

Characterising mathematical activities promoted by Fermi problems

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Source: *For the Learning of Mathematics*, November 2019, Vol. 39, No. 3 (November 2019), pp. 10-13

Published by: FLM Publishing Association

Stable URL: <https://www.jstor.org/stable/10.2307/26854427>

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of closure with publication of a result, and can move on to new problems, the teacher often teaches the same topic to different classes in parallel or in different semesters. This offers many opportunities to question her approach, to repeat her teaching with small variations over and over. This might improve the quality of her teaching, but be detrimental to herself. The parallel with OCD offers a new lens through which to examine with tension in teaching.

If, as I claim, mathematical creativity can be seen as a sort of Obsessive Compulsive Disorder, could the experiences of mathematicians inform the treatments of psychologists, and could the work of psychologists assist the productive work of mathematicians? Could methods employed by mathematicians to control their anxiety when encountering obstacles in their research be used by psychologists to ameliorate analogous situations occasioning symptoms of OCD?

I wrote this communication as an exploration of an idea, based on parallels I saw between mathematical activity and feature of OCD. By chance, as I was writing it I read of a clear, if extreme, example of the parallel I was imagining. According to a BBC online article (Keating, 2019), in 2002 Jason Padgett was mugged by two men, resulting in a brain injury that brought on OCD. Unable to leave his home, and finding his perceptions of the world around him fundamentally altered, he developed an interest in mathematics and physics, subjects he had dismissed as useless before he was attacked. Through the internet he developed this interest and took to trying to capture in drawings the way he now saw the world around him. "I had literally a thousand or more drawings of circles, fractals, every shape that I could manage to draw. It was the only way I could manage to communicate effectively what I was seeing." A chance meeting with a physicist led Padgett to enroll in a mathematics course, and through mathematics he became able to venture out into the world.

This story is unusual, but shows how the interrelatedness between the OCD and the mathematical productivity might be usefully explored by both mathematicians and psychologists. Mathematics might improve the lives of people who suffer from mental disorders, and psychology might enhance mathematical research.

Acknowledgement

I am grateful to Alf Coles for providing me with resources and encouragement to submit this communication.

Note

[1] American Psychiatric Association (2013) *Diagnostic and Statistical Manual of Mental Disorders*. Washington, DC: American Psychiatric Association Publishing.

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Characterising mathematical activities promoted by Fermi problems

LLUÍS ALBARRACÍN, JONAS B. ÄRLEBÄCK

Some questions that arise in everyday situations, such as 'How long will it take to get to the airport?' can be solved by making a quick estimate. In other cases, we can ask ourselves about situations that we have never considered before, where we are interested in obtaining a first rough answer. This would be the case if we ask how many planes are flying all over the world at a specific time are, or the amount of CO₂ emissions we could avoid in a city if gasoline-powered cars were replaced by electric ones. These two questions are examples of Fermi problems. They owe their name to Enrico Fermi, who used this particular type of problems both in his scientific work and as a university teacher. Fermi problems, being smaller, more well-defined and delimited contextualised problems and not real-world problems in all their complexity, have been considered to be miniature-modeling problems (Robinson, 2008).

Fermi problems can be solved by decomposing the original problem into simpler sub-problems, solving the original question through reasonable estimates. This way of solving realistic problems has been used in mathematics education to introduce mathematical modelling to primary education students (Peter-Koop, 2009), to use real-world knowledge to support mathematical learning with secondary education students (Albarracín & Gorgorió, 2014; Ärleback, 2009) and at university level to assess modelling processes, such as validating models (Czocher, 2018).

Although educational research on Fermi problems generally emphasises estimation, it has been suggested that the activity of estimation can be replaced by other (classroom) activities to find the numerical information needed to solve the problem (Sriraman & Knott, 2009). In this communication we identify different types of mathematical activities that can be derived from the use of Fermi problems in the existing research, with the goal to develop a characterisation framework that can be used both for designing research interventions and as an analytical tool in research.

Fermi problem activity templates

In this section we present a characterisation tool called the *Fermi problem Activity Template* (FpAT) that focuses on the structure of the Fermi problem. It can be used to describe either (a) the result of an a priori analysis of the Fermi problem split into interconnected subproblems needed to be solved; or (b) a structure that describes the activities students actually engage in when solving a Fermi problem. Hence, FpAT can be used both in the design of research interventions and as an analytical tool.

Part of our inspiration for how we have come to represent the FpAT categorisation comes from Anderson and Sherman (2010). They proposed a representation based on an a priori analysis of the problem at hand that structures the solving

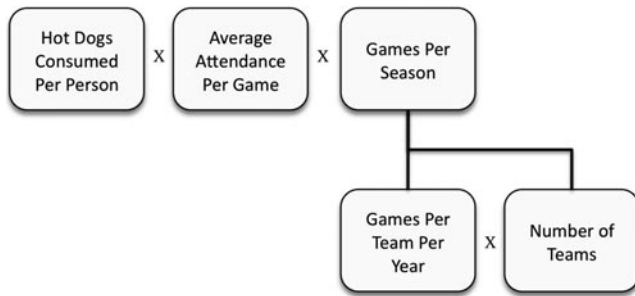


Figure 1. The MBL hotdog Fermi problem as presented in Anderson and Sherman (2010, p. 37).

process by explicitly differentiating the sub-problems the students have to engage in to solve the problem. The example they discuss in great detail is about estimating the number of hotdogs consumed at the Major League Baseball (MLB) games each season in the US. Differentiating between the values needed to be estimated (such as the number of hotdogs consumed per person and per game) from given values that can be looked up (such as the number of games in a season), Anderson and Sherman (2010) presented the breakdown of the problem as in Figure 1.

One can consider engaging in different types of activities to achieve the unknown numerical values needed in solving the different sub-problems. For example, rather than using a standard way of estimating based on previous experiences, consulting the MLB website might provide the number of games each team plays per year, and to find the number of spectators at a typical game one can turn to newspaper reports. In the case of finding the number of hotdogs consumed per person and game, a survey that generates sufficient data to determine a reasonable estimate can be used. The decision and reason for using one activity rather than another can depend on the solver's understanding of the precision required in the solution of the problem, or be a conscious choice made in the design of the activity in order for the students to work toward given certain mathematical contents and curricular goals.

Mathematical activities promoted by Fermi problems

Some studies have used the so-called MAD framework (Ärleback, 2009) to analyse the activities students engage in when solving Fermi problems. This framework characterises the process of solving a Fermi problem in terms of the following sub-modelling activities: *reading, making a model, estimating, calculating, validating* and *writing*. Estimating is the activity originally seen as key to a Fermi problem, but we concur with the suggestion of Sriraman and Knott (2009) that it can be generalised. In the following, we argue and illustrate that it can incorporate other ways for students to get the necessary numerical information they need to answer the problem or related sub-problems. The four types of mathematical activities that we propose are *guesstimation, experimentation, data search* and *statistical data collection*.

To visualise the structure of the problem and the different types of mathematical activities that can be used in solving

Table 1. The components of the Fermi problem Activity Template framework (FpAT).

Activity / representation	Students obtain the quantity by engaging in ...
 Guesstimation	... a mental process giving a rough solution through guessing and making comparisons based on previous experiences and intuition.
 Experimentation	... in-and-out-of-school experimentations and investigations, including making measurements.
 Looking for data	... searching for numerical information in external sources.
 Statistical data collection	... suitable ways of selecting, collecting and analysing statistical data.

the problem, our FpAT framework uses different graphical representations for the four (sub-)activities (see Table 1). The purpose of using different geometrical shapes for the different activities is to enhance readability.

Figure 2 shows the FpAT framework applied to the structure of the Fermi problem presented in Figure 1, illustrating that a statistical survey will be used to find the average number of hotdogs consumed per person in a game, that how many people attend a game will be determined using an estimate, and that the rest of the needed numerical values will be sought in appropriate sources. The different sub-problems identified in the problem are delimited using square brackets in order to clearly convey the structure of the problem.

Using FpAT as an analytic tool

In Albarracín and Gorgorió (2013) one of the problems studied asked individual students for an estimate of the total number of SMS messages sent in a day in Catalonia. Secondary students (12-16 years old) were asked to provide a detailed and concrete plan for estimating the quantities asked for rather than solving the problem. The following excerpt presents one of the proposed solutions given by a



Figure 2. Structure of the MBL Hotdog problem characterised using FpAT.

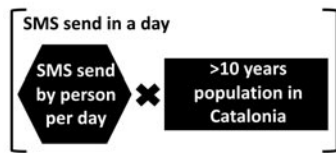


Figure 3. Structure of student A1's proposed solution to the SMS problem represented using FpAT.

student (A1). Figure 3 shows the student's solution represented using FpAT.

I would make a survey of the people I know about how many messages they send per day, trying to take a balanced number of women and men, as well as young people, adults and the elderly. Then, I would make an average and then multiply this by the number of inhabitants of Catalonia between the age ranges that might have a mobile phone.

Student A1's proposal is to use a survey to find out the number of SMS messages sent by the inhabitants of Catalonia in a day. The student considers that the participants surveyed must comply with some restrictions in order for the sample to be adequate, but does not provide any details on how to realise this. Then, when the average number of SMS messages sent in a day per person is known, the student proposes to multiply this number by the number of mobile users in Catalonia.

Figure 4 represents student A2's proposed solution:

First, it [the survey] will take 10 teenagers, 10 adults, 10 senior citizens, from here I would ask them how many SMS they send per day to each group and I would find the average. Then I would calculate the total average respecting the percentage of people of each type and multiply it by the total number of people in Catalonia.

Student A2 also suggests using a survey to collect data as the basis for his solution, but in contrast to Student A1's proposal suggests stratifying the sample according to his experiences and knowledge about the different age groups' usage of SMS messages. As illustrated by the FpAT in Figure 4, however, Student A2's proposal does not necessarily

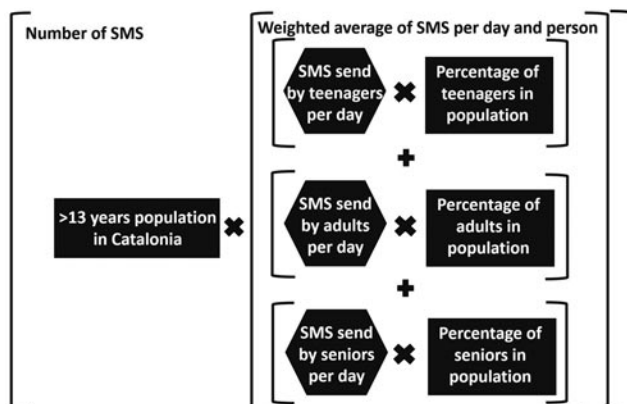


Figure 4. Structure of student A2's proposed solution to the SMS problem represented using FpAT.

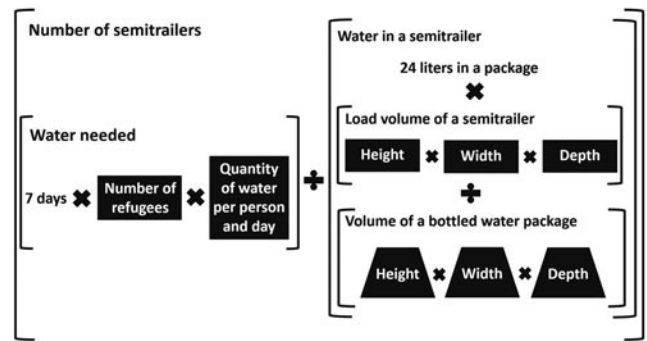


Figure 5. Structure of problem discussed in Taggart *et al.* (2007).

use this stratification in each population group, but rather suggests using the collected data to derive a weighted average. These two examples illustrate that FpAT as an analytical tool of students' work facilitates a straightforward and compact representation of students' (proposed) solutions, as well as making subtle differentiating aspects of the processes (potentially) involved in solving the problem visible, such as in the case of Student A2's suggestion of using a weighted average.

Using FpAT as a tool for task design

To illustrate the potential of the FpAT framework in designing research interventions, we apply the framework to an *a priori* analysis of an activity proposed in Taggart *et al.* (2007). It asks students "How many semitrailer loads of bottled water [16.9 fluid ounces] were needed by New Orleans' refugees in the week following Hurricane Katrina?" (p. 166). Taggart *et al.* discuss this question and list a number of sub-questions that need to be answered in order to answer the original question.

Figure 5 presents the FpAT based on the sub-questions in Taggart *et al.*, and we can see that the problem is structured around determining two main quantities: the total amount of water needed and the amount of water that a semi-trailer can transport. However, these two quantities must be determined from a succession of estimates and calculations using more specific quantities. This *a priori* analysis highlights the different types of activities students can productively engage in when tackling the problem. Hence, given as a modelling problem in a research intervention, and depending on the focus of the research, it can support the research design by preparing and supporting the research setting.

Final thoughts

Fermi problems have a long tradition in certain educational contexts. However, despite various recommendations in the mathematics education literature, this type of problem has not yet to any large degree been implemented in classrooms, nor has its full potential been exploited or researched. As we have argued here, Fermi problems in a modelling context promote multiple types of mathematical activities allowing for a broad range of solutions, classroom discussions and learning at all educational levels, in multiple subjects. Indeed, given that the contexts of Fermi problems often come from the STEM disciplines, Fermi problems have the potential to function as

integrating activities, allowing for the connection and transfer of strategies and applications of the different activities from other disciplines into mathematical classrooms (and vice versa). In addition, the different types of activities in the FpAT framework are accessible entrance points to introduce and work with various types of technologies that can be used to obtain or process data, allowing Fermi problems to also be a facilitator of students' ICT literacy. With this in mind, we are hopeful that the FpAT framework can be used as a research tool, both for designing interventions and analysis, especially focusing on what demands, roles, functions and effects different activities have on the problem-solving process, the solutions, and students' learning.

Acknowledgements

Lluís Albarracín is Serra-Hünter Fellow in Universitat Autònoma de Barcelona. Research supported by the projects EDU2017-82427-R (Ministerio de Economía, Industria y Competitividad, Spain), 2017 SGR 497 (AGAUR, Generalitat de Catalunya) and José Castillejo program (CAS17/00289 reference).

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Signed numbers and signed letters in algebra

AMIR ASGHARI

The Glossary of the Common Core State Standards for Mathematics (2010) defines an integer by *a way that it can be expressed*.

Integer. A number expressible in the form a or $-a$ for some whole number a .

This definition hints at three mathematical objects, a signed number, a signed letter and an integer, that it is useful to dis-

tinguish. A *signed number* is a specific natural number prefixed by the sign $-$ or $+$ [1]. A *signed letter* is a letter prefixed by the sign $-$ or $+$ to denote an unspecified negative or positive quantity. A letter may stand for a parameter or a variable, and I will use the expressions 'signed parameter' and 'signed variable' to reflect the role of the letter in the situation under discussion. In both signed numbers and signed letters, the $+$ sign is often omitted (as it is in the Common Core definition), and here my focus is on the $-$ sign. Signed numbers and signed letters have two parts, the number or letter that indicates a specified or unspecified number greater than 0, and the sign. An integer, on the other hand, is a unity, a number that, when added to another integer, gives the sum 0.

Thus for example, -3 as a signed number is the number 3 prefixed by the sign $-$. But, as an integer, -3 is the solution of $? + 3 = 0$; "all we need to know about -3 is that when you add 3 to it you get 0" (Gowers, p. 26). Similarly, $-a$ as a signed letter is the natural number a prefixed by the sign $-$. But, as an integer, $-a$ is the solution of $? + a = 0$, and a itself might be less than 0.

The purpose of this short communication is to provide some historical evidence to show the role of signed letters in the development of algebra. In particular, some ingenious and correct ways that mathematicians used signed letters for expressing generality are shown, that now seem cumbersome and inefficient. It is suggested that we should be more aware and explicit about the use of signed letters and numbers in schools today.

Negative numbers at the advent of algebra

The historical rejection of integers as numbers is well known. Eventually, their utility was one of the most important reasons for their acceptance and development. For example, as Hefendehl-Hebeker (1991) writes "using negative numbers one can quickly and effectively solve problems that make sense only for positive numbers and have only positive solutions" (p. 32). An important factor in the gradual acceptance of negative numbers was due to the usefulness of the rules of signs applied to signed numbers and signed letters. According to Nathaniel Hammond's 1742 *The Elements of Algebra in a New and Easy Method* [2] the rules are as follows :

When the quantities to be multiplied have like signs, that is, they are both affirmative, or both negative, then set or join the letters together, and to them prefix the sign $+$, which will be the product required. (Art. 9, p. 18)

When the signs of the two quantities that are to be multiplied are one affirmative and the other negative, then multiply the quantities as before directed, but to their product prefix the negative sign, or $-$. (Art. 16, p. 26)

With a slight change of language (*e.g.*, using 'positive' instead of 'affirmative'), this is "the rule for multiplying" given by the most popular dedicated educational site in the UK [3]:

When the signs are different the answer is negative.

When the signs are the same the answer is positive.

Signed numbers and signed letters with the correct use of the rules of signs suffice for getting the correct answer in