$\mathbb{Z}_2\mathbb{Z}_4$-Additive Cyclic Codes: Kernel and Rank

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***Abstract*—A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is called cyclic if the set of coordinates can be partitioned into two subsets, the set of $\mathbb{Z}_2$ coordinates and the set of $\mathbb{Z}_4$ coordinates, such that any cyclic shift of the coordinates of both subsets leaves the code invariant. Let $\Phi(\mathcal{C})$ be the binary Gray map image of $\mathcal{C}$. We study the rank and the dimension of the kernel of a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code $\mathcal{C}$, that is, the dimensions of the binary linear codes $\langle\Phi(\mathcal{C})\rangle$ and $\ker(\Phi(\mathcal{C}))$. We give upper and lower bounds for these parameters. It is known that the codes $\langle\Phi(\mathcal{C})\rangle$ and $\ker(\Phi(\mathcal{C}))$ are binary images of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes that we denote by $\mathcal{R}(\mathcal{C})$ and $\mathcal{K}(\mathcal{C})$, respectively. Moreover, we show that $\mathcal{R}(\mathcal{C})$ and $\mathcal{K}(\mathcal{C})$ are also cyclic and we determine the generator polynomials of these codes in terms of the generator polynomials of the code $\mathcal{C}$.**

Keywords $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes, Gray map, kernel, rank.

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I. Introduction

Denote by $\mathbb{Z}_2$ and $\mathbb{Z}_4$ the rings of integers modulo 2 and modulo 4, respectively. We denote the space of n-tuples over these rings as $\mathbb{Z}_2^n$ and $\mathbb{Z}_4^n$. A binary code is any non-empty subset C of $\mathbb{Z}_2^n$, and if that subset is a vector space then we say that it is a linear code. Any non-empty subset $\mathcal{C}$ of $\mathbb{Z}_4^n$ is a quaternary code and a submodule of $\mathbb{Z}_4^n$ is called a linear code over $\mathbb{Z}_4$.

In 1994, Hammons et al. discovered that some good non-linear binary codes can be seen as the Gray map images of linear codes over $\mathbb{Z}_4$, [10]. From then on, the study of codes over $\mathbb{Z}_4$ and other finite rings has been developing and the construction of Gray maps has been a topic of study.

In Delsarte's 1973 paper (see [6]), he defined additive codes as subgroups of the underlying abelian group in a translation association scheme. For the binary Hamming scheme, namely when the underlying abelian group is of order 2^n, the only structures for the abelian group are those of the form $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, with $\alpha+2\beta = n$. This means that the subgroups $\mathcal{C}$ of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ are the only additive codes in a binary Hamming scheme. Hence, the study of codes in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ became important. These codes are called $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. The main properties, structure and duality of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are studied in [3]. In [1], $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes are introduced and in [4], the generator polynomials and duality are studied. Conditions for the linearity of the binary images of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes are established in [5].

Let C be a binary code on length n. We define the rank of C as $\mathrm{rank}(C) = \dim(\langle C \rangle)$ and the kernel of C as $\ker(C) = \{v \in \mathbb{Z}_2^n \mid v + C = C\}$ [2]. When the binary code is not linear, the rank and the dimension of the kernel give measures of "nonlinearity" of the code. It is well known (see [15]) that a nonlinear binary code containing the all-zero vector is the union of cosets of its kernel. In general, it is harder to deal with nonlinear codes than with linear ones. However, the representation of nonlinear binary codes as the union of cosets of the kernel allows some efficient algorithms to work with these codes as it is shown in [17].

Two binary codes, C_1 and C_2, are said to be equivalent if there is a vector $v \in \mathbb{Z}_2^n$ and a permutation of coordinates π such that $C_2 = \{v + \pi(c) \mid c \in C_1\}$. If C_1 and C_2 are equivalent, then they have the same rank and the same dimension of the kernel. Therefore, both, the rank and the dimension of the kernel, are invariants that are used to distinguish between nonequivalent binary codes and, therefore, to classify them. These invariants are studied for many different well-known families of codes, for example, perfect codes in [16] or binary Hadamard codes in [13]. In fact, for some families of codes, they are used to give a complete classification as in the case of $\mathbb{Z}_4$-linear Hadamard codes in [11], [14].

Both parameters have been the topic of study for $\mathbb{Z}_4$ and $\mathbb{Z}_2\mathbb{Z}_4$-additive codes in several papers (e.g. [8], [9], [11], [14]) and also for cyclic codes over $\mathbb{Z}_4$ in [7]. In this paper, we study the rank and the kernel of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes when such codes are also cyclic, taking into account the known results for general codes over $\mathbb{Z}_4$ and $\mathbb{Z}_2\mathbb{Z}_4$-additive codes [7]–[9].

The paper is organized as follows. In Section II, we recall the necessary concepts and properties on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. In Section III, we give the main results of the paper about the rank and the kernel of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. We prove that both the binary span and the kernel are binary images of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes, we determine the possible values of the dimensions of the corresponding binary images, and we compute the generator polynomials of these codes.

II. Preliminaries

A. $\mathbb{Z}_2\mathbb{Z}_4$-*additive codes*

A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ (see [3]). Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, it is also isomorphic to an additive group of the form $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ and it has $|\mathcal{C}| = 2^{\gamma+2\delta}$ codewords.

Let X (respectively Y) be the set of $\mathbb{Z}_2$ (respectively $\mathbb{Z}_4$) coordinate positions, so $|X| = \alpha$ and $|Y| = \beta$. Unless otherwise stated, the set X corresponds to the first α coordinates and Y corresponds to the last β coordinates. Let $\mathcal{C}_X$ be the binary punctured code of $\mathcal{C}$ formed by deleting the coordinates outside X. Define similarly the quaternary code $\mathcal{C}_Y$.

Let $\mathcal{C}_b$ be the subcode of $\mathcal{C}$ generated by all order two codewords and let κ be the dimension of $(\mathcal{C}_b)_X$, which is a binary linear code. For the case $\alpha = 0$, we write $\kappa = 0$. With all these parameters, we say that a code $\mathcal{C}$ is of type $(\alpha, \beta; \gamma, \delta; \kappa)$.

For a vector $\mathbf{u} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ we write $\mathbf{u} = (u \mid u')$, where $u = (u_0, \dots, u_{\alpha-1}) \in \mathbb{Z}_2^\alpha$ and $u' = (u'_0, \dots, u'_{\beta-1}) \in \mathbb{Z}_4^\beta$.

In [3], it is shown that a $\mathbb{Z}_2\mathbb{Z}_4$-additive code is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with standard generator matrix of the form

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \mathbf{0} & S_b & S_q & R & I_\delta \end{array} \right), \qquad (1)$$

where I_k is the identity matrix of size $k \times k$; T_b, S_b are matrices over $\mathbb{Z}_2$; T_1, T_2, R are matrices over $\mathbb{Z}_4$ with all entries in $\{0,1\} \subset \mathbb{Z}_4$; and S_q is a matrix over $\mathbb{Z}_4$.

A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is said to be separable if $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$. Otherwise the code is said to be non-separable.

Let $u' = (u'_0, \ldots, u'_{n-1})$ be an element of $\mathbb{Z}_4^n$ such that $u'_i = \tilde{u}'_i + 2\hat{u}'_i$, for $i = 0, \ldots, n-1$ and with $\tilde{u}'_i, \hat{u}'_i \in \{0,1\}$. As in [10], the *Gray map* ϕ from $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^{2n}$ is defined by

$$\phi(u') = (\hat{u}'_0, \ldots, \hat{u}'_{n-1}, \tilde{u}'_0 + \hat{u}'_0, \ldots, \tilde{u}'_{n-1} + \hat{u}'_{n-1}).$$

The *extended Gray map* Φ is the map from $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ to $\mathbb{Z}_2^{\alpha+2\beta}$ given by

$$\Phi(u \mid u') = (u \mid \phi(u')).$$

B. $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes

Cyclic codes have been a primary area of study for coding theory, [12]. Recently, the class of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes has been defined in [1].

Let $\mathbf{u} = (u \mid u') \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, then the cyclic shift π is given by $\pi(\mathbf{u}) = (\pi(u) \mid \pi(u'))$ where $\pi(u) = \pi(u_0, u_1, \ldots, u_{\alpha-1}) = (u_{\alpha-1}, u_0, u_1, \ldots, u_{\alpha-2})$ and $\pi(u') = (u'_{\beta-1}, u'_0, u'_1, \ldots, u'_{\beta-2})$. We say that a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is cyclic if $\pi(\mathcal{C}) = \mathcal{C}$.

There exists a bijection between $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and $R_{\alpha,\beta} = \mathbb{Z}_2[x]/(x^\alpha - 1) \times \mathbb{Z}_4[x]/(x^\beta - 1)$ given by

$$(u_0, u_1, \ldots, u_{\alpha-1} \mid u'_0, \ldots, u'_{\beta-1}) \mapsto$$
$$(u_0 + u_1 x + \cdots + u_{\alpha-1}x^{\alpha-1} \mid u'_0 + \cdots + u'_{\beta-1}x^{\beta-1}).$$

Therefore, as usual in the study of cyclic codes, any codeword is identified as a vector or as a polynomial.

From now on, the binary reduction of a polynomial $p(x) \in \mathbb{Z}_4[x]$ will be denoted by $\tilde{p}(x)$. Let $p(x) \in \mathbb{Z}_4[x]$ and $(b(x) \mid a(x)) \in R_{\alpha,\beta}$ and consider the following multiplication $p(x) \star (b(x) \mid a(x)) = (\tilde{p}(x)b(x) \mid p(x)a(x))$. From [1], $R_{\alpha,\beta}$ is a $\mathbb{Z}_4[x]$-module with respect to this multiplication.

Let $u'(x) = \tilde{u}'(x) + 2\hat{u}'(x)$ be the polynomial representation of $u' \in \mathbb{Z}_4^n$. Then, the polynomial version of the Gray map is $\phi(u'(x)) = (\hat{u}'(x), \tilde{u}'(x) + \hat{u}'(x))$. In the following, a polynomial $p(x) \in \mathbb{Z}_2[x]$ or $\mathbb{Z}_4[x]$ will be denoted simply by p.

Using the polynomial representation, an equivalent definition of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes is the following.

Definition 2.1 ([1]): A subset $\mathcal{C} \subseteq R_{\alpha,\beta}$ is called a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code if $\mathcal{C}$ is a $\mathbb{Z}_4[x]$-submodule of $R_{\alpha,\beta}$.

From [1], if β is odd, we know that if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code then it is of the form

$$\langle (b \mid 0), (\ell \mid fh + 2f) \rangle, \tag{2}$$

where $fhg = x^\beta - 1$ in $\mathbb{Z}_4[x]$, b divides $x^\alpha - 1$ in $\mathbb{Z}_2[x]$, and we can assume that $deg(\ell) < deg(b)$. The polynomials satisfying these conditions are said to be in standard form. In this case, we have that $|C| = 2^{\alpha - deg(b)} 4^{deg(g)} 2^{deg(h)}$. From now on, we assume that β is odd. Then f, g and h are pairwise coprime polynomials. Since h and g are coprime, there exist polynomials λ and μ, that will be used later along the paper, such that

$$\lambda h + \mu g = 1. \tag{3}$$

Lemma 2.2 ([4, Corollary 2]): Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$. Then, b divides $\frac{x^\beta - 1}{\tilde{f}} \gcd(b, \ell)$ and b divides $\tilde{h} \gcd(b, \ell\tilde{g})$.

We can put the generator matrix (1) in the following form, called the standard form:

$$\left(\begin{array}{ccc|ccc} I_{\kappa_1} & T & T_{b_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{\kappa_2} & T_{b_2} & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-(\kappa_1+\kappa_2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S' & S & R & I_\delta \end{array} \right). \tag{4}$$

The next theorem relates the parameters of the type of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code to its generator polynomials.

Theorem 2.3 ([4, Theorem 5 and Proposition 6]): Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code of type $(\alpha, \beta; \gamma, \delta = \delta_1 + \delta_2; \kappa = \kappa_1 + \kappa_2)$ with $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$, where $fhg = x^\beta - 1$. Then,

$$\gamma = \alpha - \deg(b) + \deg(h),$$

$$\delta = \deg(g),$$

$$\kappa = \alpha - \deg(\gcd(\ell\tilde{g}, b)),$$

and

$$\kappa_1 = \alpha - \deg(b), \quad \kappa_2 = \deg(b) - \deg(\gcd(b, \ell\tilde{g})),$$

$$\delta_1 = \deg(\gcd(b, \ell\tilde{g})) - \deg(\gcd(b, \ell)), \text{ and}$$

$$\delta_2 = \deg(g) - \delta_1.$$

It is well known that if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then the $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ is not linear in general. The linearity of these codes was studied in [9]. The key to establish this linearity was the fact that

$$\Phi(\mathbf{v} + \mathbf{w}) = \Phi(\mathbf{v}) + \Phi(\mathbf{w}) + \Phi(2\mathbf{v} * \mathbf{w}), \qquad (5)$$

where $*$ denotes the component-wise product. It follows immediately that $\Phi(\mathcal{C})$ is linear if and only if $2\mathbf{v}*\mathbf{w} \in \mathcal{C}$, for all $\mathbf{v}, \mathbf{w} \in \mathcal{C}$.

It is shown in [5] that, for a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code $\mathcal{C}$, $\mathcal{C}_X$ is a binary cyclic code and $\mathcal{C}_Y$ is a linear cyclic code over $\mathbb{Z}_4$. Moreover, if $\Phi(\mathcal{C})$ is linear, then $\phi(\mathcal{C}_Y)$ is also linear but the converse is not true in general. The characterization of linear cyclic codes over $\mathbb{Z}_4$ of odd length whose Gray map images are linear binary codes was given in [18]. Let p be a divisor of $x^n - 1$ in $\mathbb{Z}_2[x]$ with n odd and let ξ be a primitive nth root of unity over $\mathbb{Z}_2$. The polynomial $(p \otimes p)$ is defined as the divisor of $x^n - 1$ in $\mathbb{Z}_2[x]$ whose roots are the products $\xi^i \xi^j$ such that ξ^i and ξ^j are roots of p.

Theorem 2.4 ([18, Theorem 20]): Let $\mathcal{D} = \langle fh + 2f \rangle$ be a $\mathbb{Z}_4$-additive cyclic code of odd length n and where $fhg = x^n - 1$. The following properties are equivalent:

1) $\gcd(\tilde{f}, (\tilde{g} \otimes \tilde{g})) = 1$ in $\mathbb{Z}_2[x]$;
2) $\phi(\mathcal{D})$ is a binary linear code of length $2n$.

This result was generalized for $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with β odd.

Theorem 2.5 ([5]): Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code of length $\alpha + \beta$, β odd, and where $fhg = x^\beta - 1$. The following properties are equivalent:

1) $\gcd(\frac{\tilde{f}b}{\gcd(b, \ell\tilde{g})}, (\tilde{g} \otimes \tilde{g})) = 1$ in $\mathbb{Z}_2[x]$;
2) $\Phi(\mathcal{C})$ is a binary linear code of length $\alpha + 2\beta$.

As a result, it is completely characterized when a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ has binary linear image under the Gray map, just considering its generator polynomials. The next step is study the rank and the dimension of the kernel for those codes whose image $\Phi(\mathcal{C})$ is not linear.

III. Kernel and Rank of $\mathbb{Z}_2\mathbb{Z}_4$-Additive Cyclic Codes

For an additive code $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, the kernel of $\Phi(\mathcal{C})$ is defined as $\ker(\Phi(\mathcal{C})) = \{v \in \mathbb{Z}_2^{\alpha+2\beta} \mid v + \Phi(\mathcal{C}) = \Phi(\mathcal{C})\}$. Define $\mathcal{K}(\mathcal{C}) = \{\mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \mid \Phi(\mathbf{v}) \in \ker(\Phi(\mathcal{C}))\}$. Let $\operatorname{rank}(\Phi(\mathcal{C})) = \dim(\langle \Phi(\mathcal{C}) \rangle)$ and $\mathcal{R}(\mathcal{C}) = \{\mathbf{v} \mid \mathbf{v} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta, \Phi(\mathbf{v}) \in \langle \Phi(\mathcal{C}) \rangle\}$. It is clear that $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{C} \subseteq \mathcal{R}(\mathcal{C})$.

It is known that if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then $\langle \Phi(\mathcal{C}) \rangle$ and $\ker(\Phi(\mathcal{C}))$ are both $\mathbb{Z}_2\mathbb{Z}_4$-linear codes ([9]). Therefore, $\mathcal{R}(\mathcal{C})$ and $\mathcal{K}(\mathcal{C})$ are both $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. In the next sections we will see that if the code $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, then $\mathcal{R}(\mathcal{C})$ and $\mathcal{K}(\mathcal{C})$ are also $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic. Therefore, the following proposition will be useful to relate the generator polynomials of $\mathcal{C}$ to the generator polynomials of $\mathcal{R}(\mathcal{C})$ and $\mathcal{K}(\mathcal{C})$.

Proposition 3.1: Let $\mathcal{C}_0 = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ and $\mathcal{C}_1 = \langle (b' \mid 0), (\ell' \mid f'h' + 2f') \rangle$ be $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes with $\mathcal{C}_0 \subseteq \mathcal{C}_1$. Then,

1) f' divides f;

2) $\gcd(b', \ell')$ divides $\gcd(b, \ell)$.

Proof: Since $\mathcal{C}_0 \subseteq \mathcal{C}_1$, we have that $(\mathcal{C}_0)_Y = \langle fh + 2f\rangle \subseteq (\mathcal{C}_1)_Y = \langle f'h' + 2f'\rangle$. Therefore, by [7, Theorem 3], f' divides f.

As $\mathcal{C}_0$ and $\mathcal{C}_1$ are cyclic $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, clearly $(\mathcal{C}_0)_X = \langle \gcd(b, \ell)\rangle$ and $(\mathcal{C}_1)_X = \langle \gcd(b', \ell')\rangle$. Finally, $(\mathcal{C}_0)_X \subseteq (\mathcal{C}_1)_X$ implies that $\gcd(b', \ell')$ divides $\gcd(b, \ell)$. □

In order to study the rank and the kernel of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$, it is necessary to consider the code $\mathcal{C}_b$.

Proposition 3.2 ([5]): Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code with $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f)\rangle$. Then, $\mathcal{C}_b = \langle (b \mid 0), (\tilde{\mu}\ell\tilde{g} \mid 2f)\rangle$.

Note that, if $\mathcal{C} = \mathcal{C}_b$, (5) is satisfied and the code $\Phi(\mathcal{C})$ is linear. In this case, $\delta = 0$ and, by Theorem 2.3, $g = 1$. Therefore $\mathcal{C}_b = \langle (b \mid 0), (\ell \mid 2f)\rangle$.

A. Kernel of $\mathbb{Z}_2\mathbb{Z}_4$-Additive Cyclic Codes

In this section, we will study the kernel of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. We will prove that, for a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code $\mathcal{C}$, the code $\mathcal{K}(\mathcal{C})$ is also cyclic and we will establish some properties of its generator polynomials. We will show that there does not exist a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code for all the possible values of the dimension of the kernel as for general $\mathbb{Z}_2\mathbb{Z}_4$-additive codes.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. By (5), we can give the following definition of $\mathcal{K}(\mathcal{C})$ (see [9]):

$$\mathcal{K}(\mathcal{C}) = \{\mathbf{v} \in \mathcal{C} \mid 2\mathbf{v} * \mathbf{w} \in \mathcal{C}, \forall \mathbf{w} \in \mathcal{C}\}.$$

Lemma 3.3: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Then, $\mathcal{K}(\mathcal{C})_Y \subseteq \mathcal{K}(\mathcal{C}_Y)$.

Proof: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic. Let $\mathbf{v} = (v \mid v') \in \mathcal{C}$. We have that $\mathbf{v} \in \mathcal{K}(\mathcal{C})$ if and only if $2\mathbf{v} * \mathbf{w} \in \mathcal{C}$, $\forall \mathbf{w} = (w \mid w') \in \mathcal{C}$. Since $2\mathbf{v} * \mathbf{w} = (0 \mid 2v' * w')$, we have that if $\mathbf{v} \in \mathcal{K}(\mathcal{C})$ then $v' \in \mathcal{K}(\mathcal{C}_Y)$ and the statement follows. □

Example 1: Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code in $\frac{\mathbb{Z}_2[x]}{\langle x-1\rangle} \times \frac{\mathbb{Z}_4[x]}{\langle x^3-1\rangle}$ generated by $\langle (1 \mid x+1)\rangle$, where $f = 1$ and $h = x - 1$. Note that $\mathcal{C}$ is of type $(1, 3; 1, 2; 1)$ and the generator matrix of $\mathcal{C}$ in standard form, (1), is

$$\left(\begin{array}{c|ccc} 1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right).$$

Since $f = 1$, by Theorem 2.4, we know that $\mathcal{K}(\mathcal{C}_Y) = \mathcal{C}_Y = \langle (x + 1)\rangle$. We have that the generator matrix of $\mathcal{K}(\mathcal{C})$ in standard form is

$$\left(\begin{array}{c|ccc} 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{array}\right)$$

and therefore $\mathcal{K}(\mathcal{C}) = \langle (1 \mid 2)\rangle$. Hence, $\mathcal{K}(\mathcal{C})_Y \subsetneq \mathcal{K}(\mathcal{C}_Y)$.

The following theorems determine an upper and a lower bound for the kernel of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code and that there exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ for all possible values of the kernel.

Theorem 3.4 ([9]): Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with parameters $(\alpha, \beta; \gamma, \delta; \kappa)$. Then $\gamma + \delta \leq \dim(\ker(\Phi(\mathcal{C})) \leq \gamma + 2\delta$.

Theorem 3.5 ([9]): Let $\alpha, \beta, \gamma, \delta, \kappa$ be integers satisfying

$$\begin{array}{c} \alpha, \beta, \gamma, \delta, \kappa \geq 0, \quad \alpha + \beta > 0, \\ 0 < \delta + \gamma \leq \beta + \kappa \quad \text{and} \quad \kappa \leq \min(\alpha, \gamma). \end{array} \tag{6}$$

Then, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $\dim(\ker(C)) = \gamma + 2\delta - \bar{k}$ if and only if

$$\begin{cases} \bar{k} = 0, & \text{if} \quad s = 0, \\ \bar{k} \in \{0\} \cup \{2, \dots, \delta\} \text{ and } \bar{k} \text{ even}, & \text{if} \quad s = 1, \\ \bar{k} \in \{0\} \cup \{2, \dots, \delta\}, & \text{if} \quad s \geq 2, \end{cases}$$

where $s = \beta - (\gamma - \kappa) - \delta$.

We will see that not all possible values for the kernel of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ are possible if $\mathcal{C}$ is cyclic. First, we will determine some properties of the kernel of a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code.

Proposition 3.6: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. Then,

$$\alpha - \deg(b) + \deg(h) + \deg(g) \leq |\mathcal{K}(\mathcal{C})| \leq \alpha - \deg(b) + \deg(h) + 2\deg(g).$$

Proof: Straightforward from Theorems 3.4 and 2.3. □

Note that the upper bound is sharp when the code has binary linear image, i.e., $\mathcal{C} = \mathcal{K}(\mathcal{C})$. Moreover, the lower bound is tight when $\mathcal{K}(\mathcal{C})$ only has the all-zero vector and all order two codewords; that is, $\mathcal{C} = \mathcal{C}_b$.

Proposition 3.7: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, then $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$.

Proof: Let $\mathbf{v} = (v, v') \in \mathcal{K}(\mathcal{C})$. For all $\mathbf{w} = (w, w') \in \mathcal{C}$, $2\mathbf{v} * \mathbf{w} \in \mathcal{C}$ since $\mathbf{v}$ is in the kernel. Then $v \in \mathcal{C}_X$ and $2v' * w' \in \mathcal{C}_Y$, for all $w' \in \mathcal{C}_Y$ which gives $v' \in \mathcal{K}(\mathcal{C}_Y)$ and $\mathbf{v} \in \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$ and the result follows. □

Proposition 3.8: If $\mathcal{C}$ is a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code then $\mathcal{K}(\mathcal{C}) = \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$.

Proof: Let $\mathbf{v} \in \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$, then $2\mathbf{v} * \mathbf{w} = (\mathbf{0} \mid 2v' * w')$ for all $\mathbf{w} \in \mathcal{C}$. Since $v' \in \mathcal{K}(\mathcal{C}_Y)$, we have $2v' * w' \in \mathcal{C}_Y$. Moreover, since $\mathcal{C}$ is a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, $2\mathbf{v} * \mathbf{w} = (\mathbf{0} \mid 2v' * w') \in \mathcal{C}$. Therefore, $\mathbf{v} \in \mathcal{K}(\mathcal{C})$ and $\mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y) \subseteq \mathcal{K}(\mathcal{C})$.

Finally, by Proposition 3.7, we have $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$. □

The following example shows that if the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is non-separable, then the kernel is not necessarily $\mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$.

Example 2: Let $\mathcal{C} = \langle (1 \mid x+1) \rangle$, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code of Example 1, with $\mathcal{C}_X = \langle 1 \rangle$ and $\mathcal{C}_Y = \langle x+1 \rangle$. We have seen that $\mathcal{K}(\mathcal{C}_X) = \mathcal{C}_X$, $\mathcal{K}(\mathcal{C}_Y) = \mathcal{C}_Y$ and $\mathcal{K}(\mathcal{C}) = \langle (1 \mid 2) \rangle$. Therefore $\mathcal{K}(\mathcal{C}) \subsetneq \mathcal{C}_X \times \mathcal{K}(\mathcal{C}_Y)$.

Therefore, if the code is non-separable, the equality is not satisfied in general.

From Proposition 3.7, we obtain that $\dim(\ker(\Phi(\mathcal{C}))) \leq \kappa + \dim(\ker(\phi(\mathcal{C}_Y)))$. However, we can give a bound that is more accurate.

Proposition 3.9: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then $\dim(\ker(\Phi(\mathcal{C}))) \leq \kappa_1 + \dim(\ker(\phi(\mathcal{C}_Y)))$.

Proof: Define $\mathcal{C}_0 = \{\mathbf{v} = (v \mid v') \in \mathcal{C} \mid v' = \mathbf{0}\}$. We have that $\mathcal{C}_0 \subseteq \mathcal{K}(\mathcal{C})$, and $\dim(\ker(\mathcal{C}_0)) = \kappa_1$.

Let $\mathbf{v} = (v \mid v') \in \mathcal{K}(\mathcal{C})$. If $v' = \mathbf{0}$, then $\mathbf{v} \in \mathcal{C}_0$. Otherwise, $v' \in \mathcal{K}(\mathcal{C})_Y \subseteq \mathcal{K}(\mathcal{C}_Y)$ by Lemma 3.3 and, therefore, $\dim(\ker(\Phi(\mathcal{C}))) \leq \kappa_1 + \dim(\ker(\phi(\mathcal{C}_Y)))$. □

From the generator matrix G of $\mathcal{C}$ given in (4), we have that the code $\mathcal{C}_Y$ has a generator matrix of the form

$$\begin{pmatrix} 2T_2 & \mathbf{0} & \mathbf{0} \\ 2T_1 & 2I_{\gamma-(\kappa_1+\kappa_2)} & \mathbf{0} \\ S & R & I_\delta \end{pmatrix}. \quad (7)$$

By [4, Proposition 1], we know that the code $\mathcal{C}_Y$ has type $4^\delta 2^{\gamma-\kappa_1}$. The minimum value for the dimension of $\ker(\phi(\mathcal{C}_Y))$ is $\delta + \gamma - \kappa_1$.

Theorem 3.10: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. If $\mathcal{K}(\mathcal{C}_Y)$ has minimum size, then $\mathcal{K}(\mathcal{C})$ has minimum size.

Proof: If $\mathcal{K}(\mathcal{C}_Y)$ has minimum size, by Proposition 3.9, then $\dim(\ker(\Phi(\mathcal{C}))) \leq \kappa_1 + \delta + \gamma - \kappa_1 = \gamma + \delta$. □

In the previous statements, we compute an upper bound of the kernel of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ by considering the kernel of a code over $\mathbb{Z}_4$. Now we shall give the exact value of the kernel of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ in terms of $\mathcal{C}_X$ and the kernel of a linear code over $\mathbb{Z}_4$, $\mathcal{C}'$. As we have seen, in the case of a separable code $\mathcal{C}$, the code $\mathcal{C}'$ is exactly $\mathcal{C}_Y$ and the value $\dim(\ker(\Phi(\mathcal{C})))$ is $\dim(\ker(\mathcal{C}_X)) + \dim(\ker(\phi(\mathcal{C}_Y)))$, where $\mathcal{C}_Y$ is a cyclic code over $\mathbb{Z}_4$. If the code $\mathcal{C}$ is not separable, then $\mathcal{C}'$ is not necessarily cyclic.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix in the form of (4) and let $\mathcal{C}'$ be the subcode generated by

$$\left(\begin{array}{ccc|ccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-(\kappa_1+\kappa_2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S' & S & R & I_\delta \end{array}\right). \quad (8)$$

Theorem 3.11: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive with generator matrix in the form of (4) and let $\mathcal{C}'$ be the subcode generated by the matrix in (8). Then $\dim(\ker(\Phi(\mathcal{C})) = \kappa_1 + \kappa_2 + \dim(\ker(\phi(\mathcal{C}'_Y)))$.

Proof: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code with generator matrix G in the form of (4). Let $\{\mathbf{u}_i = (u_i \mid u'_i)\}_{i=1}^{\gamma}$ be the first γ rows and $\{\mathbf{v}_j = (v_j \mid v'_j)\}_{j=1}^{\delta}$ the last δ rows of G. Define the codes $\bar{\mathcal{C}} = \langle \{\mathbf{u}_i\}_{i=1}^{\kappa_1+\kappa_2} \rangle$, $\mathcal{C}' = \langle \{\mathbf{u}_i\}_{i=\kappa_1+\kappa_2+1}^{\gamma}, \{\mathbf{v}_j\}_{j=1}^{\delta} \rangle$.

By [9], $\mathbf{v} \in \mathcal{K}(\mathcal{C})$ if and only if $2\mathbf{v} * \mathbf{w} \in \mathcal{C}$ for all $\mathbf{w} \in \mathcal{C}$. We have that $\mathbf{v} \in \bar{C}$ is of order 2 and hence $2\mathbf{v} * \mathbf{w} = 0, \forall \mathbf{w} \in \mathcal{C}$. Then, $\bar{\mathcal{C}} \subseteq \mathcal{K}(\mathcal{C})$ and $\dim(\ker(\bar{\mathcal{C}})) = \kappa_1 + \kappa_2$. Let $\mathbf{v} = (v \mid v') \in \mathcal{C}'$. Since $2\mathbf{v} * \mathbf{w} = \mathbf{0}$ for all $\mathbf{w} \in \bar{\mathcal{C}}$, we have that $\mathbf{v} \in \mathcal{K}(\mathcal{C})$ if and only if $2\mathbf{v} * \mathbf{w} \in \mathcal{C}'$ for all $\mathbf{w} \in \mathcal{C}'$; that is, $\mathbf{v} \in \mathcal{K}(\mathcal{C}')$. Finally, $2\mathbf{v} * \mathbf{w} = (\mathbf{0} \mid 2v' * w') \in \mathcal{C}'$ if and only if $2v' * w' \in \mathcal{C}'_Y$, and hence $\dim(\ker(\Phi(\mathcal{C}'))) = \dim(\ker(\phi(\mathcal{C}'_Y)))$. Therefore, $\dim(\ker(\Phi(\mathcal{C}))) = \kappa_1 + \kappa_2 + \dim(\ker(\phi(\mathcal{C}'_Y)))$. □

Now we will establish the kernel of a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code taking into account its generator polynomials. In the following theorem we shall prove that if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, then $\mathcal{K}(\mathcal{C})$ is also cyclic. This result is a generalization of the case when $\mathcal{C}$ is a linear cyclic code over $\mathbb{Z}_4$ that is given in [7].

Theorem 3.12: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. Then $\mathcal{K}(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code.

Proof: We know that $\mathcal{K}(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code so we just have to show that if $\mathbf{u} = (u \mid u') \in \mathcal{K}(\mathcal{C})$ then $\pi(\mathbf{u}) \in \mathcal{K}(\mathcal{C})$. That is, we want to show that $2\pi(\mathbf{u}) * \mathbf{w} \in \mathcal{C}$, for all $\mathbf{w} \in \mathcal{C}$.

Let $\mathbf{u} \in \mathcal{K}(\mathcal{C}), \mathbf{w} \in \mathcal{C}$. Then $2\pi(\mathbf{u}) * \mathbf{w} = \pi(2\mathbf{u} * \pi^{-1}(\mathbf{w}))$. We have that $\mathbf{u} \in \mathcal{K}(\mathcal{C})$ and $\pi^{-1}(\mathbf{w}) \in \mathcal{C}$, therefore $2\mathbf{u} * \pi^{-1}(\mathbf{w}) \in \mathcal{C}$ by (5). Since the code $\mathcal{C}$ is cyclic, $\pi(2\mathbf{u} * \pi^{-1}(\mathbf{w})) \in \mathcal{C}$, which gives that $2\pi(\mathbf{u}) * \mathbf{w} \in \mathcal{C}$, and $\pi(\mathbf{u}) \in \mathcal{K}(\mathcal{C})$. □

Corollary 3.13: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where $fhg = x^\beta - 1$. Then, $\mathcal{K}(\mathcal{C}) = \langle (b_k \mid 0), (\ell_k \mid f_k h_k + 2f_k) \rangle$, where $f_k h_k g_k = x^\beta - 1$ and

1) f divides f_k;
2) $\gcd(b, \ell)$ divides $\gcd(b_k, \ell_k)$.

Proof: By Theorem 3.12, $\mathcal{K}(\mathcal{C})$ is cyclic and therefore $\mathcal{K}(\mathcal{C}) = \langle (b_k \mid 0), (\ell_k \mid f_k h_k + 2f_k) \rangle$, where $f_k h_k g_k = x^\beta - 1$. Since $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{C}$, the result follows from Proposition 3.1. □

Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where $fhg = x^\beta - 1$. In [5] it is proved that

$$\mathcal{C} = \langle (b \mid 0), (\ell' \mid fh), (\tilde{\mu}\ell\tilde{g} \mid 2f) \rangle, \quad (9)$$

where $\ell' = \ell - \tilde{\mu}\ell\tilde{g}$.

Lemma 3.14: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where $fhg = x^\beta - 1$. Let $\langle (b \mid 0), (\ell_k \mid fhk + 2f) \rangle \subset \mathcal{C}$, for $k|g$. Then $\ell_k = \tilde{k}\ell + (1 - \tilde{k})\tilde{\mu}\ell\tilde{g} \pmod b$, for μ in (3).

Proof: Let $\mathcal{C} = \langle (b \mid 0), (\ell' \mid fh), (\tilde{\mu}\ell\tilde{g} \mid 2f) \rangle$, where $\ell' = \ell - \tilde{\mu}\ell\tilde{g}$, as in (9). Since $(\ell_k \mid fhk + 2f) \in \mathcal{C}$ and $(\ell_k \mid fhk + 2f) = c_1(b \mid 0) + c_2(\ell' \mid fh) + c_3(\tilde{\mu}\ell\tilde{g} \mid 2f)$, we obtain $c_2 = k$, $c_3 = 1$ and $\ell_k = \tilde{k}\ell' + \tilde{\mu}\ell\tilde{g} \pmod b = \tilde{k}\ell + (1 - \tilde{k})\tilde{\mu}\ell\tilde{g} \pmod b$. □

Theorem 3.15: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where $fhg = x^\beta - 1$. Then, $\mathcal{K}(\mathcal{C}) = \langle (b \mid 0), (\ell_k \mid fhk + 2f) \rangle$, where k divides g and $\ell_k = \tilde{k}\ell + (1-\tilde{k})\tilde{\mu}\ell\tilde{g} \pmod b$, for μ in (3).

Proof: By Theorem 3.12, $\mathcal{K}(\mathcal{C})$ is cyclic and then $\mathcal{K}(\mathcal{C}) = \langle (b_k \mid 0), (\ell_k \mid f_k h_k + 2f_k) \rangle$. Clearly, $b_k = b$. Since $\mathcal{C}_b \subseteq \mathcal{K}(\mathcal{C}) \subseteq \mathcal{C}$, by Proposition 3.2 and Proposition 3.1, we conclude that $f_k = f$. Since $\mathcal{K}(\mathcal{C})_Y = \langle fh_k + 2f \rangle \subseteq \mathcal{C}_Y$, with an argument analogous to that of [7, Theorem 9] we obtain that $h_k = hk$ with k a divisor of g.

Let $\ell' = \ell - \tilde{\mu}\ell\tilde{g}$. By (9), $(\ell' \mid fh), (\tilde{\mu}\ell\tilde{g} \mid 2f) \in \mathcal{C}$. Therefore, $\ell_k = \tilde{k}\ell' + \tilde{\mu}\ell\tilde{g} \pmod b = \tilde{k}\ell + (1-\tilde{k})\tilde{\mu}\ell\tilde{g}$ $\pmod b$. □

Theorem 3.5 shows that there exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive code for all possible values of the kernel for a given type $(\alpha, \beta; \gamma, \delta; \kappa)$. Considering the last theorem, the next example illustrates that this result is not true for $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes; i.e., for a given type $(\alpha, \beta; \gamma, \delta; \kappa)$ there does not always exist a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code for all possible values of the kernel. Furthermore, it shows that there does not always exist a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code for a given type $(\alpha, \beta; \gamma, \delta; \kappa)$.

Example 3: By Theorem 3.5, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ of type $(2, 7;\ 2, 3;\ \kappa)$ with $\dim(\ker(\Phi(\mathcal{C}))) = k_d$, for all $k_d \in \{5, 6, 8\}$. We will see that there does not exist any cyclic $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(2, 7; 2, 3; \kappa)$, with dimension of the kernel in $\{6, 8\}$.

Let $\alpha = 2$ and $\beta = 7$. We have that $x^7 - 1 = (x-1)(x^3 + 2x^2 + x + 3)(x^3 + 3x^2 + 2x + 3)$. Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code of type $(2, 7; 2, 3; \kappa)$, where $fhg = x^7 - 1$.

By Theorem 2.3, $\deg(g) = 3$ and $\deg(b) = \deg(h) \leq 2$. Let $\{p_3, q_3\} = \{(x^3 + 2x^2 + x + 3), (x^3 + 3x^2 + 2x + 3)\}$. Assume without loss of generality that $g = p_3$ and, since $\deg(h) \leq 2$, we have that q_3 divides f. It is easy to see that $\gcd(q_3, (\tilde{p_3} \otimes \tilde{p_3})) \neq 1$ and therefore $\gcd(\frac{\tilde{f}b}{\gcd(b, \ell\tilde{g})}, (\tilde{g} \otimes \tilde{g})) \neq 1$. Hence, by Theorem 2.5, there does not exist a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(2, 7; 2, 3; \kappa)$ with linear Gray image. Thus, $\dim(\ker(\Phi(\mathcal{C}))) \neq 8$.

By Theorem 3.15, $\mathcal{K}(\mathcal{C}) = \langle (b \mid 0), (\ell_k \mid fhk + 2f) \rangle$ where k divides g. By the previous argument, $k \neq 1$ and then we have that $k = g = p_3$ and $\mathcal{K}(\mathcal{C}) = \langle (b \mid 0), (\ell_k \mid 2f) \rangle$. Therefore $\mathcal{K}(\mathcal{C})$ does not contain codewords of order 4, thus $\dim(\ker(\Phi(\mathcal{C}))) = \gamma + \delta = 5$.

Finally, we will give $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes of type $(2, 7; 2, 3; \kappa)$, for different values of κ. Recall that $\kappa \leq \min\{\alpha, \gamma\} = 2$ and $\kappa = \alpha - \deg(\gcd(b, \ell\tilde{g}))$, then

- $\kappa = 2$: $\mathcal{C} = \langle (x-1 \mid 0), (1 \mid (x^3+2x^2+x+3)(x-1) + 2(x^3+2x^2+x+3)) \rangle$, or $\mathcal{C} = \langle (x-1 \mid 0), (1 \mid (x^3+3x^2+2x+3)(x-1) + 2(x^3+3x^2+2x+3)) \rangle$.
- $\kappa = 1$: $\mathcal{C} = \langle (x-1 \mid 0), (0 \mid (x^3+3x^2+2x+3)(x-1) + 2(x^3+3x^2+2x+3)) \rangle$, or $\mathcal{C} = \langle (x-1 \mid 0), (0 \mid (x^3+2x^2+x+3)(x-1) + 2(x^3+2x^2+x+3)) \rangle$.
- $\kappa = 0$: In this case, $\deg(\gcd(b, \ell\tilde{g})) = 2$ and therefore, $\gcd(b, \ell\tilde{g}) = x^2 - 1$. Note that $\tilde{p}_3$ and $\tilde{q}_3$ are not divisors of $x^2 - 1$ over $\mathbb{Z}_2$, thus there does not exist ℓ with $\deg(\ell) < 2$ such that $\ell\tilde{g} = x^2 - 1$. There does not exist a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code of type $(2, 7; 2, 3; 0)$.

The statement in Theorem 3.15 is also true for any maximal $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic subcode of a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code $\mathcal{C}$ whose Gray image is a linear subcode of $\Phi(\mathcal{C})$.

Corollary 3.16: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a cyclic code, where $fhg = x^\beta - 1$. Then, if $\mathcal{C}_1$ is a maximal $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic subcode with $\Phi(\mathcal{C}_1)$ linear, then $\mathcal{C}_1 = \langle (b \mid 0), (\ell_k \mid fhk + 2f) \rangle$, where k divides g and $\ell_k = \tilde{k}\ell + (1-\tilde{k})\tilde{\mu}\ell\tilde{g} \pmod b$, for μ in (3).

The kernel of a binary code is the intersection of all its maximal linear subspaces. Therefore, if $\mathcal{C}_1, \mathcal{C}_2, \dots \mathcal{C}_s$ are all the maximal subcodes of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ such that $\Phi(\mathcal{C}_i)$ is a linear subcode of $\Phi(\mathcal{C})$, for $1 \leq i \leq s$, then

$$\mathcal{K}(\mathcal{C}) = \bigcap_{i=1}^{s} \mathcal{C}_i. \tag{10}$$

In [7] it is proved that if $\mathcal{C}_1 = \langle fh_1 + 2f \rangle$ and $\mathcal{C}_2 = \langle fh_2 + 2f \rangle$ are quaternary cyclic codes of odd length n, then $\mathcal{C}_1 \cap \mathcal{C}_2 = \langle f \operatorname{lcm}(h_1, h_2) + 2f \rangle$. We will give a similar result for $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes.

Proposition 3.17: Let $\mathcal{C} = \langle (b \mid 0), (l \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. Let $\mathcal{C}_1 = \langle (b \mid 0), (l_{k_1} \mid fhk_1 + 2f) \rangle$ and $\mathcal{C}_2 = \langle (b \mid 0), (l_{k_2} \mid fhk_2 + 2f) \rangle$ be $\mathbb{Z}_2\mathbb{Z}_4$-additive maximal subcodes of $\mathcal{C}$ whose images under the Gray map are linear subcodes of $\Phi(\mathcal{C})$. Then,

$$\mathcal{C}_1 \cap \mathcal{C}_2 = \langle (b \mid 0), (l_{k'} \mid fhk' + 2f) \rangle,$$

where $k' = \operatorname{lcm}(k_1, k_2)$ and $\ell_{k'} = \tilde{k}'\ell + (1 - \tilde{k}')\tilde{\mu}\ell\tilde{g} \pmod b$, for μ in (3).

Proof: Let $\mathcal{C} = \langle (b \mid 0), (l \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code and let $\mathcal{C}_i = \langle (b \mid 0), (l_{k_i} \mid fhk_i + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive maximal subcode of $\mathcal{C}$ whose image under the Gray map is a linear subcode of $\Phi(\mathcal{C})$.

We first consider $(\mathcal{C}_1)_Y \cap (\mathcal{C}_2)_Y$. By [7], $(\mathcal{C}_1)_Y \cap (\mathcal{C}_2)_Y = \langle f \operatorname{lcm}(hk_1, hk_2) + 2f \rangle = \langle fhk' + 2f \rangle$, where $k' = \operatorname{lcm}(k_1, k_2)$. Since $\langle (b|0) \rangle \in \mathcal{C}_1 \cap \mathcal{C}_2$, we have $\mathcal{C}_1 \cap \mathcal{C}_2 = \langle (b \mid 0), (l_{k'} \mid fhk' + 2f) \rangle$, where $\ell_{k'} = \tilde{k}'\ell + (1 - \tilde{k}')\tilde{\mu}\ell\tilde{g} \pmod b$, for μ in (3) by Lemma 3.14. □

Lemma 3.18: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code and let $\mathcal{D}$ be a maximal cyclic subcode with linear binary image. Then, $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{D}$.

Proof: If $\mathcal{K}(\mathcal{C}) \not\subseteq \mathcal{D}$, then consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{D}'$ generated by $\mathcal{K}(\mathcal{C}) \cup \mathcal{D} \cup \{2\mathbf{u} * \mathbf{v} \mid \mathbf{u}, \mathbf{v} \in \mathcal{K}(\mathcal{C}) \cup \mathcal{D}\}$. Since the binary image of $\mathcal{K}(\mathcal{C}) \cup \mathcal{D}$ is cyclic, $\mathcal{D}'$ is a cyclic subcode of $\mathcal{C}$. Moreover, since $2\mathbf{u} * \mathbf{v} \in \mathcal{D}'$, for all $\mathbf{u}, \mathbf{v} \in \mathcal{D}'$, we have that $\mathcal{D}'$ has linear binary image, leading to a contradiction since we are assuming that $\mathcal{D}$ is maximal. □

Theorem 3.19: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where $fhg = x^\beta - 1$. Assume that $k_1, \dots, k_s$ are the divisors of g of minimum degree such that

$$\gcd\left(\frac{\tilde{f}b}{\gcd(b, \ell\frac{\tilde{g}}{\tilde{k}_i})}, \left(\frac{\tilde{g}}{\tilde{k}_i} \otimes \frac{\tilde{g}}{\tilde{k}_i} \right) \right) = 1,$$

for $i = 1, \dots, s$. Then,

$$\mathcal{K}(\mathcal{C}) = \langle (b \mid 0), (\ell'_k \mid fhk' + 2f) \rangle,$$

where $k' = \operatorname{lcm}(k_1, \dots, k_s)$ and $\ell_{k'} = \tilde{k}'\ell + (1 - \tilde{k}')\tilde{\mu}\ell\tilde{g} \pmod b$, for μ in (3).

Proof: Assume that $k_1, \dots, k_s$ are the divisors of g of minimum degree such that

$$\gcd\left(\frac{\tilde{f}b}{\gcd(b, \ell\frac{\tilde{g}}{\tilde{k}_i})}, \left(\frac{\tilde{g}}{\tilde{k}_i} \otimes \frac{\tilde{g}}{\tilde{k}_i} \right) \right) = 1,$$

for $i = 1, \dots, s$. Let $\mathcal{D}_i$ be a cyclic subcode of $\mathcal{C}$, where $\mathcal{D}_i = \langle (b \mid 0), (\ell_{k_i} \mid fhk_i + 2f) \rangle$, for some ℓ_{k_i}. Note that $\Phi(\mathcal{D}_i)$ is linear by Theorem 2.5. Since k_i is a polynomial of minimum degree dividing g, we have that $\mathcal{D}_i$ is a maximal cyclic subcode of $\mathcal{C}$ with linear binary image. Then each $\mathcal{D}_i$ extends to $\mathcal{C}_i$ which is a maximal subcode of $\mathcal{C}$, not necessarily cyclic, with linear binary image. Note that every maximal code with linear image must contain a cyclic code with linear image, e.g., every maximal code contains $\mathcal{K}(\mathcal{C})$ that is cyclic with linear image. By Lemma 3.18, we know $\mathcal{K}(\mathcal{C}) \subseteq \mathcal{D}_i$ and, therefore $\mathcal{K}(\mathcal{C}) \subseteq \cap_i \mathcal{D}_i$. But $\cap_i \mathcal{C}_i = \mathcal{K}(\mathcal{C}) \subseteq \cap_i \mathcal{D}_i \subseteq \cap_i \mathcal{C}_i$, so $\mathcal{K}(\mathcal{C}) = \cap_i \mathcal{D}_i$. By Corollary 3.16 and Proposition 3.17, the result follows. □

Example 4: Let $x^7 - 1 = (x - 1)p_3q_3$ over $\mathbb{Z}_4$. Let $\mathcal{C} = \langle (1 \mid 0), (0 \mid f) \rangle$ of type $(1, 7; 1, 6; 1)$ with $f = (x - 1)$, $h = 1$ and $g = p_3q_3$.

Note that $\gcd(\frac{\tilde{f}b}{\gcd(b,\ell\tilde{g})}, (\tilde{g}\otimes\tilde{g})) = x-1 \neq 1$. We have that all maximal cyclic subcodes of $\mathcal{C}$ with binary linear image are $\mathcal{C}_1 = \langle (1 \mid 0), (0 \mid fp_3 + 2f)\rangle$, and $\mathcal{C}_2 = \langle (1 \mid 0), (0 \mid fq_3 + 2f)\rangle$. Clearly, $k' = \mathrm{lcm}(p_3, q_3) = p_3q_3$ and then $\mathcal{K}(\mathcal{C}) = \langle (1 \mid 0), (0 \mid fp_3q_3 + 2f)\rangle = \langle (1 \mid 0), (0 \mid 2f)\rangle$.

B. Rank of $\mathbb{Z}_2\mathbb{Z}_4$-Additive Cyclic Codes

In this section, we will study the rank of a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code $\mathcal{C}$. We will prove that $\mathcal{R}(\mathcal{C})$ is also cyclic and we will establish some properties of its generator polynomials. However, we will show that there does not exist a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code for all possible values of the rank, in contrast to what is exhibited in the following results for a $\mathbb{Z}_2\mathbb{Z}_4$-additive code.

Proposition 3.20 ([9]): Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of binary length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $\mathrm{rank}(\Phi(C)) \in \{\gamma + 2\delta, \dots, \min(\beta + \delta + \kappa,\ \gamma + 2\delta + \binom{\delta}{2})\}$.

Theorem 3.21 ([9]): Let $\alpha, \beta, \gamma, \delta, \kappa$ be integers satisfying (6). Then, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $\mathrm{rank}(\Phi(C)) = r$ if and only if

$$r \in \{\gamma + 2\delta, \dots, \min(\beta + \delta + \kappa,\ \gamma + 2\delta + \binom{\delta}{2})\}.$$

Proposition 3.22 ([9]): Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$. Let $\mathcal{G}$ be a generator matrix of $\mathcal{C}$ as in (1) and let $\{\mathbf{u}_i\}_{i=1}^{\gamma}$ be the rows of order two and $\{\mathbf{v}_j\}_{j=1}^{\delta}$ the rows of order four in $\mathcal{G}$. Then, $\langle C\rangle$ is generated by $\{\Phi(\mathbf{u}_i)\}_{i=1}^{\gamma}$, $\{\Phi(\mathbf{v}_j), \Phi(2\mathbf{v}_j)\}_{j=1}^{\delta}$ and $\{\Phi(2\mathbf{v}_j * \mathbf{v}_k)\}_{1\leq j<k\leq\delta}$. Moreover, $\langle C\rangle$ is both binary linear and $\mathbb{Z}_2\mathbb{Z}_4$-linear.

Corollary 3.23 ([7]): Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$. Let $\mathcal{G}$ be a generator matrix of $\mathcal{C}$ as in (1) and let $\{\mathbf{u}_i\}_{i=1}^{\gamma}$ be the rows of order two and $\{\mathbf{v}_j\}_{j=1}^{\delta}$ the rows of order four in $\mathcal{G}$. Then,

$$\mathcal{R}(\mathcal{C}) = \mathcal{C} \cup \langle \{2\mathbf{v}_j * \mathbf{v}_k\}_{1\leq j<k\leq\delta}\rangle.$$

Lemma 3.24: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Then, $\mathcal{R}(\mathcal{C})_Y = \mathcal{R}(\mathcal{C}_Y)$.

Proof: Let $\{\mathbf{u}_i = (u_i \mid u'_i)\}_{i=1}^{\gamma}$ be the first γ rows and $\{\mathbf{v}_j = (v_j \mid v'_j)\}_{j=1}^{\delta}$ the last δ rows of G. By Corollary 3.23, $\mathcal{R}(\mathcal{C}) = \mathcal{C} \cup \langle \{2\mathbf{v}_j * \mathbf{v}_k\}_{1\leq j<k\leq\delta}\rangle$. By the same argument, $\mathcal{R}(\mathcal{C}_Y) = \mathcal{C}_Y \cup \langle \{2v'_j * v'_k\}_{1\leq j<k\leq\delta}\rangle$. Since $2\mathbf{v}_j * \mathbf{v}_k = (0 \mid 2v'_j * v'_k)$ for all $1 \leq j < k \leq \delta$, the statement follows. $\square$

The following theorem shows that if $\mathcal{C}$ is $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, then $\mathcal{R}(\mathcal{C})$ is also cyclic. As in the case of $\mathcal{K}(\mathcal{C})$, this theorem is a generalization of the case when $\mathcal{C}$ is a linear cyclic code over $\mathbb{Z}_4$ [7].

Theorem 3.25: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. Then $\mathcal{R}(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code.

Proof: Let $\mathbf{x} \in \mathcal{R}(\mathcal{C})$. By Corollary 3.23, $\mathcal{R}(\mathcal{C})$ is generated by $\mathcal{C}$ and $\{2\mathbf{v}*\mathbf{w} \mid \mathbf{v}, \mathbf{w} \in \mathcal{C}\}$, then $\mathbf{x} = \mathbf{u} + 2\mathbf{v} * \mathbf{w}$, for some $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{C}$. As $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, $\pi(\mathbf{u}), \pi(\mathbf{v}), \pi(\mathbf{w}) \in \mathcal{C}$ and $2\pi(\mathbf{v}) * \pi(\mathbf{w}) \in \mathcal{R}(\mathcal{C})$. Thus, $\pi(\mathbf{x}) = \pi(\mathbf{u}) + 2\pi(\mathbf{v}) * \pi(\mathbf{w}) \in \mathcal{R}(\mathcal{C})$ and $\mathcal{R}(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. $\square$

The next proposition is straightforward from Theorems 3.21 and 2.3.

Proposition 3.26: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. Then

$$\alpha - \deg(b) + \deg(h) + 2\deg(g) \leq \mathrm{rank}(\Phi(\mathcal{C})) \leq$$

$$\min\Big(\alpha + \beta + \deg(g) - \deg(\gcd(b, \ell\tilde{g})),\ \alpha - \deg(b) + \deg(h) + 2\deg(g) + \binom{\deg(g)}{2}\Big).$$

For a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code $\mathcal{C}$, define $\mathcal{C}_1 = \langle (b \mid 0)\rangle$ and $\mathcal{C}_2 = \langle (\ell \mid fh + 2f)\rangle$. Since $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ and

$\mathcal{C}_1 \cap \mathcal{C}_2 = \{\mathbf{0}\}$, we have that

$$\operatorname{rank}(\Phi(\mathcal{C})) = \operatorname{rank}(\Phi(\mathcal{C}_1)) + \operatorname{rank}(\Phi(\mathcal{C}_2)). \qquad (11)$$

If the code $\mathcal{C}$ is separable, then $\ell = 0$ and $\operatorname{rank}(\Phi(\mathcal{C}_1)) = \operatorname{rank}(\mathcal{C}_X)$. Moreover, $\mathcal{C}_2 = \langle (0 \mid fh + 2f) \rangle$, and therefore $\operatorname{rank}(\Phi(\mathcal{C}_2)) = \operatorname{rank}(\phi(\mathcal{C}_Y))$. We obtain the following result.

Proposition 3.27: If $\mathcal{C}$ is a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, then $\mathcal{R}(\mathcal{C}) = \mathcal{R}(\mathcal{C}_X) \times \mathcal{R}(\mathcal{C}_Y)$ and $\operatorname{rank}(\Phi(\mathcal{C})) = \kappa_1 + \operatorname{rank}(\phi(\mathcal{C}_Y))$.

Note that, if $\mathcal{C}$ is not separable, $\operatorname{rank}(\Phi(\mathcal{C}))$ is not necessarily equal to $\kappa_1 + \operatorname{rank}(\phi(\mathcal{C}_Y))$ as it is shown in the following example.

Example 5: Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by the following matrix.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \qquad (12)$$

In [5, Example 2] it was proved that $\phi(\mathcal{C}_Y)$ is binary linear whereas $\Phi(\mathcal{C})$ is not binary linear. In this example we have that $\kappa_1 = 2$. Since $\phi(\mathcal{C}_Y)$ is linear, $\operatorname{rank}(\phi(\mathcal{C}_Y)) = 5$. Nevertheless, $\Phi(\mathcal{C})$ is not binary linear and $\operatorname{rank}(\Phi(\mathcal{C})) = 8 > 5 + 2$.

In fact, $\operatorname{rank}(\Phi(\mathcal{C}))$ is always greater or equal to $\kappa_1 + \operatorname{rank}(\phi(\mathcal{C}_Y))$.

Proposition 3.28: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, then $\operatorname{rank}(\Phi(\mathcal{C})) \geq \kappa_1 + \operatorname{rank}(\phi(\mathcal{C}_Y))$.

Proof: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. By (11), $\operatorname{rank}(\Phi(\mathcal{C})) = \operatorname{rank}(\Phi(\mathcal{C}_1)) + \operatorname{rank}(\Phi(\mathcal{C}_2)) = \kappa_1 + \operatorname{rank}(\Phi(\mathcal{C}_2))$. By Proposition 3.22, in order to determine the rank, we have to consider the set of vectors $2\mathbf{v} * \mathbf{w}$, for $\mathbf{v} = (v \mid v'), \mathbf{w} = (w \mid w') \in \mathcal{C}$. Since for all $\mathbf{v} \in \mathcal{C}$ if $\mathbf{w} \in \mathcal{C}_1$ we obtain $2\mathbf{v} * \mathbf{w} = \mathbf{0}$, we just have to consider $\mathbf{v}, \mathbf{w} \in \mathcal{C}_2$. We have that if $2v' * w' \notin \mathcal{C}_Y$ then $2\mathbf{v} * \mathbf{w} \notin \mathcal{C}$. Therefore, $\operatorname{rank}(\Phi(\mathcal{C}_2)) \geq \operatorname{rank}(\phi((\mathcal{C}_2)_Y)) = \operatorname{rank}(\phi(\mathcal{C}_Y))$ and, therefore, $\operatorname{rank}(\Phi(\mathcal{C}_1)) \geq \kappa_1 + \operatorname{rank}(\phi(\mathcal{C}_Y))$. □

Now we can determine the rank of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ as the rank of $\mathcal{C}_X$ and the rank of a linear code over $\mathbb{Z}_4$, $\mathcal{C}'$. As in the case of the kernel, when $\mathcal{C}$ is separable we have seen that $\mathcal{C}' = \mathcal{C}_Y$, but if $\mathcal{C}$ is not separable such a code $\mathcal{C}'$ may not be cyclic over $\mathbb{Z}_4$.

Theorem 3.29: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code with generator matrix in the form of (4) and let $\mathcal{C}'$ be the subcode generated by the matrix in (8). Then,

$$\operatorname{rank}(\Phi(\mathcal{C})) = \kappa_1 + \kappa_2 + \operatorname{rank}(\phi(\mathcal{C}'_Y)).$$

Proof: Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code with generator matrix G in the form of (4). Let $\{\mathbf{u}_i = (u_i \mid u'_i)\}_{i=1}^{\gamma}$ be the first γ rows and $\{\mathbf{v}_j = (v_j \mid v'_j)\}_{j=1}^{\delta}$ the last δ rows of G. Define the codes $\bar{\mathcal{C}} = \langle \{\mathbf{u}_i\}_{i=1}^{\kappa_1+\kappa_2} \rangle$ and $\mathcal{C}' = \langle \{\mathbf{u}_i\}_{i=\kappa_1+\kappa_2+1}^{\gamma}, \{\mathbf{v}_j\}_{j=1}^{\delta} \rangle$.

By Corollary 3.23 we have that

$$\begin{aligned} \mathcal{R}(\mathcal{C}) &= \langle \{\mathbf{u}_i\}_{i=1}^{\gamma}, \{\mathbf{v}_j\}_{j=1}^{\delta}, \{2\mathbf{v}_j * \mathbf{v}_k\}_{1\leq j<k\leq\delta} \rangle, \\ \mathcal{R}(\bar{\mathcal{C}}) &= \langle \{\mathbf{u}_i\}_{i=1}^{\kappa_1+\kappa_2} \rangle, \text{ and} \\ \mathcal{R}(\mathcal{C}') &= \langle \{\mathbf{u}_i\}_{i=\kappa_1+\kappa_2+1}^{\gamma}, \{\mathbf{v}_j, 2\mathbf{v}_j\}_{j=1}^{\delta}, \\ &\qquad \{2\mathbf{v}_j * \mathbf{v}_k\}_{1\leq j<k\leq\delta} \rangle. \end{aligned}$$

Note that $\mathcal{R}(\mathcal{C}) = \mathcal{R}(\bar{\mathcal{C}}) \cup \mathcal{R}(\mathcal{C}')$. Moreover, $\mathcal{R}(\bar{\mathcal{C}}) \cap \mathcal{R}(\mathcal{C}') = \{\mathbf{0}\}$ due to the fact that for all $1 \leq j < k \leq \delta$, $2\mathbf{v}_j * \mathbf{v}_k = (0 \mid 2v'_j * v'_k) \notin \bar{\mathcal{C}}$. Therefore, $\operatorname{rank}(\Phi(\mathcal{C})) = \operatorname{rank}(\Phi(\bar{\mathcal{C}})) + \operatorname{rank}(\Phi(\mathcal{C}')) = \kappa_1 + \kappa_2 + \operatorname{rank}(\Phi(\mathcal{C}'))$. Finally, by Lemma 3.24, $\operatorname{rank}(\Phi(\mathcal{C}')) = \operatorname{rank}(\phi(\mathcal{C}'_Y))$ and the statement follows. □

Theorem 3.30: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where $fhg = x^{\beta} - 1$. Then, $\mathcal{R}(\mathcal{C}) = \langle (b_r \mid 0), (\ell_r \mid fh + 2\frac{f}{r}) \rangle$, where r is a divisor of f and b_r divides b.

Proof: By Theorem 3.25, $\mathcal{R}(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, therefore $\mathcal{R}(\mathcal{C}) = \langle (b_r \mid 0), (\ell_r \mid f_r h_r + 2f_r)\rangle$. Since $(b \mid 0) \in \mathcal{R}(\mathcal{C})$, it is clear that b_r divides b. By [9, Lemma 3], $\mathcal{C}$ and $\mathcal{R}(\mathcal{C})$ have the same number of order four codewords and since $\mathcal{C} \subseteq \mathcal{R}(\mathcal{C})$ we have that $f_r h_r = fh$. Then $g_r = g$. By Proposition 3.1, we know that f_r divides f and hence there exists $r \in \mathbb{Z}_4[x]$ such that $f_r = \frac{f}{r}$ and $h_r = hr$. Therefore, $\mathcal{R}(\mathcal{C}) = \langle (b_r \mid 0), (\ell_r \mid fh + 2\frac{f}{r})\rangle$. □

Let $\mathcal{C}\langle (b \mid 0), (\ell \mid fh + 2f)\rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. In the following example, we will see that $\mathcal{R}(\mathcal{C}) = \langle (b_r \mid 0), (\ell_r \mid f_r h_r + 2f_r)\rangle$, where $b_r \neq b$, if we consider the generator polynomials of $\mathcal{R}(\mathcal{C})$ in standard form.

Example 6: Let $x^7 - 1 = (x-1)p_3 q_3$ over $\mathbb{Z}_4$. Let $\mathcal{C} = \langle ((x-1) \mid 0), (1 \mid (x-1) + 2)\rangle$, with $f = 1, h = x - 1$ and $g = p_3 q_3$. If we compute $\mathcal{R}(\mathcal{C})$ we obtain that $\mathcal{R}(\mathcal{C}) = \langle (1 \mid 0), (0 \mid (x-1) + 2)\rangle$.

As it is shown in Theorem 3.21, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive code for any possible value of the rank. Nevertheless, the following example gives a particular type $(\alpha, \beta; \gamma, \delta; \kappa)$ such that it is not possible to construct a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code with a specific, and valid, value of the rank.

Example 7: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f)\rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code of type $(2, 7; 2, 3; \kappa)$. By Theorem 3.21, $\mathrm{rank}(\Phi(\mathcal{C})) \in \{8, 9, 10, 11\}$. We will see that there does not exist any cyclic $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(2, 7; 2, 3; \kappa)$, $\mathcal{C}$, with $\mathrm{rank}(\Phi(\mathcal{C})) \in \{8, 9, 10\}$.

Let $x^7 - 1 = (x-1)p_3 q_3$ over $\mathbb{Z}_4$, with p_3 and q_3 as in Example 3. By Theorem 2.3, $\deg(g) = 3$ and $\deg(b) = \deg(h) \leq 2$. Assume without loss of generality that $g = p_3$ and, since $\deg(h) \leq 2$, we have that q_3 divides f. We have already proved, in Example 3, that there does not exist a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(2, 7; 2, 3; \kappa)$ with linear Gray image. Thus, $\mathrm{rank}(\Phi(\mathcal{C})) \neq 8$.

By Theorem 3.30, $\mathcal{R}(\mathcal{C}) = \langle (b \mid 0), (\ell_r \mid f_r h_r + 2f_r)\rangle$ where r divides f and $h_r = hr$. Since $\mathrm{rank}(\Phi(\mathcal{C})) \in \{8, 9, 10, 11\}$, we have that $\deg(r) \leq 3$ as $|\mathcal{R}(\mathcal{C})| = 4^3 2^{2+\deg(r)} \leq 2^{11}$. Since $\gcd(\tilde{q}_3, \tilde{p}_3 \otimes \tilde{p}_3) \neq 1$ we have that q_3 must divide r. Therefore $\deg(r) \geq 3$, and by the previous argument, we know that $r = q_3$. So, $\mathrm{rank}(\Phi(\mathcal{C})) \notin \{9, 10\}$.

Finally, we will give $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes of type $(2, 7; 2, 3; \kappa)$, for different values of κ, such that $\mathrm{rank}(\Phi(\mathcal{C})) = 11$. Recall that $\kappa \leq \min\{\alpha, \gamma\} = 2$ and $\kappa = \alpha - \deg(\gcd(b, \ell\tilde{g}))$, then

- $\kappa = 2$: $\mathcal{C} = \langle (x-1 \mid 0), (1 \mid (x^3+2x^2+x+3)(x-1) + 2(x^3+2x^2+x+3))\rangle$, or $\mathcal{C} = \langle (x-1 \mid 0), (1 \mid (x^3+3x^2+2x+3)(x-1) + 2(x^3+3x^2+2x+3))\rangle$.
- $\kappa = 1$: $\mathcal{C} = \langle (x-1 \mid 0), (0 \mid (x^3+3x^2+2x+3)(x-1) + 2(x^3+3x^2+2x+3))\rangle$, or $\mathcal{C} = \langle (x-1 \mid 0), (0 \mid (x^3+2x^2+x+3)(x-1) + 2(x^3+2x^2+x+3))\rangle$.
- $\kappa = 0$: As in Example 3, there does not exist a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code of type $(2, 7; 2, 3; 0)$.

Proposition 3.31: Let $\mathcal{C} = \langle fh + 2f\rangle$ be a cyclic code over $\mathbb{Z}_4$ of length n, with $fhg = x^n - 1$. Then,

$$\mathcal{R}(\mathcal{C}) = \langle fh + 2\frac{f}{r}\rangle,$$

where r is the Hensel lift of $\gcd(\tilde{f}, \tilde{g} \otimes \tilde{g})$.

Proof: From [7], we have that $\mathcal{R}(\mathcal{C}) = \langle fh + 2\frac{f}{r}\rangle$, where r divides f.

Since $\mathcal{R}(\mathcal{C})$ is the minimum cyclic code over $\mathbb{Z}_4$ containing $\mathcal{C}$ whose image under the Gray map is linear, we have that r is the polynomial of minimum degree dividing f satisfying that $\langle fh + 2\frac{f}{r}\rangle$ has linear image. This is equivalent, by [18], to the condition $\gcd\left(\frac{\tilde{f}}{\tilde{r}}, \tilde{g} \otimes \tilde{g}\right) = 1$. Therefore, the polynomial r of minimum degree dividing f satisfying this condition is the Hensel lift of $\gcd(\tilde{f}, \tilde{g} \otimes \tilde{g})$. □

Proposition 3.32: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f)\rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where $fhg = x^\beta - 1$, such that $\Phi(\mathcal{C})$ is not linear and $\phi(\mathcal{C}_Y)$ is linear. Then,

$$\mathcal{R}(\mathcal{C}) = \langle (b_r \mid 0), (\ell_r \mid fh + 2f) \rangle,$$

where $b_r = \gcd(b, \tilde{\mu}\tilde{g}\ell)$, μ is as in (3), and $\ell_r = \ell - \tilde{\mu}\tilde{g}\ell \pmod{b_r}$.

Proof: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, $\mathcal{C}_Y = \langle fh + 2f \rangle$. Let G be a generator matrix of $\mathcal{C}$ as in (1) and let $\{\mathbf{u}_i = (u_i \mid u'_i)\}_{i=1}^{\gamma}$ be the rows of order two and $\{\mathbf{v}_j = (v_j \mid v'_j)\}_{j=1}^{\delta}$ the rows of order four in G. By Corollary 3.23, $\mathcal{R}(\mathcal{C}) = \mathcal{C} \cup \langle \{2\mathbf{v}_j * \mathbf{v}_k\}_{1\leq j<k\leq\delta} \rangle$. We have that $\Phi(\mathcal{C})$ is not linear, therefore there exist $i, k \in \{1, \ldots, \delta\}$ such that $2\mathbf{v}_j * \mathbf{v}_k = (\mathbf{0} \mid 2v'_j * v'_k) \notin \mathcal{C}$. Since $\phi(\mathcal{C}_Y)$ is linear, $2v'_j * v'_k \in \langle 2fh, 2f \rangle$. Moreover, $\langle (0 \mid 2fh) \rangle \in \mathcal{C}$ and therefore $\mathcal{R}(\mathcal{C}) = \mathcal{C} \cup \langle (0 \mid 2f) \rangle$. Considering the generator polynomials of $\mathcal{R}(\mathcal{C})$ and μ as in (3), we have

$$\begin{aligned}
\mathcal{R}(\mathcal{C}) &= \langle (b \mid 0), (\ell - \tilde{\mu}\tilde{g}\ell \mid fh), (\tilde{\mu}\tilde{g}\ell \mid 2f), (0 \mid 2f) \rangle \\
&= \langle (b \mid 0), (\tilde{\mu}\tilde{g}\ell \mid 0), (\ell - \tilde{\mu}\tilde{g}\ell \mid fh + 2f) \rangle \\
&= \langle (\gcd(b, \tilde{\mu}\tilde{g}\ell) \mid 0), (\ell - \tilde{\mu}\tilde{g}\ell \mid fh + 2f) \rangle.
\end{aligned}$$

Therefore, considering the generator polynomials of $\mathcal{R}(\mathcal{C})$ in standard form, we have that $b_r = \gcd(b, \tilde{\mu}\tilde{g}\ell)$ and $\ell_r = \ell - \tilde{\mu}\tilde{g}\ell \pmod{b_r}$. □

Theorem 3.33: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code, where $fhg = x^\beta - 1$. Then,

$$\mathcal{R}(\mathcal{C}) = \langle (b_r \mid 0), (\ell_r \mid fh + 2\frac{f}{r}) \rangle,$$

where r is the Hensel lift of $\gcd(\tilde{f}, \tilde{g} \otimes \tilde{g})$, $b_r = \gcd(b, \tilde{\mu}\tilde{g}\ell)$, μ is as in (3), and $\ell_r = \ell - \tilde{\mu}\tilde{g}\ell$.

Proof: Let $\mathcal{C} = \langle (b \mid 0), (\ell \mid fh + 2f) \rangle$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code. By Theorem 3.30, $\mathcal{R}(\mathcal{C}) = \langle (b_r \mid 0), (\ell_r \mid fh + 2\frac{f}{r}) \rangle$, where r is a divisor of f and b_r divides b.

Consider the quaternary code $\mathcal{C}_Y = \langle fh + 2f \rangle$. By Lemma 3.24, we have that $(\mathcal{R}(\mathcal{C}))_Y = \mathcal{R}(\mathcal{C}_Y)$. Therefore, by Proposition 3.31, $\mathcal{R}(\mathcal{C}_Y) = \left\langle fh + 2\frac{f}{r} \right\rangle$, where r is the Hensel lift of $\gcd(\tilde{f}, \tilde{g} \otimes \tilde{g})$. Note that $\mathcal{R}(\mathcal{C}_Y) = \mathcal{C}_Y \cup \left\langle 2\frac{f}{r} \right\rangle$.

From Corollary 3.23, $\mathcal{C} = \mathcal{C} \cup \langle \{2\mathbf{v}_j * \mathbf{v}_k\}_{1\leq j<k\leq\delta} \rangle$, where $\{\mathbf{u}_i = (u_i \mid u'_i)\}_{i=1}^{\gamma}$ are the rows of order four of the generator matrix of $\mathcal{C}$ as in (1). Note that, for all $\mathbf{u}_j, \mathbf{u}_k$, $1 \leq j \leq k \leq \delta$, $2\mathbf{u}_j * \mathbf{u}_k = (0 \mid 2u'_j * u'_k)$, where $2u'_j * u'_k \in \mathcal{R}(\mathcal{C}_Y) = \mathcal{C}_Y \cup \langle 2\frac{f}{r} \rangle$ and, therefore, $(0 \mid 2u'_j * u'_k) \in \mathcal{C} \cup \langle (0 \mid 2\frac{f}{r}) \rangle$. Hence, we have that $\mathcal{R}(\mathcal{C}) = \mathcal{C} \cup \langle (0 \mid 2\frac{f}{r}) \rangle$.

Therefore, for μ as in (3),

$$\begin{aligned}
\mathcal{R}(\mathcal{C}) &= \langle (b \mid 0), (\ell - \tilde{\mu}\tilde{g}\ell \mid fh), (\tilde{\mu}\tilde{g}\ell \mid 2f), (0 \mid 2\tfrac{f}{r}) \rangle \\
&= \langle (b \mid 0), (\tilde{\mu}\tilde{g}\ell \mid 0), (\ell - \tilde{\mu}\tilde{g}\ell \mid fh + 2\tfrac{f}{r}) \rangle \\
&= \langle (\gcd(b, \tilde{\mu}\tilde{g}\ell) \mid 0), (\ell - \tilde{\mu}\tilde{g}\ell \mid fh + 2\tfrac{f}{r}) \rangle.
\end{aligned}$$

From the last equation, and considering the generator polynomials of $\mathcal{R}(\mathcal{C})$ in standard form, we have that $b_r = \gcd(b, \tilde{\mu}\tilde{g}\ell)$ and $\ell_r = \ell - \tilde{\mu}\tilde{g}\ell \pmod{b_r}$. □

Example 8: Let $\alpha = 3$ and $\beta = 7$. Consider, as in Example 3, $\mathcal{C} = \langle (x - 1 \mid 0), (0 \mid x - 1) \rangle$ where $f = x - 1$ and $h = 1$. As we have seen, $\mathcal{C}$ does not have linear binary image. Then, by Theorem 3.33, we have that $\mathcal{R}(\mathcal{C}) = \langle (1 \mid 0), (0 \mid (x - 1) + 2) \rangle$ where $r = f = x - 1$.

Example 9: Let $\alpha = 3$ and $\beta = 15$. Consider $\mathcal{C} = \langle (x - 1 \mid 0), (1 \mid fh + 2f) \rangle$ where $f = x^4 + 2x^2 + 3x + 1$ and $h = (x - 1)(x^4 + x^3 + x^2 + x + 1)$. Then, by Theorem 3.33, we have that $\mathcal{R}(\mathcal{C}) = \langle (1 \mid 0), (0 \mid fh + 2\frac{f}{r}) \rangle$ where $r = f = x^4 + 2x^2 + 3x + 1$.

IV. Conclusions

Given a $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic code $\mathcal{C}$, we have shown that the codes $\mathcal{K}(\mathcal{C})$ and $\mathcal{R}(\mathcal{C})$ are also $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic. We have computed the generator polynomials of these codes in terms of the generator polynomials of $\mathcal{C}$. Using these results, we have concluded that the dimensions of the binary images of $\mathcal{K}(\mathcal{C})$ and $\mathcal{R}(\mathcal{C})$, i.e. the dimension of the kernel and the rank of $\mathcal{C}$, cannot take all the possible values as for a general $\mathbb{Z}_2\mathbb{Z}_4$-additive code. In other words, if a $\mathbb{Z}_2\mathbb{Z}_4$-additive code is cyclic, then the set of possible values for the rank and the dimension of the kernel becomes more restrictive.

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