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ON THE DYNAMICS OF THE SZEKERES SYSTEM

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ABSTRACT. The gravitational Szekeres differential system is completely integrable with two rational first integrals and an additional analytical first integral. We describe the dynamics of the Szekeres system when one of these two rational first integrals is negative, showing that all the orbits come from the infinity of $\mathbb{R}^4$ and go to infinity.

1. Introduction and statement of the results

The Szekeres model is a four-dimensional system which are the exact solutions of the Einstein field equations when there exists irrotational dust, see for a detailed introduction to this model [15]. The limiting cases of the Szekeres models are the Lemaître-Tolman models, see [3]. The equations of motion of the Szekeres model are

$$
\begin{aligned}
\dot{\rho} &= -\theta\rho, \\
\dot{\theta} &= -\frac{1}{3}\theta^2 - 6\sigma^2 - \frac{1}{2}\rho, \\
\dot{\sigma} &= \sigma^2 - \frac{2}{3}\theta\sigma - E, \\
\dot{E} &= -3E\sigma - \theta E - \frac{1}{2}\rho\sigma,
\end{aligned} \tag{1}
$$

where ρ is the energy density, θ is the expansion scalar, σ is the shear and E is the Weyl tensor. The dot in (1) denotes derivative with respect the independent variable t, the time.

The Szekeres system is a special case of the six-dimensional system called Silent Universe system [4] when $\sigma_1 = \sigma_2 = \sigma$, being σ_1 and σ_2 the independent eigenvalues of the traceless shear tensor σ_{ab}, and $E_1 = E_2 = E$, being E_1 and E_2 the traceless components of the Weyl tensor E_{ab}.

These last years appeared a growing interest in the dynamics of the solutions of the Szekeres system, because this system was used for describing the propagation of the light in nonhomogeneous universe models, and for analyzing the evolution and formation of the structure of the Universe, see for instance [2, 9, 10, 14, 16].

The Szekeres system is a polynomial differential system of degree 2 in $\mathbb{R}^4$. In order to control the dynamics of this polynomial differential system near the infinity of $\mathbb{R}^4$ we shall use the Poincaré compactification, see section 2. Every direction for going to or coming from the infinity of $\mathbb{R}^4$ can be identified with a point of the

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3-dimensional sphere $\mathbb{S}^3$. Thus, roughly speaking, the Poincaré compactification of the polynomial differential system given by the Szekeres system can be identified with the closed ball $B = \{(x_1, x_2, x_3, x_4) : \sum_{i=1}^4 x_i^2 \leq 1\}$ in such a way that $\mathbb{R}^4$ (the phase space of the Szekeres system) is identified with the interior of B and its infinity $\mathbb{S}^3$ is the boundary of B. Moreover, the Szekeres system which initially is defined in the interior of the ball B extends analytically to the boundary of B, i.e. to the infinity of $\mathbb{R}^4$, and the extended flow on the infinity $\mathbb{S}^3$ is invariant, i.e. a solution of the extended flow starting at a point of $\mathbb{S}^3$ remains in $\mathbb{S}^3$. In this way we can study the dynamics of the Szekeres system in a compact phase space B instead of study it in an open phase space $\mathbb{R}^4$. This extension not only allows to study easily the dynamics of the orbits in a neighborhood of the infinity, also allows that all the orbits of the extended flow are defined for all time $t \in \mathbb{R}$, for more details on this last fact see for instance Chapter 1 of [6].

It was proved in [7] that this system is completely integrable (i.e. it has three functional independent first integrals) with two rational first integrals and a third analytical one found using the Jacobi's last multiplier method. The two rational first integrals are

$$F = \frac{(-18E - 3\rho + (\theta + 3\sigma)^2)^3}{(6E + \rho)^2},$$
$$H = \frac{(3\theta\sigma(2E + \rho) - E(18E + 2\theta^2 + 3\rho) + 9\sigma^2(4E + \rho))^3}{\rho^3(\rho + 6E)^2}.$$

Setting $F = f$ and $H = h$ we will describe the dynamics of system (1) restricted to the invariant set defined by $F = f$ and $H = h$ with $f < 0$. More precisely we will investigate the α-limit and ω-limit of all the solutions of system (1) with $f < 0$, i.e. the initial and final evolutions of all the orbits of the Szekeres system having $f < 0$.

We must mention that Gierzkiewicz and Golda were the first in finding two independent first integrals of the Szekeres system, they found the first integral F and another first integral

$$G = \frac{(6E + \rho)(9E + \theta^2 - 3\rho - 3\theta\sigma - 18\sigma^2)^3}{\rho^3}.$$

We note that there exists the functional relation

$$3H^{1/3} - F^{1/3} + G^{1/3} = 0$$

between the first integrals F, H and G.

Let $q \in B$ and denote by $\phi_t(q)$ the solution of the extended flow in B of the Szekeres system. Then we recall that a point $p \in B$ is an *ω-limit* point of q if there are points $\phi_{t_1}(q), \phi_{t_2}(q), \ldots$ in the orbit of q such that $t_k \to \infty$ and $\phi_{t_k}(q) \to p$ as $k \to \infty$. A point $p \in B$ is an *α-limit* if there are points $\phi_{t_1}(q), \phi_{t_2}(q), \ldots$ in the orbit of q such that $t_k \to -\infty$ and $\phi_{t_k}(q) \to p$ as $k \to \infty$.

Our main result is the following one.

Theorem 1. *All solutions of the Szekeres system* (1) *with $f < 0$ come from the infinity and go to infinity. More precisely, when $f < 0$ the α-limit and the ω-limit of every orbit of the Szekeres system* (1) *extended to B is at the infinity, i.e. at $\mathbb{S}^3$.*

Theorem 1 is proved in section 3.

2. Preliminaries and basic results

In this section we present some basic results and notations which are necessary for stating and proving our results.

2.1. Singular points of differential systems in $\mathbb{R}^2$.

Consider a differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{2}$$

in $\mathbb{R}^2$. A singular point or an equilibrium point of the differential system (2) is a point $(x_0,y_0) \in \mathbb{R}^2$ such that $P(x_0,y_0) = Q(x_0,y_0) = 0$.

We recall that a *hyperbolic singular point* is a singular point (x_0,y_0) such that the eigenvalues of the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial P}{\partial x}(x_0,y_0) & \dfrac{\partial P}{\partial y}(x_0,y_0) \\ \dfrac{\partial Q}{\partial x}(x_0,y_0) & \dfrac{\partial Q}{\partial y}(x_0,y_0) \end{pmatrix} \tag{3}$$

have nonzero real part. It is well known that the local phase portrait of a hyperbolic singular point (x_0,y_0) is diffeomorphic to the local phase portrait of the linear differential system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial P}{\partial x}(x_0,y_0) & \dfrac{\partial P}{\partial y}(x_0,y_0) \\ \dfrac{\partial Q}{\partial x}(x_0,y_0) & \dfrac{\partial Q}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

for more details see Theorem 2.15 of [6] which is the Hartman-Grobman Theorem in dimension two.

We say that a singular point (x_0,y_0) is *linearly zero* if the matrix (3) is identically zero. The local phase portraits of the linearly zero singular points must be studied using the change of variables called blow-ups, see for instance [1].

2.2. Poincaré compactification.

Let $p_i(x_1,\dots,x_n)$ be a real polynomial in the variables $x_1,x_2,\dots,x_n$ for $i=1,\dots,n$, and let $X = (p_1,\dots,p_n)$ be a polynomial vector field of degree d in $\mathbb{R}^n$, being d the maximum of the degrees of the polynomials p_i for $i=1,\dots,n$ (see all the details in [5] or in Chapter 5 of [6]).

The *Poincaré sphere* is defined as $\mathbb{S}^n = \{y=(y_1,\dots,y_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} y_i^2 = 1\}$ and its tangent space at the point $y \in \mathbb{S}^n$ is denoted by $T_y\mathbb{S}^n$. We identify the space $\mathbb{R}^n$ where is defined the polynomial vector field X with the tangent space $T_{(0,\dots,0,1)}\mathbb{S}^n$. We define the central projection $f : T_{(0,\dots,0,1)}\mathbb{S}^n \longrightarrow \mathbb{S}^n$ as follows: to each point $q \in T_{(0,\dots,0,1)}\mathbb{S}^n$ the central projection f associates to q the two intersection points of the sphere $\mathbb{S}^n$ with the straight line which connects the points q with the origin of coordinates. Note that the infinity of $\mathbb{R}^n \equiv T_{(0,\dots,0,1)}\mathbb{S}^n$ corresponds to the equator $\mathbb{S}^{n-1} = \{y \in \mathbb{S}^n : y_{n+1} = 0\}$ of $\mathbb{S}^n$.

This central projection f provides two copies $Df \circ X$ of the polynomial vector field X in $\mathbb{S}^n$, one in the northern hemisphere and the other in the southern. Denote by X' the vector field defined by these two copies of X into the sphere $\mathbb{S}^n$ minus its equator $\mathbb{S}^{n-1}$. We can extend the vector field X' on $\mathbb{S}^n \setminus \mathbb{S}^{n-1}$ to an analytic vector field $p(X)$ on $\mathbb{S}^n$ defining $p(X) = y_{n+1}^{d+1}\mathcal{X}'$. Let $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ the projection $\pi(y_1, \ldots, y_{n+1}) = (y_1, \ldots, y_n)$. Then the projection by π of the closed northern hemisphere of $\mathbb{S}^n$ is the *Poincaré ball* B. Note that the interior of B is diffeomorphic to $\mathbb{R}^n$ and its boundary $\mathbb{S}^{n-1}$ is the infinity of $\mathbb{R}^n$, and $\pi(p(X))$ is the extension of the polynomial vector field X to the Poincaré ball B.

For the case $n = 2$, i.e. for a polynomial vector field in $\mathbb{R}^2$, the Poincaré ball B is called the *Poincaré disc* D.

For computing the analytic expression of $p(X)$ we consider the sphere $\mathbb{S}^n$ as a differentiable manifold. We consider $U_i = \{y \in \mathbb{S}^n : y_i > 0\}$ and $V_i = \{y \in \mathbb{S}^n : y_i < 0\}$ for $i = 1, \ldots, n+1$. The corresponding coordinates maps are given by $F_i\colon U_i \to \mathbb{R}^n$ and $G_i\colon V_i \to \mathbb{R}^n$ by

$$F_i(y) = G_i(y) = \frac{1}{y_i}(y_{j_1}, y_{j_2}, \ldots, y_{j_n})$$

with $1 \le j_1 < j_2 < \cdots < j_n \le n+1$ and $j_k \ne i$ for $k = 1, \ldots, n$. After a rescaling in the independent variable in the local chart (U_1, F_1) the expression of $p(X)$ is

$$z_n^d(-z_1p_1 + p_2, -z_2p_1 + p_3, \ldots, -z_{n-1}p_1 + p_n, -z_np_1),$$

where $p_i = p_i(1/z_n, z_1/z_n, \ldots, z_{n-1}/z_n)$ for $i = 1, \ldots, n$.

In a similar manner we can deduce the expressions of $p(X)$ in the local charts $(U_2, F_2), \ldots, (U_n, F_n)$. These are

$$\begin{aligned} &z_n^d(-z_1p_2 + p_1, -z_2p_2 + p_3, \ldots, -z_{n-1}p_2 + p_n, -z_np_2),\\ &\quad p_i = p_i(z_1/z_n, 1/z_n, \ldots, z_{n-1}/z_n), \text{ for } i = 1, \ldots, n;\\ &\qquad\qquad\vdots\\ &z_n^d(-z_1p_n + p_1, -z_2p_n + p_3, \ldots, -z_{n-1}p_n + p_{n-1}, -z_np_n),\\ &\quad p_i = p_i(z_1/z_n, z_2/z_n, \ldots, z_{n-1}/z_n, 1/z_n), \text{ for } i = 1, \ldots, n; \end{aligned}$$

respectively. Finally for the local chart (U_{n+1}, F_{n+1}) the expression of $p(X)$ is

$$z_n^{d+1}(p_1, \ldots, p_n) \text{ for } p_i = p_i(z_1, \ldots, z_n).$$

In the chart (V_i, G_i) the expression of $p(X)$ is the same than in the chart (U_i, F_i) multiplied by $(-1)^d$ for $i = 1, \ldots, n+1$.

2.3. **Topological equivalent polynomial vector fields.** We recall that two polynomial vector fields X and Y on $\mathbb{R}^n$ are *topologically equivalent* if there is a homeomorphism on the Poincaré ball B preserving the infinity $\mathbb{S}^{n-1}$ and carrying orbits of the flow of $\pi(p(X))$ into trajectories of the flow of $\pi(p(Y))$, either preserving or reversing the sense of all the orbits.

In the rest of this section we only consider polynomial vector fields X in $\mathbb{R}^2$.

A *separatrix* of the Poincaré compactification $\pi(p(X))$ is a trajectory which is either an equilibrium point, or a limit cycle, or an orbit on the boundary of a hyperbolic sector at an equilibrium point, finite or infinity, or any orbit contained at the infinity $\mathbb{S}^1$. The closed set (see Neumann [11]) formed by all separatrices of $\pi(p(X))$ is denoted by Σ_X.

A *canonical region* of $\pi(p(X))$ is an open connected component of $D \setminus \Sigma_X$. The union of Σ_X plus one orbit chosen from each canonical region is the *separatrix configuration* of $\pi(p(X))$, denoted by Σ'_X. Two separatrix configurations Σ'_X and Σ'_Y are *equivalent* if there is a homeomorphism in B preserving the infinity $\mathbb{S}^{n-1}$ carrying orbits ofΣ'_X into orbits of Σ'_Y, either preserving or reversing the sense of all orbits.

Markus [8], Neumann [11] and Peixoto [13] characterized the topologically equivalence between two Poincaré compactified vector fields by showing that two Poincaré compactified polynomial vector fields $\pi(p(X))$ and $\pi(p(Y))$ with finitely many separatrices are topologically equivalent if and only if their separatrix configurations Σ'_X and Σ'_Y are equivalent.

3. Proofs

From system (1) we can write

$$\theta = -\frac{\dot\rho}{\rho}, \quad \sigma = \frac{2(-\rho\dot E + E\dot\rho)}{\rho(\rho + 6E)}.$$

Now introducing these two variables in the equations of $\dot\theta$ and $\dot\sigma$ in (1) and solving them with respect to $\ddot\rho$ and $\ddot E$ we obtain a differential system of the form $\ddot\rho = f(\rho, E)$, $\ddot E = g(\rho, E)$. Doing to this system the change of variables

$$\rho = \frac{6}{(1-x)y^3}, \quad E = -\frac{x}{(1-x)y^3},$$

we have the differential system (1) rewritten as

$$\ddot x + \frac{2\dot x\dot y}{y} - \frac{3x}{y^3} = 0, \quad \ddot y + \frac{1}{y^2} = 0. \tag{4}$$

This reduction from system (1) to system (4) was done in [12]. Note that if $x = 1$ or $y = 0$ then $\rho \to \infty$ and $E \to \infty$.

We can rewrite the two differential equations of second order (4) as the following four differential equations of first order

$$\begin{aligned} \dot x &= z, \\ \dot y &= w, \\ \dot z &= -\frac{2zw}{y} + \frac{3x}{y^3}, \\ \dot w &= -\frac{1}{y^2}. \end{aligned} \tag{5}$$

System (5) does not have any equilibria. Note that in this new variables the first integrals F and H provide the first integrals

$$\mathcal{F} = w^2 - \frac{2}{y} \quad \text{and} \quad \mathcal{H} = xw^2 + \frac{x}{y} + wyz$$

of system (5). It is easy to check that $F < 0$ if and only if $\mathcal{F} < 0$.

Setting $\mathcal{F} = f$ and $\mathcal{H} = h$ we get

$$y = \frac{2}{w^2 - f} \quad \text{and} \quad x = -\frac{2(fh + 2wz - hw^2)}{(f - w^2)(f - 3w^2)}.$$

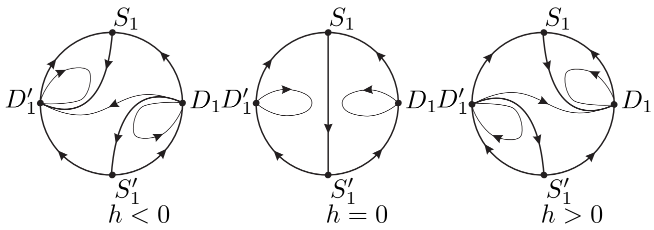

FIGURE 1. Phase portraits of system (7) with $f<0$.

Note that here the expressions for y and x are well defined when $(f-w^2)(f-3w^2) \neq 0$. So if $f<0$ then they are always well defined.

The dynamics of the differential system (5) restricted to the invariant set defined by $\mathcal{F}=f$ and $\mathcal{H}=h$ can be written as

$$
\begin{aligned}
z' &= \frac{(f-w^2)}{4(f-3w^2)}(3f^2h+10fzw-6fhw^2-18zw^3+3hw^4),\\
w' &= -\frac{1}{4}(f-w^2)^2.
\end{aligned} \tag{6}
$$

We do the reparametrization of time $dt=(f-3w^2)ds/(f-w^2)$, and we rewrite system (6) as the new system

$$
\begin{aligned}
\dot{z} &= 3f^2h+10fzw-6fhw^2-18zw^3+3hw^4,\\
\dot{w} &= -\frac{1}{4}(f-w^2)(f-3w^2),
\end{aligned} \tag{7}
$$

where now the dot means derivative with respect to the new independent variable s. Now we describe the global phase portrait of system (7) on the Poincaré disc (see subsection 2.2) when $f<0$.

Theorem 2. *The global phase portraits of system* (7) *with* $f<0$ *are topologically equivalent to the ones described in Figure 1.*

Proof. There are no finite singular points of system (7) when $f<0$.

On the local chart U_1 system (7) writes

$$
\begin{aligned}
\dot{u} &= \frac{1}{4}(69u^4-36fu^2v^2-f^2v^4-12hu^5+24fhu^3v^2-12f^2huv^4),\\
\dot{v} &= -v(-18u^3+10fuv^2+3hu^4-6fhu^2v^2+3f^2hv^4).
\end{aligned} \tag{8}
$$

The singular points at infinity in the local chart U_1 are $(u,v)=(0,0)$ and $(u,v)=(23/(4h),0)$. So, if $h\neq 0$ we have two singular points on U_1, and if $h=0$ we have only the origin as a singular point on U_1.

If $h\neq 0$, computing the eigenvalues of the linear part of the differential system at the point $(23/(4h),0)$ we get that they are $-839523/(256h^3)$ and $36501/(256h^3)$. So this point is a saddle.

On the other hand computing the linear part of the differential system at the origin $(0,0)$ we get that it is linearly zero. We need to do blow ups (see for more details subsection 2.1). We introduce the new variable $w = v/u$. In the new variables (u, w) we can rewrite system (8) as

$$\begin{aligned} \dot{u} &= -\frac{u^4}{4}(-69 + 12hu + 36fw^2 - 24fhuw^2 + f^2w^4 + 12f^2huw^4), \\ \dot{w} &= \frac{u^3w}{4}(fw^2 - 3)(fw^2 - 1). \end{aligned} \tag{9}$$

Now doing a rescaling of the independent variable we eliminate the term u^3 in system (9) and we obtain

$$\begin{aligned} \dot{u} &= -\frac{u}{4}(-69 + 12hu + 36fw^2 - 24fhuw^2 + f^2w^4 + 12f^2huw^4), \\ \dot{w} &= \frac{w}{4}(fw^2 - 3)(fw^2 - 1). \end{aligned}$$

The solutions on $w = 0$ of $\dot{u} = 0$ are precisely $u = 0$ and $u = 23/(4h)$. Computing the eigenvalues of the linear part of the differential system at the origin we get that they are $69/4$ and $3/4$. So $(0, 0)$ is an unstable node, and the eigenvalues of the linear part of the differential system at the point $(23/(4h), 0)$ are $-69/4$ and $3/4$. Hence $(23/(4h), 0)$ is a saddle. Now going back through the changes of variables to system (8) we get that the origin of U_1 is formed by two elliptic and two parabolic sectors separated by the straight line of the infinity.

On the local chart U_2 system (7) becomes

$$\begin{aligned} \dot{u} &= \frac{1}{4}(12h - 69u - 24fhv^2 + 36fuv^2 + 12f^2hv^4 + f^2uv^4), \\ \dot{v} &= \frac{1}{4}v(fv^2 - 3)(fv^2 - 1). \end{aligned}$$

The origin of U_2 is a singular point if and only if $h = 0$. In this last case computing the eigenvalues of the linear part of the differential system at the origin we get that they are $-69/4$ and $3/4$. So the origin of U_2 is a saddle.

Gluing all these information together we get that the global phase portrait of system (7) in the Poincaré disc $\mathbb{D}$ is topologically equivalent to the left one, the central one or the right one of Figure 1 according with $h < 0$, $h = 0$ or $h > 0$, respectively. $\square$

The proof of Theorem 1 will come interpreting the results provided in Theorem 2 on the gravitational Szekeres system (1).

Proof of Theorem 1. Note that

$$z = \dot{x}, \quad w = \dot{y}, \quad y = \frac{2}{w^2 - f}, \quad x = -\frac{2(fh + 2wz - hw^2)}{(f - 3w^2)(f - w^2)}. \tag{10}$$

Moreover we have that

$$\rho = \frac{6}{(1-x)y^3}, \quad \theta = \frac{z}{x-1} + 3\frac{w}{y}, \quad \sigma = -\frac{z}{3(1+x)}, \quad E = -\frac{x}{(1-x)y^3}. \tag{11}$$

In what follows we assume $f < 0$.

Case $h > 0$. In this case on the Poincaré disc we have four infinite singular points $D_1 = (z, w) = (+\infty, 0)$, $D_1' = (z, w) = (-\infty, 0)$, $S_1 = (z, w) = (+\infty, +\infty)$ but satisfying $w/z = 23/(4h)$ and $S_1' = (-\infty, -\infty)$ also satisfying $w/z = 23/(4h)$. Now we rewrite these points in the variables $(\rho, \theta, \sigma, E)$ and we call them $\tilde{D}_1, \tilde{D}_1', \tilde{S}_1$ and $\tilde{S}_1'$, respectively.

Note that using equations (10) in the case of the point D_1 taking into account that $z = \infty$ and $w = 0$ we get that $y = -2/f > 0$ and we have that x is essentially $-4wz/f^2$. Now introducing this value of x in σ we get that if x is finite then $\sigma \to \infty$ because $z \to +\infty$, and if $x \to \infty$ then

$$-\frac{z}{3(x+1)} \to \frac{zf^2}{12wz} = \frac{f^2}{12w} \to \infty,$$

because $w \to 0$. Proceeding in the same way we get that $\theta \to \infty$. The same can be done for $\tilde{D}_1'$. So the points $\tilde{D}_1$ and $\tilde{D}_1'$ are the infinity $\mathbb{S}^3$ of the Poincaré ball B where the extended Skezeres system is defined. Now we consider the point S_1. Note that in this case using (10) and taking into account that $z = +\infty$ and $w = +\infty$ we get that $x = 0$ and $y = 0$. It follows from (11) that in $\tilde{S}_1$ the variables ρ, θ and σ tend to ∞. The same happens for $\tilde{S}_1'$.

From the phase portrait of Figure 1 with $h > 0$ we see that there are three generic behaviors of the orbits: the orbits that go from $\tilde{D}_1$ to itself (would be homoclinic), the orbits that go from $\tilde{D}_1'$ to itself (also would be homoclinic) and the orbits that go from $\tilde{D}_1'$ to $\tilde{D}_1$ (that would be heteroclinic). Moreover, we have two non-generic behavior of the orbits which are the two separatrices connecting the points $\tilde{S}_1$ to $\tilde{D}_1$ and the points $\tilde{D}_1'$ to $\tilde{S}_1'$.

Additionally to these behaviors we also have to note that if one of the previous orbits is such that x takes the value 1 or y takes the value 0, then this particular orbit reaches the infinity in the variables ρ and E before reaching any of the points $\tilde{D}_1, \tilde{D}_1', \tilde{S}_1'$. In any case all the orbits with $h > 0$ start and end at some point at the infinity.

Case $h < 0$. Similar behaviors hold for the case $h < 0$. That is, at the beginning we find three generic behaviors of the orbits: the orbits that go from $\tilde{D}_1$ to itself, the orbits that go from $\tilde{D}_1'$ to itself and the orbits that go from D_1 to $\tilde{D}_1'$, and two non-generic behaviors of the orbits which are the two separatrices connecting the points $\tilde{S}_1'$ to $\tilde{D}_1'$ and the points $\tilde{D}_1$ to $\tilde{S}_1$. Again these three kind of orbits can be splitted in more orbits coming from and going to infinity if they intersect $x = 1$ or $y = 0$.

Case $h = 0$. In the case on the Poincaré disc we have four distinguished points $D_1 = (z, w) = (+\infty, 0)$, $D_1' = (z, w) = (-\infty, 0)$, $S_2 = (z, w) = (0, +\infty)$ and $S_2' = (0, -\infty)$. Now we rewrite these two last points in the variables $(\rho, \theta, \sigma, E)$ and we call them $\tilde{S}_2$ and $\tilde{S}_2'$, respectively. Using (10) and taking into account that $z = 0$ and $w = +\infty$ we get that $x = 0$ and $y = 0$. It follows from (11) that in $\tilde{S}_2$ the variables ρ and θ tend to ∞. The same happens for $\tilde{S}_2'$. Hence we have two generic behaviors of the orbits: the orbits that go from $\tilde{D}_1$ to itself and the orbits that go from $\tilde{D}_1'$ to itself, and one non-generic behavior of the orbit which is the separatrix connecting the point $\tilde{S}_2$ to $\tilde{S}_2'$. Again if there exists an orbit such that x takes the value 1 or y takes the value 0 then this particular orbit reaches the infinity in the variables ρ and E before reaching any of the points $\tilde{D}_1, \tilde{D}_1', \tilde{S}_2'$. This completes the proof of the theorem. □

4. Conclusion

The gravitational Szekeres differential system

$$\begin{aligned}\dot{\rho} &= -\theta\rho,\\ \dot{\theta} &= -\frac{1}{3}\theta^2 - 6\sigma^2 - \frac{1}{2}\rho,\\ \dot{\sigma} &= \sigma^2 - \frac{2}{3}\theta\sigma - E,\\ \dot{E} &= -3E\sigma - \theta E - \frac{1}{2}\rho\sigma,\end{aligned}$$

is completely integrable with two rational first integrals and an additional analytical first integral. One of these two rational first integrals is

$$F = \frac{(-18E - 3\rho + (\theta + 3\sigma)^2)^3}{(6E + \rho)^2}.$$

We describe completely the dynamics of the Szekeres system when the first integral $F < 0$, showing that all the orbits come from the infinity of $\mathbb{R}^4$ in the variables $(\rho, \theta, \sigma, E)$ and go to infinity. In other words all orbits of the gravitational Szekeres differential system born at infinity and end at infinity when the first integral $F < 0$.

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