Phi(p_i)$. We first define Ψ on $\mathcal{N}^{(0)}$ by

$$\Psi(v_i) = p_i.$$

If two vertices v_i and v_j are connected by an edge $[v_i, v_j]$, we join p_i and p_j in P by any minimizing geodesic denoted by $\gamma_{i,j}$. The map Ψ is then defined on the edge $[v_i, v_j]$ to the arc $\gamma_{i,j}$ in the obvious way

$$\Psi : [v_i, v_j] \longrightarrow \gamma_{i,j}.$$

This defines Ψ on the 1-skeleton $\mathcal{N}^{(1)}$. Remark that the curve $\gamma_{i,j}$ has length less than $4 \cdot R_0$ (v_i and v_j are connected by an edge if and only if $2B_i \cap 2B_j \neq \emptyset$).

Next we consider a 2-simplex $\tau = [v_i, v_j, v_k]$ of \mathcal{N} . The concatenation $\gamma_{i,j} \star \gamma_{j,k} \star \gamma_{k,i}$ is a closed curve of P of length less than $12 \cdot R_0 < 1$. So it is contractible and any contraction of this curve into a point gives rise to an extension of the map Ψ to τ . We get this way a map

$$\Psi : \mathcal{N}^{(2)} \rightarrow P.$$

Observe that the restriction of Ψ to $\mathcal{N}^{(1)}$ is unique up to homotopy.

By construction, $\Phi(p_i) = v_i$ for any $i = 1, \dots, N$, and if $[v_i, v_j]$ denotes an edge of \mathcal{N} and p belongs to the corresponding geodesic $\gamma_{i,j}$, $\Phi(p) \in \text{St}([v_i, v_j])$ where $\text{St}([v_i, v_j])$ denotes the star of $[v_i, v_j]$. This implies that $\Phi \circ \Psi : [v_i, v_j] \longrightarrow \mathcal{N}$ is homotopically equivalent to the identity relatively to $\{v_i, v_j\}$. So $\Phi \circ \Psi : \mathcal{N}^{(1)} \longrightarrow \mathcal{N}$ is homotopically equivalent to the identity relatively to $\mathcal{N}^{(0)}$. From this, we get that the induced morphism $\Phi_{\#} \circ \Psi_{\#} : \pi_1(\mathcal{N}) \rightarrow \pi_1(\mathcal{N})$ is the identity and so $\Psi_{\#} : \pi_1(\mathcal{N}) \rightarrow \pi_1(P)$ is injective.

It remains to prove that Ψ_{\sharp} is onto. Consider a geodesic loop δ based at the center p_1 of the ball B_1 and whose length is minimal in its own homotopy class. We complete p_1 into a finite family $\{p_{i_j}\}_{j \in \mathbb{Z}_n}$ of points of P such that

- each p_{i_j} is the center of some ball B_{i_j} of \mathcal{U} ;
- the family $\{2B_{i_j}\}_{j \in \mathbb{Z}_n}$ covers δ ;
- $2B_{i_j} \cap 2B_{i_{j+1}} \neq \emptyset$.

For each $j \in \mathbb{Z}_n$, fix any point $x_j \in 2B_{i_j} \cap \delta$ and denote by δ_j the part of the loop δ joining x_j and x_{j+1} and contained in $2B_{i_j} \cup 2B_{i_{j+1}}$. By construction, the curve δ_j has length less than $8 \cdot R_0$. Fix a minimizing geodesic β_j joining p_{i_j} and x_j . The concatenation

$$\gamma_{i_j, i_{j+1}} \star \beta_{j+1} \star (\delta_j)^{-1} \star (\beta_j)^{-1}$$

is a closed curve of length less than $24 \cdot R_0 < 1$ and thus contractible. So δ is homotopic to $\gamma_{1,2} \star \gamma_{2,3} \star \dots \star \gamma_{n,1}$ with based point p_1 fixed. This proves the surjectivity of Ψ_{\sharp} and completes the proof. \square

So $\pi_1(\mathcal{N}) \simeq G$, and we deduce the lower bound

$$s_2(\mathcal{N}) \geq \kappa(G).$$

We now estimate the number $s_2(\mathcal{N})$ by the systolic area of (P, g) .

Recall that B_1, \dots, B_N denote pairwise disjoint α -admissible balls of radii R_1, \dots, R_N satisfying $R_i \geq R_{i+1}$. For $i = 1, \dots, N-1$ denote by I_i the set of $j > i$ such that $2B_i \cap 2B_j \neq \emptyset$, and observe that if $j \in I_i$ then $B_j \subset 5B_i$. Then

$$\begin{aligned} \sigma(P, g) = \text{area}(P, g) &\geq \sum_{i=1}^N |B_i| \geq \alpha^{-1} \sum_{i=1}^N |5B_i| \\ &\geq \alpha^{-1} \sum_{i=1}^N \sum_{j \in I_i} |B_j| \\ &\geq \alpha^{-2} \sum_{i=1}^N \sum_{j \in I_i} |5B_j| \\ &\geq \alpha^{-2} \sum_{i=1}^N \sum_{j \in I_i} \sum_{k \in I_j} |B_k| \\ &\geq \alpha^{-2} \sum_{i=1}^N \sum_{j \in I_i} \sum_{k \in I_j} A. \end{aligned}$$

But the set of $i < j < k$ such that $2B_i \cap 2B_j \neq \emptyset$ and $2B_j \cap 2B_k \neq \emptyset$ includes the set of $i < j < k$ such that $2B_i \cap 2B_j \cap 2B_k \neq \emptyset$, thus its cardinal is at least equal to $s_2(\mathcal{N})$, and we deduce

$$s_2(\mathcal{N}) \leq \alpha^2 \cdot \frac{\sigma(P, g)}{A}.$$

So

$$\kappa(G) \leq \alpha^2 \cdot \frac{(\sigma(G) + \varepsilon)}{\frac{1}{50^2} \left(\frac{1}{25}\right)^{\frac{\ln 50^2 (\sigma(G) + \varepsilon)}{\ln(\alpha/25)}}} \leq \alpha^2 \cdot 50^2 \cdot 25^{\frac{\ln 50^2 (\sigma(G) + \varepsilon)}{\ln(\alpha/25)}} \cdot (\sigma(G) + \varepsilon).$$

From the equality

$$\ln \frac{\alpha}{25} = \sqrt{\ln(50^2 \cdot \sigma(G))},$$

we then compute that

$$\kappa(G) \leq 25^2 \cdot e^{2\sqrt{\ln(50^2 \cdot \sigma(G))}} \cdot e^{\ln 25 \cdot \frac{\ln 50^2 (\sigma(G) + \varepsilon)}{\sqrt{\ln(50^2 \cdot \sigma(G))}}} \cdot 50^2 (\sigma(G) + \varepsilon).$$

This finally implies the result by letting $\varepsilon \rightarrow 0$.

4. FINITUDE RESULTS FOR SIMPLICIAL COMPLEXITY

We focus in this section on finitude problems for the invariant κ : how to estimate the (obviously finite) number of isomorphism classes of groups whose simplicial complexity is at most T ? Recall that given a positive integer T the set $\mathcal{G}_\kappa(T)$ is defined as the isomorphism classes of groups G with free index zero such that $\kappa(G) \leq T$ while $\mathcal{A}_\kappa(T)$ denotes the subset corresponding to finite abelian groups.

4.1. Upper bound for $|\mathcal{G}_\kappa(T)|$. We start with the proof of the upper bound contained in Theorem 1.3, namely

$$|\mathcal{G}_\kappa(T)| \leq 2^{6T \log_2 T}.$$

Start with any simplicial complex X of dimension 2 minimal for G . Because G is of free index zero and the simplicial complex X is minimal recall that

(M_1) : any edge of X is adjacent to at least two 2-simplices,

(M_2) : any vertex of X is adjacent at least to four 2-simplices.

We derive from (M_1) and from (M_2) the following upper bounds on the number of 0- and 1-simplices:

$$s_0(X) \leq \frac{3T}{4} \text{ and } s_1(X) \leq \frac{3T}{2}.$$

Consider the barycentric subdivision $\text{sd}(X)$ of X and color the vertices as follows.

- The original vertices of X are colored in black,
- barycenters of edges of X are colored in green,
- barycenters of faces of X are colored in red.

Consider the 1-skeleton of $\text{sd}(X)$ and erase the edges joining red and black vertices (see Figure 5). We denote by b , g and r the number of respectively black, green and red vertices. The 1-dimensional simplicial complex thus obtained is a 3-colored graph which satisfies the following properties:

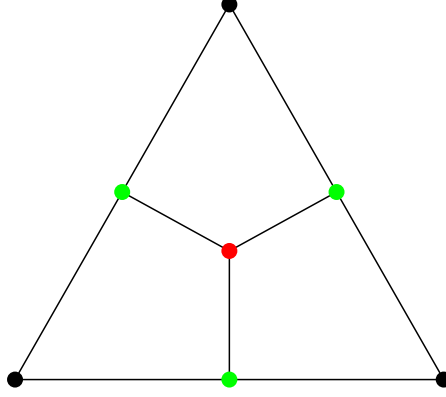


FIGURE 5. The 3-colored graph associated to a face.

$$(P_1) : b \leq \frac{3T}{4}, g \leq \frac{3T}{2} \text{ and } r \leq T,$$

(P_2) : any green vertex is adjacent to exactly two black vertices,

(P_3) : any red vertex is adjacent to exactly three green vertices,

(P_4) : no pair of red and black vertices are adjacent.

It is straightforward to check that to any minimal complex of a group with free index zero and simplicial complexity at most T corresponds a unique 3-colored graph satisfying properties (P_1) to (P_4). Observe that there exist 3-colored graphs satisfying properties (P_1) to (P_4) which does not correspond to any minimal complex, and even does not correspond to any simplicial complex.

The number of 3-colored graphs satisfying properties (P_1) to (P_4) can be estimated using their incidence matrix which has the following form

$$\begin{pmatrix} 0 & A^t & 0 \\ A & 0 & B^t \\ 0 & B & 0 \end{pmatrix}$$

where A is a $g \times b$ matrix with each row containing exactly two non zero coefficients equal to 1, and B is a $r \times g$ matrix for which each row contains exactly three non zero coefficient equal to 1. Thus the number of such matrices A is at most

$$\left(\frac{b(b-1)}{2} \right)^g$$

and the number of such matrices B is at most

$$\left(\frac{g(g-1)(g-2)}{6} \right)^r.$$

From this we compute that the number of 3-colored graphs satisfying properties (P_1) to (P_4) is at most

$$\frac{9T^3}{8} \cdot \left(\frac{1}{2} \cdot \frac{3T}{4} \cdot \left(\frac{3T}{4} - 1 \right) \right)^{\frac{3T}{2}} \cdot \left(\frac{1}{6} \cdot \frac{3T}{2} \cdot \left(\frac{3T}{2} - 1 \right) \cdot \left(\frac{3T}{2} - 2 \right) \right)^T$$

which is less than T^{6T} for $T \geq 2$. This concludes the proof.

4.2. Simplicial complexity for finite abelian groups. In this subsection we shall see that the subset $\mathcal{A}_\kappa(T)$ of *finite abelian* groups with simplicial complexity bounded by T is already large.

Recall that a finite abelian group G decomposes in a direct sum

$$G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_s}$$

where $n_1 | n_2 | \dots | n_s$. These numbers are called the *invariant factors* of G and are uniquely defined by the group. This decomposition can be used to estimate the simplicial complexity of G as follows.

Theorem 4.1. *Any finite abelian group G satisfies the double inequality*

$$2 \log_3 |G| \leq \kappa(G) \leq 14 \log_2 |G| + 7s^2 - 4s,$$

where s is the number of invariant factors of G .

Remark 4.1. In particular

$$\kappa(G) \leq 7(\log_2 |G|)^2 + 10 \log_2 |G|$$

for any finite abelian group. The order of this upper bound is asymptotically sharp as shown by the following example. If

$$G = \bigoplus_{i=1}^s (\mathbb{Z}_2)_i,$$

we see that $|G| = 2^s$ and $H_2(G, \mathbb{Z}_2)$ is a \mathbb{Z}_2 -vector space of dimension $\frac{s(s+1)}{2}$. So

$$\kappa(G) \geq \frac{s(s+1)}{2} \geq \frac{1}{2}(\log_2 |G|)^2.$$

From Theorem 4.1 we directly derive that for any $m \geq 2$

$$2 \log_3 m \leq \kappa(\mathbb{Z}_m) \leq 14 \log_2 m + 3.$$

In particular we get the lower bound announced in Theorem 1.3:

Corollary 4.1. *For any positive T*

$$|\mathcal{A}_\kappa(T)| \geq \left\lceil 2^{\frac{T-3}{14}} \right\rceil$$

where $[x]$ denotes the integral part of the number x .

The rest of the subsection is devoted to the proof of Theorem 4.1.

Proof of Theorem 4.1. From Proposition 2.2

$$\kappa(G) \geq 2 \log_3 |\text{tors}H_1(G, \mathbb{Z})|$$

which gives the left-hand side inequality of Theorem 4.1 as $|\text{tors}H_1(G, \mathbb{Z})| = |G|$ for a finite abelian group.

The proof of the right-hand side inequality of Theorem 4.1 relies on the following estimate for the simplicial complexity of \mathbb{Z}_m :

Lemma 4.1.

$$\kappa(\mathbb{Z}_m) \leq 14 \log_2 m + 3.$$

Proof of the Lemma. Start with a Moebius strip denoted by \mathcal{M} and fix (see Figure 6)

- a point P of its boundary $\partial\mathcal{M}$,
- a simple loop γ based at P such that $\gamma \setminus \{P\}$ lies in the interior of \mathcal{M} .

In particular

$$\{\partial\mathcal{M}\} = 2\{\gamma\} \in \pi_1(\mathcal{M})$$

and the class of γ generates the fundamental group of \mathcal{M} .

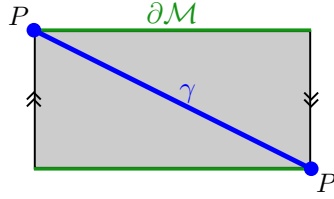


FIGURE 6. The Moebius strip.

Let $\{\mathcal{M}_k\}_{k \in \mathbb{N}}$ be an infinite number of copies of the Moebius strip \mathcal{M} where the point corresponding to P and the loop corresponding to γ in the copy \mathcal{M}_n are denoted respectively by P_n and γ_n . We define a *Moebius telescope* \mathcal{T}_n of height n as follows. Start with $\mathcal{T}_1 = \mathcal{M}_0$ and then define by induction

$$\mathcal{T}_{n+1} = \mathcal{T}_n \cup_{\varphi_n} \mathcal{M}_n$$

where φ_n is a homeomorphism between γ_n and $\partial\mathcal{M}_{n-1}$ that sends P_n to P_{n-1} . All gluing homeomorphisms will be chosen to be piecewise linear in the sequel.

Observe that

- $\mathcal{T}_1 \subset \dots \subset \mathcal{T}_{n-1} \subset \mathcal{T}_n$,
- all points $P_i \in \mathcal{M}_i$ for $i = 0, \dots, n-1$ glue onto a same point denoted P ,

- γ_0 is a deformation retract of \mathcal{T}_n , and thus $\pi_1(\mathcal{T}_n) = \mathbb{Z}$,
- $\{\gamma_i\} = 2^i\{\gamma_0\}$ for $i = 0, \dots, n-1$.

Fix a positive integer m and define n to be the smallest integer such that $m < 2^{n+1}$. The dyadic decomposition of m writes $m = 2^{n_1} + \dots + 2^{n_s}$ for some integers $0 \leq n_1 < \dots < n_s = n$. In particular we have $s \leq n+1$. Let $\xi(m) = \gamma_{n_1} \star \gamma_{n_2} \star \dots \star \gamma_{n_{s-1}} \star \partial\mathcal{M}_{n-1} \in \mathcal{T}_n$ be the loop based at P (\star denoting the concatenation operation for based loops). Consider the 2-cell complex

$$X_m = \mathcal{T}_n \cup_{\xi(m)} D^2$$

where the boundary of the 2-disk D^2 is glued along the curve $\xi(m)$. Because $\{\xi(m)\} = m\{\gamma_0\}$ we get $\pi_1(X_m) = \mathbb{Z}_m$.

We shall construct an economic triangulation for \mathcal{T}_n , and then for X_m . Start with the minimal triangulation of $\mathbb{R}P^2$ which consists of 10 triangles (see Figure 1 and compare with Figure 7). We fix a vertex P and choose a simplicial loop γ based at P which generates $\pi_1(\mathbb{R}P^2)$ such as in Figure 7. By deleting the interior of a triangle one of whose vertices is P but which is not adjacent to any edge of the loop γ , we obtain a triangulation of \mathcal{M} with 9 triangles, see Figure 7.

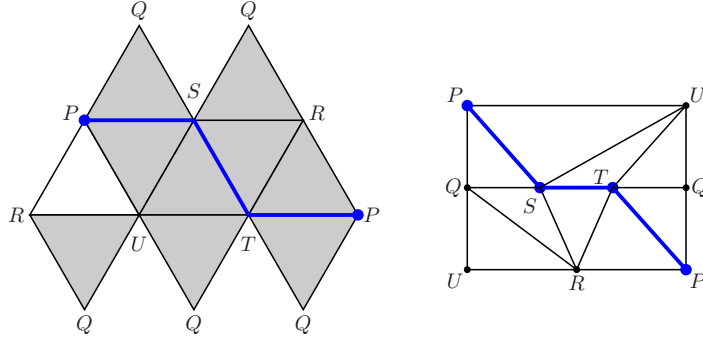


FIGURE 7. A special triangulation of the Moebius strip.

If each Moebius strip of the Moebius telescope of height n is endowed with this triangulation, we get a triangulation of \mathcal{T}_n with $9n$ triangles. Observe that the loop $\xi(m)$ consists of exactly $3s$ edges. Now triangulate the 2-disk D^2 by using at most $5s - 2$ triangles as in Figure 8 (compare with the proof of Proposition 2.1).

The triangulation of X_m thus obtained satisfies

$$s_2(X_m) \leq 9n + 5s - 2 \leq 14n + 3 \leq 14 \log_2 m + 3$$

which concludes the proof of the lemma. \square

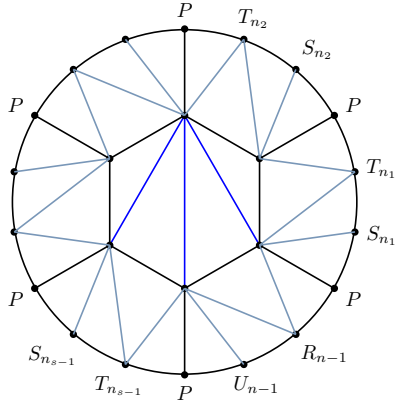


FIGURE 8. Triangulation of D^2 for $s = 6$.

Remark 4.2. The group \mathbb{Z}_m can be realized as the fundamental group of a 2-cell complex with only one 2-cell. It can be shown that this complex can not be triangulated with less than $3m$ triangles, the boundary of the 2-cell being mapped into the 1-skeleton by a PL-map of degree m . The Moebius telescope thus shows that to get an economic simplicial complex whose fundamental group is \mathbb{Z}_m we first have to start with a 2-cell complex with roughly $\log_2 m$ cells of dimension 2.

Remark 4.3. For $m = 2^n + 1$, the proof of Lemma 4.1 implies that

$$\kappa(\mathbb{Z}_m) \leq 9 \log_2(m - 1) + 8.$$

For $m = 3$ this upper bound is sharp (see the table in Example 1) and the Moebius telescope gives thus the minimal complex for \mathbb{Z}_3 . It is natural to ask whether $\kappa(\mathbb{Z}_m) = 9 \log_2(m - 1) + 8$ for $m = 2^n + 1$.

We now prove the general upper bound of Theorem 4.1. Consider the decomposition

$$G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_s}$$

where $n_1 | n_2 | \dots | n_s$.

For $k = 1, \dots, s$ take the economic 2-simplicial complex X_{n_k} constructed in the proof of Lemma 4.1 with fundamental group \mathbb{Z}_{n_k} . By gluing all the points $P_k \in X_{n_k}$ for $k = 1, \dots, s$ into the same point P , we obtain a 2-simplicial complex

$$\bigvee_{k=1}^s X_{n_k}$$

with at most

$$\sum_{k=1}^s (14 \log_2 n_k + 3) = 14 \log_2 |G| + 3s$$

2-simplices.

The fundamental group of this wedge sum is the group

$$\mathbb{Z}_{n_1} * \dots * \mathbb{Z}_{n_s}.$$

In order to get a simplicial complex whose fundamental group is G we fix for each $k = 1, \dots, s$ a loop $\alpha_k \subset X_{n_k}$ based at P , consisting of three edges and generating the fundamental group $\pi_1(X_{n_k}) \simeq \mathbb{Z}_{n_k}$. For each $1 \leq k < l \leq s$ we glue a minimal triangulated 2-torus to our 2-simplicial complex by identifying the pair of loops (α_k, α_l) with a pair of loops (α', α'') of the minimal 2-torus as depicted in Figure 2. We thus get a new 2-simplicial complex X_G with fundamental group G as each pair (α_k, α_l) now commutes. Because we add $7s(s-1)$ triangles (14 triangles for each minimal 2-torus) we have that

$$s_2(X_G) \leq 14 \log_2 |G| + 3s + 7s(s-1) = 14 \log_2 |G| + 7s^2 - 4s,$$

which concludes the proof of Theorem 4.1. \square

5. FROM SIMPLICIAL COMPLEXITY TO SYSTOLIC AREA FOR GROUPS

In this section we first prove Theorem 1.1. Then we show how to derive inequality (1.5).

5.1. Proof of Theorem 1.1. We start with the proof of the right-hand side inequality in Theorem 1.1.

By the right-hand side inequality of Theorem 1.2 we know that

$$\mathcal{G}_\sigma(T) \subset \mathcal{G}_\kappa \left(625 \cdot (50^2 T)^{1 + \frac{2 + \ln 25}{\sqrt{\ln(50^2 T)}}} \right).$$

This together with the right-hand side inequality in Theorem 1.3 gives that

$$|\mathcal{G}_\sigma(T)| \leq e^{6K \ln K},$$

where $K = 625 \cdot (50^2 T)^{1 + \frac{2 + \ln 25}{\sqrt{\ln(50^2 T)}}$. If $T \geq 2$, then

$$\frac{2 + \ln 25}{\sqrt{\ln(50^2 T)}} \leq 2$$

and we compute that

$$\begin{aligned} 6K \ln K &= 6 \cdot 625 \cdot (50^2 T)^{1 + \frac{2 + \ln 25}{\sqrt{\ln(50^2 T)}}} \cdot \ln \left(625 (50^2 T)^{1 + \frac{2 + \ln 25}{\sqrt{\ln(50^2 T)}}} \right) \\ &\leq 6 \cdot 625 \cdot 50^6 \cdot T^{1 + \frac{2 + \ln 25}{\sqrt{\ln(50^2 T)}}} \cdot 4 \ln(50^2 T) \\ &\leq 6 \cdot 625 \cdot 50^6 \cdot T^{1 + \frac{39 + \ln 25}{\sqrt{\ln(50^2 T)}}}. \end{aligned}$$

This implies the right-hand side inequality in Theorem 1.1 with $B = e^{6 \cdot 625 \cdot 50^6}$, $B' = 39 + \ln 25$ and $B'' = 50^2$.

We now turn to the proof of the right-hand side inequality in Theorem 1.1.

The first strategy we can try is the following. According to Theorem 1.2 we have

$$\sigma(G) \leq \frac{\kappa(G)}{2\pi}$$

which implies the inclusion

$$\mathcal{A}_\kappa(T) \subset \mathcal{A}_\sigma\left(\frac{T}{2\pi}\right).$$

Thus for any positive T

$$\left| \mathcal{A}_\sigma\left(\frac{T}{2\pi}\right) \right| \geq |\mathcal{A}_\kappa(T)| \geq \left[2^{\frac{T-3}{14}} \right]$$

according to Corollary 4.1. But this lower bound is not as good as the left-hand side of the inequality stated in Theorem 1.1. To improve our estimate we proceed as follows.

We construct a metric version of the Moebius telescope. First consider the Moebius strip endowed with the Riemannian metric of constant curvature 1 given by the quotient of a spherical strip by the central symmetry like in Figure 9.

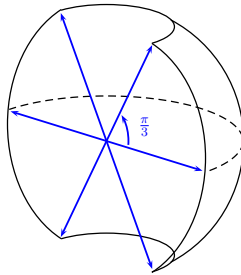


FIGURE 9. Spherical strip.

The height of this spherical strip is chosen to be $2\pi/3$ and its area is thus equal to $2\pi\sqrt{3}$. The equatorial curve and the boundary have the same length π .

The metric version of the Moebius telescope is constructed in the same way as in Lemma 4.1 using this special metric on each Moebius strip except that at each step the boundary of \mathcal{M}_{k-1} is identified with the equatorial curve of \mathcal{M}_k instead of with γ_k . Here γ_k is realized as half a great circle whose intersection with the equatorial curve is a point denoted by p_k and its intersection with the boundary curve another point denoted by q_k . Observe that the length of γ_k is equal to π and that γ_k is tangent to the boundary at q_k . We glue together the Moebius strips in such a way that the point q_k on $\partial\mathcal{M}_k$ coincides with the point p_{k+1} on the equatorial curve of \mathcal{M}_{k+1} for $k = 0, \dots, n-1$.

Fix a positive integer m and define n to be the smallest integer such that $m < 2^{n+1}$. The dyadic decomposition of m writes $m = 2^{n_1} + \dots + 2^{n_s}$ for some integers $0 \leq n_1 < \dots < n_s = n$. We define an analog of $\xi(m)$ by first opening up the closed curves $\gamma_{n_1}, \gamma_{n_2}, \dots, \gamma_{n_{s-1}}, \partial\mathcal{M}_{n-1}$ at the points $q_{n_1}, p_{n_2}, q_{n_2}, p_{n_3}, q_{n_3}, \dots, p_{n_{s-1}}, q_{n_{s-1}}$ and p_n , and then connect each pair of points $(q_{n_i}, p_{n_{i+1}})$ (appearing twice) for $i = 1, \dots, s-1$ through a length minimizing arc. It may happen that $q_{n_i} = p_{n_{i+1}}$ in which case our procedure reduces to the concatenation operation at this point of the curves γ_{n_i} and $\gamma_{n_{i+1}}$. As the distance between p_k and q_k is equal to $\pi/2$, we get that the curve $\xi(m)$ thus obtained is of length at most $\pi(n+1)$. We then glue a disk D^2 of constant curvature 1 and radius $\frac{\pi}{2}$ having at its center a conical singularity. The angle of the conical singularity is chosen such that the length of the boundary equals the length of the curve $\xi(m)$. We denote by Y_m the resulting 2-dimensional simplicial complex endowed with this piecewise smooth Riemannian metric. Because the length of $\xi(m)$ is at most equal to $\pi(n+1)$, the area of the 2-disk is also at most equal to $\pi(n+1)$. This implies that

$$\text{area}(Y_m) \leq (1 + 2\sqrt{3})\pi(n+1) \leq (1 + 2\sqrt{3})\pi(1 + \log_2 m).$$

Next we show the following.

Lemma 5.1. $\text{sys}(Y_m) = \pi$.

Proof. Let γ be a systolic loop, that is a non-contractible and simple closed curve of length equal to the systole. If γ meets the conical singularity of D^2 then its length is at least equal to π . If γ does not touch the conical singularity but crosses the interior of D^2 , we can continuously deform this portion of γ to the boundary of D^2 without increasing its length.

So we now assume that $\gamma \subset \mathcal{T}_n$. If γ entirely lies inside some Moebius band \mathcal{M}_k then it is straightforward to check that its length is at least equal to π . If not, let denote by \mathcal{M}_k a Moebius band whose interior is crossed by γ and consider a maximal non-empty connected subarc c of $\gamma \cap \mathcal{M}_k$. There are three alternatives.
a) The endpoints of c both lie on the equatorial curve of \mathcal{M}_k . Because c is a geodesic arc, it is a half of a great circle and its length is at least π ;

- b) The endpoints of c both lie on $\partial\mathcal{M}_k$. This subarc being a portion of a great circle, it necessarily cuts the equatorial curve and its length is thus at least equal to $2\pi/3$. The remaining part of the curve must both start and end transversally to the equatorial curve of \mathcal{M}_{k+1} . If not, the entire curve would lie inside \mathcal{M}_k , case which has been excluded (in particular the case where $k = n$ is ruled out). Thus either the remaining curve is entirely contained in \mathcal{M}_{k+1} in which case its length is equal to π , or its length is at least twice the distance from the equatorial curve to the boundary of \mathcal{M}_{k+1} , that is at least $2\pi/3$. Thus the length of γ is always at least equal to π ;
- c) One of the endpoints of c lies on the equatorial curve of \mathcal{M}_k and the other on $\partial\mathcal{M}_k$. This subarc has length at least equal to $\pi/3$. By arguing in the same way as above, we see that the remaining part of γ has length at least equal to $2\pi/3$ thus concluding the proof. \square

Because $\pi_1(Y_m) = \mathbb{Z}_m$, this implies that

$$\sigma(\mathbb{Z}_m) \leq \frac{1 + 2\sqrt{3}}{\pi} (1 + \log_2 m).$$

We thus obtain the right-hand side of inequality (1.5), and the left-hand side inequality in Theorem 1.1:

$$|\mathcal{A}_\sigma(T)| \geq \left\lceil 2^{\frac{\pi}{1+2\sqrt{3}}T} \right\rceil.$$

5.2. Systolic area for finite abelian groups. For a finite abelian group G , the behaviour of $\sigma(G)$ in terms of the number $|G|$ of its elements and the number of its invariant factors can be described by combining Theorems 1.2 and 4.1. First we easily derive the following.

Theorem 5.1. *Let G be a finite abelian group with s invariant factors. Then*

$$\sigma(G) \leq \frac{1}{2\pi} (14 \log_2 |G| + 7s^2 - 4s).$$

Now recall that given a group G with free index zero we have

$$\kappa(G) \leq 625 (50^2 \cdot \sigma(G))^{1 + \frac{2 + \ln 25}{\sqrt{\ln(50^2 \cdot \sigma(G))}}}$$

according to Theorems 1.2. We get

$$\ln \left(\frac{\kappa(G)}{625} \right) \leq (2 + \ln 25) \sqrt{\ln(50^2 \cdot \sigma(G))} + \ln(50^2 \cdot \sigma(G))$$

which implies with $C = 2 + \ln 25 \simeq 5.2$ that

$$\sqrt{\ln(50^2 \cdot \sigma(G))} \geq \frac{\sqrt{C^2 + 4 \ln \left(\frac{\kappa(G)}{625} \right)} - C}{2}$$

provided $\kappa(G) \geq 625 \cdot e^{-\frac{C^2}{4}} \simeq 0,69$. This condition being always fulfilled as $\kappa(G) \neq 0$, we deduce the following result.

Corollary 5.1. *Let G be a group with free index zero. Then*

$$\sigma(G) \geq \frac{1}{50^2} \left(\frac{\kappa(G)}{625} \right)^{1-\varphi\left(\frac{\kappa(G)}{625}\right)},$$

where $\varphi : [1, \infty[\rightarrow \mathbb{R}$ denotes the decreasing function given by the formula

$$\varphi(x) = \frac{C^2}{2 \ln x} \left(\sqrt{1 + \frac{4 \ln x}{C^2}} - 1 \right).$$

Because a finite abelian group G satisfies

$$\kappa(G) \geq 2 \log_3 |G|,$$

we derive the following lowerbound.

Theorem 5.2. *A finite abelian group G satisfies the inequality*

$$\sigma(G) \geq \frac{1}{50^2} \left(\frac{2 \log_3(|G|)}{625} \right)^{1-\varphi\left(\frac{2 \log_3(|G|)}{625}\right)}.$$

Finally, observe that $\varphi(x) \sim \frac{C}{\sqrt{\ln x}}$ for large values of x , which leads in particular to the almost logarithmic asymptotic lower bound on $\sigma(\mathbb{Z}_m)$ presented in the introduction (see the left-hand side of inequality (1.5)).

Corollary 5.2. *For any positive ε ,*

$$\sigma(\mathbb{Z}_m) \geq (\log_2 m)^{1-\varepsilon}$$

provided m is large enough.

In the same way we conclude that for any positive ε

$$\sigma(G) \geq (\ln |\text{Tors } H_1(G, \mathbb{Z})|)^{1-\varepsilon}$$

for groups with large torsion in homology according to Proposition 2.2.

6. APPLICATIONS TO SYSTOLIC VOLUME OF HOMOLOGY CLASSES

In this section we first recall the definition of systolic volume associated to a homology class of a group, and then explain how to derive an interesting lower bound for the systolic volume in terms of the 1-torsion of the group using the notion of simplicial complexity.

Let $\mathbf{a} \in H_n(G, \mathbb{Z})$ be an n -dimensional homology class of a group G where n denotes some positive integer. A *geometric cycle* (X, f) representing the class \mathbf{a} is a pair (X, f) where X is an orientable pseudomanifold X of dimension n and $f : X \rightarrow K(G, 1)$ a continuous map such that $f_*[X] = \mathbf{a}$ where $[X]$ denotes the

fundamental class of X and $K(G, 1)$ is the Eilenberg-MacLane space of G . The representation is said to be *normal* if in addition the induced map $f_{\#} : \pi_1(X) \rightarrow G$ is an epimorphism. Given a geometric cycle (X, f) and a piecewise smooth metric g on X , the *relative homotopic systole* $\text{sys}_f(X, g)$ is defined as the least length of a loop γ of X such that $f \circ \gamma$ is not contractible. The *systolic volume* of (X, f) is then the number

$$\mathfrak{S}_f(X) := \inf_g \frac{\text{vol}(X, g)}{\text{sys}_f(X, g)^n},$$

where the infimum is taken over all piecewise smooth metrics g on X and $\text{vol}(X, g)$ denotes the n -dimensional volume of X . When $f : X \rightarrow K(\pi_1(X), 1)$ is the classifying map (induced by an isomorphism between the fundamental groups), we simply denote by $\mathfrak{S}(X)$ the systolic volume of the pair (X, f) . From [Gro83, Section 6] we know that for any dimension n

$$\mathfrak{S}_n := \inf_{(X, f)} \mathfrak{S}_f(X) > 0,$$

the infimum being taken over all geometric cycles (X, f) representing a non trivial homology class of dimension n . The following notion was introduced by Gromov in [Gro83, Section 6].

Definition 6.1. The *systolic volume* of a homology class $\mathbf{a} \in H_n(G, \mathbb{Z})$ is defined as the number

$$\mathfrak{S}(\mathbf{a}) := \inf_{(X, f)} \mathfrak{S}_f(X)$$

where the infimum is taken over all geometric cycles (X, f) representing the class \mathbf{a} .

Observe that for any homology class $\mathbf{a} \in H_2(G, \mathbb{Z})$ we have

$$\mathfrak{S}(\mathbf{a}) \geq \sigma(G).$$

Recently the systolic volume of homology classes has been extensively studied in [BB15] where the reader can find numerous results on this invariant.

6.1. A lower bound of systolic volume by 1-torsion. We now define the 1-torsion of a homology class and explain how to use it to bound from below its systolic volume.

Definition 6.2. The *1-torsion of a homology class* $\mathbf{a} \in H_n(G, \mathbb{Z})$ is defined as the number

$$t_1(\mathbf{a}) = \inf_{(X, f)} |\text{Tors } H_1(X, \mathbb{Z})|$$

where the infimum is taken over the set of geometric cycles (X, f) representing the class \mathbf{a} and $|\text{Tors } H_1(X, \mathbb{Z})|$ denotes the number of torsion elements in the first integral homology group of X .

We now present the main result of this section.

Theorem 6.1. *Let G be a group and $\mathbf{a} \in H_n(G, \mathbb{Z})$. Then*

$$\mathfrak{S}(\mathbf{a}) \geq C_n (\ln t_1(\mathbf{a}))^{1 - \frac{C'_n}{\sqrt{\ln(\ln t_1(\mathbf{a}))}}},$$

where C_n and C'_n are two positive numbers depending only on n .

In particular,

$$\mathfrak{S}(\mathbf{a}) \geq (\ln t_1(\mathbf{a}))^{1-\varepsilon}$$

for any $\varepsilon > 0$ provided $t_1(\mathbf{a})$ is large enough. It is important to remark that there is no hope in dimension ≥ 3 to prove a universal lower bound of the type

$$\mathfrak{S}(\mathbf{a}) \geq C \ln t_1(\mathbf{a})$$

for some positive constant C . Indeed, for any positive integer s , the Eilenberg-MacLane space of the group

$$G_s := \underbrace{\mathbb{Z}_2 * \dots * \mathbb{Z}_2}_s$$

is the complex $\bigvee_{i=1}^s \mathbb{R}P_i^\infty$. If n is a positive integer and $\mathbb{R}P_i^{2n+1} \subset \mathbb{R}P_i^\infty$ denotes the skeleton of odd dimension $2n+1$ of the i -th component, we consider the sequence of homology classes

$$\mathbf{a}_s = \sum_{i=1}^s [\mathbb{R}P_i^{2n+1}] \in H_{2n+1}(G_s, \mathbb{Z}).$$

We can see that $t_1(\mathbf{a}_s) = 2^s$ as $|\text{Tors } H_1(X, \mathbb{Z})| \geq 2^s$ for any representation (X, f) of \mathbf{a} . According to [BB15, Theorem 5.4] this implies that

$$\mathfrak{S}(\mathbf{a}_s) \leq \mathfrak{S}(\#_{i=1}^s \mathbb{R}P_i^{2n+1}) \leq C \cdot \frac{\ln t_1(\mathbf{a}_s)}{\ln \ln t_1(\mathbf{a}_s)}$$

for some positive constant C . For even dimension, we consider the sequence of classes $\tilde{\mathbf{a}}_s = [S^1] \times \mathbf{a}_s \in H_{2n+2}(\mathbb{Z} \times G_s, \mathbb{Z})$ for which the same upper bound holds.

Proof of Theorem 6.1. Let G be a finitely presentable group and $\mathbf{a} \in H_n(G, \mathbb{Z})$ a homology class. Recall that the *simplicial height* $h(\mathbf{a})$ of a homology class \mathbf{a} is the minimum number of simplexes (of any dimension) of a simplicial complex representing the class \mathbf{a} . According to [Gro83, 6.4.C'''] (see also [Gro96, 3.C.3]) there exists two positive numbers c_n and c'_n depending only on the dimension n such that

$$\mathfrak{S}(\mathbf{a}) \geq c_n \cdot \frac{h(\mathbf{a})}{\exp(c'_n \sqrt{\ln h(\mathbf{a})})}.$$

We conclude using Proposition 2.2 which asserts that

$$h(\mathbf{a}) \geq 2 \log_3 t_1(\mathbf{a}).$$

□

6.2. Application to lens spaces. In general the 1-torsion of a class is difficult to compute. In the case of $G = \mathbb{Z}_m$, we can estimate from below the 1-torsion of any generator by the number m as follows.

Lemma 6.1. *Let \mathbf{a} be a generator of $H_{2n+1}(\mathbb{Z}_m, \mathbb{Z})$. Then*

$$t_1(\mathbf{a}) \geq m.$$

Proof. Let (X, f) be a geometric cycle representing \mathbf{a} . As \mathbf{a} is a generator of $H_{2n+1}(\mathbb{Z}_m, \mathbb{Z}) \simeq \mathbb{Z}_m$, the map f induces an isomorphism

$$(6.1) \quad f^* : H^{2n+1}(\mathbb{Z}_m, \mathbb{Z}_m) \rightarrow H^{2n+1}(X, \mathbb{Z}_m).$$

Let

$$\beta : H^1(\mathbb{Z}_m, \mathbb{Z}_m) \rightarrow H^2(\mathbb{Z}_m, \mathbb{Z}).$$

denotes the Bockstein homomorphism and

$$j : H^2(\mathbb{Z}_m, \mathbb{Z}) \rightarrow H^2(\mathbb{Z}_m, \mathbb{Z}_m)$$

the morphism of reduction modulo m . In our case, j is an isomorphism. A generator of $H^{2n+1}(\mathbb{Z}_m, \mathbb{Z}_m)$ (not necessarily dual to \mathbf{a}) can be chosen as the element $\mathbf{u} \cup (j \circ \beta(\mathbf{u}))^n$ where $\mathbf{u} \in H^1(\mathbb{Z}_m, \mathbb{Z}_m)$ is some generator. Now consider $f^*(\beta(\mathbf{u})) = \beta(f^*(\mathbf{u})) \in H^2(X, \mathbb{Z})$. Taking into account the isomorphism (6.1) we see that $f^*(\mathbf{u}) \cup (f^*(j \circ \beta(\mathbf{u})))^n$ is an element of order m in $H^{2n+1}(X, \mathbb{Z}_m)$. This implies that the order of $f^*(j \circ \beta(\mathbf{u})) \in H^2(X, \mathbb{Z}_m)$ is m , and thus that the order of $\beta(f^*(\mathbf{u})) \in H^2(X, \mathbb{Z})$ is also m . By the duality (2.7) we get the result. \square

Remark that the statement of this lemma as well as its proof holds in the case of a simplicial complex X representing the class \mathbf{a} . In this more general case note that the map (6.1) is only injective.

Given two integers $n \geq 0$ and $m \geq 2$ let $L_n(m)$ denote a lens space of dimension $2n + 1$ with fundamental group \mathbb{Z}_m : there exist integers q_1, \dots, q_m coprime with m and an isometry A of order m of the form

$$A(z_1, \dots, z_n) = (e^{2\pi i \frac{q_1}{m}} z_1, \dots, e^{2\pi i \frac{q_n}{m}} z_n)$$

such that

$$L_n(m) := \{Z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \dots + |z_n|^2 = 1\} / \sim_A \simeq S^{2n+1} / \mathbb{Z}_m,$$

where $Z \sim Z'$ if and only if $Z = A^k Z'$. Observe that the fundamental class of a lens space $L_n(m)$ realizes a generator \mathbf{a} of the homology group $H_{2n+1}(\mathbb{Z}_m, \mathbb{Z})$.

Combining Lemma 6.1 and Theorem 6.1 we derive the following result.

Theorem 6.2. *For any integer $m \geq 2$ we have*

$$\mathfrak{S}(L_n(m)) \geq C_n (\ln m)^{1 - \frac{C'_n}{\sqrt{\ln(\ln m)}}}$$

where C_n and C'_n are two positive numbers depending only on the dimension n .

We remark that this lower bound is of the same type as that for $\sigma(\mathbb{Z}_m)$ in Corollary 5.2. Now we turn to the proof of the new upper bound for systolic volume of lens spaces.

Theorem 6.3. *For any integer $m \geq 2$ we have*

$$\mathfrak{S}(L_n(m)) \leq D_n m^n$$

where D_n is a positive number depending only on the dimension n .

As observed in the introduction, this polynomial lower bound is better than the one obtained by computing the systolic volume for the round metric (which is roughly $\approx m^{2n}$).

Proof. We first decompose the manifold $L_n(m)$ into $(2n + 1)$ -dimensional cubes, and then use this decomposition to construct a metric for which we control the systolic volume in terms of the number of these cubes, compare with [BB05] and [BB15].

We decompose $L_n(m)$ into $(2n + 1)$ -cubes as follows. Start with the standard cellular decomposition of $L_n(m)$ denoted by Θ (see [Hat02, p.145] generalizing for the case $n > 1$ the construction of [ST80]). This decomposition Θ has exactly one k -cell denoted by e^k in each dimension $k = 0, 1, \dots, 2n + 1$. Denote by q_k its center. We subdivide the cellular decomposition Θ in such a way that each new cell admits a simplicial structure. We proceed by induction as follows. For $k = 0$ there is nothing to do but observe that $e^0 = \{q_0\}$. Because e^1 is attached to e^0 we subdivide e^1 into the two arcs denoted by e_0^1 and e_1^1 connecting q_0 and q_1 . Observe at this stage that the complex thus obtained is not a simplicial complex, as two simplices may share more than one face in common. The next step consists to first remark that e^2 is attached to $e^1 \cup e^0$ through a linear map $\varphi_2 : \partial e^2 \rightarrow e^1 \cup e^0$ of degree m , and then take in e^2 the cone over the preimages by φ_2 of e_0^1 and e_1^1 with respect to the vertex q_2 . Doing so we subdivide e^2 into $2m$ new 2-cells with a natural structure of a simplex. We then proceed in that way by induction on the dimension following the structure of Θ_1 . More precisely at each step $k \geq 3$ we form the cone over the preimages by the attaching map of the $k - 1$ -dimensional new cells with respect to the center of the k -cell. The attaching maps

$$\varphi_k : \partial e^k \longrightarrow \bigcup_{s=0}^{k-1} e^s$$

having degree m if k is even, and zero if k is odd, this gives a new decomposition Θ_1 with $2^{n+1}m^n$ simplices of dimension $2n + 1$. Remark that despite the fact that Θ_1 is not a simplicial decomposition this structure is coherent in the sense that any face of a simplex is a simplex of lower dimension.

Denote by Θ_2 the barycentric subdivision of Θ_1 . The structure Θ_2 is now simplicial with $2^{n+1} \cdot (2(n + 1))! \cdot m^n$ simplices of dimension $2n + 1$. We decompose

each $(2n + 1)$ -simplex of Θ_2 into $2(n + 1)$ cubes of dimension $2n + 1$. This gives a decomposition Θ_3 of $L_n(m)$ into $2^{n+1} \cdot (2(n + 1))! \cdot 2(n + 1) \cdot m^n$ cubes of dimension $2n + 1$. Endow each cube of Θ_3 with the Euclidean metric with side length 1. We thus get a polyhedral metric g on $L_n(m)$. This metric satisfies $\text{sys}(L_n(m), g) \geq 2$ according to [BB15, Lemma 5.6]. Because $\text{vol}(L_n(m), g) = 2^{n+1}[2(n + 1)]![2(n + 1)]m^n$, this gives the result with $D_n = \frac{[2(n+1)]!(n+1)}{2^{n-1}}$. \square

6.3. Application to 3-manifolds. Let M be a closed manifold. For a covering space M' with k sheets of M , it is straightforward to check that

$$(6.2) \quad \mathfrak{S}(M) \geq \frac{1}{k} \mathfrak{S}(M').$$

Let M be a manifold of dimension 3 with finite fundamental group. Its universal cover is the sphere S^3 , and the action of the fundamental group $\pi_1(M)$ on S^3 is orthogonal. The list of finite groups which act orthogonally on S^3 can be found in [Mil57] for instance. An analysis of this list shows that $\pi_1(M)$ possesses a cyclic subgroup of index $k \leq 12$. Denote by M' the covering space corresponding to this subgroup. The manifold M' is a lens space $L_1(n)$ with $n \geq \frac{|\pi_1(M)|}{12}$. So by applying Theorem 6.2 and the inequality (6.2) with $k = 12$ we derive the last theorem of this paper.

Theorem 6.4. *There exists two positive constants a and b such that any manifold M of dimension 3 with finite fundamental group satisfies*

$$\mathfrak{S}(M) \geq a \left(\ln |\pi_1(M)| \right)^{1 - \frac{b}{\sqrt{\ln |\pi_1(M)|}}},$$

where $|\pi_1(M)|$ denotes the number of elements in $\pi_1(M)$.

Remark that finite fundamental groups of 3-manifolds can have a large number of elements but a very small torsion in $H_1(\pi_1(M), \mathbb{Z})$. A direct estimate of $\mathfrak{S}(M)$ by the torsion of $H_1(\pi_1(M), \mathbb{Z})$ is interesting only if the manifold M is a lens space.

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