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TOPOLOGICAL ENTROPY OF CONTINUOUS SELF–MAPS ON A GRAPH

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Abstract. Let G be a graph and f be a continuous self–map on G. Using the Lefschetz zeta function of f we provide a sufficient condition in order that f has positive topological entropy. Moreover, for some classes of graphs we improve this condition making it easier to check.

1. Introduction and statement of the main results

One of the ways to measure the complexity of a dynamical system is showing that its topological entropy is positive. The topological entropy is a nonnegative real number such that the larger this number, the greater the complexity of the dynamical system. The notion of topological entropy was introduced by Adler, Konheim and McAndrew in 1965, see [1]. Later on this definition was modified by Kolmogorov–Sinai introducing the metric entropy (see [22]), of course both definitions are related. After Dinaburg and Bowen [4] provided a weaker definition of topological entropy which clarifies the meaning of the topological entropy. Roughly speaking, for a system given by an iterated map (as the ones studied in this paper) the topological entropy essentially is the exponential growth rate of the number of distinguishable orbits of the iterates.

In this work a *graph* G will be a compact connected space containing a finite set of points V such that $G \setminus V$ has finitely many open connected components, each one of them homeomorphic to the interval $(0, 1)$. These components are called the *edges* of G, and the points of V are called the *vertices* of G. The edges are disjoint from the vertices, and the vertices are at the boundary of the edges.

For a graph G the *degree* of a vertex is the number of edges having this vertex in its boundary, if an edge has both boundaries in the same vertex then we compute this edge twice in the definition of the degree

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of that vertex. An *endpoint* of a graph G is a vertex of degree one. A *branching point* of a graph G is a vertex of degree at least three.

Let G be a graph and $f : G \to G$ a continuous map. A point $x \in G$ is *periodic* of *period* k if $f^k(x) = x$ and $f^i(x) \neq x$ if $0 < i < k$. If $k = 1$ then x is called a *fixed point*.

The set $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$, where by f^n we denote the composition of f with itself n times, is called the *orbit* of the point $x \in G$. Of course, if $x \in G$ is a periodic point, then the orbit defined by it is called a *periodic orbit*, this orbit is finite and its length is the period of the periodic point.

To describe the behaviour of all orbits of f is to study *the topological dynamics of the map* f. The set of all orbits of f forms the *discrete dynamical system* defined by f.

Roughly speaking the *topological entropy* $h(f)$ of a discrete dynamical system (G, f) is a non–negative real number (possibly infinite) which measures how much f mixes up the image of the space G. When $h(f)$ is positive the dynamics of the system becomes *complicated*, and the positivity of $h(f)$ is used as a measure of the so called *topological chaos*.

Here we introduce the topological entropy using the definition of Bowen [4]. Since it is possible to embed a graph G in $\mathbb{R}^3$, we consider the distance between two points of G as the distance of these two points in $\mathbb{R}^3$. Now we define the distance d_n on G by

$$d_n(x, y) = \max_{0 \leq i \leq n} d(f^i(x), f^i(y)), \quad \forall x, y \in G.$$

A finite set S is called (n, ε)*–separated with respect to* f if for different points $x, y \in S$ we have $d_n(x, y) > \varepsilon$. We denote by S_n the maximal cardinality of an (n, ε)–separated set. Define

$$h(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log S_n.$$

Then

$$h(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon)$$

is the *topological entropy* of f.

We have chosen the definition of Bowen because it is the shortest to state. The classical definition was due to Adler, Konheim and McAndrew [1]. See for instance the book of Hasselblatt and Katok [16] and [3] for other equivalent definitions and properties of the topological entropy. For more details on the topological entropy, see [1, 2, 9, 22].

The homological spaces of G with coefficients in $\mathbb{Q}$ are denoted by $H_k(G,\mathbb{Q})$. Since G is a graph the subindex $k = 0, 1$. A continuous map $f : G \to G$ induces linear maps $f_{*k} : H_k(G,\mathbb{Q}) \to H_k(G,\mathbb{Q})$. We note that $H_0(G,\mathbb{Q}) \approx \mathbb{Q}$ and we have that f_{*0} is the identity map because G is connected. A subset of G homeomorphic to a circle is a called a *circuit*. It is known that $H_1(G,\mathbb{Q}) \approx \mathbb{Q}^m$ being m the number of the independent circuits of G in the sense of the homology. Here f_{*1} is a $m \times m$ matrix $A = (a_{ij})$ with integer entries. More precisely, if $H_1(G,\mathbb{Q}) \approx \mathbb{Q}^m$ and γ_i for $i = 1, \dots, m$ are m independent oriented circuits of G, then the entry a_{ij} of the file i column j of the matrix A is the number of turns that $f(\gamma_i)$ gaves on the circuit γ_j taking into account the orientation of the circuit γ_j. For example, if $f(\gamma_i)$ covers the circuit γ_j five times in the same orientation of γ_j, and two times in the converse orientation, then $a_{ij} = 3$. For more details on this homology see for instance [21].

Independently of the fact that to study the dynamical complexity via the topological entropy of this kind of graph maps is relevant by itself for understanding their dynamics, the graph maps are relevant for studying the dynamics of some different kind of surface maps, see for instance [15, 19].

For a polynomial $H(t)$ we define $H^*(t)$ by

$$H(t) = (1-t)^\alpha (1+t)^\beta t^\gamma H^*(t),$$

where α, β and γ are non-negative integers such that $1-t$, $1+t$ and t do not divide $H^*(t)$. We also define $H^{**}(t)$ by

$$H(t) = (1-t)^\alpha (1+t)^\beta H^{**}(t),$$

where α and β are non-negative integers such that $1-t$ and $1+t$ do not divide $H^{**}(t)$.

Our results are inspired by the relation between the topological entropy and the periodic orbit structure, using as precedents the results of the papers [6, 12, 18, 20]. First, we present a general result to provide a sufficient condition in order that a continuous self–map on any graph has positive topological entropy. This condition is based in the notion of Lefschetz zeta function $\mathcal{Z}_f(t)$ for a map f, for its definition see subsection 2.1.

Theorem 1. *Let (G, f) be a discrete dynamical system induced by a continuous self–map f defined on a graph G, and let $\mathcal{Z}_f(t) = P(t)/Q(t)$ be its Lefschetz zeta function.*

(a) *Assume that $P^*(t)$ or $Q^*(t)$ has odd degree, then the topological entropy of f is positive.*

(b) *Assume that $P^{**}(t)$ or $Q^{**}(t)$ has odd degree, G is either $\mathbb{R}$ or $\mathbb{S}^1$ and f is a $\mathcal{C}^1$ map, then f has infinitely many periodic points.*

Statement (a) of Theorem 1 was known for continuous self–maps on connected surfaces in [6]. Statement (b) of Theorem 1 was already known, see statement (c) of Theorem 1 of [11].

In the next corollary, statement (b) of Theorem 1 allows to reprove in a different way a known result (see for instance [2]) for continuous circle maps in the smooth case.

Corollary 2. *Let $(\mathbb{S}^1, f)$ be a discrete dynamical system induced by a $\mathcal{C}^1$ map of degree d, then if $d \notin \{-1, 0, 1\}$ the map f has infinitely many periodic points.*

We present some improvements of Theorem 1 for some classes of graphs for which we provide more precise conditions, easy to check, in order to show that their continuous self–maps have positive topological entropy.

A *p-flower graph* is a graph with a unique branching point z and $p > 1$ edges all having a unique endpoint, the point z, equal for all of them. So this graph has p independent loops, each one is called a *petal*. See a 5-flower graph in Figure 1.

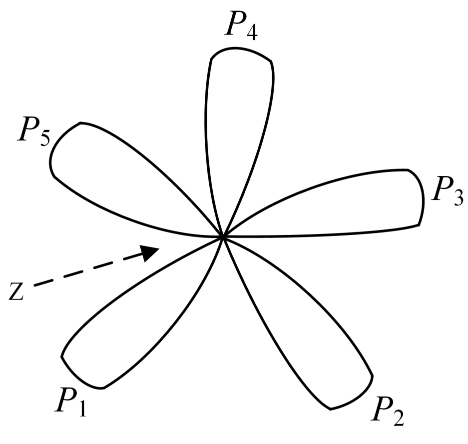

FIGURE 1. A 5-flower graph.

Theorem 3 (p-flower graph theorem)**.** *Let (G, f) be a discrete dynamical system induced by a continuous self–map f on p–flower graph G.*

(a) *If p is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then the topological entropy of f is positive.*

(b) *If p is odd and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not odd, then the topological entropy of f is positive.*

A graph with only two vertices z and w and $n \geq 1$ edges having every edge the vertices z and w as endpoints is called an *n-lips* graph and denoted by L^n. See a 7-lips graph L^7 in Figure 2.

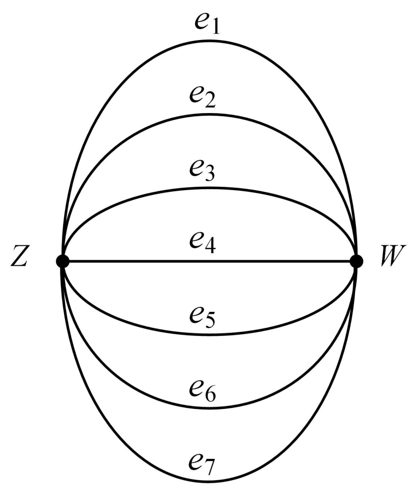

FIGURE 2. The 7-lips graph.

Theorem 4 (n-lips graph theorem)**.** *Let (G, f) be a discrete dynamical system induced by a continuous self–map f on an n–lip graph G, with $n > 1$.*

(a) *If $n-1$ is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then the topological entropy of f is positive.*

(b) *If $n-1$ is odd and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not odd, then the topological entropy of f is positive.*

A *graph* $p + r_1 L^1 + ... + r_s L^s$ is formed by p petals and $r_1 + ... + r_s$ lips where r_j lips are of type L^j for $j = 1, ..., s$. Note that a such graph has

$p+\sum_{j=1}^{s} jr_j$ edges and

$$\mathcal{L}_{p,r_2,\ldots,r_s} = p + r_2 + 2r_3 \ldots + (s-1)r_s$$

is the number of its independent circuits.

See a $(4+3L^1+2L^2+1L^3)$–graph in Figure 3, this graph has $p=4$ and six lips, three lips L^1, two lips L^2 and one lip L^3.

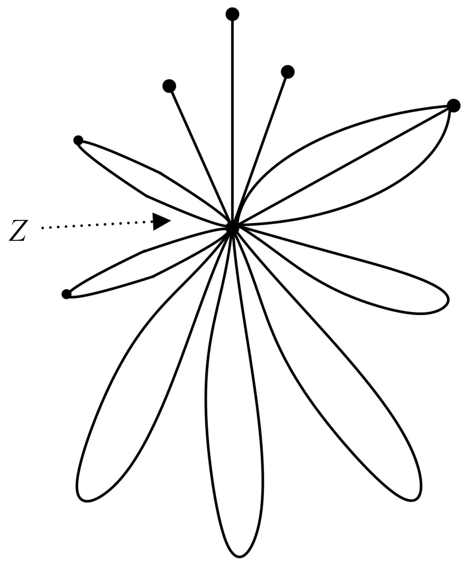

FIGURE 3. A $(4+3L^1+2L^2+1L^3)$–graph.

Theorem 5 ($(p+r_1L^1+\ldots+r_sL^s)$–graph theorem)**.** *Let (G,f) be a discrete dynamical system induced by a continuous self–map f on a $(p+r_1L^1+\ldots+r_sL^s)$–graph G.*

(a) *If $\mathcal{L}_{p,r_2,\ldots,r_s}$ is even and the number of roots of the characteristic polynomial of f_{*1} equal to ±1 or 0 taking into account their multiplicities is not even, then the topological entropy of f is positive.*

(b) *If $\mathcal{L}_{p,r_2,\ldots,r_s}$ is odd and the number of roots of the characteristic polynomial of f_{*1} equal to ±1 or 0 taking into account their multiplicities is not odd, then the topological entropy of f is positive.*

2. Preliminary results

2.1. **Lefschetz zeta function.** Given a discrete dynamical system $(\mathbb{M},f)$ where f is a continuous self–map defined on the compact n–dimensional topological space $\mathbb{M}$ the *Lefschetz number* is

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}),$$

where the induced homomorphism by f on the k–th rational homology group of $\mathbb{M}$ is $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$. We note that $H_k(\mathbb{M}, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$, and that f_{*k} is a linear map given by a matrix with integer entries. The Lefschetz Fixed Point Theorem connects the fixed point theory with the algebraic topology via the following result.

Theorem 6. *Let* $(\mathbb{M}, f)$ *be a discrete dynamical system induced by a continuous self–map* f *on a compact topological space* $\mathbb{M}$ *and* $L(f)$ *be its Lefschetz number. If* $L(f) \neq 0$ *then* f *has a fixed point.*

For a proof of Theorem 6 see for instance [5].

The sequence of the Lefschetz numbers of all iterates of f denoted by $\{L(f^m)\}_{m=0}^{\infty}$ is used for defining the Lefschetz *zeta function* of f as follows

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m \right).$$

This function contains information of all the sequence of the iterated Lefschetz numbers. Note that the function $\mathcal{Z}_f(t)$ can be computed also through

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}}, \tag{1}$$

where $n = \dim \mathbb{M}$, $n_k = \dim H_k(\mathbb{M}, \mathbb{Q})$, I_{n_k} is the $n_k \times n_k$ identity matrix, and we take $\det(I_{n_k} - t f_{*k}) = 1$ if $n_k = 0$, for more details on the function $\mathcal{Z}_f(t)$ see [7]. From (1) the Lefschetz zeta function is a rational function.

2.2. **Cyclotomic polynomials.** The n–th *cyclotomic polynomial* is defined recursively by

$$c_n(t) = \frac{1 - t^n}{\prod_{d|n} c_d(t)},$$

for a positive integer $n > 1$ and $c_1(t) = 1 - t$. Note that all the zeros of $c_n(t)$ are roots of unity. See [17] for the properties of these polynomials.

For a positive integer n the *Euler function* is $\varphi(n) = n \prod_{p|n, p \text{ prime}} \left(1 - \frac{1}{p}\right)$. It is known that the degree of the polynomial $c_n(t)$ is $\varphi(n)$. Note that $\varphi(n)$ is even for $n > 2$.

A proof of the next result can be found in [17].

Proposition 7. *Let ξ be a primitive n–th root of the unity and $P(t)$ a polynomial with rational coefficients. If $P(\xi) = 0$ then $c_n(t)|P(t)$.*

Lemma 8. *If a polynomial has integer coefficients, constant term 1 and all of whose roots have modulus 1, then all of its roots are roots of unity.*

For a proof of Lemma 8 see [23].

2.3. **Topological entropy.** As we showed in subsection 2.1, given a discrete dynamical system $(\mathbb{M}, f)$ with f a continuous self–map defined on a compact n–dimensional topological space $\mathbb{M}$, the map f induces an action on the homology groups of $\mathbb{M}$, which we denote $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$, for $k \in \{0, 1, \ldots, m\}$. The *spectral radii* of these maps are denoted $\text{sp}(f_{*k})$, and they are equal to the largest modulus of all the eigenvalues of the linear map f_{*k}. The *spectral radius of f_** is

$$\text{sp}(f_*) = \max_{k=0,\ldots,m} \text{sp}(f_{*k}).$$

The next result is due to Guaschi and Llibre [10] and Jiang [13, 14], for more details see Theorem 5.4.2 from [2].

Theorem 9. *Let $f : G \to G$ be a continuous map on the graph G. Then $\log \max\{1, \text{sp}(f_{*1})\} \leq h(f)$.*

3. Auxiliary results

We need the following results for proving our theorems. The next result is Theorem 6 from [8].

Proposition 10. *Let $\mathbb{M}$ be a smooth compact manifold and let $(\mathbb{M}, f)$ be a discrete dynamical system induced by a $\mathcal{C}^1$ self–map f such that $f(\mathbb{M}) \subseteq \text{Int}(\mathbb{M})$, and assume that f has finitely many periodic points. Then $\mathcal{Z}_f(t)$ has a finite factorization in terms of the form $(1 \pm t^r)^{\pm 1}$ with r a positive integer.*

Lemma 11. *Let (G, f) be a discrete dynamical system induced by a continuous self–map f defined on graph G. If the topological entropy of f is zero, then all the eigenvalues of the induced homomorphism f_{*1} are zero or roots of unity.*

Proof. Since the topological entropy is zero, by Theorem 9 we have $\mathrm{sp}(f_{*1}) = 1$. So all the eigenvalues of f_{*1} have modulus in the interval $[0, 1]$ and at least one of them is 1. Then the characteristic polynomial of f_{*1} is of the form $t^m p(t)$, where m is a non–negative integer, positive if the zero is an eigenvalue. And $p(t)$ is a polynomial with integer coefficients and whose independent term a_0 is non–zero. Since the product of all non-zeros eigenvalues of f_{*1} is the integer a_0 and, these eigenvalues have modulus in $(0, 1]$, we have that any of these eigenvalues can have modulus smaller than one, otherwise we are in contradiction with the fact a_0 is an integer. In short, all the non–zero eigenvalues have modulus one, and consequently $a_0 = 1$. By Lemma 8 all the roots of the polynomial $p(t)$ are roots of unity finishing the proof. □

Lemma 12. *Let $(\mathbb{M}, f)$ be a discrete dynamical system induced by a $\mathcal{C}^1$ self–map f defined on a smooth compact connected n–dimensional manifold $\mathbb{M}$. Assume that $f(\mathbb{M}) \subseteq \mathrm{Int}(\mathbb{M})$. If f has finitely many periodic points, then all the eigenvalues of the induced homomorphisms f_{*k}'s are zero or roots of unity.*

Proof. Since by Proposition 10 the Lefschetz zeta function (1) has a finite factorization in terms of the form $(1 \pm t^r)^{\pm 1}$ with r a positive integer, it follows that all the eigenvalues of f_{*1} are roots of unity. This completes the proof. □

4. Proof of Theorem 1

Proof of Theorem 1. From the definitions of a polynomial H^* and of the Lefschetz zeta function we have

$$\mathcal{Z}_f(t) = \frac{P(t)}{Q(t)} = (1-t)^a(1+t)^b t^c \frac{P^*(t)}{Q^*(t)},$$

where a, b and c are integers.

Assume now that the topological entropy $h(f) = 0$. Then by Lemma 11 all the eigenvalues of the induced homomorphisms f_{*1} are zero or roots of unity. Therefore, by (1) all the roots of the polynomials $P^*(t)$ and $Q^*(t)$ are roots of the unity different from ± 1 and zero. Hence, by Proposition 7 the polynomials $P^*(t)$ and $Q^*(t)$ are product of cyclotomic polynomials different from $c_1(t) = 1 - t$ and $c_2(t) = 1 + t$. Consequently $P^*(t)$ and $Q^*(t)$ have even degree because all the cyclotomic polynomials which appear in them have even degree due to the fact that the Euler function $\varphi(n)$ for $n > 2$ only takes even values. But this is a contradiction with the assumption that $P^*(t)$ or $Q^*(t)$ has odd degree. This completes the proof of statement (a).

For proving statement (b) we shall use Proposition 10 taking account that the unique graphs admitting $\mathcal{C}^1$ maps are the ones which are manifolds, i.e. the real line and the circle. Note that under the hypothesis of statement (b) if we assume that f has finitely many periodic points, by Lemma 12 all the eigenvalues of f_{*1} are zero or root of unity. From the definition of the polynomial H^{**} and of the Lefschetz zeta function we have

$$\mathcal{Z}_f(t) = \frac{P(t)}{Q(t)} = (1-t)^a(1+t)^b\frac{P^{**}(t)}{Q^{**}(t)},$$

where a and b are integers. By Proposition 10 all the roots of the polynomials $P^{**}(t)$ and $Q^{**}(t)$ are roots of unity different from ± 1. Therefore the rest of the proof of statement (b) follows as in the last part of the proof of statement (a). This completes the proof of the theorem. □

5. Proofs of Corollary 2 and Theorems 3, 4 and 5

Let $f : G \to G$ be a continuous map on the graph G. The homological spaces of G with coefficients in $\mathbb{Q}$ are denoted by $H_k(G, \mathbb{Q})$. Since G is a graph $k = 0, 1$ and f induces linear maps $f_{*k} : H_k(G, \mathbb{Q}) \to H_k(G, \mathbb{Q})$. Since G is a graph, then $H_0(G, \mathbb{Q}) \approx \mathbb{Q}$ and f_{*0} is the identity map. It is known that $H_1(G, \mathbb{Q}) \approx \mathbb{Q}^m$ being m the number of the independent circuits of G in the sense of the homology. Here f_{*1} is a $m \times m$ matrix A with integer entries. For more details on this homology see for instance [21].

By (1) the form of the Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - tf_{*1})}{\det(I - tf_{*0})} = \frac{\det(I - tA)}{1 - t},$$

where A is the integer matrix defined by f_{*1}, for a proof see Franks [7].

Proof of Corollary 2. Since G is the circle, $H_1(G, \mathbb{Q}) \approx \mathbb{Q}$, so the Lefschetz zeta function is

$$Z_f(t) = \frac{1 - td}{1 - t},$$

where d is the degree of f. Therefore the result follows directly from statement (b) of Theorem 1 when $d \neq -1, 0, 1$. □

Proof of Theorem 3. Since G is a p-flower, which is a graph with p independent circuits, $H_1(G, \mathbb{Q}) \approx \mathbb{Q}^p$. Thus, the Lefschetz zeta function is

$$Z_f(t) = \frac{\det(I - tA)}{1 - t},$$

where $\det(I - tA)$ is a polynomial of degree p with integer coefficients and $f_{*1} = A$ is a $p \times p$ matrix with integer entries. Note that in this case $Q(t) = 1 - t$ and $Q^*(t) = 1$. So, by Theorem 1 the main role will be played by the polynomial $P(t) = \det(I - tA)$ where $f_{*1} = A$. If p is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then $P^*(t)$ has odd degree. Therefore statement (a) follows by the application of statement (a) of Theorem 1.

On the other hand, if p is odd and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not odd, then $P^*(t)$ has odd degree and as before the proof of statement (b) follows. □

Proof of Theorem 4. The proof is the same as the proof of Theorem 3 taking account that an n–lip graph has $n - 1$ independent circuits and therefore f_{*1} is a polynomial of degree $n - 1$. □

Proof of Theorem 5. The proof follows from the arguments stated in the proof of Theorem 3 taking account that for a $(p + r_1 L^1 + ... + r_s L^s)$–graph, $\mathcal{L}_{p,r_2,...,r_s} = p + r_2 + 2r_3 + ... + (s - 1)r_s$ is the number of independent circuits. □

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