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Fernández-Córdoba, Cristina; Vela, Carlos; Villanueva, Mercè. «On Z8-linear Hadamard codes : rank and classification». IEEE Transactions on Information Theory, Vol. 66, issue 2 (Feb. 2020), p. 970-982. DOI 10.1109/TIT.2019.2952599

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On $\mathbb{Z}_8$-linear Hadamard codes: rank and classification

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***Abstract*—The $\mathbb{Z}_{2^s}$-additive codes are subgroups of $\mathbb{Z}_{2^s}^n$, and can be seen as a generalization of linear codes over $\mathbb{Z}_2$ and $\mathbb{Z}_4$. A $\mathbb{Z}_{2^s}$-linear Hadamard code is a binary Hadamard code which is the Gray map image of a $\mathbb{Z}_{2^s}$-additive code. It is known that either the rank or the dimension of the kernel can be used to give a complete classification for the $\mathbb{Z}_4$-linear Hadamard codes. However, when $s > 2$, the dimension of the kernel of $\mathbb{Z}_{2^s}$-linear Hadamard codes of length 2^t only provides a complete classification for some values of t and s. In this paper, the rank of these codes is computed for $s = 3$. Moreover, it is proved that this invariant, along with the dimension of the kernel, provides a complete classification, once $t \geq 3$ is fixed. In this case, the number of nonequivalent such codes is also established.**

Keywords Rank, Kernel, Hadamard code, $\mathbb{Z}_{2^s}$-additive code, Gray map, classification.

I. INTRODUCTION

Let $\mathbb{Z}_{2^s}$ be the ring of integers modulo 2^s with $s \geq 1$. The set of n-tuples over $\mathbb{Z}_{2^s}$ is denoted by $\mathbb{Z}_{2^s}^n$. In this paper, the elements of $\mathbb{Z}_{2^s}^n$ will also be called vectors over $\mathbb{Z}_{2^s}$ of length n. A binary code of length n is a nonempty subset of $\mathbb{Z}_2^n$, and it is linear if it is a subspace of $\mathbb{Z}_2^n$. A nonempty subset of $\mathbb{Z}_{2^s}^n$ is a $\mathbb{Z}_{2^s}$-additive code if it is a subgroup of $\mathbb{Z}_{2^s}^n$. Note that, when $s = 1$, a $\mathbb{Z}_{2^s}$-additive code is a binary linear code and, when $s = 2$, it is a quaternary linear code or a linear code over $\mathbb{Z}_4$.

Let $\mathcal{S}_n$ be the symmetric group of permutations on the set $\{1, \dots, n\}$. Two binary codes, C_1 and C_2, are said to be equivalent if there is a vector $\mathbf{a} \in \mathbb{Z}_2^n$ and a permutation of coordinates $\pi \in \mathcal{S}_n$ such that $C_2 = \{\mathbf{a} + \pi(\mathbf{c}) : \mathbf{c} \in C_1\}$. Two $\mathbb{Z}_{2^s}$-additive codes, $\mathcal{C}_1$ and $\mathcal{C}_2$, are said to be permutation equivalent if they differ only by a permutation of coordinates, that is, if there is a permutation of coordinates π such that $\mathcal{C}_2 = \{\pi(\mathbf{c}) : \mathbf{c} \in \mathcal{C}_1\}$.

The Hamming weight of a binary vector $\mathbf{u} \in \mathbb{Z}_2^n$, denoted by $\mathrm{wt}_H(\mathbf{u})$, is the number of nonzero coordinates of $\mathbf{u}$. The Hamming distance of two binary vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_2^n$, denoted by $d_H(\mathbf{u}, \mathbf{v})$, is the number of coordinates in which they differ. Note that $d_H(\mathbf{u}, \mathbf{v}) = \mathrm{wt}_H(\mathbf{v} - \mathbf{u})$. The minimum distance of a binary code C is $d(C) = \min\{d_H(\mathbf{u}, \mathbf{v}) : \mathbf{u}, \mathbf{v} \in C, \mathbf{u} \neq \mathbf{v}\}$. The Lee weight of an element $i \in \mathbb{Z}_{2^s}$ is $\mathrm{wt}_L(i) = \min\{i, 2^s - i\}$ and the Lee weight of a vector $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{Z}_{2^s}^n$ is $\mathrm{wt}_L(\mathbf{u}) = \sum_{j=1}^n \mathrm{wt}_L(u_j) \in \mathbb{Z}_{2^s}$. The Lee distance of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_{2^s}^n$ is $d_L(\mathbf{u}, \mathbf{v}) = \mathrm{wt}_L(\mathbf{v} - \mathbf{u})$. The minimum distance of a $\mathbb{Z}_{2^s}$-additive code $\mathcal{C}$ is $d(\mathcal{C}) = \min\{d_L(\mathbf{u}, \mathbf{v}) : \mathbf{u}, \mathbf{v} \in \mathcal{C}, \mathbf{u} \neq \mathbf{v}\}$.

In [15], a Gray map from $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$ is defined as $\phi(0) = (0, 0)$, $\phi(1) = (0, 1)$, $\phi(2) = (1, 1)$ and $\phi(3) = (1, 0)$. There exist different generalizations of this Gray map, which go from $\mathbb{Z}_{2^s}$ to $\mathbb{Z}_2^{2^{s-1}}$ [6], [8], [9], [16]. The one given in [8] by Carlet is the map $\phi : \mathbb{Z}_{2^s} \to \mathbb{Z}_2^{2^{s-1}}$ defined as follows:

$$\phi(u) = (u_{s-1}, \dots, u_{s-1}) + (u_0, \dots, u_{s-2})Y, \qquad (1)$$

where $u \in \mathbb{Z}_{2^s}$, $[u_0, u_1, \dots, u_{s-1}]_2$ is the binary expansion of u, that is $u = \sum_{i=0}^{s-1} 2^i u_i$ ($u_i \in \{0, 1\}$), and Y is a matrix of size $(s-1) \times 2^{s-1}$ which columns are the elements of $\mathbb{Z}_2^{s-1}$. Note that $(u_{s-1}, \dots, u_{s-1})$ and $(u_0, \dots, u_{s-2})Y$ are binary vectors of length 2^{s-1}, and that the rows of Y form a basis of a first order Reed-Muller code after adding the all-one row. This generalization can be defined in terms of the elements of a Hadamard code [16]. In this paper, we will focus on Carlet's Gray map ϕ, which is a particular case of the one presented in [16] satisfying that $\sum_{i=0}^{s-1} \lambda_i \phi(2^i) = \phi(\sum_{i=0}^{s-1} \lambda_i 2^i)$ ($\lambda_i \in \{0, 1\}$), as it was shown in [13] and will be recalled later. Then, we define $\Phi : \mathbb{Z}_{2^s}^n \to \mathbb{Z}_2^{n 2^{s-1}}$ as the component-wise Gray map ϕ.

Let $\mathcal{C}$ be a $\mathbb{Z}_{2^s}$-additive code of length n. We say that its binary image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_{2^s}$-linear code of length $2^{s-1}n$. Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_{2^s}^n$, it is isomorphic to an abelian structure $\mathbb{Z}_{2^s}^{t_1} \times \mathbb{Z}_{2^{s-1}}^{t_2} \times \cdots \times \mathbb{Z}_4^{t_{s-1}} \times \mathbb{Z}_2^{t_s}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi(\mathcal{C})$, is of type $(n; t_1, \dots, t_s)$. Note that $|\mathcal{C}| = 2^{st_1} 2^{(s-1)t_2} \cdots 2^{t_s}$. Unlike linear codes over finite fields, linear codes over rings do not have a basis, but there exists a generator matrix for these codes. If $\mathcal{C}$ is a $\mathbb{Z}_{2^s}$-additive code of type $(n; t_1, \dots, t_s)$, then a generator matrix of $\mathcal{C}$ with minimum number of rows has exactly $t_1 + \cdots + t_s$ rows.

Two structural properties of binary codes are the rank and the dimension of the kernel. The rank of a binary code C is simply the dimension of the linear span, $\langle C \rangle$, of C. The kernel of a binary code C is defined as $\mathrm{K}(C) = \{\mathbf{x} \in \mathbb{Z}_2^n : \mathbf{x} + C = C\}$ [3]. If the all-zero vector belongs to C, then $\mathrm{K}(C)$ is a linear subcode of C. Note also that if C is linear, then $K(C) = C = \langle C \rangle$. We denote the rank of a binary code C as $\mathrm{rank}(C)$ and the dimension of the kernel as $\ker(C)$. These parameters can be used to distinguish between nonequivalent binary codes, since equivalent ones have the same rank and dimension of the kernel.

A binary code of length n, $2n$ codewords and minimum distance $n/2$ is called a Hadamard code. Hadamard codes can be constructed from Hadamard matrices [2], [19]. Note

Manuscript received Month day, year; revised Month day, year.

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This work was supported in part by the Spanish MINECO under Grants TIN2016-77918-P (AEI/FEDER, UE) and MTM2015-69138-REDT, and by the Catalan AGAUR under Grant 2017SGR-00463.

The material in this paper was presented in part at the 2nd IMA Conference on Theoretical and Computational Discrete Mathematics [12].

that linear Hadamard codes are in fact first order Reed-Muller codes, or equivalently, the dual of extended Hamming codes [19, Ch.13 §3]. The $\mathbb{Z}_{2^s}$-additive codes that, under the Gray map Φ, give a Hadamard code are called $\mathbb{Z}_{2^s}$-additive Hadamard codes and the corresponding binary images are called $\mathbb{Z}_{2^s}$-linear Hadamard codes.

The $\mathbb{Z}_4$-linear Hadamard codes of length 2^t can be classified by using either the rank or the dimension of the kernel [17], [20]. Specifically, it is known that for a $\mathbb{Z}_4$-linear Hadamard code C of type $(2^{t-1};t_1,t_2)$, $\ker(C)=t_1+t_2+1$ if $t_1>2$, and $\ker(C)=2t_1+t_2$ if $t_1=1$ or 2, where $t_2=t+1-2t_1$. For any integer $t\geq 3$ and each $t_1\in\{1,\dots,\lfloor(t+1)/2\rfloor\}$, there is a unique (up to equivalence) $\mathbb{Z}_4$-linear Hadamard code of type $(2^{t-1};t_1,t+1-2t_1)$, and all these codes are pairwise nonequivalent, except for $t_1=1$ and $t_1=2$, where the codes are equivalent to the linear Hadamard code [17]. Therefore, the number of nonequivalent $\mathbb{Z}_4$-linear Hadamard codes of length 2^t is $\lfloor\frac{t-1}{2}\rfloor$ for all $t\geq 3$, and it is 1 for $t=1$ and $t=2$.

Linear codes over $\mathbb{Z}_{p^s}$, which are a generalization of $\mathbb{Z}_{2^s}$-additive codes, were studied by Blake [4] and Shankar [21] in 1975 and 1979, respectively. Nevertheless, the study of codes over rings increased significantly after the publication of some good properties of linear codes over $\mathbb{Z}_4$ and the definition of the Gray map [15]. After that, $\mathbb{Z}_{2^s}$-additive codes and their images under the Gray map are deeply studied, for example, in [8], [14], and [22]. In [16], Krotov studied $\mathbb{Z}_{2^s}$-linear Hadamard codes and their dual codes by using different generalizations of the Gray map. Recently, in [1], considering Carlet's generalization of the Gray map, two-weight $\mathbb{Z}_{2^s}$-linear codes are studied. Note that $\mathbb{Z}_{2^s}$-linear Hadamard codes are in fact a particular case of these two-weight codes.

In [13], the dimension of the kernel of $\mathbb{Z}_{2^s}$-linear Hadamard codes is given. It is shown that the kernel do not classify these codes, since there are nonequivalent codes having the same dimension of the kernel. As a consequence, a partial classification for these codes is established. In this paper, in order to classify the $\mathbb{Z}_8$-linear Hadamard codes of length 2^t, for any $t\geq 3$, we compute the rank of these codes. Moreover, we prove that this invariant, along with the dimension of the kernel, provides a complete classification, once we fix $t\geq 3$. Note that, unlike for $s=2$, in the case $s=3$, it is necessary to use both invariants.

This correspondence is organized as follows. In Section II, we describe the recursive construction of the $\mathbb{Z}_{2^s}$-linear Hadamard codes of type $(n;t_1,\dots,t_s)$, introduced in [13]. In Section III, we give some known results and prove new ones related to the Carlet's generalized Gray map. In Section IV, we compute the rank of the $\mathbb{Z}_8$-linear Hadamard codes in terms of the parameters t_1, t_2 and t_3, by finding a set of linear independent vectors of the span. In Section V, we show that, for $s=3$, a complete classification can be given by using both invariants: the rank and dimension of the kernel. Finally, in Section VI, we give some conclusions and further research on this topic.

II. Recursive Construction of $\mathbb{Z}_{2^s}$-Linear Hadamard Codes

The description of generator matrices having minimum number of rows for $\mathbb{Z}_4$-additive Hadamard codes, as long as recursive constructions of these matrices, are given in [17]. In [13], [16], these results are generalized for any $s>2$. In this section, we describe the recursive construction of the generator matrices of the $\mathbb{Z}_{2^s}$-additive Hadamard codes, introduced in [13].

Let $T_i=\{j\cdot 2^{i-1} : j\in\{0,1,\dots,2^{s-i+1}-1\}\}$ for all $i\in\{1,\dots,s\}$. Note that $T_1=\{0,\dots,2^s-1\}$. Let t_1, t_2,…,t_s be nonnegative integers with $t_1\geq 1$. Consider the matrix $A^{t_1,\dots,t_s}$ whose columns are of the form $\mathbf{z}^T$, $\mathbf{z}\in\{1\}\times T_1^{t_1-1}\times T_2^{t_2}\times\cdots\times T_s^{t_s}$.

Example 2.1: For $s=3$, for example, we have the following matrices:

$$A^{1,0,1}=\begin{pmatrix}1\,1\\0\,4\end{pmatrix},\quad A^{1,1,0}=\begin{pmatrix}1\,1\,1\,1\\0\,2\,4\,6\end{pmatrix},$$

$$A^{2,0,0}=\begin{pmatrix}1\,1\,1\,1\,1\,1\,1\,1\\0\,1\,2\,3\,4\,5\,6\,7\end{pmatrix},$$

$$A^{1,1,1}=\begin{pmatrix}1\,1\,1\,1\;\;1\,1\,1\,1\\0\,2\,4\,6\;\;0\,2\,4\,6\\0\,0\,0\,0\;\;4\,4\,4\,4\end{pmatrix},$$

$$A^{2,0,1}=\begin{pmatrix}1\,1\,1\,1\,1\,1\,1\,1\;\;1\,1\,1\,1\,1\,1\,1\,1\\0\,1\,2\,3\,4\,5\,6\,7\;\;0\,1\,2\,3\,4\,5\,6\,7\\0\,0\,0\,0\,0\,0\,0\,0\;\;4\,4\,4\,4\,4\,4\,4\,4\end{pmatrix},$$

$$A^{2,1,0}=$$

$$\begin{pmatrix}1\,1\,1\,1\,1\,1\,1\,1\;1\,1\,1\,1\,1\,1\,1\,1\;1\,1\,1\,1\,1\,1\,1\,1\;1\,1\,1\,1\,1\,1\,1\,1\\0\,1\,2\,3\,4\,5\,6\,7\;0\,1\,2\,3\,4\,5\,6\,7\;0\,1\,2\,3\,4\,5\,6\,7\;0\,1\,2\,3\,4\,5\,6\,7\\0\,0\,0\,0\,0\,0\,0\,0\;2\,2\,2\,2\,2\,2\,2\,2\;4\,4\,4\,4\,4\,4\,4\,4\;6\,6\,6\,6\,6\,6\,6\,6\end{pmatrix}.$$

Let $\mathbf{0},\mathbf{1},\mathbf{2},\dots,\mathbf{2^s-1}$ be the vectors having the elements $0,1,2,\dots,2^s-1$ from $\mathbb{Z}_{2^s}$ repeated in each coordinate, respectively. The order of a vector $\mathbf{u}$ over $\mathbb{Z}_{2^s}$, denoted by $\mathrm{ord}(\mathbf{u})$, is the smallest positive integer m such that $m\mathbf{u}=\mathbf{0}$.

Any matrix $A^{t_1,\dots,t_s}$ can be obtained by applying the following recursive construction. We start with $A^{1,0,\dots,0}=(1)$. Then, if we have a matrix $A=A^{t_1,\dots,t_s}$, for any $i\in\{1,\dots,s\}$, we may construct the matrix

$$A_i=\begin{pmatrix}A & A & \cdots & A\\ 0\cdot\mathbf{2^{i-1}} & 1\cdot\mathbf{2^{i-1}} & \cdots & (2^{s-i+1}-1)\cdot\mathbf{2^{i-1}}\end{pmatrix}. \tag{2}$$

Finally, permuting the rows of A_i, we obtain a matrix $A^{t'_1,\dots,t'_s}$, where $t'_j=t_j$ for $j\neq i$ and $t'_i=t_i+1$. Note that any permutation of columns of A_i gives also a matrix $A^{t'_1,\dots,t'_s}$.

Example 2.2: From the matrix $A^{1,0,0}=(1)$, we obtain the matrix $A^{2,0,0}$; and from $A^{2,0,0}$ we can construct $A^{2,0,1}$, where $A^{2,0,0}$ and $A^{2,0,1}$ are the matrices given in Example 2.1. Note that we can also generate another matrix $A^{2,0,1}$ as follows: from $A^{1,0,0}=(1)$ we obtain the matrix $A^{1,0,1}$ given in Example 2.1, and from $A^{1,0,1}$ we can construct the matrix

$$A_1=\begin{pmatrix}1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\\0\,4\,0\,4\,0\,4\,0\,4\,0\,4\,0\,4\,0\,4\,0\,4\\0\,0\,1\,1\,2\,2\,3\,3\,4\,4\,5\,5\,6\,6\,7\,7\end{pmatrix}.$$

Then, after permuting the rows of A_1, we have the matrix

$$A^{2,0,1} = \begin{pmatrix} 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1 \\ 0\,0\,1\,1\,2\,2\,3\,3\,4\,4\,5\,5\,6\,6\,7\,7 \\ 0\,4\,0\,4\,0\,4\,0\,4\,0\,4\,0\,4\,0\,4\,0\,4 \end{pmatrix},$$

which is different to the matrix $A^{2,0,1}$ of Example 2.1. These two matrices $A^{2,0,1}$ generate permutation equivalent codes.

Along this paper, we consider that the matrices $A^{t_1,t_2,\dots,t_s}$ are constructed recursively starting from $A^{1,0,\dots,0}$ in the following way. First, we add $t_1 - 1$ rows of order 2^s, up to obtain $A^{t_1,0,\dots,0}$; then t_2 rows of order 2^{s-1} up to generate $A^{t_1,t_2,0,\dots,0}$; and so on, until we add t_s rows of order 2 to achieve $A^{t_1,t_2,\dots,t_s}$.

Let $\mathcal{H}^{t_1,\dots,t_s}$ be the $\mathbb{Z}_{2^s}$-additive code generated by the matrix $A^{t_1,\dots,t_s}$, where $t_1,\dots,t_s \geq 0$ with $t_1 \geq 1$. Let $n = 2^{t-s+1}$, where $t = (\sum_{i=1}^{s}(s-i+1)\cdot t_i)-1$. It is easy to see that $\mathcal{H}^{t_1,\dots,t_s}$ is of length n and has $|\mathcal{H}^{t_1,\dots,t_s}| = 2^s n = 2^{t+1}$ codewords. Note that this code is of type $(n; t_1, t_2, \dots, t_s)$. Let $H^{t_1,\dots,t_s} = \Phi(\mathcal{H}^{t_1,\dots,t_s})$ be the corresponding $\mathbb{Z}_{2^s}$-linear code.

Theorem 2.1: [16] [13] Let $t_1,\dots,t_s$ be nonnegative integers with $t_1 \geq 1$. The $\mathbb{Z}_{2^s}$-linear code $H^{t_1,\dots,t_s}$ of type $(n; t_1, t_2, \dots, t_s)$ is a binary Hadamard code of length 2^t, with $t = (\sum_{i=1}^{s}(s-i+1)\cdot t_i) - 1$ and $n = 2^{t-s+1}$.

Let $\mathcal{G}$ be a generator matrix of a $\mathbb{Z}_{2^s}$-additive code $\mathcal{C}$ of length n. Then, $(\mathcal{G}\cdots\mathcal{G})$ is a generator matrix of the r-fold replication code of $\mathcal{C}$, $(\mathcal{C},\dots,\mathcal{C}) = \{(\mathbf{c},\dots,\mathbf{c}) : \mathbf{c} \in \mathcal{C}\}$, of length $r\cdot n$.

Let $\mathcal{H}^{t_1,t_2,t_3}$ be a $\mathbb{Z}_8$-additive Hadamard code, which is generated by A^{t_1,t_2,t_3}. Let $\mathbf{w}_i$ be the ith row of A^{t_1,t_2,t_3}, $1 \leq i \leq t_1$. If $t_2 = t_3 = 0$, we also denote $\mathbf{w}_i$ by $\mathbf{w}_i^{t_1}$. Note that $\mathbf{w}_i^{t_1} \in \mathcal{H}^{t_1,0,0}$ is the $8^{t_1-\ell}$-fold replication of $\mathbf{w}_i^{\ell} \in \mathcal{H}^{\ell,0,0}$ for all $1 \leq \ell \leq t_1$ and $i \leq \ell$.

Remark 2.1: Let $\mathcal{H}^{t_1,0,0}$ be a $\mathbb{Z}_8$-additive Hadamard code of type $(n; t_1, 0, 0)$. Let

$$W = \begin{pmatrix} \mathbf{w}_{i_1} \\ \vdots \\ \mathbf{w}_{i_q} \end{pmatrix},$$

where $2 \leq i_1 < \cdots < i_q \leq t_1$. By construction, we have that each one of the 8^q elements of $\mathbb{Z}_8^q$ appears $\frac{8^{t_1-1}}{8^q} = 8^{t_1-q-1}$ times as a column of W. Therefore, there exist a permutation of coordinates $\rho \in \mathcal{S}_n$ such that

$$\rho(W) = \begin{pmatrix} \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_{q+1} \end{pmatrix}.$$

Note also that $\mathbf{w}_i$ is the 8^{t_1-q-1}-fold replication of $\mathbf{w}_i^{q+1}$ for all $2 \leq i \leq q+1$.

III. Properties of the Generalized Gray Map

In this section, we give some known results on the Carlet's generalized Gray map and we present new results, which will be used in the next section to establish the rank of the $\mathbb{Z}_8$-linear Hadamard codes.

Let e_i be the vector that has 1 in the ith position and 0 otherwise. Let $u, v \in \mathbb{Z}_{2^s}$ and $[u_0, u_1, \dots, u_{s-1}]_2$, $[v_0, v_1, \dots, v_{s-1}]_2$ be the binary expansions of u and v, respectively. The operation "$\odot$" on $\mathbb{Z}_{2^s}$ is defined as $u \odot v = \sum_{i=0}^{s-1} 2^i u_i v_i$. Note that the binary expansion of $u \odot v$ is $[u_0v_0, u_1v_1, \dots, u_{s-1}v_{s-1}]_2$.

Proposition 3.1: [22] Let $u, v \in \mathbb{Z}_{2^s}$. Then, $\phi(u) + \phi(v) = \phi(u + v - 2(u \odot v))$.

Corollary 3.1: [8], [22] Let $u \in \mathbb{Z}_{2^s}$. Then, $\phi(u) + \phi(2^{s-1}) = \phi(u + 2^{s-1})$.

Lemma 3.1: [13] Let $u \in \{2^{s-2}, \dots, 2^{s-1}-1\} \cup \{3\cdot 2^{s-2}, \dots, 2^s - 1\} \subset \mathbb{Z}_{2^s}$. Then, $\phi(u) + \phi(2^{s-2}) = \phi(u + 2^{s-2} + 2^{s-1})$.

Proposition 3.2: [8] Let $u, v \in \mathbb{Z}_{2^s}$. Then, $d_H(\phi(u), \phi(v)) = \mathrm{wt}_H(\phi(u - v))$.

Corollary 3.2: [13] Let $u, v \in \mathbb{Z}_{2^s}$. Then, $d_H(\phi(u), \phi(v + 2^{s-1})) + d_H(\phi(u), \phi(v)) = 2^{s-1}$.

All the remaining results, given in this section, are only proved for $s = 3$, that is, for $\mathbb{Z}_8$-linear Hadamard codes. In this case, the generalized Gray map $\phi : \mathbb{Z}_8 \to \mathbb{Z}_2^4$ is defined by

$$\begin{array}{ll} \phi(0) = (0,0,0,0) & \phi(4) = (1,1,1,1) \\ \phi(1) = (0,1,0,1) & \phi(5) = (1,0,1,0) \\ \phi(2) = (0,0,1,1) & \phi(6) = (1,1,0,0) \\ \phi(3) = (0,1,1,0) & \phi(7) = (1,0,0,1). \end{array}$$

Lemma 3.2: Let q be a positive integer and $[q_0, q_1, q_2, \dots]_2$ its binary expansion. Then, $\binom{q-1}{3} + q_0\binom{q-1}{2} + (q_0+q_1)(q-1) + q_0(q_0+q_1) \equiv 1 \bmod 2$.

Proof. If $q \equiv 0 \bmod 4$, then $q_0 = q_1 = 0$ and $\binom{q-1}{3} \equiv 1 \bmod 2$ since $(q-2)/2$, $q-1$ and $q-3$ are odd numbers. Similarly, if $q \equiv 1 \bmod 4$, then $q_0 = 1$, $q_1 = 0$ and $\binom{q-1}{3} + \binom{q-1}{2} + (q-1) + 1 \equiv 0+0+0+1 \equiv 1 \bmod 2$. If $q \equiv 2 \bmod 4$, then $q_0 = 0$, $q_1 = 1$ and $\binom{q-1}{3} + (q-1) \equiv 0 + 1 \equiv 1 \bmod 2$. Finally, if $q \equiv 3 \bmod 4$, then $q_1 = 1$, $q_1 = 1$ and $\binom{q-1}{3} + \binom{q-1}{2} \equiv 0 + 1 \equiv 1 \bmod 2$. □

Lemma 3.3: Let q be a positive integer and $[q_0, q_1, q_2, \dots]_2$ its binary expansion. Then,

(i) $q - 4 \equiv q_0 \bmod 2$,
(ii) $\binom{q-4}{2} \equiv q_1 \bmod 2$,
(iii) $\binom{q-3}{2} \equiv q_0 + q_1 \bmod 2$,
(iv) $\binom{q-2}{3} \equiv q_0(q_0 + q_1) \bmod 2$.

Proof. These congruences can be proved easily considering the different values of q modulo 4, as in the proof of Lemma 3.2. □

Lemma 3.4: Let $\mathcal{H}^{t_1,0,0}$ be a $\mathbb{Z}_8$-additive Hadamard code of type $(n; t_1, 0, 0)$. Let $E \subseteq \{1, \dots, t_1\}$, $q = |E|$ and $[q_0, q_1, q_2, \dots]_2$ the binary expansion of q. Let $\mathbf{w}_i$ be the ith row of $A^{t_1,0,0}$, $i \in E$. Then,

$$\begin{aligned} \Phi(\sum_{i\in E}\mathbf{w}_i) = & \sum_{\substack{i,j,k,p\in E \\ i<j<k<p}} \Phi(\mathbf{w}_i + \mathbf{w}_j + \mathbf{w}_k + \mathbf{w}_p) \\ & + q_0(\sum_{\substack{i,j,k\in E \\ i<j<k}} \Phi(\mathbf{w}_i + \mathbf{w}_j + \mathbf{w}_k)) + \\ & + (q_0+q_1)(\sum_{\substack{i,j\in E \\ i<j}} \Phi(\mathbf{w}_i + \mathbf{w}_j)) + q_0(q_0+q_1)(\sum_{i\in E}\Phi(\mathbf{w}_i)). \end{aligned}$$

Proof. First, assume $E \subseteq \{2,\ldots,t_1\}$, and let $q = |E|$. By Remark 2.1, without loss of generality, we can assume that $E = \{2,\ldots,q+1\}$. Now, we prove this lemma by induction on the integer $q \geq 1$.

For $q \leq 5$, it is easy to check that the result holds. Note that, for $q = 5$, it is enough to check the result for $\mathbf{w}_2^6,\ldots,\mathbf{w}_6^6$. Suppose that it is true for $|E| = q-1$. Consider $\sum_{i=2}^{q+1}\mathbf{w}_i = \sum_{i=2}^{q}\mathbf{w}_i + \mathbf{w}_{q+1}$. Let $\mathbf{y} = \sum_{i=2}^{q}\mathbf{w}_i^q$. We have that $\sum_{i=2}^{q}\mathbf{w}_i = (\mathbf{y},\ldots,\mathbf{y})$ is the 8^{t_1-q-2}-fold replication of $\mathbf{y}$. Then, $\sum_{i=2}^{q+1}\mathbf{w}_i$ is the 8^{t_1-q-1}-fold replication of $(\mathbf{y}+\mathbf{0},\mathbf{y}+\mathbf{1},\ldots,\mathbf{y}+\mathbf{7})$. The result holds if

$$\begin{aligned}\Phi(\sum_{i=2}^{q}\mathbf{w}_i^q+\mathbf{k}) = &\sum_{2\le i<j<k<p\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_k^q+\mathbf{w}_p^q)+\\ &\sum_{2\le i<j<k\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_k^q+\mathbf{k})+\\ &q_0\big(\sum_{2\le i<j<k\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_k^q)+\\ &\sum_{2\le i<j\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{k})\big)+\\ &(q_0+q_1)\big(\sum_{2\le i<j\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q)+\sum_{i=2}^{q}\Phi(\mathbf{w}_i^q+\mathbf{k})\big)+\\ &q_0(q_0+q_1)\big(\sum_{i=2}^{q}\Phi(\mathbf{w}_i^q)+\Phi(\mathbf{k})\big)\end{aligned} \tag{3}$$

for all $k \in \{0,\ldots,7\}$.

Let $\pi_8 = \prod_{i=0}^{8^{t_1-2}-1}(8i+1, 8i+2, 8i+3, 8i+4, 8i+5, 8i+6, 8i+7, 8i+8) \in \mathcal{S}_n$ be a permutation of coordinates. Let π_8^k be the composition of π_8, k times, i.e., $\pi_8^k = \pi_8 \circ \overset{(k)}{\cdots} \circ \pi_8$. Note that $\pi_8^k(\mathbf{w}_2) = \mathbf{w}_2+\mathbf{k}$ and $\pi_8^k(\mathbf{w}_i) = \mathbf{w}_i$ for all $i \in \{3,\ldots,q\}$. Let $\tilde{\pi}_8^k \in \mathcal{S}_{4n}$ be a permutation such that $\Phi\circ\pi_8^k = \tilde{\pi}_8^k\circ\Phi$.

Now, we have that $\Phi(\sum_{i=2}^{q}\mathbf{w}_i^q+\mathbf{k}) = \Phi(\pi_8^k(\sum_{i=2}^{q}\mathbf{w}_i^q)) = \tilde{\pi}_8^k(\Phi(\sum_{i=2}^{q}\mathbf{w}_i^q))$. By induction, taking into account that $(q-1)_0 \equiv q_0+1 \bmod 2$ and $(q-1)_1 \equiv q_0+q_1+1 \bmod 2$, and using again the properties of π_8^k and the fact that $\Phi\circ\pi_8^k = \tilde{\pi}_8^k\circ\Phi$, we have that

$$\begin{aligned}\Phi(\sum_{i=2}^{q}\mathbf{w}_i+\mathbf{k}) = &\sum_{3\le i<j<r<p\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_r^q+\mathbf{w}_p^q)+\\ &\sum_{3\le i<j<r\le q}\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_r^q+\mathbf{k})+\\ &(q_0+1)\sum_{3\le i<j<r\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_r^q)+\\ &(q_0+1)\sum_{3\le i<j\le q}\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{k})+\\ &q_1\sum_{3\le i<j\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q)+\\ &q_1\sum_{i=3}^{q}\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q+\mathbf{k})+q_1(q_0+1)\sum_{i=3}^{q}\Phi(\mathbf{w}_i^q)+\\ &q_1(q_0+1)\Phi(\mathbf{w}_2+\mathbf{k}).\end{aligned} \tag{4}$$

By applying again the induction hypothesis to $\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_r^q+\mathbf{k})$, and noting that for any $\mathbf{z} \in \mathbb{Z}_8^n$ we have $\sum_{3\le i<j<r\le q}\sum_{x,y\in\{i,j,r\},\ x<y}\Phi(\mathbf{z}+\mathbf{w}_x^q+\mathbf{w}_y^q) = (q-4)\sum_{3\le i<j\le q}\Phi(\mathbf{z}+\mathbf{w}_i^q+\mathbf{w}_j^q)$ and $\sum_{3\le i<j<r\le q}\sum_{x\in\{i,j,r\}}\Phi(\mathbf{z}+\mathbf{w}_x^q) = \binom{q-3}{2}\sum_{i=3}^{q}\Phi(\mathbf{z}+\mathbf{w}_i^q)$, we obtain that

$$\begin{aligned}\sum_{3\le i<j<r\le q}&\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_r^q+\mathbf{k}) =\\ &\sum_{3\le i<j<r\le q}\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_r^q)+\\ &(q-4)\sum_{3\le i<j\le q}\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{k})+\\ &\sum_{3\le i<j<r\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_r^q+\mathbf{k})+\\ &\sum_{3\le i<j<r\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{w}_r^q)+\\ &(q-4)\sum_{3\le i<j\le q}\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q+\mathbf{w}_j^q)+\\ &\binom{q-3}{2}\sum_{i=3}^{q}\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q+\mathbf{k})+\\ &(q-4)\sum_{3\le i<j\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q+\mathbf{k})+\\ &(q-4)\sum_{3\le i<j\le q}\Phi(\mathbf{w}_i^q+\mathbf{w}_j^q)+\binom{q-3}{2}\sum_{i=3}^{q}\Phi(\mathbf{w}_2^q+\mathbf{w}_i^q)+\\ &\binom{q-3}{2}\sum_{i=3}^{q}\Phi(\mathbf{w}_i^q+\mathbf{k})+\binom{q-2}{3}\Phi(\mathbf{w}_2+\mathbf{k})+\\ &\binom{q-3}{2}\sum_{i=3}^{q}\Phi(\mathbf{w}_i^q)+\binom{q-2}{3}\Phi(\mathbf{w}_2)+\binom{q-2}{3}\Phi(\mathbf{k}).\end{aligned} \tag{5}$$

By replacing (5) into expression (4), and using items (i), (iii) and (iv) of Lemma 3.3, we have that (3) holds.

Finally, consider $1 \in E$. By Remark 2.1, we can assume that $E = \{1,\ldots,q\}$. Then, $\Phi(\sum_{i\in E}\mathbf{w}_i) = \Phi(\sum_{i=2}^{q}\mathbf{w}_i+\mathbf{1})$, and we can apply the same arguments as above. □

Lemma 3.5: Let $\mathcal{H}^{t_1,0,0}$ be a $\mathbb{Z}_8$-additive Hadamard code of type $(n; t_1, 0, 0)$. Let $q \in \mathbb{Z}$ and $[q_0, q_1, q_2, \ldots]_2$ its binary expansion. Let $\mathbf{w}_i$ be the ith row of $A^{t_1,0,0}$. Then,

$$\begin{aligned}\Phi(\sum_{i=1}^{q}\mathbf{s}_i) = &\sum_{1\le i<j<k<p\le q}\Phi(\mathbf{s}_i+\mathbf{s}_j+\mathbf{s}_k+\mathbf{s}_p)+\\ &q_0(\sum_{1\le i<j<k\le q}\Phi(\mathbf{s}_i+\mathbf{s}_j+\mathbf{s}_k))+\\ &+(q_0+q_1)(\sum_{1\le i<j\le q}\Phi(\mathbf{s}_i+\mathbf{s}_j))+q_0(q_0+q_1)(\sum_{i=1}^{q}\Phi(\mathbf{s}_i)),\end{aligned}$$

where $\mathbf{s}_i \in \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{t_1}\}$ for all $i \in \{1, 2, \ldots, q\}$.

Proof. We prove this lemma by induction on the integer $q \geq 1$. It is easy to check by computer that for $q \leq 5$ the result holds. Suppose that $q \geq 6$ and the statement is true for all positive integers until $q-1$.

Let r_i be the multiplicity of $\mathbf{w}_i$, $i \in \{1,\ldots,t_1\}$, that is, the number of elements $\mathbf{w}_i$ that appear in the multiset $S =$

$\{\mathbf{s}_1, \dots, \mathbf{s}_q\}$. If there is an element $\mathbf{w}_i$ with multiplicity $r_i \geq 4$, then we may consider that $\mathbf{s}_q = \mathbf{s}_{q-1} = \mathbf{s}_{q-2} = \mathbf{s}_{q-3} = \mathbf{w}_i$. Note that the right-hand side of the equation of the statement can be easily rewritten by replacing q by $q-4$ and adding $\Phi(4\mathbf{w}_j)$. Moreover, by Corollary 3.1, the left-hand side of the equation is $\Phi(\sum_{i=1}^{q-4} \mathbf{s}_i) + \Phi(4\mathbf{w}_j)$. Therefore, we may assume that $r_i \leq 3$ for all $i \in \{1, \dots, t_1\}$.

Let W be the set containing the elements of S without repetition. On the one hand, if $\mathbf{w}_1 \notin S$, taking into account the multiplicity of each element in W and Remark 2.1, we may assume that $W = \{\mathbf{w}_2, \dots, \mathbf{w}_d\}$, where $r_2 \leq \dots \leq r_d$ and $\mathbf{s}_1 = \dots = \mathbf{s}_{r_2} = \mathbf{w}_2, \dots, \mathbf{s}_{q-r_d+1} = \dots = \mathbf{s}_q = \mathbf{w}_d$. On the other hand, if $\mathbf{w}_1 \in S$, we assume that $q > r_1 + r_2$. Otherwise, if $q = r_1 + r_2$, since $q \geq 6$ and $r_1, r_2 \leq 3$, then we have to show that the statement is true for $\Phi(\mathbf{w}_2 + \mathbf{w}_2 + \mathbf{w}_2 + \mathbf{w}_1 + \mathbf{w}_1 + \mathbf{w}_1)$, which can be checked easily. Since $q > r_1 + r_2$, we can order all elements $\mathbf{s}_1, \dots, \mathbf{s}_q$ as above, placing the r_1 vectors $\mathbf{w}_1$ just before the r_d vectors $\mathbf{w}_d$.

Consider $\sum_{i=1}^{q} \mathbf{s}_i = \sum_{i=1}^{q-(r_1+r_d)} \mathbf{s}_i + \sum_{i=1}^{r_1} \mathbf{w}_1 + \sum_{i=1}^{r_d} \mathbf{w}_d$. Let $\mathbf{y} = \sum_{i=1}^{q-(r_1+r_d)} \mathbf{s}_i^{d-1}$. We have that $\sum_{i=1}^{q-(r_1+r_d)} \mathbf{s}_i = (\mathbf{y}, \dots, \mathbf{y})$ is a fold replication of $\mathbf{y}$. Then, $\sum_{i=1}^{q} \mathbf{s}_i$ is a fold replication of

$$(\mathbf{y} + r_1\mathbf{w}_1^{d-1} + \mathbf{0}, \mathbf{y} + r_1\mathbf{w}_1^{d-1} + \mathbf{1} + \overset{(r_d)}{\cdots} + \mathbf{1}, \dots,$$
$$\mathbf{y} + r_1\mathbf{w}_1^{d-1} + \mathbf{7} + \overset{(r_d)}{\cdots} + \mathbf{7}) =$$
$$(\mathbf{y} + r_1\mathbf{1}, \mathbf{y} + (r_1 + r_d)\mathbf{1}, \dots, \mathbf{y} + (r_1 + 7r_d)\mathbf{1}).$$

The result holds if the statement is true for $\Phi(\sum_{i=1}^{q-(r_1+r_d)} \mathbf{s}_i^{d-1} + (r_1 + k \cdot r_d)\mathbf{1})$ for all $k \in \{0, \dots, 7\}$. Moreover, as before, we may assume that $(r_1 + k \cdot r_d) < 4$, so we have to check that the statement is true for $\Phi(\sum_{i=1}^{q-(r_1+r_d)} \mathbf{s}_i^{d-1} + \bar{r}\mathbf{w}_1^{d-1})$, where $\bar{r} = (r_1 + k \cdot r_d) \bmod 4$, or equivalently for $\Phi(\sum_{i=1}^{q-(r_1+r_d)+\bar{r}} \mathbf{s}_i^{d-1})$, where $\mathbf{s}_i = \mathbf{w}_1$ for all $i \in \{q-(r_1+r_d)+1, \dots, q-(r_1+r_d)+\bar{r}\}$ if $\bar{r} \geq 1$.

If $r_1 + r_d - \bar{r} > 0$, we can apply the induction hypothesis to obtain the result. Otherwise, let $\pi_8 = \prod_{i=0}^{8^{t_1-2}-1}(8i+1, 8i+2, 8i+3, 8i+4, 8i+5, 8i+6, 8i+7, 8i+8) \in \mathcal{S}_n$ be a permutation of coordinates. Note that $\pi_8(\mathbf{w}_2) = \mathbf{w}_2 + \mathbf{1}$ and $\pi_8(\mathbf{w}_j) = \mathbf{w}_j$ for all $j \in \{3, \dots, d\}$. Let $\tilde{\pi}_8 \in \mathcal{S}_{4n}$ be a permutation such that $\Phi \circ \pi_8 = \tilde{\pi}_8 \circ \Phi$. Therefore, we have that $\Phi(\sum_{i=1}^{r_2} \mathbf{w}_2^{d-1} + \sum_{i=r_2+1}^{q-(r_1+r_d)} \mathbf{s}_i^{d-1} + (\bar{r} - r_2)\mathbf{1} + r_2\mathbf{1}) = \Phi(\pi_8(\sum_{i=1}^{r_2} \mathbf{w}_2^{d-1} + \sum_{i=r_2+1}^{q-(r_1+r_d)} \mathbf{s}_i^{d-1} + (\bar{r}-r_2)\mathbf{1})) = \tilde{\pi}_8(\Phi(\sum_{i=1}^{r_2} \mathbf{w}_2^{d-1} + \sum_{i=r_2+1}^{q-(r_1+r_d)} \mathbf{s}_i^{d-1} + (\bar{r}-r_2)\mathbf{1}))$. Note that $\bar{r} \geq r_1 + r_d \geq r_d \geq r_2$. Then, considering $\mathbf{s}_i^{d-1} = \mathbf{w}_1$ for all $i \in \{q-(r_1+r_d)+1, \dots, q-(r_1+r_d)+(\bar{r}-r_2)\}$ if $\bar{r}-r_2 \geq 1$, it is enough to show the statement for $\tilde{\pi}_8(\Phi(\sum_{i=1}^{r_2} \mathbf{w}_2^{d-1} + \sum_{i=r_2+1}^{q-(r_1+r_d-\bar{r}+r_2)} \mathbf{s}_i^{d-1})) = \tilde{\pi}_8(\Phi(\sum_{i=1}^{q-r^*} \mathbf{s}_i^{d-1}))$, where $r^* = r_1 + r_2 + r_d - \bar{r}$.

Now, in order to be able to apply the hypothesis induction to $\Phi(\sum_{i=1}^{q-r^*} \mathbf{s}_i^{d-1})$, we have to verify that $r^* > 0$. First, note that if $r_i \in \{0, 1\}$ for all $i \in \{1, \dots, t_1\}$, then the statement is true by Proposition 3.4. Therefore, we can assume that for some $i \in \{1, \dots, t_1\}$, $r_i \geq 2$, so at least one of r_1 or r_d must be greater than 1. We also have that $r_2, r_d \in \{1, 2, 3\}$ and $r_1 \in \{0, 1, 2, 3\}$. On the one hand, if $r_1 = 0$, we have that $r_d \in \{2, 3\}$. Then, if $\bar{r} < 3$, clearly $r^* > 0$; and if $\bar{r} = 3$, $k \cdot r_d = 3 \bmod 4$ which implies that $r_d = 3$ and $r^* > 0$. On the other hand, if $r_1 > 0$, $r_d \in \{1, 2, 3\}$ and $r_1 + r_2 + r_d > 3$ which also gives that $r^* > 0$.

In order to verify the statement, we consider $\tilde{\pi}_8(\Phi(\sum_{i=1}^{q-r^*} \mathbf{s}_i^{d-1}))$ under different cases depending on the value of $r_2 \in \{1, 2, 3\}$. First, consider that $r_2 = 1$, i.e., $\mathbf{s}_1 = \mathbf{w}_2$ and $\mathbf{s}_i \neq \mathbf{w}_2$ for all $i \in \{2, 3, \dots, q\}$. Then, by using the same arguments as in the proof of Lemma 3.4, we have that the result holds. Next, consider that $r_2 = 2$. By induction hypothesis, taking into account that $(q-2)_0 \equiv q_0 \bmod 2$ and $(q-2)_1 \equiv q_1 + 1 \bmod 2$, and using again the properties of π_8 and the fact that $\Phi \circ \pi_8 = \tilde{\pi}_8 \circ \Phi$, we have that

$$\begin{aligned}
&\Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \sum_{i=3}^{q-2} \mathbf{s}_i^q + \mathbf{1} + \mathbf{1}) = \\
&\sum_{3 \leq i<j<k<p \leq q-2} \Phi(\mathbf{s}_i^d + \mathbf{s}_j^d + \mathbf{s}_k^d + \mathbf{s}_p^d) + \\
&\sum_{3 \leq i<j \leq q-2} \Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{s}_i^d + \mathbf{s}_j^d + \mathbf{1} + \mathbf{1}) + \\
&q_0 \Big[\sum_{3 \leq i<j<k \leq q-2} \Phi(\mathbf{s}_i^d + \mathbf{s}_j^d + \mathbf{s}_k^d) + \\
&\sum_{3 \leq i \leq q-2} \Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{s}_i^d + \mathbf{1} + \mathbf{1}) \Big] + \\
&(q_0 + q_1 + 1) \Big[\sum_{3 \leq i<j \leq q-2} \Phi(\mathbf{s}_i^d + \mathbf{s}_j^d) + \Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{1} + \mathbf{1}) \Big] + \\
&q_0(q_0 + q_1 + 1) \sum_{3 \leq i \leq q-2} \Phi(\mathbf{s}_i^d). \quad (6)
\end{aligned}$$

By applying again the induction hypothesis to the terms of (6) having more than four addends, that is, $\Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{s}_i^d + \mathbf{1} + \mathbf{1})$ and $\Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{s}_i^d + \mathbf{s}_j^d + \mathbf{1} + \mathbf{1})$, we obtain that

$$\begin{aligned}
&\sum_{3 \leq i \leq q-2} \Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{s}_i^d + \mathbf{1} + \mathbf{1}) = \sum_{3 \leq i \leq q-2} \Phi(\mathbf{s}_i^d) + \\
&\sum_{3 \leq i \leq q-2} \Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{s}_i^d) + \sum_{3 \leq i \leq q-2} \Phi(\mathbf{s}_i^d + \mathbf{1} + \mathbf{1}) + \\
&(q-4) \Big[\Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{1} + \mathbf{1}) + \Phi(\mathbf{w}_2^d + \mathbf{w}_2^d) + \Phi(\mathbf{1} + \mathbf{1}) \Big] \quad (7)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{3 \leq i<j \leq q-2} \Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{s}_i^d + \mathbf{s}_j^d + \mathbf{1} + \mathbf{1}) = \\
&\sum_{3 \leq i<j \leq q-2} \Phi(\mathbf{s}_i^d + \mathbf{s}_j^d) + \\
&\sum_{3 \leq i<j \leq q-2} \Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{s}_i^d + \mathbf{s}_j^d) + \\
&\sum_{3 \leq i<j \leq q-2} \Phi(\mathbf{s}_i^d + \mathbf{s}_j^d + \mathbf{1} + \mathbf{1}) + \\
&\binom{q-4}{2} \Big[\Phi(\mathbf{w}_2^d + \mathbf{w}_2^d + \mathbf{1} + \mathbf{1}) + \Phi(\mathbf{w}_2^d + \mathbf{w}_2^d) + \Phi(\mathbf{1} + \mathbf{1}) \Big]. \quad (8)
\end{aligned}$$

By replacing (7) and (8) into expression (6), and using items

(i) and (ii) of Lemma 3.3, we have that (6) is equal to

$$\sum_{3\le i<j<k<p\le q-2} \Phi(\mathbf{s}_i^d+\mathbf{s}_j^d+\mathbf{s}_k^d+\mathbf{s}_p^d)+ \sum_{3\le i<j\le q-2} \Phi(\mathbf{w}_2^d+\mathbf{w}_2^d+\mathbf{s}_i^d+\mathbf{s}_j^d)+$$
$$\sum_{3\le i<j\le q-2} \Phi(\mathbf{s}_i^d+\mathbf{s}_j^d+\mathbf{1}+\mathbf{1})+\Phi(\mathbf{w}_2^d+\mathbf{w}_2^d+\mathbf{1}+\mathbf{1})+$$
$$q_0\Big[\sum_{3\le i<j\le q-2} \Phi(\mathbf{s}_i^d+\mathbf{s}_j^d)+\sum_{3\le i\le q-2} \Phi(\mathbf{w}_2^d+\mathbf{w}_2^d+\mathbf{s}_i^d)+ \sum_{3\le i\le q-2} \Phi(\mathbf{s}_i^d+\mathbf{1}+\mathbf{1})\Big]+$$
$$(q_0+q_1)\Big[\sum_{3\le i<j\le q-2} \Phi(\mathbf{s}_i^d+\mathbf{s}_j^d)+\Phi(\mathbf{w}_2^d+\mathbf{w}_2^d)+\Phi(\mathbf{1}+\mathbf{1})\Big]+ q_0(q_0+q_1)\sum_{3\le i\le q-2}\Phi(\mathbf{s}_i^d).$$

Note that all the terms that are missing in order to obtain the result appear repeated in pairs, so they sum to zero. The case with $r_2=3$ can also be proved by using similar arguments. Therefore, the result holds. □

Lemma 3.6: Let $\mathcal{H}^{t_1,0,0}$ be a $\mathbb{Z}_8$-additive Hadamard code of type $(n;t_1,0,0)$. Let $\mathbf{w}_i$ be the ith row of $A^{t_1,0,0}$, $1\le i\le t_1$. Then, given $i,j,k\in\{1,\dots,t_1\}$,

$$\Phi(2\mathbf{w}_i+\mathbf{w}_j+\mathbf{w}_k)+\Phi(\mathbf{w}_i+2\mathbf{w}_j+\mathbf{w}_k)= \Phi(\mathbf{w}_i)+\Phi(\mathbf{w}_j)+\Phi(2\mathbf{w}_i)+\Phi(2\mathbf{w}_j)+\Phi(\mathbf{w}_i+\mathbf{w}_k)+ \Phi(\mathbf{w}_j+\mathbf{w}_k)++\Phi(2\mathbf{w}_i+\mathbf{w}_k)+\Phi(2\mathbf{w}_j+\mathbf{w}_k)+ \Phi(2\mathbf{w}_j+\mathbf{w}_i)+\Phi(2\mathbf{w}_i+\mathbf{w}_j). \quad (9)$$

Proof. Suppose that $2\le i<j<k$. By Remark 2.1, it is enough to see that (9) holds for $\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_4$. In fact, it is enough to show that it is true for $\mathbf{w}_2^3,\mathbf{w}_3^3,\mathbf{k}$ for all $k\in\{0,1,\dots,7\}$. Let A be the right-hand side of (9).

On the one hand, if $\mathbf{w}_k=\mathbf{k}$, we need to show that

$$\Phi(2\mathbf{w}_2^3+\mathbf{w}_3^3+\mathbf{k})+\Phi(\mathbf{w}_2^3+2\mathbf{w}_3^3+\mathbf{k})= \Phi(\mathbf{w}_2^3)+\Phi(\mathbf{w}_3^3)+\Phi(2\mathbf{w}_2^3)+\Phi(2\mathbf{w}_3^3)+\Phi(\mathbf{w}_2^3+\mathbf{k})+ \Phi(\mathbf{w}_3^3+\mathbf{k})++\Phi(2\mathbf{w}_2^3+\mathbf{k})+\Phi(2\mathbf{w}_3^3+\mathbf{k})+\Phi(2\mathbf{w}_3^3+\mathbf{w}_2^3)+ \Phi(2\mathbf{w}_2^3+\mathbf{w}_3^3) \quad (10)$$

for all $k\in\{0,1,\dots,7\}$. Let A_1 be the right-hand side of (10). First, for $k=0$, it is easy to see that (10) holds. Note that, by Proposition 3.1, $\Phi(2\mathbf{w}_i+\mathbf{1})=\Phi(2\mathbf{w}_i)+\Phi(\mathbf{1})$ for all $1\le i\le t_1$. For $k=1$, $A_1=\Phi(\mathbf{w}_2^3)+\Phi(\mathbf{w}_3^3)+\Phi(\mathbf{w}_2^3+\mathbf{1})+\Phi(\mathbf{w}_3^3+\mathbf{1})+\Phi(2\mathbf{w}_3^3+\mathbf{w}_2^3)+\Phi(2\mathbf{w}_2^3+\mathbf{w}_3^3)$. By the same proposition, we also have that $\Phi(\mathbf{w}_i)+\Phi(\mathbf{w}_i+2\mathbf{w}_j)=\Phi(2\mathbf{w}_j)+\Phi(-2(\mathbf{w}_i\odot 2\mathbf{w}_j))$ for all $i,j\in\{2,3\}$. Thus,

$$A_1=\Phi(2\mathbf{w}_2^3)+\Phi(2\mathbf{w}_3^3)+\Phi(\mathbf{w}_2^3+\mathbf{1})+\Phi(\mathbf{w}_3^3+\mathbf{1})+ \Phi(-2(2\mathbf{w}_3^3\odot\mathbf{w}_2^3))+\Phi(-2(2\mathbf{w}_2^3\odot\mathbf{w}_3^3)).$$

Again, by Proposition 3.1, $\Phi(2\mathbf{w}_i)+\Phi(\mathbf{w}_j+\mathbf{1})=\Phi(2\mathbf{w}_i+\mathbf{w}_j+\mathbf{1})+\Phi(-2(2\mathbf{w}_i\odot(\mathbf{w}_j+\mathbf{1})))$ for all $i,j\in\{2,3\}$, so

$$A_1=\Phi(2\mathbf{w}_2^3+\mathbf{w}_3^3+\mathbf{1})+\Phi(\mathbf{w}_2^3+2\mathbf{w}_3^3+\mathbf{1})+ \Phi(-2(2\mathbf{w}_3^3\odot\mathbf{w}_2^3))+\Phi(-2(2\mathbf{w}_2^3\odot\mathbf{w}_3^3))+ +\Phi(-2(2\mathbf{w}_3^3\odot(\mathbf{w}_2^3+\mathbf{1})))+\Phi(-2(2\mathbf{w}_2^3\odot(\mathbf{w}_3^3+\mathbf{1}))).$$

Let $\mathbf{x}=(0,0,4,4,0,0,4,4)$, $\mathbf{y}=(0,4,4,0,0,4,4,0)$, and $\mathbf{z}=(0,4,0,4,0,4,0,4)$. It is easy to check that

$$\begin{array}{rcl} -2(2\mathbf{w}_3^3\odot\mathbf{w}_2^3) &=& (\mathbf{0},\mathbf{x},\mathbf{0},\mathbf{x},\mathbf{0},\mathbf{x},\mathbf{0},\mathbf{x})\\ -2(2\mathbf{w}_3^3\odot(\mathbf{w}_2^3+\mathbf{1})) &=& (\mathbf{0},\mathbf{y},\mathbf{0},\mathbf{y},\mathbf{0},\mathbf{y},\mathbf{0},\mathbf{y})\\ -2(2\mathbf{w}_2^3\odot\mathbf{w}_3^3) &=& (\mathbf{0},\mathbf{0},\mathbf{z},\mathbf{z},\mathbf{0},\mathbf{0},\mathbf{z},\mathbf{z})\\ -2(2\mathbf{w}_2^3\odot(\mathbf{w}_3^3+\mathbf{1})) &=& (\mathbf{0},\mathbf{z},\mathbf{z},\mathbf{0},\mathbf{0},\mathbf{z},\mathbf{z},\mathbf{0}). \end{array} \quad (11)$$

The sum of the four vectors in (11) is a zero vector, since $\mathbf{x}+\mathbf{y}+\mathbf{z}=\mathbf{0}$, so $A_1=\Phi(2\mathbf{w}_2^3+\mathbf{w}_3^3+\mathbf{1})+\Phi(\mathbf{w}_2^3+2\mathbf{w}_3^3+\mathbf{1})$ and (10) holds. For $k=2$, it is easy to see that the result holds by Lemma 3.5. For $k=3$, it follows also from Lemma 3.5, the previous result for $k=1$, and the fact that $\Phi(2\mathbf{w}_i+\mathbf{1})=\Phi(2\mathbf{w}_i)+\Phi(\mathbf{1})$ for all $1\le i\le t_1$. Finally, for the rest of the cases, if $\mathbf{w}_k=\mathbf{k}+\mathbf{4}$, $k\in\{0,1,2,3\}$, then $\Phi(2\mathbf{w}_2+\mathbf{w}_3+\mathbf{k}+\mathbf{4})+\Phi(\mathbf{w}_2+2\mathbf{w}_3+\mathbf{k}+\mathbf{4})=\Phi(2\mathbf{w}_2+\mathbf{w}_3+\mathbf{k})+\Phi(\mathbf{w}_2+2\mathbf{w}_3+\mathbf{k})$ and the result holds since $\mathbf{w}_k$ appears 4 times in A.

On the other hand, if $\mathbf{w}_i=\mathbf{k}$ (or $\mathbf{w}_j=\mathbf{k}$), we need to show that

$$\Phi(2\mathbf{k}+\mathbf{w}_2^3+\mathbf{w}_3^3)+\Phi(\mathbf{k}+2\mathbf{w}_2^3+\mathbf{w}_3^3)= \Phi(\mathbf{k})+\Phi(\mathbf{w}_2^3)+\Phi(2\mathbf{k})+\Phi(2\mathbf{w}_2^3)+\Phi(\mathbf{k}+\mathbf{w}_3^3)+ \Phi(\mathbf{w}_2^3+\mathbf{w}_3^3)++\Phi(2\mathbf{k}+\mathbf{w}_3^3)+\Phi(2\mathbf{w}_2^3+\mathbf{w}_3^3)+\Phi(2\mathbf{w}_2^3+\mathbf{k})+ \Phi(2\mathbf{k}+\mathbf{w}_2^3) \quad (12)$$

for all $k\in\{0,1,\dots,7\}$. Let A_2 be the right-hand side of (12). First, for $k=0$, it is easy to see that (12) holds. For $k=1$, by applying Proposition 3.1 to $\Phi(\mathbf{w}_2^3+\mathbf{w}_3^3)+\Phi(\mathbf{2})$ and $\Phi(\mathbf{w}_3^3+\mathbf{1})+\Phi(2\mathbf{w}_2^3)$, we have that

$$A_2=\Phi(\mathbf{2}+\mathbf{w}_2^3+\mathbf{w}_3^3)+\Phi(\mathbf{1}+2\mathbf{w}_2^3+\mathbf{w}_3^3) +\Phi(-2((\mathbf{w}_2^3+\mathbf{w}_3^3)\odot\mathbf{2}))+\Phi(-2((\mathbf{w}_3^3+\mathbf{1})\odot 2\mathbf{w}_2^3))+ \Phi(\mathbf{1})+\Phi(\mathbf{w}_2^3)+\Phi(\mathbf{2}+\mathbf{w}_3^3)+\Phi(2\mathbf{w}_2^3+\mathbf{w}_3^3)+ \Phi(2\mathbf{w}_2^3+\mathbf{1})+\Phi(\mathbf{2}+\mathbf{w}_2^3).$$

Again, applying Proposition 3.1 to the terms $\Phi(\mathbf{2}+\mathbf{w}_3^3)$, $\Phi(2\mathbf{w}_2^3+\mathbf{w}_3^3)$, $\Phi(2\mathbf{w}_2^3+\mathbf{1})$ and $\Phi(\mathbf{2}+\mathbf{w}_2^3)$ of A_2, we obtain that

$$A_2=\Phi(\mathbf{2}+\mathbf{w}_2^3+\mathbf{w}_3^3)+\Phi(\mathbf{1}+2\mathbf{w}_2^3+\mathbf{w}_3^3) +\Phi(-2((\mathbf{w}_2^3+\mathbf{w}_3^3)\odot\mathbf{2}))+\Phi(-2((\mathbf{w}_3^3+\mathbf{1})\odot 2\mathbf{w}_2^3)) +\Phi(-2(\mathbf{2}\odot\mathbf{w}_3^3))+\Phi(-2(2\mathbf{w}_2^3\odot\mathbf{w}_3^3))+\Phi(-2(\mathbf{2}\odot\mathbf{w}_2^3)).$$

It is easy to check that

$$\begin{array}{c} \Phi(-2((\mathbf{w}_2^3+\mathbf{w}_3^3)\odot\mathbf{2}))=\\ (\mathbf{x},\mathbf{y},\mathbf{x}+\mathbf{4},\mathbf{y}+\mathbf{4},\mathbf{x},\mathbf{y},\mathbf{x}+\mathbf{4},\mathbf{y}+\mathbf{4})\\ \Phi(-2((\mathbf{w}_3^3+\mathbf{1})\odot 2\mathbf{w}_2^3))=(\mathbf{0},\mathbf{z},\mathbf{z},\mathbf{0},\mathbf{0},\mathbf{z},\mathbf{z},\mathbf{0})\\ \Phi(-2(2\mathbf{w}_2^3\odot\mathbf{w}_3^3))=(\mathbf{0},\mathbf{0},\mathbf{z},\mathbf{z},\mathbf{0},\mathbf{0},\mathbf{z},\mathbf{z})\\ \Phi(-2(\mathbf{2}\odot\mathbf{w}_3^3))=(\mathbf{0},\mathbf{0},\mathbf{4},\mathbf{4},\mathbf{0},\mathbf{0},\mathbf{4},\mathbf{4})\\ \Phi(-2(\mathbf{2}\odot\mathbf{w}_2^3))=(\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x},\mathbf{x}). \end{array} \quad (13)$$

The sum of the five vectors in (13) is a zero vector, since $\mathbf{x}+\mathbf{y}+\mathbf{z}=\mathbf{0}$, so $A_2=\Phi(\mathbf{2}+\mathbf{w}_2^3+\mathbf{w}_3^3)+\Phi(\mathbf{1}+2\mathbf{w}_2^3+\mathbf{w}_3^3)$ and (12) holds. For $k\in\{2,3\}$, it is easy to see that the result holds by Lemma 3.5. Finally, if $\mathbf{w}_i=\mathbf{k}+\mathbf{4}$, $k\in\{0,1,2,3\}$, then $\Phi(2\mathbf{k}+\mathbf{8}+\mathbf{w}_2+\mathbf{w}_3)+\Phi(\mathbf{k}+\mathbf{4}+2\mathbf{w}_2+\mathbf{w}_3)=\Phi(\mathbf{4})+\Phi(2\mathbf{k}+\mathbf{w}_2+\mathbf{w}_3)+\Phi(\mathbf{k}+2\mathbf{w}_2+\mathbf{w}_3)$ and the result follows since $\mathbf{w}_i$ appears 3 times in A.

Now, suppose that some of the elements i,j,k are equal. If $i=j=k$ or $i=j$, then (9) holds trivially. If $i=k$ (or $j=k$), then it is enough to show that

$$\Phi(3\mathbf{k}+\mathbf{w}_2^2)+\Phi(2\mathbf{k}+2\mathbf{w}_2^2)=\Phi(\mathbf{k})+\Phi(\mathbf{w}_2^2)+ \Phi(2\mathbf{w}_2^2)+\Phi(3\mathbf{k})+\Phi(\mathbf{k}+\mathbf{w}_2^2)+\Phi(2\mathbf{k}+\mathbf{w}_2^2) \quad (14)$$

for all $k\in\{0,1,\dots,7\}$. Let A_3 be the right-hand side of (14). First, for $k=0$, it is easy to see that (14) holds. For $k=1$, note that, by Proposition 3.1, $\Phi(\mathbf{2})=\Phi(\mathbf{3})+\Phi(\mathbf{1})$ and $\Phi(\mathbf{w}_2)+\Phi(\mathbf{2})+\Phi(\mathbf{w}_2+\mathbf{2})=\Phi(-2(\mathbf{w}_2\odot\mathbf{2}))$. Therefore,

$$A_3=\Phi(2\mathbf{w}_2^2)+\Phi(\mathbf{w}_2^2+\mathbf{1})+\Phi(-2(\mathbf{w}_2^2\odot\mathbf{2}))= \Phi(2\mathbf{w}_2^2)+\Phi(\mathbf{2})+\Phi(\mathbf{w}_2^2+\mathbf{1})+\Phi(\mathbf{2})+\Phi(-2(\mathbf{w}_2^2\odot\mathbf{2})).$$

Again, by Proposition 3.1, we have that

$$A_3=\Phi(2\mathbf{w}_2^2+\mathbf{2})+\Phi(\mathbf{w}_2^2+\mathbf{3})+\Phi(-2(\mathbf{w}_2^2\odot\mathbf{2}))+ \Phi(-2(2\mathbf{w}_2^2\odot\mathbf{2}))+\Phi(-2((\mathbf{w}_2^2+\mathbf{1})\odot\mathbf{2})).$$

It is easy to check that the sum of the three last terms is $(0,0,4,4,0,0,4,4)+(0,4,0,4,0,4,0,4)+(0,4,4,0,0,4,4,0)=\mathbf{0}$. In a similar way, it holds for $k=3$. The rest of the cases, $k\in\{2,4,5,6,7\}$, can also be checked easily, so (14) holds.

Now, we consider that, at least one of i,j,k is equal to 1. If $i=j=k=1$, or $i=j=1$, then the result is trivial. If $i=k=1$ (or $j=k=1$), the result is equivalent to prove (14) with $\mathbf{k}=\mathbf{1}$. Finally, if $k=1$, it is equivalent to (10) with $\mathbf{k}=\mathbf{1}$, and if $i=1$ (or $j=1$), it is equivalent to (12) with $\mathbf{k}=\mathbf{1}$. Therefore, the result holds. □

Lemma 3.7: Let $\mathcal{H}^{t_1,0,0}$ be a $\mathbb{Z}_8$-additive Hadamard code of type $(n;t_1,0,0)$. Let $\mathbf{w}_i$ be the ith row of $A^{t_1,0,0}$, $1\le i\le t_1$. Then, given $i,j,k\in\{1,\dots,t_1\}$,

$$\Phi(\mathbf{w}_i+\mathbf{w}_j+\mathbf{1})=\Phi(2\mathbf{w}_i)+\Phi(2\mathbf{w}_j)+\Phi(\mathbf{1})+\Phi(\mathbf{w}_i+\mathbf{1})+ +\Phi(\mathbf{w}_j+\mathbf{1})+\Phi(\mathbf{w}_i+\mathbf{w}_j)+\Phi(2\mathbf{w}_i+\mathbf{w}_j)+\Phi(\mathbf{w}_i+2\mathbf{w}_j),$$

$$\Phi(\mathbf{w}_i+\mathbf{w}_j+\mathbf{w}_k+\mathbf{1})=\Phi(\mathbf{w}_i+\mathbf{1})+\Phi(\mathbf{w}_i+\mathbf{w}_j)+ \Phi(\mathbf{w}_i+2\mathbf{w}_j)++\Phi(2\mathbf{w}_i+\mathbf{w}_j)+\Phi(\mathbf{w}_j+\mathbf{1})+\Phi(\mathbf{w}_k)+ \Phi(2\mathbf{w}_k)+\Phi(\mathbf{w}_k+\mathbf{1})++\Phi(2\mathbf{w}_i+\mathbf{w}_k)+ \Phi(2\mathbf{w}_j+\mathbf{w}_k)+\Phi(\mathbf{w}_i+\mathbf{w}_j+2\mathbf{w}_k)+\Phi(\mathbf{w}_i+\mathbf{w}_j+\mathbf{w}_k).$$

Proof. First, if $2\le i<j<k$, by Remark 2.1, the above equations can be showed to be true by checking that they hold for $\mathbf{w}_2^3,\mathbf{w}_3^3,\mathbf{k}$ for all $k\in\{0,1,\dots,7\}$. It is also easy to see that they hold if some of the elements i,j,k are equal, or at least one of them is equal to 1. □

Lemma 3.8: Let $\mathcal{H}^{t_1,t_2,t_3}$ be a $\mathbb{Z}_8$-additive Hadamard code of type $(n;t_1,t_2,t_3)$. Let $\mathbf{w}$ be a row of $A^{t_1,0,0}$. Then,

$$\Phi(3\mathbf{w})=\Phi(\mathbf{3})+\Phi(\mathbf{w})+\Phi(\mathbf{w}+\mathbf{1})+\Phi(\mathbf{w}+\mathbf{2}).$$

Proof. Let $A=\Phi(\mathbf{3})+\Phi(\mathbf{w})+\Phi(\mathbf{w}+\mathbf{1})+\Phi(\mathbf{w}+\mathbf{2})$. By Proposition 3.1, we have that $\Phi(\mathbf{w}+\mathbf{1})+\Phi(\mathbf{w}+\mathbf{2})=\Phi(2\mathbf{w}+\mathbf{3}-2((\mathbf{w}+\mathbf{1})\odot(\mathbf{w}+\mathbf{2})))$. It easy to check that $\mathrm{ord}(-2((\mathbf{w}+\mathbf{1})\odot(\mathbf{w}+\mathbf{2})))=2$, so $A=\Phi(\mathbf{3})+\Phi(\mathbf{w})+\Phi(2\mathbf{w}+\mathbf{3})+\Phi(-2((\mathbf{w}+\mathbf{1})\odot(\mathbf{w}+\mathbf{2})))$. Now, by applying Lemma 3.5 to the term $\Phi(2\mathbf{w}+\mathbf{3})$ and using that $\Phi(\mathbf{1})+\Phi(\mathbf{2})=\Phi(\mathbf{3})$, we obtain that $A=\Phi(\mathbf{3})+\Phi(\mathbf{w})+\Phi(-2((\mathbf{w}+\mathbf{1})\odot(\mathbf{w}+\mathbf{2})))+\Phi(2\mathbf{w}+\mathbf{2})+\Phi(2\mathbf{w}+\mathbf{1})+\Phi(2\mathbf{w})$. By Proposition 3.1, we have that $\Phi(2\mathbf{w})+\Phi(\mathbf{w})=\Phi(3\mathbf{w})+\Phi(-2(\mathbf{w}\odot 2\mathbf{w}))$, thus

$$A=\Phi(\mathbf{3})+\Phi(3\mathbf{w})+\Phi(2\mathbf{w}+\mathbf{2})+\Phi(2\mathbf{w}+\mathbf{1})+ \Phi(-2(\mathbf{w}\odot 2\mathbf{w}))+\Phi(-2((\mathbf{w}+\mathbf{1})\odot(\mathbf{w}+\mathbf{2}))).$$

It easy to check that $\Phi(-2(\mathbf{w}\odot 2\mathbf{w}))+\Phi(-2((\mathbf{w}+\mathbf{1})\odot(\mathbf{w}+\mathbf{2})))=\Phi(4\mathbf{w})$. Finally, since $\Phi(-2((2\mathbf{w}+\mathbf{1})\odot(2\mathbf{w}+\mathbf{2})))=\mathbf{0}$, we have that $\Phi(2\mathbf{w}+\mathbf{2})+\Phi(2\mathbf{w}+\mathbf{1})=\Phi(4\mathbf{w}+\mathbf{3})=\Phi(4\mathbf{w})+\Phi(\mathbf{3})$. Therefore, $A=\Phi(3\mathbf{w})$ and the result holds. □

Lemma 3.9: Let $w,v\in\mathbb{Z}_{2^s}$ such that $\mathrm{ord}(v)=2^i$ with $i<s$. Then, $2^{i-1}((w+v)\odot 2^{s-i})=2^{i-1}(w\odot 2^{s-i})+2^{i-1}(v\odot 2^{s-i})$.

Proof. The binary expansion of v and $w+v$ are $[0,\dots,0,1,v_{s-i+1},\dots,v_{s-1}]_2$ and $[w_0,\dots,w_{s-i}+1,(w+v)_{s-i+1},\dots,(w+v)_{s-1}]_2$, respectively. Then, we have that the binary expansion of $w\odot 2^{s-i},v\odot 2^{s-i}$ and $(w+v)\odot 2^{s-i}$ are $[0,\dots,w_{s-i},0,\dots,0]_2$, $[0,\dots,0,1,0,\dots,0]_2$ and $[0,\dots,0,w_{s-i}+1,0,\dots,0]_2$, respectively. Note that, by multiplying by 2^{i-1}, the binary expansions are $[0,\dots,0,w_{s-i}]_2$, $[0,\dots,0,1]_2$ and $[0,\dots,0,w_{s-i}+1]_2$, respectively. Therefore, $2^{i-1}(w\odot 2^{s-i})+2^{i-1}(v\odot 2^{s-i})=2^{i-1}((w+v)\odot 2^{s-i})$. □

In order to simplify the notation in the following results, we define $\mu(\mathbf{w})=-2(\mathbf{w}\odot\mathbf{2})$ for any $\mathbf{w}\in\mathbb{Z}_8^n$. Note that $\mathrm{ord}(\mu(\mathbf{w}))=2$ if $\mathbf{w}\neq\mathbf{0}$.

Lemma 3.10: Let $\mathbf{w},\mathbf{v}\in\mathbb{Z}_8^n$ such that $\mathrm{ord}(\mathbf{v})<8$. Then, $\mu(\mathbf{w}+\mathbf{v})=\mu(\mathbf{w})+\mu(\mathbf{v})$.

Proof. We may assume that $\mathbf{v}\neq\mathbf{0}$. If $\mathrm{ord}(\mathbf{v})=4$, then $2((\mathbf{w}+\mathbf{v})\odot 2)=2(\mathbf{w}\odot 2)+2(\mathbf{v}\odot 2)$ by Lemma 3.9, so the result follows. Finally, if $\mathrm{ord}(\mathbf{v})=2$, then the result also holds since $\mathbf{v}\odot 2=\mathbf{0}$ and $(\mathbf{w}+\mathbf{v})\odot 2=\mathbf{w}\odot 2$. □

Lemma 3.11: Let $\mathcal{H}^{t_1,0,0}$ be a $\mathbb{Z}_8$-additive Hadamard code of type $(n;t_1,0,0)$. Let $\mathbf{w}_i$ be the ith row of $A^{t_1,0,0}$, $1\le i\le t_1$. Then,

$$\mu(\mathbf{w}_i+\mathbf{w}_j+\mathbf{w}_k)=\mu(\mathbf{w}_i+\mathbf{w}_j)+\mu(\mathbf{w}_i+\mathbf{w}_k)+ \mu(\mathbf{w}_j+\mathbf{w}_k)+\mu(\mathbf{w}_i)+\mu(\mathbf{w}_j)+\mu(\mathbf{w}_k) \quad (15)$$

for all $1\le i<j<k\le t_1$. Furthermore, for all $2\le i<j\le t_1$ and $k\in\mathbb{Z}_8$,

$$\mu(\mathbf{k}+\mathbf{w}_i+\mathbf{w}_j)=\mu(\mathbf{k}+\mathbf{w}_i)+\mu(\mathbf{k}+\mathbf{w}_j)+ \mu(\mathbf{w}_i+\mathbf{w}_j)+\mu(\mathbf{k})+\mu(\mathbf{w}_i)+\mu(\mathbf{w}_j).$$

Proof. First, consider the $\mathbb{Z}_8$-additive Hadamard code $\mathcal{H}^{4,0,0}$. In this case, it is easy to check that $\mu(\mathbf{w}_i^4+\mathbf{w}_j^4+\mathbf{w}_k^4)=\mu(\mathbf{w}_i^4+\mathbf{w}_j^4)+\mu(\mathbf{w}_i^4+\mathbf{w}_k^4)+\mu(\mathbf{w}_j^4+\mathbf{w}_k^4)+\mu(\mathbf{w}_i^4)+\mu(\mathbf{w}_j^4)+\mu(\mathbf{w}_k^4)$ for all $1\le i<j<k\le 4$. Then, the result follows by Remark 2.1 and the fact that $\mathbf{w}_1,\dots,\mathbf{w}_4\in\mathcal{H}^{t_1,0,0}$ are an

8^{t_1-4}-fold replication of $\mathbf{w}_1^4, \ldots, \mathbf{w}_4^4 \in \mathcal{H}^{4,0,0}$, respectively. By using the same argument, the second equation also holds. □

Lemma 3.12: Let $\mathcal{H}^{t_1,0,0}$ be a $\mathbb{Z}_8$-additive Hadamard code of type $(n; t_1, 0, 0)$. Let $\mathbf{w}_i$ be the ith row of $A^{t_1,0,0}$ for $1 \leq i \leq t_1$. Let $E \subseteq \{1, \ldots, t_1\}$. Then,

$$\mu(\sum_{i\in E} \mathbf{w}_i) = \sum_{\substack{i,j\in E\\ i<j}} \mu(\mathbf{w}_i + \mathbf{w}_j) + (|E| \bmod 2) \sum_{i\in E} \mu(\mathbf{w}_i).$$

Proof. Assume $E \subseteq \{2, \ldots, t_1\}$, and let $q = |E|$. By Remark 2.1, without loss of generality, we can assume that $E = \{2, \ldots, q+1\}$. Now, we prove this lemma by induction on the integer $q \geq 1$.

For $q = 1$ the result holds. Let $q \geq 2$ and suppose that it is true for $q-1$. Consider $\sum_{i=2}^{q+1} \mathbf{w}_i = \sum_{i=2}^{q} \mathbf{w}_i + \mathbf{w}_{q+1}$. Let $\mathbf{y} = \sum_{i=2}^{q} \mathbf{w}_i^q$. We have that $\sum_{i=2}^{q} \mathbf{w}_i = (\mathbf{y}, \ldots, \mathbf{y})$ is the 8^{t_1-q-2}-fold replication of $\mathbf{y}$. Then, $\sum_{i=2}^{q+1} \mathbf{w}_i$ is the 8^{t_1-q-1}-fold replication of $(\mathbf{y}+\mathbf{0}, \mathbf{y}+\mathbf{1}, \ldots, \mathbf{y}+\mathbf{7})$. The result holds if

$$\mu(\sum_{i=2}^{q} \mathbf{w}_i + \mathbf{w}_{q+1}) = \sum_{2\leq i<j\leq q+1} \mu(\mathbf{w}_i + \mathbf{w}_j) + (q \bmod 2) \sum_{i=2}^{q+1} \mu(\mathbf{w}_i).$$

That is, for all $k \in \{0, \ldots, 7\}$, we have to prove that

$$\mu(\mathbf{y}+\mathbf{k}) = \mu(\sum_{i=2}^{q} \mathbf{w}_i^q + \mathbf{k}) = \sum_{2\leq i<j\leq q} \mu(\mathbf{w}_i^q + \mathbf{w}_j^q) + \sum_{i=2}^{q} \mu(\mathbf{w}_i^q + \mathbf{k}) + (q \bmod 2)(\mu(\mathbf{k}) + \sum_{i=2}^{q} \mu(\mathbf{w}_i^q)). \quad (16)$$

Note that, by the induction hypothesis, the statement holds for $\sum_{i=2}^{q} \mathbf{w}_i = (\mathbf{y}, \ldots, \mathbf{y})$ and hence,

$$\mu(\mathbf{y}) = \sum_{2\leq i<j\leq q} \mu(\mathbf{w}_i^q + \mathbf{w}_j^q) + ((q-1) \bmod 2) \sum_{i=2}^{q} \mu(\mathbf{w}_i^q). \quad (17)$$

Let $\pi_8 = \prod_{i=0}^{8^{t_1-2}-1}(8i+1, 8i+2, 8i+3, 8i+4, 8i+5, 8i+6, 8i+7, 8i+8) \in \mathcal{S}_n$ be a permutation of coordinates. Let π_8^k be the composition of π_8, k times, i.e., $\pi_8^k = \pi_8 \circ \overset{(k)}{\cdots} \circ \pi_8$. Note that $\pi_8^k(\mathbf{w}_2) = \mathbf{w}_2 + \mathbf{k}$ and $\pi_8^k(\mathbf{w}_i) = \mathbf{w}_i$ for all $i \in \{1, 3, \ldots, q\}$. Moreover, it is also easy to see that $\pi_8^k \circ \mu = \mu \circ \pi_8^k$.

Now, we have that $\mu(\mathbf{y}+\mathbf{k}) = \pi_8^k(\mu(\mathbf{y}))$. By applying (17), $\mu(\mathbf{y}+\mathbf{k}) = \sum_{2\leq i<j\leq q} \pi_8^k(\mu(\mathbf{w}_i^q + \mathbf{w}_j^q)) + ((q-1) \bmod 2) \sum_{i=2}^{q} \pi_8^k(\mu(\mathbf{w}_i^q))$. By using the properties of π_8^k, we have that

$$\mu(\mathbf{y}+\mathbf{k}) = \sum_{3\leq i<j\leq q} \mu(\mathbf{w}_i^q + \mathbf{w}_j^q) + \sum_{i=3}^{q} \mu(\mathbf{w}_i^q + \mathbf{w}_2^q + \mathbf{k}) + ((q-1) \bmod 2)(\sum_{i=3}^{q} \mu(\mathbf{w}_i^q) + \mu(\mathbf{w}_2^q + \mathbf{k})).$$

By Lemma 3.11, we have that $\mu(\mathbf{w}_i^q + \mathbf{w}_2^q + \mathbf{k}) = \mu(\mathbf{w}_2^q + \mathbf{k}) + \mu(\mathbf{w}_i^q + \mathbf{k}) + \mu(\mathbf{w}_2^q + \mathbf{w}_i^q) + \mu(\mathbf{w}_2^q) + \mu(\mathbf{w}_i^q) + \mu(\mathbf{k})$. Therefore, $\mu(\mathbf{y}+\mathbf{k}) = \sum_{2\leq i<j\leq q} \mu(\mathbf{w}_i^q + \mathbf{w}_j^q) + \sum_{i=2}^{q} \mu(\mathbf{w}_i^q + \mathbf{k}) + (q \bmod 2)(\mu(\mathbf{k}) + \sum_{i=2}^{q} \mu(\mathbf{w}_i^q))$ and (16) holds. □

Corollary 3.3: Let $\mathcal{H}^{t_1,t_2,t_3}$ be a $\mathbb{Z}_8$-additive Hadamard code of type $(n; t_1, t_2, t_3)$. Let $\mathbf{w}_i$ be the ith row of A^{t_1,t_2,t_3}, $1 \leq i \leq t_1$. Let $E \subseteq \{1, \ldots, t_1\}$. Then,

$$\mu(\sum_{i\in E} \mathbf{w}_i) = \sum_{\substack{i,j\in E\\ i<j}} \mu(\mathbf{w}_i + \mathbf{w}_j) + (|E| \bmod 2) \sum_{i\in E} \mu(\mathbf{w}_i).$$

Proof. Note that $\mathcal{H}^{t_1,t_2,t_3}$ contains the $2^{2t_2+t_3}$-fold replication code of $\mathcal{H}^{t_1,0,0}$. Therefore, the result follows from Lemma 3.12. □

IV. Rank of $\mathbb{Z}_8$-Linear Hadamard Codes

The rank of a $\mathbb{Z}_4$-linear Hadamard code of type $(2^{t-1}; t_1, t_2)$, where $t+1 = 2t_1 + t_2$, is $2t_1 + t_2 + \binom{t_1-1}{2}$ if $t_1 > 2$, and $2t_1 + t_2$ if $t_1 = 1$ or 2 [20]. In this section, we establish the rank of a $\mathbb{Z}_8$-linear Hadamard code of type $(2^{t-2}; t_1, t_2, t_3)$, where $t+1 = 3t_1 + 2t_2 + t_3$, in terms of the parameters t_1, t_2 and t_3 by finding a set of linear independent vectors that generate the span of the code.

Proposition 4.1: Let $t_1, t_2, \ldots, t_s$ be nonnegative integers with $t_1 \geq 1$.

Then, $\text{rank}(\Phi(\mathcal{H}^{t_1,\ldots,t_s})) = t_s + \text{rank}(\Phi(\mathcal{H}^{t_1,\ldots,t_{s-1},0}))$.

Proof. Let $\mathcal{H}' = \mathcal{H}^{t_1,\ldots,t_s}$ and $\mathcal{H} = \mathcal{H}^{t_1,\ldots,t_{s-1},t_s-1}$. We prove this result by induction on the integer $t_s \geq 0$. First, for $t_s = 0$, the result holds trivially.

Let $t_s \geq 1$ and suppose that the result is true for $t_s - 1$. By the recursive construction (2), $\mathcal{H}'$ can be seen as the union of two cosets, that is, $\mathcal{H}' = C_0 \cup C_1$, where $C_0 = (\mathcal{H}, \mathcal{H})$ and $C_1 = (\mathcal{H}, \mathcal{H}) + (\mathbf{0}, \mathbf{2^{s-1}})$. By Corollary 3.1, we have that $\Phi((\mathcal{H}, \mathcal{H}) + (\mathbf{0}, \mathbf{2^{s-1}})) = \Phi((\mathcal{H}, \mathcal{H})) + \Phi((\mathbf{0}, \mathbf{2^{s-1}}))$, so $\text{rank}(\Phi(\mathcal{H}')) = 1 + \text{rank}(\Phi(\mathcal{H}))$. By the induction hypothesis, $\text{rank}(\Phi(\mathcal{H}')) = 1 + t_s - 1 + \text{rank}(\Phi(\mathcal{H}^{t_1,\ldots,t_{s-1},0})) = t_s + \text{rank}(\Phi(\mathcal{H}^{t_1,\ldots,t_{s-1},0}))$. □

Example 4.1: Let $B^{1,1,0} = \{\Phi(\mathbf{b}_i) : 1 \leq i \leq 5\}$ be a basis of $\langle \Phi(\mathcal{H}^{1,1,0}) \rangle$. For example, we can consider the one given in Example 4.2. Then, by the proof of Proposition 4.1, a basis of $\langle \Phi(\mathcal{H}^{1,1,1}) \rangle$ is $B^{1,1,1} = \{\Phi(\mathbf{b}_i, \mathbf{b}_i) : 1 \leq i \leq 5\} \cup \{\Phi(\mathbf{0}, \mathbf{4})\}$; and a basis of $\langle \Phi(\mathcal{H}^{1,1,2}) \rangle$ is $B^{1,1,2} = \{\Phi(\mathbf{b}_i, \mathbf{b}_i, \mathbf{b}_i, \mathbf{b}_i) : 1 \leq i \leq 5\} \cup \{\Phi(\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}), \Phi(\mathbf{0}, \mathbf{0}, \mathbf{4}, \mathbf{4})\}$. Since the rank of $\Phi(\mathcal{H}^{1,1,0})$ is 5, we have that $\text{rank}(\Phi(\mathcal{H}^{1,1,1})) = 6$ and $\text{rank}(\Phi(\mathcal{H}^{1,1,2})) = 7$.

Proposition 4.2: Let t_1 and t_2 be nonnegative integers with $t_1 \geq 1$. Then, $\text{rank}(\Phi(\mathcal{H}^{t_1,t_2+1,0})) = \text{rank}(\Phi(\mathcal{H}^{t_1,t_2,0})) + 2t_1 + t_2 + \binom{t_1-1}{2}$.

Proof. By (2), the generator matrix of $\mathcal{H}' = \mathcal{H}^{t_1,t_2+1,0}$ is

$$A^{t_1,t_2+1,0} = \begin{pmatrix} A & A & A & A \\ \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{6} \end{pmatrix},$$

where $A = A^{t_1,t_2,0}$ is the generator matrix of $\mathcal{H} = \mathcal{H}^{t_1,t_2,0}$. Note that $\mathcal{H}'$ can be seen as the union of four cosets of the 4-fold replication code of $\mathcal{H}$, $(\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H})$, which are

$$\begin{array}{lllll} C_0: & (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) & & \\ C_1: & (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) & + & (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}) \\ C_2: & (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) & + & (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}) \\ C_3: & (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) & + & (\mathbf{0}, \mathbf{6}, \mathbf{4}, \mathbf{2}). \end{array}$$

We have that $\text{rank}(\Phi(C_0)) = \text{rank}(\Phi(\mathcal{H})) = r$. Let $\{\Phi(\mathbf{g}_1), \ldots, \Phi(\mathbf{g}_r)\}$ be a basis of $\langle H \rangle$. Then, a basis of $\langle \Phi(C_0) \rangle$ is $\{\Phi(\mathbf{g}'_1), \ldots, \Phi(\mathbf{g}'_r)\}$, where $\mathbf{g}'_i = (\mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i)$ for all $i \in \{1, \ldots, r\}$. By Corollary 3.1, we have that $\langle \Phi(C_0 \cup C_2) \rangle = \langle \Phi(\mathbf{g}'_1), \ldots, \Phi(\mathbf{g}'_r), \Phi((\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})) \rangle$. Note that, if $\mathbf{u}' \in C_3$, then $\mathbf{u}' = (\mathbf{u}, \mathbf{u}+\mathbf{6}, \mathbf{u}+\mathbf{4}, \mathbf{u}+\mathbf{2}) = (\mathbf{u}, \mathbf{u}+\mathbf{2}, \mathbf{u}+\mathbf{4}, \mathbf{u}+\mathbf{6}) + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ with $\mathbf{u} \in \mathcal{H}$. Thus, it is easy to see that $\langle \Phi(\mathcal{H}') \rangle = \langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_3) \rangle = \langle \Phi(C_0 \cup C_1 \cup C_2) \rangle$, again by Corollary 3.1.

Let $\mathbf{u}' = (\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}) \in C_0$ and $\mathbf{v}' = (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$, so $\mathbf{u} \in \mathcal{H}$. By Proposition 3.1, we know that $\Phi(\mathbf{u}') + \Phi(\mathbf{v}') = \Phi(\mathbf{u}' + \mathbf{v}' - 2(\mathbf{u}' \odot \mathbf{v}'))$. Since $-2(\mathbf{u}' \odot \mathbf{v}')$ is a vector of order 2, we have that $\Phi(\mathbf{u}' + \mathbf{v}') = \Phi(\mathbf{u}') + \Phi(\mathbf{v}') + \Phi(-2(\mathbf{u}' \odot \mathbf{v}'))$ by Corollary 3.1. Let $M' = \{-2(\mathbf{u}' \odot \mathbf{v}') : \mathbf{u}' \in C_0\} = \{(\mathbf{0}, \mu(\mathbf{u}), \mathbf{0}, \mu(\mathbf{u})) : \mathbf{u} \in \mathcal{H}\}$. Then, $\langle \Phi(\mathcal{H}') \rangle = \langle \Phi(\mathbf{g}'_1), \ldots, \Phi(\mathbf{g}'_r), \Phi((\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})), \Phi(\mathbf{v}'), \Phi(M') \rangle$. Note that, if $\mathbf{u} = \mathbf{2} \in \mathcal{H}$, then $\mathbf{u}' = \mathbf{2} \in C_0$ and $-2(\mathbf{u}' \odot \mathbf{v}') = (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}) \in M'$. Thus, $\langle \Phi(\mathcal{H}') \rangle = \langle \Phi(\mathbf{g}'_1), \ldots, \Phi(\mathbf{g}'_r), \Phi(\mathbf{v}'), \Phi(M') \rangle$. It is easy to see that $\Phi(\mathbf{v}')$ and the elements of $\{\Phi(\mathbf{g}'_1), \ldots, \Phi(\mathbf{g}'_r)\}$ and $\Phi(M')$ are linearly independent, because of the form of every $\mathbf{g}'_i$, $i \in \{1, \ldots, r\}$, and the elements of M'. Therefore, $\text{rank}(\Phi(\mathcal{H}')) = r + 1 + \dim(\langle \Phi(M') \rangle)$. Since $M' = \{(\mathbf{0}, \mu(\mathbf{u}), \mathbf{0}, \mu(\mathbf{u})) : \mathbf{u} \in \mathcal{H}\}$, $\dim(\langle \Phi(M') \rangle) = \dim(\langle \Phi(M) \rangle)$, where $M = \{\mu(\mathbf{u}) : \mathbf{u} \in \mathcal{H}\}$.

Let $\mathcal{B}_2 = \{\mathbf{w}_1, \ldots, \mathbf{w}_{t_1}, \mathbf{v}_1, \ldots, \mathbf{v}_{t_2}, 2\mathbf{w}_1, \ldots, 2\mathbf{w}_{t_1}, 2\mathbf{v}_1, \ldots, 2\mathbf{v}_{t_2}, 4\mathbf{w}_1, \ldots, 4\mathbf{w}_{t_1}\}$ be a 2-basis of $\mathcal{H}$ and recall that $\text{ord}(\mathbf{w}_i) = 8$ and $\text{ord}(\mathbf{v}_j) = 4$ for all $i \in \{1, \ldots, t_1\}$ and $j \in \{1, \ldots, t_2\}$. Let $\mathbf{u} \in \mathcal{H}$. We know that $\mathbf{u} = \sum_{i=1}^{3t_1+2t_2} \lambda_i \mathbf{b}_i$, where $\mathbf{b}_i \in \mathcal{B}_2$ is the ith element of $\mathcal{B}_2$ and $\lambda_i \in \{0, 1\}$. By Lemma 3.10 and the fact that $\mu(\mathbf{2v}_j) = \mu(\mathbf{4w}_i) = \mathbf{0}$ for all $i \in \{1, \ldots, t_1\}$ and $j \in \{1, \ldots, t_2\}$, we have that $\mu(\mathbf{u}) = \mu(\sum_{i=1}^{t_1} \lambda_i \mathbf{b}_i) + \sum_{i=t_1+1}^{2t_1+t_2} \mu(\lambda_i \mathbf{b}_i)$. Let $E = \{1 \leq i \leq t_1 : \lambda_i \neq 0\}$. Since $\mathbf{b}_i = \mathbf{w}_i$ for all $i \in \{1, \ldots, t_1\}$, by Corollary 3.3,

$$\mu(\sum_{i=1}^{t_1} \lambda_i \mathbf{b}_i) = \sum_{\substack{i,j \in E \\ i<j}} \mu(\mathbf{w}_i + \mathbf{w}_j) + (|E| \bmod 2) \sum_{i \in E} \mu(\mathbf{w}_i).$$

Moreover, since $\mathbf{w}_1 = \mathbf{1}$, we have that $\mu(\mathbf{w}_1) = \mathbf{0}$ and it is easy to check that $\mu(\mathbf{w}_1 + \mathbf{w}_i) = \mu(\mathbf{w}_i) + \mu(2\mathbf{w}_i) = \mu(\mathbf{b}_i) + \mu(\mathbf{b}_{t_1+t_2+i})$ for all $i \in \{2, \ldots, t_1\}$. Therefore,

$$\mu(\mathbf{u}) = \sum_{\substack{i,j \in E \setminus \{1\} \\ i<j}} \mu(\mathbf{b}_i + \mathbf{b}_j) + \sum_{i=2}^{2t_1+t_2} \mu(\lambda'_i \mathbf{b}_i)$$

for some $\lambda'_i \in \{0, 1\}$. Let $M_1 = \{\mu(\mathbf{b}_i + \mathbf{b}_j) : 2 \leq i < j \leq t_1\}$ and $M_2 = \{\mu(\mathbf{b}_i) : 2 \leq i \leq 2t_1 + t_2\}$. Recall that $\text{ord}(\mu(\mathbf{w})) = 2$ for all $\mathbf{w} \neq \mathbf{0}$, so we can apply Corollary 3.1. Then, $\dim(\langle \Phi(M) \rangle) = \dim(\langle \Phi(M_1), \Phi(M_2) \rangle)$. Since the elements in $\Phi(M_1) \cup \Phi(M_2)$ are linearly independent, we have that $\text{rank}(\Phi(\mathcal{H}')) = r + 1 + 2t_1 + t_2 - 1 + \binom{t_1-1}{2} = r + 2t_1 + t_2 + \binom{t_1-1}{2}$. □

Example 4.2: Let $B^{1,0,0} = \{\Phi(1), \Phi(2), \Phi(4)\}$ be a basis of $\langle \Phi(\mathcal{H}^{1,0,0}) \rangle$. Then, by the proof of Proposition 4.2, a basis of $\langle \Phi(\mathcal{H}^{1,1,0}) \rangle$ is $B^{1,1,0} = \{\Phi(1,1,1,1), \Phi(2,2,2,2), \Phi(4,4,4,4), \Phi(0,2,4,6)\} \cup \{\Phi(0, m_i, 0, m_i) : m_i \in M_1 \cup M_2\}$, where $M_1 = \emptyset$ and $M_2 = \{\mu(2)\} = \{4\}$. Similarly, from $B^{1,1,0}$, we obtain that a basis of $\langle \Phi(\mathcal{H}^{1,2,0}) \rangle$ is

$$\begin{aligned} B^{1,2,0} = \{&\Phi(\mathbf{1}), \Phi(\mathbf{2}), \Phi(\mathbf{4}), \Phi(0,2,4,6,0,2,4,6), \\ &\Phi(0,4,0,4,0,4,0,4)\} \\ &\cup \{\Phi(0,0,0,0,\mathbf{m}_i,0,0,0,0,\mathbf{m}_i) : \mathbf{m}_i \in M_1 \cup M_2\}, \end{aligned}$$

where $M_1 = \emptyset$ and $M_2 = \{\mu(0,2,4,6), \mu(2,2,2,2)\} = \{(0,4,0,4), (4,4,4,4)\}$. Since the rank of $\Phi(\mathcal{H}^{1,0,0})$ is 3, we have that $\text{rank}(\Phi(\mathcal{H}^{1,1,0})) = 5$ and $\text{rank}(\Phi(\mathcal{H}^{1,2,0})) = 6$.

Proposition 4.3: Let t_1 be a positive integer. Then, $\text{rank}(\Phi(\mathcal{H}^{t_1+1,0,0})) = \text{rank}(\Phi(\mathcal{H}^{t_1,0,0})) + 4t_1 + 2\binom{t_1-1}{2} + 1 + \binom{t_1-1}{3}$.

Proof. By (2), the generator matrix of $\mathcal{H}' = \mathcal{H}^{t_1+1,0,0}$ is

$$A^{t_1+1,0,0} = \begin{pmatrix} A & A & A & A & A & A & A & A \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} \end{pmatrix},$$

where $A = A^{t_1,0,0}$ is the generator matrix of $\mathcal{H} = \mathcal{H}^{t_1,0,0}$. Note that $\mathcal{H}'$ can be seen as the union of eight cosets of the 8-fold replication code of $\mathcal{H}$, $(\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H})$, which are

$$\begin{aligned}
&C_0 : (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) \\
&C_1 : (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) + (\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}) \\
&C_2 : (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) + (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}) \\
&C_3 : (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) + (\mathbf{0}, \mathbf{3}, \mathbf{6}, \mathbf{1}, \mathbf{4}, \mathbf{7}, \mathbf{2}, \mathbf{5}) \\
&C_4 : (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}) \\
&C_5 : (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) + (\mathbf{0}, \mathbf{5}, \mathbf{2}, \mathbf{7}, \mathbf{4}, \mathbf{1}, \mathbf{6}, \mathbf{3}) \\
&C_6 : (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) + (\mathbf{0}, \mathbf{6}, \mathbf{4}, \mathbf{2}, \mathbf{0}, \mathbf{6}, \mathbf{4}, \mathbf{2}) \\
&C_7 : (\mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}, \mathcal{H}) + (\mathbf{0}, \mathbf{7}, \mathbf{6}, \mathbf{5}, \mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{1}) \quad .
\end{aligned}$$

Note that $\text{rank}(\Phi(C_0)) = \text{rank}(\Phi(\mathcal{H})) = r$. Let $\{\Phi(\mathbf{g}_1), \ldots, \Phi(\mathbf{g}_r)\}$ be a basis of $\langle H \rangle$. Then, a basis of $\langle \Phi(C_0) \rangle$ is $\{\Phi(\mathbf{g}'_1), \ldots, \Phi(\mathbf{g}'_r)\}$, where $\mathbf{g}'_i = (\mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i, \mathbf{g}_i)$ for all $i \in \{1, \ldots, r\}$. Let $\mathbf{w}' = (\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7})$. By the proof of Proposition 4.2, we have that $\langle \Phi(C_0 \cup C_2 \cup C_4 \cup C_6) \rangle = \langle \Phi(C_0 \cup C_2 \cup C_4) \rangle = \langle \Phi(\mathbf{g}'_1), \ldots, \Phi(\mathbf{g}'_r), \Phi(2\mathbf{w}'), \Phi(M') \rangle$, where M' is defined as in the mentioned proof using $2\mathbf{w}' = (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ instead of $\mathbf{v}' = (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$.

Note that, if $\mathbf{u}' \in C_5$, then $\mathbf{u}' = (\mathbf{u}, \mathbf{u}+\mathbf{5}, \mathbf{u}+\mathbf{2}, \mathbf{u}+\mathbf{7}, \mathbf{u}+\mathbf{4}, \mathbf{u}+\mathbf{1}, \mathbf{u}+\mathbf{6}, \mathbf{u}+\mathbf{3}) = (\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}) + \mathbf{w}' + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ with $\mathbf{u} \in \mathcal{H}$. Similarly, if $\mathbf{u}' \in C_7$, then $\mathbf{u}' = (\mathbf{u}, \mathbf{u}+\mathbf{7}, \mathbf{u}+\mathbf{6}, \mathbf{u}+\mathbf{5}, \mathbf{u}+\mathbf{4}, \mathbf{u}+\mathbf{3}, \mathbf{u}+\mathbf{2}, \mathbf{u}+\mathbf{1}) = (\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}) + 3\mathbf{w}' + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ with $\mathbf{u} \in \mathcal{H}$. Thus, it is easy to see that $\langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_7) \rangle = \langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4) \rangle$, by Corollary 3.1. Now, we will find a basis for $\langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_4) \rangle$ by extending the given basis for $\langle \Phi(C_0 \cup C_2 \cup C_4) \rangle$. After that, we will see that $\langle \Phi(C_3) \rangle$ is linearly dependent of $\langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_4) \rangle$.

Let $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{t_1}, 2\mathbf{w}_1, \ldots, 2\mathbf{w}_{t_1}, 4\mathbf{w}_1, \ldots, 4\mathbf{w}_{t_1}\}$ be a 2-basis of $\mathcal{H}$ and recall that $\text{ord}(\mathbf{w}_i) = 8$ for all $i \in \{1, \ldots, t_1\}$. Let $\mathbf{u} \in \mathcal{H}$. We know that $\mathbf{u} = \sum_{i=1}^{3t_1} \lambda_i \mathbf{b}_i$, where $\mathbf{b}_i \in \mathcal{B}_2$ is the ith element of $\mathcal{B}_2$ and $\lambda_i \in \{0, 1\}$. Let

$E = \{1 \le i \le 3t_1 : \lambda_i \neq 0\}$, $E_1 = \{1 \le i \le t_1 : i \in E\} \cup \{1 \le i \le t_1 : t_1 + i \in E\} \cup \{1 \le i \le t_1 : t_1 + i \in E\}$ as a multiset, and $E_4 = \{1 \le i \le t_1 : 2t_1 + i \in E\}$. Let $\mathbf{u}' = (\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u})$ and $\mathbf{w}'_i = (\mathbf{w}_i, \mathbf{w}_i, \mathbf{w}_i, \mathbf{w}_i, \mathbf{w}_i, \mathbf{w}_i, \mathbf{w}_i, \mathbf{w}_i)$ for all $i \in \{1, \dots, t_1\}$. Let $\mathbf{s}_i$ be the ith element of the ordered multiset $\{\mathbf{w}'_i : i \in E_1\}$. Now, we consider the element $\mathbf{u}' + \mathbf{w}' \in C_1$. By Corollary 3.1, $\Phi(\mathbf{u}' + \mathbf{w}') = \Phi(\sum_{i \in E_1} \mathbf{w}'_i + \mathbf{w}') + \sum_{i \in E_4} \Phi(4\mathbf{w}'_i)$.

Therefore, by Lemma 3.5, we have that

$$\begin{aligned}\Phi(\mathbf{u}' + \mathbf{w}') = &\sum_{i \in E_4} \Phi(4\mathbf{w}'_i) + \\ &\sum_{i<j<k<p<q} \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k + \mathbf{s}_p) + \\ &\sum_{i<j<k<q} \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k + \mathbf{w}') \\ &+ q_0\big(\sum_{i<j<k<q} \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k) + \sum_{i<j<q} \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{w}')\big) \\ &+ (q_0 + q_1)\big(\sum_{i<j<q} \Phi(\mathbf{s}_i + \mathbf{s}_j) + \sum_{i<q} \Phi(\mathbf{s}_i + \mathbf{w}')\big) \\ &+ q_0(q_0 + q_1)\big(\sum_{i<q} \Phi(\mathbf{s}_i) + \Phi(\mathbf{w}')\big),\end{aligned}$$

where $q = |E_1| + 1$ and $[q_0, q_1, \dots]_2$ is the binary expansion of q. We know that $\sum_{i \in E_4} \Phi(4\mathbf{w}'_i)$, $\Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k + \mathbf{s}_p)$, $\Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k)$, $\Phi(\mathbf{s}_i + \mathbf{s}_j)$, and $\Phi(\mathbf{s}_i)$ belong to $\langle \Phi(C_0) \rangle$.

We will see that $\Phi(\mathbf{u}' + \mathbf{w}') - \sum_{i \in E_4} \Phi(4\mathbf{w}'_i) \in \langle \Phi(C_0 \cup C_2) \cup L_1 \cup L_2 \cup L_3 \cup \{\Phi(\mathbf{w}')\} \rangle$, where $L_1 = \{\Phi(\mathbf{w}'_i + \mathbf{w}') : 1 \le i \le t_1\} \cup \{\Phi(2\mathbf{w}'_i + \mathbf{w}') : 1 \le i \le t_1\}$, $L_2 = \{\Phi(\mathbf{w}'_i + \mathbf{w}'_j + \mathbf{w}') : 2 \le i < j \le t_1\}$, and $L_3 = \{\Phi(\mathbf{w}'_i + \mathbf{w}'_j + \mathbf{w}'_k + \mathbf{w}') : 2 \le i < j < k \le t_1\}$. First, it is clear that $\Phi(\mathbf{s}_i + \mathbf{w}') \in L_1$. Now, we consider the terms of the form $A = \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{w}')$. If $A = \Phi(2\mathbf{w}'_i + \mathbf{w}')$, then $A \in L_1$; if $A = \Phi(\mathbf{1} + \mathbf{w}'_i + \mathbf{w}')$ with $2 \le i \le t_1$, then $A \in \langle \Phi(C_0 \cup C_2) \cup L_1 \rangle$ by Lemma 3.7; and if $A = \Phi(\mathbf{w}'_i + \mathbf{w}'_j + \mathbf{w}')$ with $2 \le i < j \le t_1$, then $A \in L_2$. Next, we consider the terms of the form $B = \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k + \mathbf{w}')$. If $B = \Phi(2\mathbf{w}'_i + \mathbf{w}' + \mathbf{w}'_k)$, then $B \in \langle \Phi(C_0 \cup C_2) \cup L_1 \cup \{\Phi(\mathbf{w}')\} \rangle$ by using Lemma 3.6 and taking $\Phi(\mathbf{w}'_i + 2\mathbf{w}' + \mathbf{w}'_k) \in \Phi(C_2)$ as the other addend in the left-hand side of the equation of the lemma. If $B = \Phi(\mathbf{1} + \mathbf{w}'_i + \mathbf{w}'_j + \mathbf{w}')$ with $2 \le i < j \le t_1$, then $B \in \langle \Phi(C_0 \cup C_2) \cup L_1 \cup L_2 \cup \{\Phi(\mathbf{w}')\} \rangle$ by Lemma 3.7. Finally, if $B = \Phi(\mathbf{w}'_i + \mathbf{w}'_j + \mathbf{w}'_k + \mathbf{w}')$ with $2 \le i < j < k \le t_1$, then $B \in L_3$.

The elements of L_1, L_2 and L_3 are linearly independent from each other. Therefore, the elements of $L_1 \cup L_2 \cup L_3 \cup \{\Phi(\mathbf{w}')\}$ are linearly independent and $\text{rank}(\langle L_1 \cup L_2 \cup L_3 \cup \{\Phi(\mathbf{w})\} \rangle) = 2t_1 + \binom{t_1 - 1}{2} + \binom{t_1 - 1}{3} + 1$. It is also easy to see that they are linearly independent from the elements in $\langle \Phi(C_0 \cup C_2 \cup C_4) \rangle$, so $\text{rank}(\langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_4) \rangle) = r + 4t_1 + 2\binom{t_1 - 1}{2} + 1 + \binom{t_1 - 1}{3}$ by Proposition 4.2.

Finally, we will show that $\langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4) \rangle = \langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_4) \rangle$. We consider the element $\mathbf{u}' + 3\mathbf{w}' \in C_3$. Again, by Corollary 3.1, $\Phi(\mathbf{u}' + 3\mathbf{w}') = \Phi(\sum_{i \in E_1} \mathbf{w}'_i + 3\mathbf{w}') + \sum_{i \in E_4} \Phi(4\mathbf{w}'_i)$. Therefore, by Lemma 3.5, we have that

$$\begin{aligned}\Phi(\mathbf{u}' + 3\mathbf{w}') = &\sum_{i \in E_4} \Phi(4\mathbf{w}'_i) + \\ &\sum_{i<j<k<p \le q-3} \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k + \mathbf{s}_p) + \\ &\sum_{i<j<k \le q-3} \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k + \mathbf{w}') \\ &+ \sum_{i<j \le q-3} \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{w}' + \mathbf{w}') + \sum_{i \le q-3} \Phi(\mathbf{s}_i + \mathbf{w}' + \mathbf{w}' + \mathbf{w}') \\ &+ q_0\big(\sum_{i<j<k \le q-3} \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k) + \sum_{i<j \le q-3} \Phi(\mathbf{s}_i + \mathbf{s}_j + \mathbf{w}') \\ &+ \sum_{i \le q-3} \Phi(\mathbf{s}_i + \mathbf{w}' + \mathbf{w}') + \Phi(\mathbf{w}' + \mathbf{w}' + \mathbf{w}')\big) \\ &+ (q_0 + q_1)\big(\sum_{i<j \le q-3} \Phi(\mathbf{s}_i + \mathbf{s}_j) + \\ &\sum_{i \le q-3} \Phi(\mathbf{s}_i + \mathbf{w}') + \Phi(\mathbf{w}' + \mathbf{w}')\big) \\ &+ q_0(q_0 + q_1)\big(\sum_{i \le q-3} \Phi(\mathbf{s}_i) + \Phi(\mathbf{w}')\big),\end{aligned}$$

where $q = |E_1| + 3$. All the addends belong to $\langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_4) \rangle$, except the ones of the form $\Phi(\mathbf{w}' + \mathbf{w}' + \mathbf{w}')$ and $\Phi(\mathbf{s}_i + \mathbf{w}' + \mathbf{w}' + \mathbf{w}')$. First, we have that $\Phi(3\mathbf{w}') \in \langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_4) \rangle$ by Lemma 3.8. Finally, by using Lemma 3.6 with $\Phi(\mathbf{s}_i + 2\mathbf{w}' + \mathbf{w}')$ and $\Phi(2\mathbf{s}_i + \mathbf{w}' + \mathbf{w}') \in \Phi(C_2)$, we have that $\Phi(\mathbf{s}_i + \mathbf{w}' + \mathbf{w}' + \mathbf{w}') \in \langle \Phi(C_0 \cup C_1 \cup C_2 \cup C_4) \rangle$. Therefore, the result holds. □

Example 4.3: Let $B^{1,0,0} = \{\Phi(1), \Phi(2), \Phi(4)\}$ be a basis of $\langle \Phi(\mathcal{H}^{1,0,0}) \rangle$. Then, by the proof of Proposition 4.3 and the basis of $\langle \Phi(\mathcal{H}^{1,1,0}) \rangle$, $B^{1,1,0} = \{\Phi(1,1,1,1), \Phi(2,2,2,2), \Phi(4,4,4,4), \Phi(0,2,4,6), \Phi(0,4,0,4)\}$, obtained in Example 4.2, we have that a basis of $\langle \Phi(\mathcal{H}^{2,0,0}) \rangle$ is

$$B^{2,0,0} = \{\Phi(\mathbf{1}), \Phi(\mathbf{2}), \Phi(\mathbf{4}), \Phi(0,2,4,6,0,2,4,6),$$
$$\Phi(0,4,0,4,0,4,0,4), \Phi(\mathbf{w}')\} \cup L_1 \cup L_2 \cup L_3,$$

where $\mathbf{w}' = (0,1,2,3,4,5,6,7)$, $L_1 = \{\Phi(\mathbf{1} + \mathbf{w}'), \Phi(\mathbf{2} + \mathbf{w}')\} = \{\Phi(1,2,3,4,5,6,7,0), \Phi(2,3,4,5,6,7,0,1)\}$, and $L_2 = L_3 = \emptyset$. Since the rank of $\Phi(\mathcal{H}^{1,0,0})$ is 3, we have that $\text{rank}(\Phi(\mathcal{H}^{2,0,0})) = 8$.

Lemma 4.1: Let $t, k \in \mathbb{N}$. Then,

$$\sum_{i=1}^{t} \binom{i}{k} = \frac{(t+1-k)\binom{t+1}{k} + (k-1)\binom{1}{k}}{k+1}.$$

Proof. Straightforward by induction on the integer t. □

Corollary 4.1: Let t_1 be a positive integer. Then,

$$\text{rank}(\Phi(\mathcal{H}^{t_1,0,0})) = \frac{t_1^4}{24} - \frac{t_1^3}{12} + \frac{35t_1^2}{24} + \frac{7t_1}{12} + 1.$$

Proof. We know that $\text{rank}(\Phi(\mathcal{H}^{1,0,0})) = 3$. By applying Proposition 4.3 recursively, we have that

$$\text{rank}(\Phi(\mathcal{H}^{t_1,0,0})) = 3 + 4\sum_{i=1}^{t_1-1} i + 2\sum_{i=1}^{t_1-2} \binom{i}{2} +$$

$$(t_1-1)+\sum_{i=1}^{t_1-2}\binom{i}{3}.$$

Finally, by Lemma 4.1, it is easy to see that the result holds. □

Corollary 4.2: Let t_1 and t_2 be nonnegative integers with $t_1 \geq 1$. Then,

$$\mathrm{rank}(\Phi(\mathcal{H}^{t_1,t_2,0}))=\mathrm{rank}(\Phi(\mathcal{H}^{t_1,0,0}))+\frac{t_2}{2}(t_1^2+t_1+t_2+1).$$

Proof. By applying Proposition 4.2 recursively, it is easy to see that

$$\mathrm{rank}(\Phi(\mathcal{H}^{t_1,t_2,0}))=\mathrm{rank}(\Phi(\mathcal{H}^{t_1,0,0}))+ t_2\left(2t_1+\binom{t_1-1}{2}+\frac{t_2-1}{2}\right).$$

Since $t_2(2t_1+\binom{t_1-1}{2}+\frac{t_2-1}{2})=\frac{t_2}{2}(t_1^2+t_1+t_2+1)$, the result follows. □

Theorem 4.1: Let $\mathcal{H}=\mathcal{H}^{t_1,t_2,t_3}$ be a $\mathbb{Z}_8$-additive Hadamard code. Then,

$$\mathrm{rank}(\Phi(\mathcal{H}))= \frac{t_1^4}{24}-\frac{t_1^3}{12}+\frac{35t_1^2}{24}+\frac{7t_1}{12}+\frac{t_2}{2}(t_1^2+t_1+t_2+1)+t_3+1.$$

Proof. Straightforward from Proposition 4.1 and Corollaries 4.1 and 4.2. □

V. Classification of $\mathbb{Z}_8$-Linear Hadamard Codes

The classification of the $\mathbb{Z}_4$-linear Hadamard codes of length 2^t, for any $t \geq 3$, can be established by using either the rank or the dimension of the kernel [17], [20]. In [13], it is shown that, in general, for $s>2$, the dimension of the kernel is not enough to establish a complete classification of the $\mathbb{Z}_{2^s}$-linear Hadamard codes of length 2^t, for any $t \geq 3$. In this section, we show that for $s=3$, a complete classification can be given by using both invariants: the dimension of the kernel and the rank, computed in [13] and in the previous section, respectively.

First, recall that the dimension of the kernel for $\mathbb{Z}_8$-linear Hadamard codes is given by the following result:

Proposition 5.1: [13] Let $\mathcal{H}=\mathcal{H}^{t_1,t_2,t_3}$ be a $\mathbb{Z}_8$-additive Hadamard code. If $\Phi(\mathcal{H})$ is nonlinear, then $\ker(\Phi(\mathcal{H}))=t_1+t_2+t_3+\sigma_{t_1}$, where $\sigma_{t_1}=1$ if $t_1\geq 2$ and $\sigma_{t_1}=2$ if $t_1=1$.

In [13], it is also shown that, in order to obtain a complete classification of nonlinear $\mathbb{Z}_{2^s}$-linear Hadamard codes of length 2^t, it is enough to focus on $t\geq 5$, since all $\mathbb{Z}_{2^s}$-linear Hadamard codes of length 2^t are linear for $t<5$. It is also mentioned in [13] that, at least for any $3\leq t\leq 11$, these codes can be fully classified by using only the values of the rank (see Table I). Then, this pointed out that, maybe, it was possible to obtain a complete classification for any $t \geq 5$ by using just this invariant. However, the following example shows us that both invariants, the rank and the dimension of the kernel, are necessary in some cases.

Example 5.1: Consider the $\mathbb{Z}_8$-linear Hadamard codes of length $2^{17}=131072$, ($t=17$), $H^{2,6,0}$ and $H^{4,1,4}$. By Theorem 4.1, we have that $\mathrm{rank}(H^{2,6,0})=\mathrm{rank}(H^{4,1,4})=47$. However, since $\ker(H^{2,6,0})=9$ and $\ker(H^{4,1,4})=10$ by Proposition 5.1, they are not equivalent even though they have the same rank. The rest of 23 nonlinear such codes of length 2^{17} have a different rank, so we have that there are exactly 26 nonequivalent $\mathbb{Z}_8$-linear Hadamard codes of length 2^{17}.

Although we cannot completely classify the $\mathbb{Z}_8$-linear Hadamard codes by using only the rank, the following result shows that if two such codes have the same dimension of the kernel, then their values of the rank are different.

Theorem 5.1: Let $5\leq t\in\mathbb{Z}$. Then, for every pair, H^{t_1,t_2,t_3} and $H^{t'_1,t'_2,t'_3}$, of nonlinear $\mathbb{Z}_8$-linear Hadamard codes of length 2^t with $(n;t_1,t_2,t_3)\neq(n;t'_1,t'_2,t'_3)$ and $\ker(H^{t_1,t_2,t_3})=\ker(H^{t'_1,t'_2,t'_3})$, we have that $\mathrm{rank}(H^{t_1,t_2,t_3})\neq\mathrm{rank}(H^{t'_1,t'_2,t'_3})$.

Proof. Let $k=\ker(H^{t_1,t_2,t_3})=\ker(H^{t'_1,t'_2,t'_3})$. By Proposition 5.1, we have that $k=t_1+t_2+t_3+\sigma_{t_1}$. Moreover,

$$\left.\begin{array}{rcl} t_1+t_2+t_3+\sigma_{t_1} &=& k\\ 3t_1+2t_2+t_3 &=& t+1\end{array}\right\}\Longleftrightarrow \left\{\begin{array}{rcl} t_2 &=& \sigma_{t_1}-k+t+1-2t_1\\ t_3 &=& t_1+2k-2\sigma_{t_1}-t-1.\end{array}\right. \quad (18)$$

By replacing the formulas in (18) into the expression of the rank, given by Theorem 4.1, we have that

$$\mathrm{rank}(H^{t_1,t_2,t_3})=\mathrm{rank}(t_1,t,k)= \frac{t_1^4}{24}-\frac{t_1^3}{12}+\frac{35t_1^2}{24}+\frac{7t_1}{12}+ +\frac{1}{2}(\sigma_{t_1}-k+t+1-2t_1)(t_1^2+t_1+\sigma_{t_1}-k+t+1-2t_1+1)+ t_1+2k-2\sigma_{t_1}-t-1+1,$$

which is equal to

$$\mathrm{rank}(t_1,t,k)=\frac{t_1^4}{24}-\frac{13}{12}t_1^3+(\frac{71}{24}+\frac{1}{2}(t-k+\sigma_{t_1}))t_1^2- -(\frac{11}{12}+\frac{3}{2}(t-k+\sigma_{t_1}))t_1+\frac{1}{2}((t-k+\sigma_{t_1})^2+t+k-\sigma_{t_1}+2).$$

Now, we suppose that $\mathrm{rank}(H^{t_1,t_2,t_3})=\mathrm{rank}(H^{t'_1,t'_2,t'_3})$ for $(n;t_1,t_2,t_3)\neq(n;t'_1,t'_2,t'_3)$ or, equivalently, $\mathrm{rank}(t_1,t,k)=\mathrm{rank}(t'_1,t,k)$ for $t_1\neq t'_1$. Without loss of generality, we can assume that $t'_1<t_1$. Note that if $t'_1=t_1$, then $t_2=t'_2$ and $t_3=t'_3$, so both codes are equal.

First, we consider that $2\leq t'_1<t_1$. In this case, we have to see that $\mathrm{rank}(t_1,t,k)-\mathrm{rank}(t'_1,t,k)\neq 0$. Since $t_1,t'_1\geq 2$, $\sigma_{t_1}=\sigma_{t'_1}=1$ and we have that

$$\mathrm{rank}(t_1,t,k)-\mathrm{rank}(t'_1,t,k)= \frac{t_1^4}{24}-\frac{13}{12}t_1^3+(\frac{71}{24}+\frac{1}{2}(t-k+1))t_1^2-(\frac{11}{12}+\frac{3}{2}(t-k+1))t_1+ -\frac{t_1'^4}{24}+\frac{13}{12}t_1'^3-(\frac{71}{24}+\frac{1}{2}(t-k+1))t_1'^2+(\frac{11}{12}+\frac{3}{2}(t-k+1))t'_1.$$

By using the identity $x^2-y^2=(x+y)(x-y)$, we have that

$$\begin{aligned}&\operatorname{rank}(t_1,t,k)-\operatorname{rank}(t'_1,t,k)=\\&\quad\frac{1}{24}\big[(t_1+t'_1)(t_1^2+t_1'^2)-26(t_1^2+t_1t'_1+t_1'^2)+\\&\quad(t_1+t'_1)(83+12(t-k))-58-36(t-k)\big]=\\&\frac{1}{24}\big[(t_1+t'_1)(t_1^2+t_1'^2+83)-26(t_1^2+t_1t'_1+t_1'^2)-58+\\&\qquad 12(t-k)(t_1+t'_1-3)\big],\end{aligned}\tag{19}$$

which can be written as $\operatorname{rank}(t_1,t,k)-\operatorname{rank}(t'_1,t,k)=f(t_1,t'_1)+(t-k)g(t_1,t'_1)$, where $f(t_1,t'_1)=1/24\big[(t_1+t'_1)(t_1^2+t_1'^2+83)-26(t_1^2+t_1t'_1+t_1'^2)-58\big]$ and $g(t_1,t'_1)=1/2(t_1+t'_1-3)$. Note that $(t-k)g(t_1,t'_1)\geq 0$ for all integer pairs $(t_1,t'_1)\in D$, where $D=\{(t_1,t'_1):2\leq t'_1<t_1\}$. It is easy to see that $f(t_1,t'_1)>0$ for all $t'_1\geq 26$, since we can rewrite this expression in the following form:

$$t_1^2(t_1+t'_1)+t_1'^2(t_1+t'_1)+83(t_1+t'_1)>26t_1^2+26t'_1(t_1+t'_1)-58,\tag{20}$$

and we can observe that $t_1^2(t_1+t'_1)>26t_1^2$, $t_1'^2(t_1+t'_1)\geq 26t'_1(t_1+t'_1)$ and $83(t_1+t'_1)>-58$.

Similarly, $f(t_1,t'_1)>0$ for all $t_1\geq 26$, considering the left-hand side of (20) as $26t_1(t_1+t'_1)+26t_1'^2-58$. Therefore, if there exists a pair of integers (t_1,t'_1) such that $\operatorname{rank}(t_1,t,k)-\operatorname{rank}(t'_1,t,k)=0$, this pair has to be in $R=\{(t_1,t'_1):2\leq t'_1<t_1,t'_1<26,t_1<26\}\subset D$. There are $1+2+\cdots+23=276$ pairs $(t_1,t'_1)\in R$, and it can be checked that any of them is a solution of the equation.

Finally, we consider that $1=t'_1<t_1$. In this case, we have to prove that $\operatorname{rank}(t_1,t,k)-\operatorname{rank}(t'_1,t,k)\neq 0$. Then, since $t_1\geq 2$ and $t'_1=1$, $\sigma_{t_1}=1$, $\sigma_{t'_1}=2$, and we obtain that

$$\begin{aligned}&\operatorname{rank}(t_1,t,k)-\operatorname{rank}(1,t,k)=\\&\frac{t_1^4}{24}-\frac{13}{12}t_1^3+(\frac{71}{24}+\frac{1}{2}(t-k+1))t_1^2-(\frac{11}{12}+\frac{3}{2}(t-k+1))t_1+\\&\quad\frac{1}{2}((t-k+1)^2+t+k+1)-\frac{1}{24}+\frac{13}{12}-\frac{71}{24}-\\&\frac{1}{2}(t-k+2)+\frac{11}{12}+\frac{3}{2}(t-k+2)-\frac{1}{2}((t-k+2)^2+t+k).\end{aligned}$$

By simplifying, we have that

$$\begin{aligned}&\operatorname{rank}(t_1,t,k)-\operatorname{rank}(1,t,k)=\\&\quad\frac{1}{24}\big[t_1^4-26t_1^3+83t_1^2-58t_1+12(t-k)(t_1^2-3t_1)\big].\end{aligned}$$

Let $f(t_1,t,k)=\operatorname{rank}(t_1,t,k)-\operatorname{rank}(1,t,k)$. We know that $t-k=2t_1+t_2-2\geq 2t_1-2$. Since $12(t-k)(t_1^2-3t_1)\geq 0$ for $t_1\geq 3$, we have that $f(t_1,t,k)\geq g(t_1)=\frac{1}{24}[t_1^4-26t_1^3+83t_1^2-58t_1+12(2t_1-2)(t_1^2-3t_1)]=\frac{1}{24}[t_1(t_1^3-2t_1^2-13t_1+14)]$. By computing the zeros of the polynomial $g(t_1)$ and analyzing its behavior, we have that $f(t_1,t,k)\geq g(t_1)>0$ for $t_1\geq 5$. Therefore, we just need to compute $f(t_1,t,k)$ when $t_1\in\{2,3,4\}$. For these cases, we have that

$$\begin{aligned}f(2,t,k)&=1-(t-k)=0\Leftrightarrow t-k=1\\f(3,t,k)&=-2\\f(4,t,k)&=2(t-k)-13=0\Leftrightarrow t-k=13/2.\end{aligned}$$

Note that if $t_1=2$, then $t-k=t_2+2\geq 2$, so $t-k\neq 1$. Therefore, for $t_1\in\{2,3,4\}$, $f(t_1,t,k)\neq 0$ and the result holds. □

Recall that it is already known that there are $\lfloor\frac{t-1}{2}\rfloor$ nonequivalent $\mathbb{Z}_4$-linear Hadamard codes of length 2^t, $t\geq 3$ [17]. Now, we establish how many nonequivalent $\mathbb{Z}_8$-linear Hadamard codes of length 2^t there are, once the length 2^t if fixed, for $t\geq 5$. In [13], some upper and lower bounds are given for certain values of t. By Theorems 4.1 and 5.1, we know that if H^{t_1,t_2,t_3} and $H^{t'_1,t'_2,t'_3}$ are nonlinear $\mathbb{Z}_8$-linear Hadamard codes of the same length with $(t_1,t_2,t_3)\neq(t'_1,t'_2,t'_3)$, then their corresponding pairs, (r,k), where r is the rank and k is the dimension of the kernel, are different. Then, we have the following result:

Theorem 5.2: Let $\mathcal{A}_{t,3}$ be the number of nonequivalent $\mathbb{Z}_8$-linear Hadamard codes of length 2^t. Then, for any $t\geq 5$,

$$\mathcal{A}_{t,3}=\Big\lfloor\frac{t+1}{3}\Big\rfloor+\sum_{i=1}^{\lfloor(t+1)/3\rfloor}\Big\lfloor\frac{t+1-3i}{2}\Big\rfloor-1.$$

Proof. In [13, Theorem 5.3], an upper bound is given for the amount of different nonequivalent $\mathbb{Z}_{2^s}$-linear Hadamard codes for any $t\geq 3$ and $2\leq s\leq t-1$. In particular, when $s=3$, we have the following bound:

$$\begin{aligned}&\mathcal{A}_{t,3}\leq\\&|\{(t_1,t_2,t_3)\in\mathbb{N}^3:t=3t_1+2t_2+t_3-1,\ t_1\geq 1\}|-1.\end{aligned}$$

By Theorems 4.1 and 5.1, we know that this bound is tight. Therefore, we just have to see that

$$\begin{aligned}&|\{(t_1,t_2,t_3)\in\mathbb{N}^3:t=3t_1+2t_2+t_3-1,\ t_1\geq 1\}|=\\&\qquad\Big\lfloor\frac{t+1}{3}\Big\rfloor+\sum_{i=1}^{\lfloor(t+1)/3\rfloor}\Big\lfloor\frac{t+1-3i}{2}\Big\rfloor.\end{aligned}$$

This means that we need to compute the amount of different solutions, (t_1,t_2,t_3), of the equation $t=3t_1+2t_2+t_3-1$ with $t_1\geq 1$.

It is easy to see that $1\leq t_1\leq\Big\lfloor\dfrac{t+1}{3}\Big\rfloor$. Once the value of t_1 is fixed, we can see that t_2 is bounded by $0\leq t_2\leq\Big\lfloor\dfrac{t+1-3t_1}{2}\Big\rfloor$. Note that, once t_1 and t_2 are fixed, there is a unique value for t_3. Then, the amount of different solutions of $t=3t_1+2t_2+t_3-1$ with $t_1\geq 1$, or equivalently $|\{(t_1,t_2,t_3)\in\mathbb{N}^3:t=3t_1+2t_2+t_3-1,\ t_1\geq 1\}|$, is

$$\begin{aligned}&\sum_{i=1}^{\lfloor(t+1)/3\rfloor}\Big(\Big\lfloor\frac{t+1-3i}{2}\Big\rfloor+1\Big)=\\&\qquad\Big\lfloor\frac{t+1}{3}\Big\rfloor+\sum_{i=1}^{\lfloor(t+1)/3\rfloor}\Big\lfloor\frac{t+1-3i}{2}\Big\rfloor,\end{aligned}$$

so the result holds. □

Example 5.2: Table I shows all possible values of (t_1, t_2, t_3) for which there exists a nonlinear $\mathbb{Z}_8$-linear code H^{t_1,t_2,t_3} of length 2^t for $5 \leq t \leq 11$, together with the values of (r, k), where r is the rank and k the dimension of the kernel. Since for these values of t the rank gives a complete classification, the value of $A_{t,3}$ obtained from Theorem 5.2 coincides with the number of different (t_1, t_2, t_3) in the second column of the table, increased by one to add also the $\mathbb{Z}_8$-linear code that is linear.

	(t_1, t_2, t_3)	(r, k)	$A_{t,3}$
$t = 5$	$(2, 0, 0)$	(8,3)	2
$t = 6$	$(1, 2, 0)$	(8,5)	3
	$(2, 0, 1)$	(9,4)	
$t = 7$	$(1, 2, 1)$	(9,6)	4
	$(2, 0, 2)$	(10,5)	
	$(2, 1, 0)$	(12,4)	
$t = 8$	$(1, 2, 2)$	(10,7)	6
	$(1, 3, 0)$	(12,6)	
	$(2, 0, 3)$	(11,6)	
	$(2, 1, 1)$	(13,5)	
	$(3, 0, 0)$	(17,4)	
$t = 9$	$(1, 2, 3)$	(11,8)	7
	$(1, 3, 1)$	(13,7)	
	$(2, 0, 4)$	(12,7)	
	$(2, 1, 2)$	(14,6)	
	$(2, 2, 0)$	(17,5)	
	$(3, 0, 1)$	(18,5)	
$t = 10$	$(1, 2, 4)$	(12,9)	9
	$(1, 3, 2)$	(14,8)	
	$(1, 4, 0)$	(17,7)	
	$(2, 0, 5)$	(13,8)	
	$(2, 1, 3)$	(15,7)	
	$(2, 2, 1)$	(18,6)	
	$(3, 0, 2)$	(19,6)	
	$(3, 1, 0)$	(24,5)	
$t = 11$	$(1, 2, 5)$	(13,10)	11
	$(1, 3, 3)$	(15,9)	
	$(1, 4, 1)$	(18,8)	
	$(2, 0, 6)$	(14,9)	
	$(2, 1, 4)$	(16,8)	
	$(2, 2, 2)$	(19,7)	
	$(2, 3, 0)$	(23,6)	
	$(3, 0, 3)$	(20,7)	
	$(3, 1, 1)$	(25,6)	
	$(4, 0, 0)$	(32,5)	

Table I
TYPE, RANK AND DIMENSION OF THE KERNEL FOR ALL NONLINEAR $\mathbb{Z}_8$-LINEAR HADAMARD CODES OF LENGTH 2^t FOR $5 \leq t \leq 11$.

VI. CONCLUSIONS

The $\mathbb{Z}_4$-linear Hadamard codes can be classified by using just one of the invariants, the rank or the dimension of the kernel [17], [20]. In general, for $\mathbb{Z}_{2^s}$-linear Hadamard codes of length 2^t, the kernel and its dimension were studied in [13], where it was also proved that this invariant is not enough to obtain a full classification of these codes, once s and t are fixed. In this paper, we focus on $s = 3$. We study the rank of the $\mathbb{Z}_8$-linear Hadamard codes of length 2^t, and also give an explicit construction of the linear independent vectors that generate the span. Through Example 5.1, we observe that the rank, by itself, is not enough to obtain a complete classification. However, we prove that it is really possible by using both of them, the rank and dimension of the kernel. We also provide the amount of nonequivalent $\mathbb{Z}_8$-linear Hadamard codes of length 2^t for a given t. It would be interesting to generalize these results to any $s \geq 4$, or prove that it is necessary to consider other invariants to classify such codes. Another further research on this topic would be to fix only the parameter t and find the number of nonequivalent $\mathbb{Z}_{2^s}$-linear Hadamard codes having the same length 2^t. In this sense, it is already proved that there are $\mathbb{Z}_4$-linear codes which are equivalent to a $\mathbb{Z}_8$-linear Hadamard code [13].

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