Students from grade 2 to grade 10 solving a Fermi problem: analysis of emerging models

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Abstract

104 students aged 8 to 16 worked on one Fermi problem involving estimating the number of people that can fit in their school playground. We present a qualitative analysis of the different mathematical models developed by the students. The analysis of the students' written productions is based on the identification of the model of elements distribution and the strategy used. The results show how the students adapt their solutions in order to tackle the problem from their available knowledge. Indeed, younger students have important difficulties to deal with two-dimensional mathematical contents, but they overcome them by simplifying the problem. Finally, we also discuss the possibilities of using the proposed problem as part of a sequence to promote mathematical modelling in each educational stage, in basis of the potentialities identified in our analysis.

Keywords: Fermi problems, mathematical modelling, modelling sequences

Introduction

Since ICMI 14 (Blum 2002), a movement has been developed in mathematical education with the main idea of creating classroom activities that evidence the close relationship between mathematics and the world around us. From this perspective, mathematical modelling is the use of mathematics to describe (i.e., model) a real-world situation and infer information about the situation through mathematical calculations and analysis.

The relevance of applying mathematics to problems in different real-world contexts is captured by several curricula worldwide, such as Australia (ACARA, 2016) or the USA (CCSI 2010). In the particular case of Spain, curricula have recently explicitly included the practice of mathematization and modelling processes (Royal Decree 1105/2014, in Real Decreto 1105/ 2014), both in real life and in strictly mathematical contexts. Mathematical modelling can also support the learning of mathematics in terms of motivation, comprehension, and retention, and, in particular, mathematical modelling demonstrates what mathematics are and how can be used (Blum and Borromeo Ferri 2009). Nevertheless, some studies have identified that, even though curricula emphasize the use of modelling, teachers have difficulties related to the design and implementation of modelling tasks (Borromeo Ferri and Blum 2013), whereby the knowledge on how models are constructed by students at each age will prove useful to design activities that are adequately suited to each level.

In this work we use Fermi problems as mathematical modelling activities that allow us to study the models generated by the students. Such tasks are named after Enrico Fermi (1901-1954) a physicist and Nobel Prize winner who used them in his lectures to promote the reasoning needed to work effectively in the laboratory. Ärlebäck (2009) defines Fermi problems as "*open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations* (p. 331)". In previous investigations, Fermi problems have been used as a type of activity to introduce mathematical modelling in primary and secondary schools (Albarracín and Gorgorió 2014; Ärlebäck 2009; Peter-Koop 2004; Ferrando, Albarracín, Gallart, García-Raffi and Gorgorió 2017). Borromeo Ferri (2018) stresses that Fermi problems are useful to introduce modelling in the classroom since they are a type of task that allows students to develop their own problem-solving strategies based on laying out questions about real-life aspects without providing much information on the phenomena to be studied. For instance, this kind of problems fosters mathematical thinking (Sriraman and Knott 2009). Following Ärlebäck (2009), Fermi problems stand out for their approachability as educational resources in the sense that they can be solved by students of different ages, and that they can generate answers at different levels of complexity. However, there are no available studies to back it up, nor do we have specific knowledge of how the aforementioned complexity differences arise. Determining the potentiality of a Fermi problem to promote the creation of different mathematical models in students of different ages should provide us with enough knowledge to design educational sequences that allow students to develop modelling competences.

In previous studies (Albarracín and Gorgorió 2014), we have analysed the way students of secondary education (12 to 16 years of age) solve Fermi problems involving the estimation of the number of elements in an enclosure. Other studies suggest the existence of differences in the concepts that support the models developed by students aged 12 to 16 in secondary education (Ferrando, Albarracín, Gallart, García-Raffi and Gorgorió 2017) from those developed by students of ages 10 to 12 in primary education (Albarracín and Gorgorió 2019). That is why, we are driven to analyse the type of models that students between 8 and 16 years of age are able to develop when working on a single Fermi problem.

In the context of solving a Fermi problem as a one-off modelling activity without teacher support, the present work sets out one main objective aimed at answering the following question: What models emerge when 8 to 16-year-old students solve the same Fermi problem? Our purpose is to characterize the models developed by the students in order to understand how they approach problem solving and to establish the role of the activity as a modelling task.

Conceptual framework: mathematical modelling

Applying mathematical knowledge in order to understand real phenomena or to be able to make predictions implies a process of mathematization of reality. The processes of mathematization of reality are far from simple for the students (Geiger, Stillman, Brown, Galbraith and Niss 2018), and that is why there is a wide discussion in the literature on the use of mathematical modelling in classrooms. In practice, different points of view regarding the nature of the modelling processes coexist, depending on their educational objectives and the understanding of the relationship between mathematics and reality (Blomhøj 2009; Kaiser and Sriraman 2006). During the modelling process a mathematical model is developed and is one of the products obtained from modelling activity. This mathematical model can be used later to solve new mathematical problems in other situations that can be described in similar terms.

Following Lesh and Harel (2003), we consider a mathematical model to include concepts and procedures interrelated, even though students in the educational stages studied herein would explain the concepts they base their solutions on with great difficulty. The definition we work with in our analysis is the following:

> "Models are conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience based metaphors. Their purposes are to construct, describe or explain other system(s).
>
> Models include both: (a) a conceptual system for describing or explaining the relevant mathematical objects, relations, actions, patterns, and regularities that are attributed to the problem-solving situation; and (b) accompanying procedures for generating useful constructions, manipulations, or predictions for achieving clearly recognized goals." (Lesh and Harel 2003, p. 159)

With this viewpoint, we understand that a mathematical model is a construct in which we can identify two types of elements, conceptual and procedural (Hiebert and Lefevre 1986), with the purpose of describing a system –that is generally complex– and that will be expressed using different types of language.

In our work, we place modelling activities within the framework of *contextual modelling* (Lesh and Doerr 2003). Such activities are constructed upon a series of instructional principles (Lesh et al. 2000): the activity must appear meaningful to the students; it should allow for the creation and evaluation of models; the work of the students has to be adequately documented; and the constructed models should be able to be generalized. In this type of activities, it is expected that the students –working in small groups (Zawojewski, Lesh and English 2003), and facing a significant and relevant situation– must come up with, expand and perfect their own mathematical constructs.

Modelling activities can be embedded in sequences that allow students to develop and refine their models. In this paper we focus on the characteristics of the models generated by students of different ages for a single activity, but we are interested in interpreting the results from the perspective of using the problem studied in a broader sequence appropriate to each educational level. In Lesh et al. (2003), the authors describe the principles of so-called model development sequences. Model development sequences consist of three structurally related activities: model eliciting activities (MEA) which aim is to encourage students to generate descriptions, explanations and constructions that reveal how they interpret situations; model-exploration activities (MXA) that focus on the structure of the models built by students; and model application activities (MAA) that lead the students to transform the previously constructed models in order to afford a more complex task (McLean and Doerr, 2016). In this work we just focus our analysis on the solution of a single Fermi problem, but our aim is that this analysis highlights the role of this particular task in the design of a model development sequence. In addition, Ärlebäck and Doerr (2015) emphasized that special attention should be given to the evolving learning space promoted by the initial MEA. Therefore, the precise determination of the different options to be followed by the students in this activity is a key aspect of the sequence design. Indeed, the activities that follow the MEA should allow students to refine –or even redefine- their models by contrasting the weaknesses and strengths of different types of representations; using language in a precise manner; and using representations purposely and productively.

Research background: Fermi problems

According to Efthimiou and Llewellyn (2007), Fermi problems often have statements with diffuse formulations that provide little information. However, with a more detailed analysis, they can be decomposed into simpler problems in order to attain the answer to the original question. From the viewpoint of mathematical education, the need to establish which factors are most relevant to a phenomenon studied -establishing the sub-problems to be solved- in a Fermi problem is connected to aspects studied in the field of mathematical modelling. The available research on Fermi problems as tools to promote modelling includes a small number of investigations. We outline in this section those references on which we have based our work.

Peter-Koop (2004, 2009) used Fermi problems with students aged 9 and 10 to investigate the solution strategies employed by the students. Her study shows that students of these ages are able to approach the problems presented and give meaning to the solutions obtained. She also observed that the students developed new mathematical knowledge from a solution that contains a multi-cyclical modelling process. Ärlebäck (2009) studied the way secondary school students solve Fermi problems. In his analysis, Ärlebäck used an analysis tool (Modelling Activity Diagrams) to identify the stages at which the students develop the mathematical model that will solve the problem. This author has pointed out that students use a large amount of extra-mathematical information when carrying out the estimations required, and also when validating the results obtained.

Based on the pioneering proposals of Peter-Koop (2004, 2009) and Ärlebäck (2009) on the study of modelling processes arisen from the solution of Fermi problems, our research team has developed several investigations in order to study the use of such problems to introduce modelling in Spanish primary and secondary school classrooms. To this end, we have used problems that require the estimation of large quantities as a sub-class of Fermi problems, given that their nature forces students to generate mathematical processes to solve them, offering no possibility to utilize exhaustive counting.

In a previous work (Albarracín and Gorgorió 2014), we have analysed the way students (grades 7 to 10) solve tasks involving the estimation of the number of elements in an enclosure as Fermi problems. The following is a list of the elements identified in the outputs of the students when they solve this kind of tasks. We give some details because this is our starting point in the analysis of the students' productions.

- Exhaustive counting: the student suggests counting all the elements of the group of objects one by one.
- External source: the student passes on the responsibility of answering the problem to a third party, who supposedly has the necessary information.
- Concentration measures: the student bases the solution on determining the numerical proportion that reflects the relationship between two magnitudes, such as population density.
- Reference point: a reference point is an object which is a suitable unit of measurement according to the estimator.
- Grid distribution: the student uses a mental image of the distribution of objects over a grid, then estimates the total number of objects found in each unit area, and finally applies the rule of the product to establish the final result.

This study should lead us to better understand how, according to their age - and therefore, their mathematical and extra-mathematical knowledge - students create mathematical models. This is particularly interesting for students below the age of 10, whose modelling competences are not extensively covered in the literature (Stohlmann and Albarracín 2016), although it has been noted they used to apply their informal and personal knowledge to realistic mathematical problems (see for instance English & Watters 2005; English 2013; and Kazak et al. 2018). Indeed, the realistic context of a modelling activity can sometimes help students develop a model but it can also be an obstacle (English 2015).

Methodology

Design of the Experience and Data Collection

The experience was carried out in a primary school and a high school of the same town. We specifically collected data of the classroom work done by students in second, fourth and sixth grade of primary education, and from second and fourth grade of secondary education, covering the age range from 8 to 16. None of the students had previous experience solving modelling problems neither estimation tasks, the participation in the experience was compulsory for all the students of the natural groups involved. In each grade, the students, working in group of four, solved the same Fermi problem consisting of estimating the number of people that could fit in the school playground during a concert. Table 1 displays the number of students and working groups by grade that participated in the study:

Table 1. Summary of the participant of the experience

Grade	Number of students	Number of groups
2 (8 years)	23	6
4 (10 years)	19	4
6 (12 years)	20	5
8 (14 years)	20	5
10 (16 years)	22	7
Total	104	26

The statement of the problem provided to the students was the same, regardless of their age. The translation of the statement is the following:

> The school is organizing a concert to celebrate the year-end party on the 22nd of June. To do so, we need your help to choose the most suitable space to hold the concert. The preferred option is to hold it in the school playground, but we are not sure how many people it can fit. Considering this, how many people do you think could fit in the playground?

It is necessary to point out that the students worked in situ and, consequently, the characteristics of each school's playground are part of the problem. However, in both schools the playground has a rectangular shape. Data collection was carried out in sessions of 90 minutes for each grade, during which the teacher restricted himself to clarifying aspects of the problem statement. Firstly, the problem statement was handed to the students in the classroom and they were allowed to discuss their solution proposals in small groups of four students, before going to the playground to collect any necessary data. When there were collected, the students went back to the classroom to solve the problem and to write out their solution reports, where they were required to specify the methods used and results obtained. Such conditions were the same in all the natural groups involved in the experience. Thus, the data available to us from this study are the written reports of each group of four students.

Data Analysis

The analysis of the data is based on the qualitative analysis of the students' written productions. To characterise mathematical models developed by students, we have based on previous research. Ferrando, Albarracín, Gallart, García-Raffi & Gorgorió (2017) present an analysis tool based on the model definition established by Lesh and Harel (2003). The written solution collected in each of the groups' work shows a series of actions and previous decisions leading to the use of specific mathematical procedures and concepts.

For example, in Figure 2 we have the written production of a group of 2nd grade students. The students have represented the schoolyard where the concert is to be held. Since there are a series of fitted rectangles represented and we see some people drawn on one of the sides, we can deduce that they want those attending the concert to fill the space by progressively covering the perimeter until covering all (5 fitted rectangles are perceived). Furthermore, the students have made an operation (5x118=590). This allows us to understand that, indeed, the procedure is based on a particular distribution of the elements and on the (erroneous) hypothesis that all the fitted rectangles have the same perimeter.

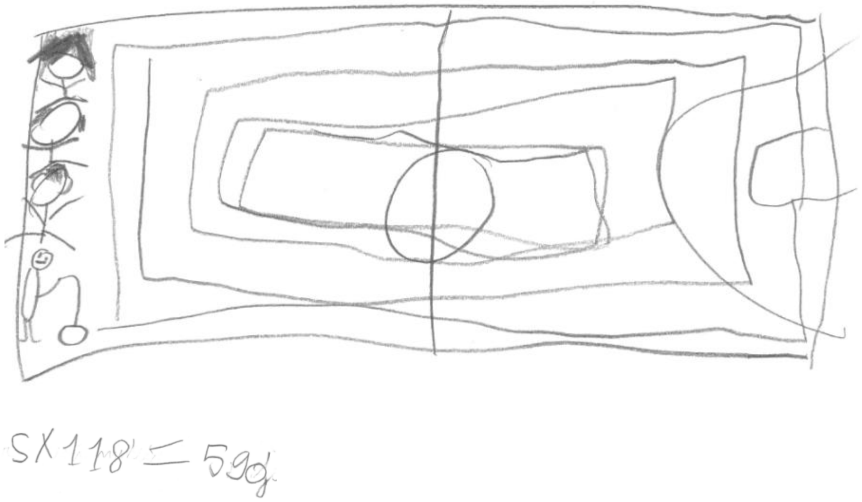

Figure 1. Production of a group from 2nd grade students.

Hence, the analysis consists on identifying clearly in the students' work a developed chain of procedures. We interpret that the chosen procedures are related to a conceptual system chosen to represent the studied phenomenon. In this way, the action plan executed as a succession of procedures evidences a model.

As we will explain in the next section, this analysis provides us with an objective view on the complexity depending on the models used by students to represent the problem situation. Indeed, the problem presented certain degree of complexity, that is why the students have to choose between different strategies, determined by the chosen model for the arrangement of people in the playground. Therefore, each solution includes of two main elements: a) an emerging model that refers to an idealized arrangement of people of which we want to estimate the number that can fit in the rectangular playground; and b) the necessary strategies to obtain an estimate of the number.

To ensure that the analysis is reliable, an iterative qualitative analysis has been carried out. In fact, once all productions have been characterised, we have gone back and, in some cases, modified the initial categorisations. Moreover, the analysis was carried out in a first iteration independently by three members of the research team and then collaboratively in order to reach a consensus on all possible dissenting aspects. Categorizations were constantly checked using the whole data set.

Characterizing Elements of the Students' Output

Emerging Models to Represent the Distribution of Attendants

The problem is set in a two-dimensional situation. Indeed, students are expected to argue for the estimation of the number of elements in a rectangular surface. However, when analysing the students' output, we have observed that many of them approach the problem from a one-dimensional perspective.

The following example of one of the analysed productions (see Figure 2) can clarify this idea. These students evidenced a specific procedure to carry out the estimation based on lining up and counting to determine how many people are necessary to fill the playground in its two dimensions, establishing two similar problems, in order to end up calculating the product of these two values. These students idealized the distribution of people in the playground in rows and columns, creating an emerging model that actually corresponds to a *grid distribution* (Albarracín and Gorgorió 2014). We understand that the students did not consider the surface area of the playground as a two dimensional whole and, by counting the number of people in rows and columns, their route to the final answer passed through breaking the initial problem into two one-dimensional problems: How many people can fit on the width of the playground? And, how many people can fit on the length of the playground?

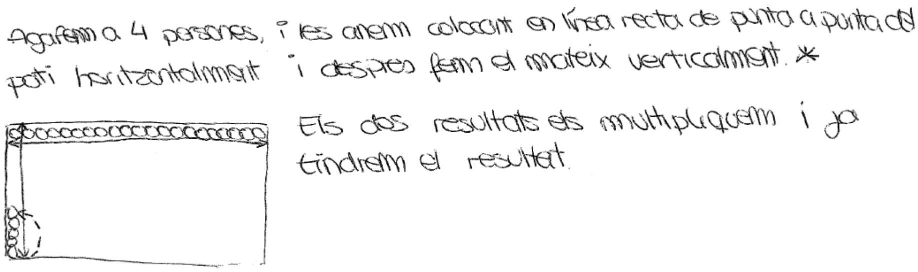

Figure 2. Proposal of 10th grade students. The students wrote: “You take four people and place them in a straight line from one side of the playground to the other. Then we do the same vertically. Then we multiply and obtain the result.”

Another identified model is that of *fitted rectangles*. In this case, as shown in Figure 1, the students idealize the arrangement of people as a series of fitted rectangles that decrease progressively in size. Although, as we have already commented, students are not able to transfer this size reduction when trying to obtain the result of the estimation.

Both of these emerging models, *grid distribution* and *fitted rectangles,* are defined by geometric patterns. We include them in a broader category type, denominated as *regular distribution,* in order to represent the distribution of people on the playground. In both cases, as we will see in the following section, the students reduce the two-dimensional problem to one dimension.

We have also found emerging models that correspond to distributions that don’t follow any particular pattern, the so-called *irregular* distributions. The use of models based on irregular distributions implies, in all cases, finding the surface area and, therefore, we will consider them to be two-dimensional models. We have found two kind of models based on irregular distributions, we present and illustrate them with examples in the following paragraphs. The first model consists of a *homogeneous irregular distribution*. This comprises the students whose reasoning was based on the space occupied by one or several people (that implies the use of *concentration measurements*), or those whose reasoning utilized the concept of density. In this type of model, we will only include the students who considered the distribution to be irregular (i.e. arranged without following any patterns) but homogeneous in terms of density or on the estimated space occupied by one or more people. In Quotation 1 we reproduce the translation of the solution developed by a group of 10th grade students. We can observe the reasoning of a group of students who, considering the surface area of the premises, intends to reach the total number of people on the basis of the space occupied by four people:

Quotation 1.

Firstly, we measure the length of the playground. After that, we measure its width and multiply it by the length, to obtain the total space of the playground. Then, we calculate the space taken up by the stage and we would subtract that from the previously obtained total.

After that, we would have four people stand, separated from one another, with enough space to move and dance a bit. We would calculate the space occupied by these people and we would divide the total space available by the number we have just obtained.

We would divide the resulting number by four to know how many people would fit in the playground.

The last model identified corresponds to the *heterogeneous irregular distribution* model, in the sense that it employs differentiated or variable densities or space occupied by one -or more. People. In Quotation 2 we reproduce the translation of the solution developed by a group of 10th grade ‘students:

Quotation 2.

Tracing a square of 1 by 1 metre, we have estimated that it should contain between 5 and 8 people, with enough space to be comfortable but without it being too empty. The number would also be variable, because more people would be in the centre than on the sides. This means an average of 6.5 people per square metre.

As previously mentioned, each of the emerging models of the distribution of the attendance in the rectangular enclosure requires carrying out different strategies associated to the model, in order to reach an estimate of the total number. The following section contains a detailed description of these strategies.

Strategies associated to the emerging models

The output of the students who chose models that reduce the problem to one single dimension –by establishing a *regular distribution*– evidences strategies associated to the counting of people arranged in line along a defined length. In these cases, the students explain the proposed strategy using sentences, graphs or even calculations, though they are not always able to carry it out successfully, or in some cases are not even able to approach it.

The simplest strategy of linear counting involves carrying out an *exhaustive count* (count A, for short). Figure 3 displays how this strategy is represented by students of 2nd grade of primary education. However, other students chose to establish an *effective count* (count B, for short), using the number of group members, as shown in Figure 1, where students count 4 by 4.

Figure 3. Exhaustive count strategy, 2nd grade.

Other groups chose to avoid direct counting and made use of measuring tools. These groups simply obtained measurements –in width or length– of a standing person and therefore, they divided the total length of one of the sides by the obtained measurement for one person.

Others used strategies that imply the use of intensive magnitudes (quotient of two magnitudes, in this case number of people in a linear metre) and, therefore they multiplied this value by the total length of one of the sides of the playground. In both cases, in order to obtain the final result, they just make arithmetic operations (multiplication and or additions). In the following we show an example of each strategy.

In Figure 4 we find an example of the so-called *linear reference point* (LRP, for short) strategy performed by students of 6th grade. In this case, the students supposed that a person takes up 30 cm in a line, and thus dividing the length of each side measured in centimetres by 30 obtained the number of people that can be arranged along either side. The algorithms show the calculations for 4100:30=136 and 2100:30=70.

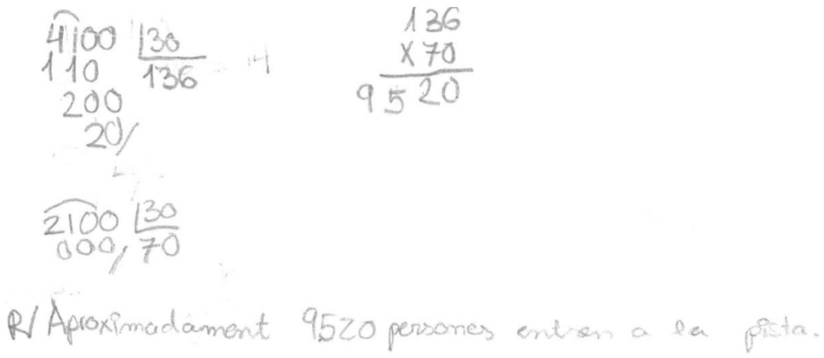

Figure 4. Solution of a group in 6th grade corresponding to grid model and LRP strategy

As for the strategy based on finding out the number of people that could fit in a linear metre, we present (see Figure 5) the production of a group of students that measured the width and length of the rectangular playground in metres and, considering that a metre can fit 2 people in line, they deduced that each side can fit 94 and 50 people, respectively, and multiplied both numbers. In this solution, there is an intuitive idea of density, however, used on a one-dimensional intensive magnitude –people/linear metre– which we will call a strategy of *linear concentration measure* (LCM, for short).

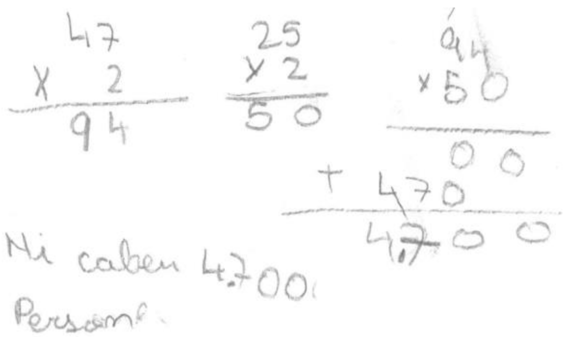

Figure 5. 4th grade production corresponding to grid model and LRP strategy.

In the two-dimensional models (corresponding to the models of *homogeneous or heterogeneous irregular distribution*), the students are forced to calculate the area of the playground. In this case, being a rectangular area, the calculation does not lead to significant difficulties (once students understand the concept of area) to the problem. Therefore, we do not find any strategies related to it –as we may find for the case of irregular shapes that may require, for instance, splitting up into sub-areas. However, this model could be approached using different strategies: either by measuring the area occupied by one or several people (*reference point*, RP for short), or using *concentration measure* (CM, for short), i.e. based on the hypothesis of the number of people that fit into a defined space (usually a square metre). Hence, the students that use the strategy of RP and CM strategies do so directly starting from the surface area. It is not possible, in this case, to use intermediate counting strategies. In quotations 1 and 2 previously we find, respectively, RP and CM strategies. Nevertheless, for the sake of completeness, we add the extract of another solution. In Figure 6 we can observe a part of the production of a group of 10th grade students that deduced the total number of people from the area defined, based on the hypothesis that one square metre can fit 6 people.

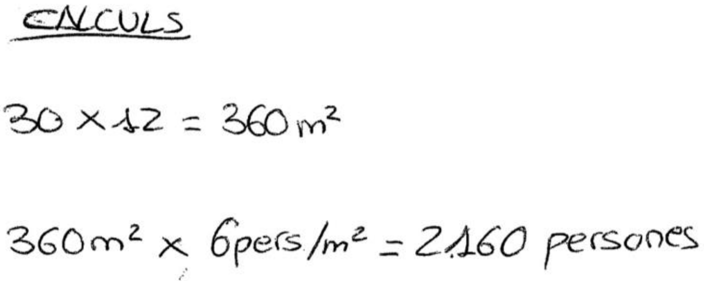

Figure 6. Production of a group in 10th grade group of students corresponding to irregular heterogeneous model and CM strategy

These estimations are carried out experimentally in some cases, while in others they are estimated without any explicit justification. With respect to the *RP strategy*, we found a production where, first (see Figure 7), the students based themselves on the space occupied by four people. However, when moving on to performing calculations, they observed that it was easier to operate with the space occupied by one person.

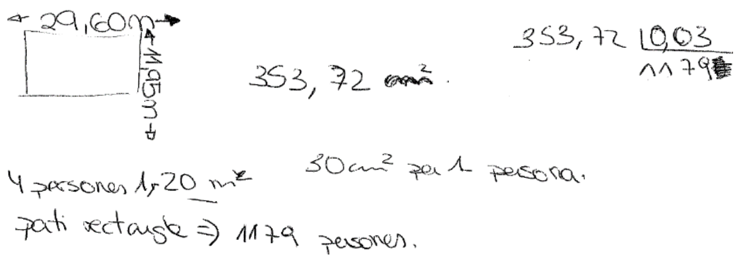

Figure 7. Production of a group in 10^{th} grade group of students corresponding to irregular heterogeneous model and RP strategy.

Results: Characterization of models

Once we defined the criteria for analysis, we moved on to describing the results. Since the emerging models are linked to strategies, we will show the evolution of emerging models and strategies in the same table. At the lowest levels (2^{nd} grade), not all groups were able to approach the creation of an emerging model for the distribution of people in the playground. However, though incomplete, such solutions are interesting because the students suggested different strategies to count the number of people that could fit in a certain length. This is why we have included two rows that are exclusively destined to the counting strategy.

In Table 2 we show the number of emerging model and the corresponding strategy used in each grade. We can observe that as the students' level increases, the amount of productions based on two-dimensional reasoning also increases and that the students develop different models that avoid confronting the two-dimensionality of the surface of the playground by establishing one-dimensional emerging models. In table 2 we show the number of emerging model and the corresponding strategy used in each grade. We can observe that as the students' level increases, the amount of productions based on two-dimensional reasoning also increases and that the students develop different models that avoid confronting the two-dimensionality of the surface of the playground by establishing one-dimensional emerging models.

Table 2. Emerging models and strategies used in each grade

	Model	Strategy	2^{nd}	4^{th}	6^{th}	8^{th}	10^{th}
1-dimensional	Only counting	Exhaustive	3				
		Effective	1				
	Fitted rectangles		1				
	Grid distribution	Exhaustive	1				
		Effective		2	1		1
		Linear Reference Point		1	2	1	1
		Linear Concentration Measurement		1	1	1	
2-dimensional	Irregular homogeneous	Reference Point			1	1	2
		Concentration Measurement				2	2

	Irregular heterogeneous	Concentration Measurement	1

In terms of the use of intensive quantities, we have already mentioned that they are not exclusive of two-dimensional solutions, but are also found in solutions that correspond to, for instance, the grid distribution model. In these cases, the students used the magnitude of people per linear metre, based on the length in metres, to find the number of people that fit on one of the sides of the field. The grid distribution model appears to be the most widespread proposal among students, although distinctions can be made in the specific strategy for dealing with the measures included in it. We also observe that models based on reference point and concentration measures, which previous studies have identified as predominant in secondary education (Albarracín and Gorgorió 2014), appear as strategies for one-dimensional models, as in the case of the Linear Concentration Measurement.

A global view of the observed models allows us to observe two relevant aspects in order to establish the role of the problem as a modelling activity. In the first place, we observe that, from grade 2, new strategies emerge at each higher grade. This is the case of the grade 4 group that uses a Linear Concentration Measurement or the grade 6 group that develops a two-dimensional model. The second aspect we highlight is that in each class group there is a wide variety of models. This is because the problem is an open task and students use different characteristics of reality to solve it. At the same time, this variety opens the options for classroom discussions among students about different solution methods and their adequacy to the problem. This fact is what allows us to discuss in the following section the role of the problem in a modelling sequence adapted to each educational level.

The Role of the Problem of the People in the Playground as an Activity to Promote Modelling

In this section we discuss the role that the analysed problem could play in a model development sequence with the purpose of promoting modelling with students of different ages. The study presented in the previous sections shows that the posed problem allows different approaches and reveals a trend of predominant emerging models developed at each educational stage. The problem we used specifically discloses the differences between students of different ages in the way they approach expanding their knowledge on the concept of surface area and the processes related to estimating the number of elements that can fit on a defined surface. Initially, we understand that the students have to elaborate the model based on the knowledge they have already established, otherwise they would be unable to construct a useful model by themselves. On the other hand, our results show that the students tend to use emerging models that are not necessarily the simplest or that are not always backed up by knowledge they have assimilated completely, see for instance the production shown in Figure 1 or in Figure 8. In this case students have some difficulties to use CM strategy, because they begin with a division instead of a multiplication. Generally, the students at higher educational stages do not use procedures that correspond to models developed by younger students. Indeed, 8 of 12 groups in grades 8 and 10 used two-dimensional models, which no groups in grades 2 and 4 used. This shows that, this task and the analysis of the models (and associated strategies) produced by students are particularly effective in uncovering students' conceptual knowledge.

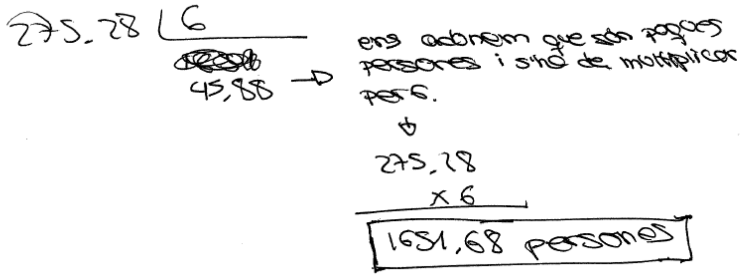

Figure 8. Production of a group in 10th grade group of students corresponding to irregular heterogeneous model and CM strategy

Indeed, the students in their first years of primary education predominantly use linear approximations to solve the given problem. We understand that this fact stems from the difficulties encountered when confronting a problem in which dealing with the distribution of people over a given surface area is unavoidable. This fact leads us to believe that the use of the studied problem on 8-year-old students cannot be the start (MEA) of a modelling activity. However, as shown in table 2, the 6 groups in 2nd grade were able to tackle the problem of estimating the number of people that fit in a line, some even suggesting strategies involving *effective counting*. We therefore suggest that the students do preliminary work on problems that prompt them to count the number of people that can be placed in a line; for instance, we can ask them how many people is needed to surround the playground. Moreover, in order to somehow encourage the students to use more sophisticated strategies to find the estimated number of elements in a line, we may suggest doing a similar activity as MXA task –counting the number of elements that fit in one dimension– but with elements that cannot be measured individually. One possible option would be to propose estimating the number of sheets of paper that are piled up on a table. Since it is neither simple nor manageable to reason using the space occupied by a single unit, the students would be forced to use the strategies of the linear reference point –for instance, using the space occupied by 10 sheets in the pile as main reasoning– or the linear concentration measure –reasoning based on the number of sheets that can fit in a 10-cm line. Therefore, in a sequence that is directed towards 2nd grade students, the natural role of the problem of estimating the number of people that can fit in a rectangular enclosure is an MAA for the students to get acquainted with the grid distribution model and the relation between surface area and multiplication.

The students in 4th and 6th grade established the appropriate number of people based on the grid distribution model, though after different procedures to determine the number of people that can fit along one of the dimensions of the playground. We understand that the studied problem allows the students to come up with the distribution grid model, which links up to the construction of measuring schemes of lengths and surfaces that are dealt with at such educational stages. Furthermore, solutions that include intensive magnitudes in one-dimensional reasoning are already dealt with in 4th grade. Therefore, the studied problem may act as an MEA (for students in 4th grade) and may be accompanied by activities that act upon the need to adequately measure irregular surfaces or complex or undefined spaces (such as in a demonstration). Students in grade 6 can work on the introduction of volume counting using the strategies developed in the activity in problems such as determining how many coins can fit in a cubic metre, which has shown that the students are able to develop a three-dimensional version of the model (Albarracín and Gorgorió 2014).

It is not until grade 6 that we find a group that deals with the surface of the playground as a whole. We therefore understand that it is in secondary education (that, in Spain, starts at grade 7) when working with the chosen problem allows for the treatment of an area as a whole and it is then that the model of the iteration of a unit can be introduced in a natural way to help the students to generate a more general and adaptable model such as that of population density. In this sense, the problem of counting the people in the playground should act as an MEA, allow to establish the basis of procedures and models to carry out object and surface counts in different contexts in which different conditions are changed, such as the shape of the space where objects are placed or the need to use different population densities in the same problem. Another option to be considered is the use of estimation problems involving recounting elements other than people, such as trees in a forest or cars on a road, which fosters specific strategies (such as, in this case, the use of reference point strategy).

Finally, we understand that coordinating the students' educational material at different educational levels and updating it with time should allow them to revisit a problem after several years with subsequent improvements in relation to cognition and skill maturity and knowledge. This means that not only it would be possible to work on the constructed models at each educational stage, but it would also enable to reflect on the improvement and development of the students' own knowledge, thus handling the self-concept of the students as mathematical problem solvers (Marsh et al. 2014). Therefore, we emphasize the need for students to work in the direction of connecting and coordinating previously established mathematical knowledge in order to generate useful mathematical models that are also transferable and generalizable, following the proposal of Ärlebäck and Doerr (2015).

Conclusions

The presented study establishes the basis of modelling activity sequences related to the treatment of magnitudes on a closed surface. We can state that the solution strategies identified and the models they are supported on feature a faithful characterization of the output expected of students in different age groups. In this sense, the results obtained in the first part of this work are coherent with the results of previous studies and are related to the difficulties in conferring meaning to the two-dimensionality of the surface area of flat figures (Fernández, De Bock, Verschaffel & Van Dooren 2014). We have subsequently observed that younger students are able to generate different types of geometric pattern distributions, thus widening the range of results obtained in previous studies, in which research had been conducted on students below the age of 10.

If we observe the distribution of the models generated by students in different age groups, we realise that the present study has limitations that can only be overcome by conducting extensive data collections in different schools. Given the exploratory nature of this study, it does include some limitations with respect to students developing different models to those identified based on previous experiences or cultural or curricular differences between countries. Despite these limitations, we understand that the obtained results displayed in table 2 evidence one of the main characteristics expected of the emerging models generated by the students and can be understood as a conceptual progression of the different models that can be used to solve the problem. Indeed, the analysis carried out clearly shows that the models obtained by the students give us truthful information about their conceptual knowledge that manifests itself from the procedures they choose to solve the problem by interpreting that concepts and procedures are combined in the elaboration of a mathematical model (Lesh and Harel 2003). The comparison of ideas generated by the students offers an opportunity to relate concepts. Moreover, the students are faced to complex situations that do not have only one single way of being solved. Along these lines, the results of this study confirm Ärlebäck's (2009) description of Fermi problems when he presents them as accessible to students of different educational levels and a source of discussion during lessons.

Additionally, the results obtained offer a refinement of students' models analysis made in previous work (Albarracín and Gorgorió 2014; Ferrando, Albarracín, Gallart, García-Raffi and Gorgorió 2017) and facilitate the design of modelling activity sequences to encourage learners to construct complex mathematical models and learn to adapt them to new situations. Thus, the analysis of the students' output is found to be a key tool to identify a problem or an activity's potentiality to promote modelling. In this way, we understand that analysing in isolation the productions of students of different ages for the same problem provides a tool for the design of modelling sequences that can allow for the integration of open activities with work related to specific curricular contents that can be identified in a previous experimentation.

Focusing on classroom work, the characterization of students' mathematical models can be used as knowledge for teachers to provide strategic-content help to students when they solve complex modelling problems (Stender & Kaiser 2015) taking into account that it has been observed that teachers need to develop knowledge on scaffolding students towards more sophisticated models instead of concluding the modelling process after having developed a simple model (Ng 2013). This study shows that the teacher can anticipate that ideas of different natures will arise in the classroom and that confronting them is an option for later developing more complex models that can be adapted to other situations to be studied. Finally, the differences observed in the models generated by students of different ages at the same problem suggest the possibility of studying how to revisit a problem can be a way of evidencing and valuing the learning of the students themselves.

Acknowledgements

Research supported by the projects EDU2017-84377-R and EDU2017-82427-R (Ministerio de Economía, Industria y Competitividad, Spain) and also 2017 SGR 497 (AGAUR, Generalitat de Catalunya).

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