

CENTERS OF PLANAR GENERALIZED ABEL EQUATIONS

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ABSTRACT. We deal with the differential equation

$$\dot{r} = \frac{dr}{d\theta} = a(\theta)r^n + b(\theta)r^m,$$

where (r, θ) are the polar coordinates in the plane \mathbb{R}^2 , m and n are integers such that $m > n \geq 2$, and a, b are C^1 functions. Note that when $n = 2$ and $m = 3$ we have an Abel differential equation. For this class of generalized Abel equations we characterize a new family of centers.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Consider the generalized Abel equation

$$(1) \quad \dot{r} = \frac{dr}{d\theta} = a(\theta)r^n + b(\theta)r^m,$$

defined in the plane $(r, \theta) \in [0, +\infty) \times \mathbb{S}^1$ in polar coordinates where $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Here m and n are integers such that $m > n \geq 2$, $\theta \in [-\pi, \pi]$ and a, b are C^1 -functions.

The origin of the plane is a *center* for the differential equation (1) if there is a neighborhood of it where all the solutions are periodic except the equilibrium point at the origin.

When $n = 2$ and $m = 3$ the differential equation (1) is a particular family of Abel equations. In fact the Abel equations are of the form

$$\dot{r} = a(\theta) + b(\theta)r + c(\theta)r^2 + d(\theta)r^3,$$

and they appeared by the first time in the works of Niels Henryk Abel, see [7]. Today there are more than 1400 papers in MathSciNet with the name “Abel equation” in their title, see for instance the papers [1, 2, 3, 5, 6, 8] for results on centers in the Abel equations and the references quoted therein.

The main objective of this work is to provide a new family of centers in the generalized Abel equation (1). Thus, our main result is the following.

Theorem 1. *If $a(\theta)$ and $b(\theta)$ are C^1 odd functions and $m \geq 2n - 1$, then the origin $r = 0$ of the differential equation (1) is a center.*

The proof of Theorem 1 is given in the next section. We note that Theorem 1 in the particular case $n = 2$ and $m = 3$, i.e. when equation (1) is an Abel equation, already was obtained in [4] by Araujo, Lemos and Alves.

2010 *Mathematics Subject Classification.* 34C25, 34A34.

Key words and phrases. Centers, generalized Abel equations.

2. PROOF OF THEOREM 1

In order to prove Theorem 1 we introduce some auxiliary results that will be used in its proof.

Proposition 2. *The origin $r = 0$ of equation (1) is a center if and only if*

$$(2) \quad \int_{-\pi}^{\pi} a(\theta) d\theta = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} b(\theta)r(\theta;\rho)^{m-n} d\theta = 0,$$

for $|\rho| < \rho_0$ with ρ_0 sufficiently small where $r(\theta;\rho)$ is the solution of equation (1) such that $r(-\pi;\rho) = \rho$.

Proof. First we prove sufficiency. Note that dividing equation (1) by r^n and integrating it, we obtain

$$-\frac{1}{n-1}r^{1-n}(\theta;\rho) = \int_{-\pi}^{\theta} a(s) ds + \int_{-\pi}^{\theta} b(s)r(s;\rho)^{m-n} ds - \frac{1}{(n-1)\rho^{n-1}},$$

where ρ is the initial condition. So we have

$$r^{n-1}(\theta;\rho) = \frac{\rho^{n-1}}{1 - (n-1)\rho^{n-1}[\int_{-\pi}^{\theta} a(s) ds + \int_{-\pi}^{\theta} b(s)r(s;\rho)^{m-n} ds]},$$

which yields

$$(3) \quad r(\theta;\rho) = \frac{\rho}{(1 - (n-1)\rho^{n-1}[\int_{-\pi}^{\theta} a(s) ds + \int_{-\pi}^{\theta} b(s)r(s;\rho)^{m-n} ds])^{\frac{1}{n-1}}}.$$

Taking $\theta = \pi$ in the previous equation we have

$$r(\pi;\rho) = \frac{\rho}{(1 - (n-1)\rho^{n-1}[\int_{-\pi}^{\pi} a(s) ds + \int_{-\pi}^{\pi} b(s)r(s;\rho)^{m-n} ds])^{\frac{1}{n-1}}}.$$

If (2) holds, then $x(\pi;\rho) = \rho$, and the sufficiency in the theorem follows.

Assume now that equation (1) has a center on $r = 0$. We first note that any solution of equation (1), $r(\theta;\rho)$ can be expanded in power series in ρ for $|\rho| < \rho_0$ with ρ_0 sufficiently small and $\theta \in [-\pi, \pi]$ in the form

$$(4) \quad r(\theta;\rho) = r_0(\theta) + r_1(\theta)\rho + r_2(\theta)\rho^2 + \dots$$

Clearly $r_0(\theta) = 0$ because $r(\theta;0) = 0$. Substituting (4) into equation (1) we get $\dot{r}_1(\theta) = 0$ because $n \geq 2$, and since $r(\pi;\rho) = \rho$ we must have $r_1(\theta) = 1$. Hence we have

$$(5) \quad r(\theta;\rho) = \rho + r_2(\theta)\rho^2 + \dots$$

Note that a sufficient and necessary condition for $r = 0$ to be a center of (1) is that

$$r_i(\pi;\rho) = 0 \quad \text{for } i = 2, 3, \dots$$

Substituting (4) into (1) and computing the coefficients in ρ^j for $j \geq 2$ we get that

$$\dot{r}_i(\theta;\rho) = 0 \quad \text{for } i = 2, \dots, n-1.$$

Since $r_i(\pi;\rho) = 0$ we must have $r_i(\theta;\rho) = 0$ for $i = 2, \dots, n-1$. On the other hand

$$\dot{r}_n(\theta;\rho) = a(\theta) \quad \text{that is} \quad r_n(\theta;\rho) = \int_{-\pi}^{\theta} a(\theta) d\theta.$$

Since $r_n(\pi; \rho) = 0$ we obtain that $\int_{-\pi}^{\pi} a(s) ds = 0$. But then, from (3) we readily get that $\int_{-\pi}^{\pi} b(s)r(s; \rho)^{m-n} ds = 0$, which proves the necessity. This concludes the proof of the proposition. \square

Note that by equation (3) a solution of equation (1) is equivalent to a solution of the integral equation

$$(6) \quad r(\theta; \rho) = \frac{\rho}{\left(1 - (n-1)\rho^{n-1} \int_{-\pi}^{\theta} (a(s) + b(s)r(s; \rho)^{m-n}) ds\right)^{\frac{1}{n-1}}},$$

for $\theta \in [-\pi, \pi]$, where $r(-\pi; \rho) = \rho$. We denote

$$(7) \quad y(\theta) = r^{n-1}(\theta; \rho) \quad \text{and} \quad \ell = \frac{m-n}{n-1}.$$

Then equation (6) becomes

$$(8) \quad y(\theta) = \frac{\rho^{n-1}}{\left(1 - (n-1)\rho^{n-1} \int_{-\pi}^{\theta} (a(s) + b(s)y(s)^{\ell}) ds\right)}.$$

We define the operator $T: B_M \rightarrow C([- \pi, \pi])$ by

$$T(y)(\theta) = \frac{\rho^{n-1}}{1 - (n-1)\rho^{n-1} \int_{-\pi}^{\theta} (a(s) + b(s)y(s)^{\ell}) ds},$$

for $\theta \in [-\pi, \pi]$, where

$$B_M = \{y \in E : \|y\|_{\infty} \leq M\},$$

being E the closed subspace of $C([- \pi, \pi])$ defined by

$$E = \{y \in C([- \pi, \pi]) : y \text{ is an even function}\}.$$

Note that if the operator T has a unique fixed point, i.e., a unique $y \in B_M$ such that $T(y)(\theta) = y$ then (8) has an even solution. As usual $\|r\|_{\infty} = \max_{\theta \in [-\pi, \pi]} |r(\theta)|$.

We take the notation

$$J_{\theta}(y) = \frac{\rho^{n-1}}{1 - (n-1)\rho^{n-1} \int_{-\pi}^{\theta} (a(s) + b(s)y(s)^{\ell}) ds}.$$

We define

$$A = \max_{\theta \in [-\pi, \pi]} |a(\theta)|, \quad B = \max_{\theta \in [-\pi, \pi]} |b(\theta)|.$$

Proposition 3. *For*

$$(9) \quad 0 \leq \rho < \min \left\{ \frac{1}{(4\pi(n-1)(A + BM^{\ell}))^{1/(n-1)}}, \left(\frac{M}{2}\right)^{1/(n-1)} \right\},$$

the operator $T: B_M \rightarrow C([- \pi, \pi])$ is continuous and compact. Moreover $T(B_M) \subset B_M$.

Proof. For each $y_1, y_2 \in C([- \pi, \pi])$, we have

$$|T(y_1)(\theta) - T(y_2)(\theta)| = |(n-1)J_{\theta}(y_1)J_{\theta}(y_2)| \left| \int_{-\pi}^{\theta} b(s)(y_1(s)^{\ell} - y_2(s)^{\ell}) ds \right|,$$

for $\theta \in [-\pi, \pi]$. Note that since

$$0 \leq \rho \leq \frac{1}{(4\pi(n-1)(A+BM^\ell))^{1/(n-1)}},$$

we have

$$|(n-1)\rho^{n-1} \int_{-\pi}^{\theta} (a(s) + b(s)y(s)^\ell) ds| \leq 2\pi(n-1)\rho^{n-1}(A+BM^\ell) < \frac{1}{2}.$$

Therefore

$$(10) \quad 1 - (n-1)\rho^{n-1} \int_{-\pi}^{\theta} (a(s) + b(s)y(s)^\ell) ds > 1/2,$$

which yields $J_\theta(y) \leq 2\rho^{n-1}$. Moreover, since $m \geq 2n-1$ we have that $\ell \geq 1$ and by the Mean Value theorem we get

$$|y_1(s)^\ell - y_2(s)^\ell| \leq \ell M^{\ell-1} \|y_1 - y_2\|_\infty.$$

Hence

$$\begin{aligned} |T(y_1)(\theta) - T(y_2)(\theta)| &\leq 4(n-1)\rho^{2n-2} \int_{-\pi}^{\theta} |b(s)(y_1(s)^\ell - y_2(s)^\ell)| ds \\ &\leq 8\pi(n-1)\rho^{2n-2} B \ell M^{\ell-1} \|y_1 - y_2\|_\infty, \end{aligned}$$

for each $\theta \in [-\pi, \pi]$. So,

$$\|T(y_1) - T(y_2)\|_\infty \leq 8\pi(n-1)\rho^{2n-2} B \ell M^{\ell-1} \|y_1 - y_2\|_\infty$$

for all $y_1, y_2 \in C([-\pi, \pi])$. Hence the operator T is continuous.

For proving that the operator T is compact, we shall see that T is bounded and equicontinuous. Now we prove that it is bounded. Indeed, by (10) and the condition in ρ in (9) we have

$$|T(y)(\theta : -\pi)| \leq 2\rho^{n-1}, \quad \text{for all } y \in B_M \text{ and } \theta \in [-\pi, \pi]$$

and so

$$(11) \quad \|T(y)\|_\infty \leq 2\rho^{n-1} \quad \text{for all } y \in B_M,$$

proving that the operator T is bounded.

Now we show that T is equicontinuous. For each $\theta_1, \theta_2 \in [-\pi, \pi]$ (that without loss of generality we can assume that $\theta_2 > \theta_1$), and any $y \in B_M$, we have

$$\begin{aligned} |T(y)(\theta_1) - T(y)(\theta_2)| &= |(n-1)J_{\theta_1}(y)J_{\theta_2}(y)| \times \\ &\quad \left| \int_{-\pi}^{\theta_1} (a(s) + b(s)y(s)^\ell) ds - \int_{-\pi}^{\theta_2} (a(s) + b(s)y(s)^\ell) ds \right| \\ &\leq 4(n-1)\rho^{2n-2} \int_{\theta_1}^{\theta_2} |a(s) + b(s)y(s)^\ell| ds \\ &\leq 4(n-1)\rho^{2n-2}(A+BM^\ell)|\theta_2 - \theta_1|. \end{aligned}$$

Therefore $T(B_M)$ is an equicontinuous subset of $C([-\pi, \pi])$. By Ascoli-Arzelà Theorem (see for instance [9]) we have that $T: B_M \rightarrow C([-\pi, \pi])$ is compact.

Since the functions $a(\theta)$ and $b(\theta)$ are odd and by assumptions $y(\theta)$ is an even function (and so also $y(\theta)^\ell$ is an even function), we have that $a(s) + b(s)y(s)^\ell$ is odd. Taking into account that the integral of an odd function is an even function

we conclude that $\int_{-\pi}^{\theta} (a(s) + b(s)y(s)^{\ell}) ds$ is an even function in θ . Hence, for each $y \in E$ we have that $T(y)(\theta) = T(y)(-\theta)$ for all $\theta \in [-\pi, \pi]$. Therefore, $T(y) \in E$ for every $y \in E$. Moreover, by (9) and (11) we have that

$$\|T(y)\|_{\infty} \leq 2\rho^{n-1} < M \quad \text{for all } y \in B_M.$$

So $T: B_M \rightarrow B_M$ is well defined. This concludes the proof of the proposition. \square

Proposition 4. *Under the assumptions of Theorem 1 there are infinitely many closed even solutions $r(\theta; \rho)$ of system (1) for ρ satisfying (9).*

Proof. It follows from Proposition 3 that the operator $T: B_M \rightarrow B_M$ is well defined, continuous and compact. By the Schauder fixed point Theorem, see [9], the operator T has a fixed point y satisfying

$$T(y)(\theta) = y(\theta) = \frac{\rho}{1 - (n-1)\rho^{n-1} \int_{-\pi}^{\theta} (a(s) + b(s)y(s)^{\ell}) ds}$$

and $y(-\pi) = \rho^{n-1}$ for each ρ satisfying (9). From (7) there exists $r(\theta; \rho)$ such that

$$r(\theta; \rho) = \frac{\rho}{\left(1 - (n-1)\rho^{n-1} \int_{-\pi}^{\theta} (a(s) + b(s)r(s; \rho)^{m-n}) ds\right)^{\frac{1}{n-1}}}$$

and $r(-\pi; \rho) = \rho$. Note that $r(-\theta; \rho) = r(\theta; \rho)$ and so the solution is closed and even. In short, there are many closed even solutions of system (1) near the origin. \square

Proof of Theorem 1. To prove Theorem 1 we first show that if $\bar{r}(\theta; \rho)$ is a solution of equation (1) that satisfies $\bar{r}(-\pi; \rho) = \rho$ with ρ satisfying (9), then $\bar{r}(\theta; \rho)$ is closed and even. Indeed, by Proposition 4 there is $r(\theta; \rho)$ a closed even solution of system (1) such that $r(-\pi; \rho) = \rho$, and by the uniqueness of solutions of an ordinary differential equation, we obtain that $\bar{r}(\theta; \rho) = r(\theta; \rho)$. Hence if a and b are odd functions in the variable θ , then each solution of equation (1) with initial condition ρ satisfying (9) is a closed even solution. Hence, for any ρ satisfying (9) we have

$$\int_{-\pi}^{\pi} a(s) ds = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} b(s)r(s; \rho)^{m-n} ds = 0.$$

Therefore it follows from Proposition 2 that $r = 0$ is a center for equation (1). \square

ACKNOWLEDGEMENTS

The first author is partially supported by a FEDER-MINECO grant MTM2016-77278-P, a MINECO grant MTM2013-40998-P, and an AGAUR grant number 2014SGR-568. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

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