

LIMIT CYCLES OF A SECOND-ORDER DIFFERENTIAL EQUATION

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ABSTRACT. We provide an upper for the maximum number of limit cycles bifurcating from the periodic solutions of $\ddot{x} + x = 0$, when we perturb this system as follows

$$\ddot{x} + \varepsilon(1 + \cos^m \theta)Q(x, y) + x = 0,$$

where $\varepsilon > 0$ is a small parameter, m is an arbitrary non-negative integer, $Q(x, y)$ is a polynomial of degree n and $\theta = \arctan(y/x)$. The main tool used for proving our results is the averaging theory.

1. INTRODUCTION

To determine the number of limit cycles of a differential equation is one of the main problems in the qualitative theory of planar differential system. In 1881 Poincaré [6] defined the notion of *limit cycle* of a planar differential system as a periodic orbit isolated in the set of all periodic orbits of the differential system. And he defined the notion of a *center* of a real planar differential system, i.e. of an isolated equilibrium point having a neighborhood filled with periodic orbits. Later on one way to produce limit cycles is by perturbing the periodic orbits of a center, see for instance the papers [7, 2, 4] and the references quoted there.

In [5] Mathieu consider the second order differential equation

$$(1) \quad \ddot{x} + b(1 + \cos t)x = 0,$$

where b is a real constant. It is called Mathieu equation, which is the simplest mathematical model of an excited system depending on a parameter. The more general Emarkov-Pinney equation is Mathieu-Duffing type equations

$$(2) \quad \ddot{x} + b(1 + \cos t)x - x^\beta = 0,$$

where β is an integer and $b > 0$. These equations describe the dynamic of a system with harmonic parametric excitation and a nonlinear term corresponding to a restoring force, see the papers [9, 10, 3, 8].

We shall study the limit cycles of a kind of generalization of the second-order differential equations (1) and (2). More precisely, the objective of this paper is to consider the second-order differential equations

$$\ddot{x} + \varepsilon(1 + \cos^m \theta)Q(x, y) + x = 0,$$

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or equivalently its differential system of first order

$$(3) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(1 + \cos^m \theta)Q(x, y). \end{aligned}$$

We study the maximum number of limit cycles which can bifurcate from the center of system (3) with $\varepsilon = 0$, where ε is sufficiently small and $\theta = \arctan(y/x)$. More precisely, we consider the planar vector field

$$\chi = \chi(x, y) = (y, -x),$$

and we perturb this vector field χ as follows

$$\chi_\varepsilon = \chi(x, y) + \varepsilon(1 + \cos^m \theta)(0, Q(x, y)).$$

The main result of this paper is the following. For a definition of averaged function of first order see section 2 and [4].

Theorem 1. *Assume that the average function $f(r)$ of first order associated to the vector field χ_ε is non-zero and $\varepsilon > 0$ sufficiently small.*

- (a) *If m is odd, the maximum number of limit cycles of χ_ε bifurcating from the periodic solutions of the center χ , is at most $n - 1$ using the averaging theory of first order.*
- (b) *If m is even, the maximum number of limit cycles of χ_ε bifurcating from the periodic solutions of center χ , is at most $n - 1$ or $n - 2$, when n is odd or even, respectively.*

Moreover these upper bounds are reached.

Theorem 1 is proved in section 3. Note that the maximum number of limit cycles stated in Theorem 1 depend on the numbers m and n .

We provide a summary about the averaging theory for computing periodic solutions of vector fields that we shall use for proving Theorem 1 in section 2.

2. AVERAGING THEORY FOR DIFFERENTIAL SYSTEMS

In this section we recall some known results of the averaging theory that we shall need for proving Theorem 1. For more details on the averaging theory see [4].

Consider a non-autonomous differential equation of the form

$$(4) \quad \frac{dr}{d\theta} = \chi(r, \theta) = \varepsilon F(r, \theta) + \varepsilon^2 R(r, \theta, \varepsilon),$$

where $r \in \mathbb{R}$, $\theta \in \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. And $F : D \times \mathbb{S}^1 \rightarrow \mathbb{R}^2$, $R : D \times \mathbb{S}^1 \times (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^2$ are \mathbb{C}^2 functions, 2π -periodic in the variable θ and D is an open interval of \mathbb{R} .

The averaged function $f : D \rightarrow \mathbb{R}$ associated to system (4) is defined by

$$f(r) = \frac{1}{2\pi} \int_0^{2\pi} F(r, \theta) d\theta.$$

We called that if $r(r_0, \theta)$ is the solution of the vector field $\chi(r, \theta)$ such that $r(r_0, 0) = r_0$, then we have

$$r(r_0, 2\pi) - r_0 = \varepsilon f(r) + O(\varepsilon^2).$$

So for $\varepsilon > 0$ sufficiently small the simple zeros of the averaged function $f(r)$ provides limit cycles of vector field $\chi(r, \theta)$. In order to study the simple zeros of the function $f(r)$ we shall apply the *Descartes Theorem* (for a proof see for instance [1]).

Descartes Theorem. *Consider the real polynomial $p(r) = a_{i_1}r^{i_1} + a_{i_2}r^{i_2} + \dots + a_{i_n}r^{i_n}$ with $0 \leq i_1 < i_2 < \dots < i_n$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \dots, n\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs in m , then $p(r)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(r)$ in such a way that $p(r)$ has exactly $n - 1$ positive real roots.*

3. PROOF OF THEOREM 1

Assume that the polynomial $Q(x, y) = \sum_{i+j=0}^n a_{ij}x^i y^j$. Doing the change of variable $(x, y) \rightarrow (r, \theta)$, where $x = r \cos \theta$, $y = r \sin \theta$, with $r > 0$, system (3) associated to the vector field χ_ε in coordinates (r, θ) writes

$$\begin{aligned} \dot{r} &= -\varepsilon \sum_{i+j=0}^n R_{ij}(\theta)r^{i+j}, \\ \dot{\theta} &= -1 - \varepsilon \sum_{i+j=0}^n \Theta_{ij}(\theta)r^{i+j-1}, \end{aligned}$$

where

$$\begin{aligned} R_{ij}(\theta) &= a_{ij}(\cos^i \theta \sin^{j+1} \theta + \cos^{i+m} \theta \sin^{j+1} \theta), \\ \Theta_{ij}(\theta) &= a_{ij}(\cos^{i+1} \theta \sin^j \theta + \cos^{i+m+1} \theta \sin^j \theta). \end{aligned}$$

Taking θ as the new independent variable the previous differential system becomes the differential equation

$$(5) \quad \frac{dr}{d\theta} = \varepsilon \sum_{i+j=0}^n R_{ij}(\theta)r^{i+j} + O(\varepsilon^2) = \varepsilon F(r, \theta) + O(\varepsilon^2).$$

Note that this differential equation is written in the normal form (4) for applying the averaging theory of first order.

In our study we will use the following formulas for compute the averaged function. The first formula is

$$\int_0^{2\pi} \cos^p \theta \sin^{2q} \theta d\theta = \frac{(2q-1)!!}{(2q+p)(2q+p-2)\dots(p+2)} \int_0^{2\pi} \cos^p \theta d\theta,$$

for more details of this integral and the other ones integral see pages 152-153 of [11]. This formula is applicable for arbitrary real p and arbitrary positive integer q , except for the following negative even integers $p = -2, -4, \dots, -2n$. If p is a natural integer and $q = 0$, we have the formulas

$$\begin{aligned} \int_0^{2\pi} \cos^{2l} \theta d\theta &= \frac{(2l-1)!!}{2^l l!} 2\pi, \\ \int_0^{2\pi} \cos^{2l+1} \theta d\theta &= 0. \end{aligned}$$

We also have the formula

$$\int_0^{2\pi} \cos^p \theta \sin^{2q+1} \theta d\theta = 0.$$

This last formula is applicable for arbitrary real p and non-negative integer q , except for the following negative odd integers $p = -1, -3, \dots, -(2n+1)$. Therefore, from the previous section, we consider two cases and some subcases in order to study the averaged function associated to the differential equation (5).

Case (a). Let m be an odd. Then we consider two subcases for studying the averaged function.

Subcase (a.1). If n is an even, we have

$$\begin{aligned} f_1(r) &= \frac{1}{2\pi} \int_0^{2\pi} F(r, \theta) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i+j=0}^n \left[a_{ij} (\cos^i \theta + \cos^{i+m} \theta) \sin^{j+1} \theta \right] r^{i+j} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i+2q=2}^{n+1} \left[a_{i,2q-1} (\cos^i \theta + \cos^{i+m} \theta) \sin^{2q} \theta \right] r^{i+2q-1} d\theta \\ &= \frac{1}{2\pi} \left[\sum_{2l+1+2q=3}^{n+1} a_{2l+1,2q-1} r^{2l+2q} \int_0^{2\pi} \cos^{2l+m+1} \theta \sin^{2q} \theta d\theta \right. \\ &\quad \left. + \sum_{2l+2q=2}^n a_{2l,2q-1} r^{2l+2q-1} \int_0^{2\pi} \cos^{2l} \theta \sin^{2q} \theta d\theta \right] \\ &= \sum_{l+q=1}^{n/2} a_{2l+1,2q-1} r^{2l+2q} \frac{(2q-1)!!}{(2q+2l+1)(2q+2l-1) \cdots (2l+3)} \frac{(2l+m)!!}{2^{\frac{2l+m+1}{2}} (\frac{2l+m+1}{2})!} \\ &\quad + \sum_{l+q=1}^{n/2} a_{2l,2q-1} r^{2l+2q-1} \frac{(2q-1)!!}{(2q+2l)(2q+2l-2) \cdots (2l+2)} \frac{(2l-1)!!}{2^l l!} \\ &= \sum_{l+q=1}^{n/2} a_{2l+1,2q-1} r^{2l+2q} \frac{(2l+m)!!(2q-1)!!}{2^{\frac{2l+m+1}{2}} (\frac{2l+m+1}{2})! (2q+2l+1)(2q+2l-1) \cdots (2l+3)} \\ &\quad + \sum_{l+q=1}^{n/2} a_{2l,2q-1} r^{2l+2q-1} \frac{(2l-1)!!(2q-1)!!}{2^{l+q} l! (q+l)(q+l-1) \cdots (l+1)} \\ &= \sum_{k=1}^n A_k r^k. \end{aligned}$$

Subcase (a.2). If n is an odd, we have

$$\begin{aligned}
f_2(r) &= \frac{1}{2\pi} \int_0^{2\pi} F(r, \theta) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i+j=0}^n \left[a_{ij} (\cos^i \theta + \cos^{i+m} \theta) \sin^{j+1} \theta \right] r^{i+j} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i+2q=2}^{n+1} \left[a_{i,2q-1} (\cos^i \theta + \cos^{i+m} \theta) \sin^{2q} \theta \right] r^{i+2q-1} d\theta \\
&= \frac{1}{2\pi} \left[\sum_{2l+1+2q=3}^n a_{2l+1,2q-1} r^{2l+2q} \int_0^{2\pi} \cos^{2l+m+1} \theta \sin^{2q} \theta d\theta \right. \\
&\quad \left. + \sum_{2l+2q=2}^{n+1} a_{2l,2q-1} r^{2l+2q-1} \int_0^{2\pi} \cos^{2l} \theta \sin^{2q} \theta d\theta \right] \\
&= \sum_{l+q=1}^{(n-1)/2} a_{2l+1,2q-1} r^{2l+2q} \frac{(2l+m)!!(2q-1)!!}{2^{\frac{2l+m+1}{2}} \left(\frac{2l+m+1}{2}\right)! (2q+2l+1)(2q+2l-1) \cdots (2l+3)} \\
&\quad + \sum_{l+q=1}^{(n+1)/2} a_{2l,2q-1} r^{2l+2q-1} \frac{(2l-1)!!(2q-1)!!}{2^{l+q} l! (q+l)(q+l-1) \cdots (l+1)} \\
&= \sum_{k=1}^n \tilde{A}_k r^k.
\end{aligned}$$

Case (b). Assume m is an even, we consider the following subcases for studying the averaged function.

Subcase (b.1). If n is an even, we have

$$\begin{aligned}
f_3(r) &= \frac{1}{2\pi} \int_0^{2\pi} F(r, \theta) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i+j=0}^n \left[a_{ij} (\cos^i \theta + \cos^{i+m} \theta) \sin^{j+1} \theta \right] r^{i+j} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i+2q=2}^{n+1} \left[a_{i,2q-1} (\cos^i \theta + \cos^{i+m} \theta) \sin^{2q} \theta \right] r^{i+2q-1} d\theta \\
&= \frac{1}{2\pi} \left[\sum_{2l+2q=2}^n a_{2l,2q-1} r^{2l+2q-1} \int_0^{2\pi} (\cos^{2l} \theta + \cos^{2l+m} \theta) \sin^{2q} \theta d\theta \right] \\
&= \sum_{l+q=1}^{n/2} a_{2l,2q-1} r^{2l+2q-1} \frac{(2q-1)!!}{2^q (q+l)(q+l-1) \cdots (l+1)} \left[\frac{(2l-1)!!}{2^l l!} \right. \\
&\quad \left. + \frac{(2l+m-1)!!}{2^{\frac{2l+m}{2}} \left(\frac{2l+m}{2}\right)!} \right] \\
&= \sum_{k=1}^{n-1} B_k r^k.
\end{aligned}$$

Subcase (b.2). If n is an odd, we have

$$\begin{aligned}
f_4(r) &= \frac{1}{2\pi} \int_0^{2\pi} F(r, \theta) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i+j=0}^n \left[a_{ij} (\cos^i \theta + \cos^{i+m} \theta) \sin^{j+1} \theta \right] r^{i+j} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i+2q=2}^{n+1} \left[a_{i,2q-1} (\cos^i \theta + \cos^{i+m} \theta) \sin^{2q} \theta \right] r^{i+2q-1} d\theta \\
&= \frac{1}{2\pi} \left[\sum_{2l+2q=2}^{n+1} a_{2l,2q-1} r^{2l+2q-1} \int_0^{2\pi} (\cos^{2l} \theta + \cos^{2l+m} \theta) \sin^{2q} \theta d\theta \right] \\
&= \sum_{l+q=1}^{(n+1)/2} a_{2l,2q-1} r^{2l+2q-1} \frac{(2q-1)!!}{2^q (q+l)(q+l-1) \cdots (l+1)} \left[\frac{(2l-1)!!}{2^l l!} \right. \\
&\quad \left. + \frac{(2l+m-1)!!}{2^{\frac{2l+m}{2}} (\frac{2l+m}{2})!} \right] \\
&= \sum_{k=1}^n B_k r^k.
\end{aligned}$$

Note that the coefficients a_{ij} of the vector field χ_ε which appear in A_k (or \tilde{A}_k , or B_k) can be arbitrarily chosen. From the above expressions of the four subcases we obtain that the averaged function $f(r)$ is generated by a linear combination of a set $\mathcal{F}_l = \{r, r^2, \dots, r^l\}$ with $l \in \{n-1, n\}$. Using the Descartes Theorem, it follows that $f(r)$ can have at most $n-1$ (or at most $n-2$ when n and m are even numbers) simple zeros. Therefore for $\varepsilon > 0$ sufficiently small the vector field χ_ε can have at most $n-1$ (or $n-2$) limit cycles. This completes the proof of Theorem 1.

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