This is a preprint of: "The centers and their cyclicity for a class of polynomial differential systems of degree 7", Rebiha Benterki, Jaume Llibre, *J. Comput. Appl. Math.*, vol. 368(112456), 1–16, 2020.
DOI: [10.1016/j.cam.2019.112456]

THE CENTERS AND THEIR CYCLICITY FOR A CLASS OF POLYNOMIAL DIFFERENTIAL SYSTEMS OF DEGREE 7

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Abstract. We classify the global phase portraits in the Poincaré disc of the generalized Kukles systems

$$\dot{x} = -y, \qquad \dot{y} = x + axy^6 + bx^3y^4 + cx^5y^2 + dx^7,$$

which are symmetric with respect to both axes of coordinates. Moreover using the averaging theory up to sixth order, we study the cyclicity of the center located at the origin of coordinates, i.e. how many limit cycles can bifurcate from the origin of coordinates of the previous differential system when we perturb it inside the class of all polynomial differential systems of degree 7.

1. Introduction and statement of the main results

Two classical and difficult problems of the qualitative theory of planar polynomial differential systems are the characterization of their centers, and the study of their cyclicity, i.e. how many limit cycles can bifurcate from a center when we perturb it inside a given class of polynomial differential systems. Of course, this kind of bifurcation is called in the literature a Hopf bifurcation.

In this work we deal with planar polynomial differential systems of the form

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{1}$$

having a center at the origin, being $Q_n(x, y)$ a homogeneous polynomial of degree n. As usual the dot in system (1) denotes derivative with respect an independent variable t usually called the time. Systems of this form were called by Giné [5] *Kukles homogeneous systems*.

In 1999 Volokitin and Ivanov [16] conjectured that the systems (1) have a center at the origin if and only if they are symmetric with respect to one of the coordinate axes. For $n = 2$ and $n = 3$, the authors of the conjecture knew that it holds. Giné [5] in 2002 proved the conjecture for $n = 4$ and $n = 5$. Giné et al. [6, 7] proved the conjecture for all n under an additional assumption, that the authors believe that it is redundant.

In this work we consider the class of polynomial differential systems (1) for $n = 7$ which are symmetric with respect to both coordinate axes, i.e.

$$\dot{x} = -y, \quad \dot{y} = x + axy^6 + bx^3y^4 + cx^5y^2 + dx^7. \tag{2}$$

2010 *Mathematics Subject Classification.* Primary 34C15, 34C25.
Key words and phrases. center, phase portrait, cyclicity, limit cycle, Hopf bifurcation, averaging method, Kukles.

We note that if we look for the systems (1) which are symmetric with respect to one of the coordinate axes, we obtain that they are symmetric with respect to both coordinate axes.

The first main objective of this work is to classify the phase portraits of the polynomial differential systems (2) in the Poincaré disc. For more details on the Poincaré disc see for instance the Chapter 5 of [4].

The phase portraits of the centers of systems (1) with $n = 2$, are known because are known the phase portraits of all the center of quadratic polynomial differential systems, see Vulpe [17]. The phase portraits of cubic polynomial differential systems with a symmetry with respect to a straight line are also known and in particular those of system (1) with $n = 3$, see Buzzi et al. [3], see also Malkin [13]; Vulpe Sibirskii [18] and Żołądek [19, 20]. The phase portraits of systems (1) with $n = 4$ follows from Benterki and Llibre [2] and Llibre and Salhi [9]. In Llibre and Silva [10, 11] classified the phase portraits of the systems (1) with $n = 5, 6$.

In order to present the classification of the phase portraits of systems (2) we write the homogeneous polynomial $axy^6 + bx^3y^4 + cx^5y^2 + dx^7$ of degree 7 which appears in system (2) into the form

$$p(x, y) = x(dx^6 + cx^4y^2 + bx^2y^4 + ay^6) = x(dX^3 + cX^2Y + bXY^2 + aY^3),$$

where $X = x^2$ and $Y = y^2$. Then, according with the different kind of roots of the polynomial $dX^3 + cX^2Y + bXY^2 + aY^3$ we can consider the following 62 cases for the polynomial $p(x, y)$. For the next 14 cases the polynomial $p(x, y)$ is $ax(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2)$ with
(1) $a > 0$ and $0 < r_1 < r_2 < r_3$,
(2) $a > 0$ and $r_1 = 0 < r_2 < r_3$,
(3) $a > 0$ and $r_1 < 0 < r_2 < r_3$,
(4) $a > 0$ and $r_1 < r_2 = 0 < r_3$,
(5) $a > 0$ and $r_1 < r_2 < 0 < r_3$,
(6) $a > 0$ and $r_1 < r_2 < r_3 = 0$,
(7) $a > 0$ and $r_1 < r_2 < r_3 < 0$,
(8) $a < 0$ and $0 < r_1 < r_2 < r_3$,
(9) $a < 0$ and $r_1 = 0 < r_2 < r_3$,
(10) $a < 0$ and $r_1 < 0 < r_2 < r_3$,
(11) $a < 0$ and $r_1 < r_2 = 0 < r_3$,
(12) $a < 0$ and $r_1 < r_2 < 0 < r_3$,
(13) $a < 0$ and $r_1 < r_2 < r_3 = 0$,
(14) $a < 0$ and $r_1 < r_2 < r_3 < 0$.
For the next 10 cases the polynomial $p(x, y)$ is $ax(y^2 - r_1x^2)(y^2 - r_2x^2)^2$ with
(15) $a > 0$ and $0 < r_1 < r_2$,
(16) $a > 0$ and $r_1 = 0 < r_2$,
(17) $a > 0$ and $r_1 < 0 < r_2$,
(18) $a > 0$ and $r_1 < r_2 = 0$,
(19) $a > 0$ and $r_1 < r_2 < 0$,
(20) $a < 0$ and $0 < r_1 < r_2$,
(21) $a < 0$ and $r_1 = 0 < r_2$,
(22) $a < 0$ and $r_1 < 0 < r_2$,
(23) $a < 0$ and $r_1 < r_2 = 0$,

(24) $a < 0$ and $r_1 < r_2 < 0$.
For the next 6 cases the polynomial $p(x, y)$ is $ax(y^2 - r_1x^2)^3$ with
(25) $a > 0$ and $0 < r_1$,
(26) $a > 0$ and $r_1 = 0$,
(27) $a > 0$ and $r_1 < 0$,
(28) $a < 0$ and $0 < r_1$,
(29) $a < 0$ and $r_1 = 0$,
(30) $a < 0$ and $r_1 < 0$.
For the next 6 cases the polynomial $p(x, y)$ is $ax(y^2-r_1x^2)(y^4-2\alpha x^2y^2+(\alpha^2+\beta^2)x^4)$ with
(31) $a > 0$ and $0 < r_1$,
(32) $a > 0$ and $r_1 = 0$,
(33) $a > 0$ and $r_1 < 0$,
(34) $a < 0$ and $0 < r_1$,
(35) $a < 0$ and $r_1 = 0$,
(36) $a < 0$ and $r_1 < 0$.
For the next 10 cases the polynomial $p(x, y)$ is $bx^3(y^2 - r_1x^2)(y^2 - r_2x^2)$ with
(37) $b > 0$ and $0 < r_1 < r_2$,
(38) $b > 0$ and $r_1 = 0 < r_2$,
(39) $b > 0$ and $r_1 < 0 < r_2$,
(40) $b > 0$ and $r_1 < r_2 = 0$,
(41) $b > 0$ and $r_1 < r_2 < 0$,
(42) $b < 0$ and $0 < r_1 < r_2$,
(43) $b < 0$ and $r_1 = 0 < r_2$,
(44) $b < 0$ and $r_1 < 0 < r_2$,
(45) $b < 0$ and $r_1 < r_2 = 0$,
(46) $b < 0$ and $r_1 < r_2 < 0$.
For the next 6 cases the polynomial $p(x, y)$ is $bx^3(y^2 - r_1x^2)^2$ with
(47) $b > 0$ and $0 < r_1$,
(48) $b > 0$ and $r_1 = 0$,
(49) $b > 0$ and $r_1 < 0$,
(50) $b < 0$ and $0 < r_1$,
(51) $b < 0$ and $r_1 = 0$,
(52) $b < 0$ and $r_1 < 0$.
For the next 2 cases the polynomial $p(x, y)$ is $bx^3(y^4 - 2\alpha x^2y^2 + (\alpha^2 + \beta^2)x^4)$ with
(53) $b > 0$,
(54) $b < 0$.
For the next 6 cases the polynomial $p(x, y)$ is $cx^5(y^2 - r_1x^2)$ with
(55) $c > 0$ and $0 < r_1$,
(56) $c > 0$ and $r_1 = 0$,
(57) $c > 0$ and $r_1 < 0$,
(58) $c < 0$ and $0 < r_1$,
(59) $c < 0$ and $r_1 = 0$,
(60) $c < 0$ and $r_1 < 0$.
For the next 2 cases the polynomial $p(x, y)$ is dx^7 with
(61) $d > 0$,
(62) $d < 0$.

For a definition of topological equivalence between two phase portraits in the Poincaré disc see subsections 2.1 and 2.2.

Theorem 1. *The polynomial differential systems* (2) *have* 25 *topologically non–equivalent phase portraits in the Poincaré disc. More precisely, the phase portrait in the Poincaré disc of Figure*

1 *is realizable by the case* (1)*;*
2 *is realizable by the case* (2)*;*
3 *is realizable by the cases* (3) *and* (37)*;*
4 *is realizable by the case* (4)*;*
5 *is realizable by the cases* (5) *and* (39)*;*
6 *is realizable by the cases* (6), (18), (26), (32) *and* (40)*;*
7 *is realizable by the cases* (7), (19), (27), (33), (41), (49) *and* (53)*;*
8 *is realizable by the case* (8)*;*
9 *is realizable by the case* (9)*;*
10 *is realizable by the cases* (10) *and* (42)*;*
11 *is realizable by the cases* (11) *and* (43)*;*
12 *is realizable by the cases* (12), (28), (34), (44), (50) *and* (58)*;*
13 *is realizable by the cases* (13), (23), (29), (35), (45), (51) *and* (59)*;*
14 *is realizable by the cases* (14), (24), (30), (36), (46), (52), (54), (60) *and* (62)*;*
15 *is realizable by the case* (15)*;*
16 *is realizable by the case* (16)*;*
17 *is realizable by the case* (17)*;*
18 *is realizable by the case* (20)*;*
19 *is realizable by the case* (21)*;*
20 *is realizable by the case* (22)*;*
21 *is realizable by the cases* (25) *and* (31)*;*
22 *is realizable by the case* (38)*;*
23 *is realizable by the cases* (47) *and* (55)*;*
24 *is realizable by the cases* (48) *and* (56)*;*
25 *is realizable by the cases* (57) *and* (61).

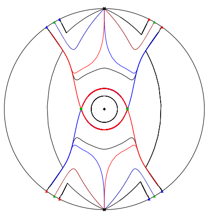


FIGURE 1

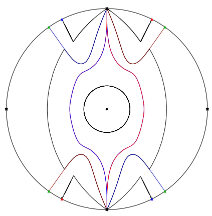


FIGURE 2

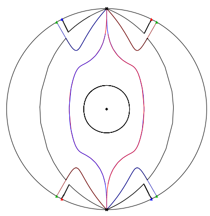
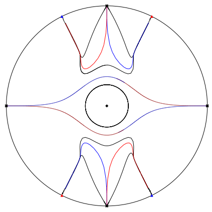


FIGURE 3

Theorem 1 is proved in section 3.

The averaging theory described in subsection 2.3 allows to study analytically the existence of limit cycles of a non–autonomous differential system, by studying the simple zeros of the averaged function $f_k = f_k(r)$. Here we shall use the



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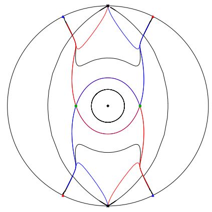

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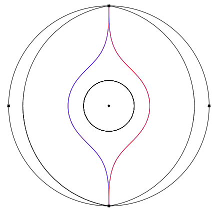
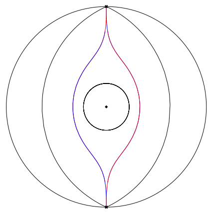


FIGURE 6



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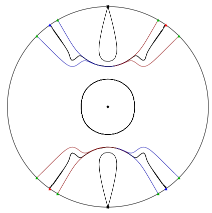

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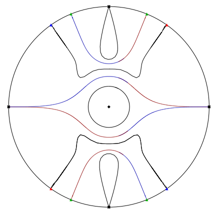

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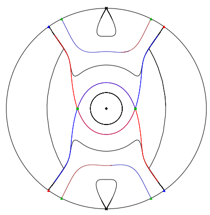

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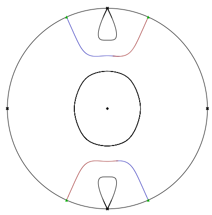

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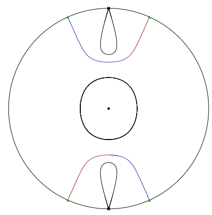


FIGURE 12

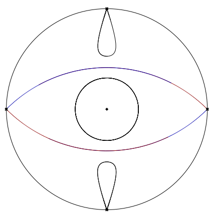


FIGURE 13

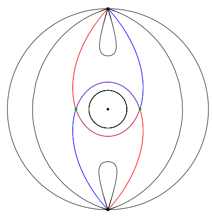


FIGURE 14

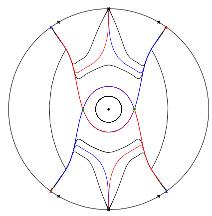


FIGURE 15

averaging theory up to sixth order for studying the number of limit cycles which can bifurcate from the center of system (2) when we perturb it inside the class of all polynomial differential systems of degree 7. More precisely, we deal with the

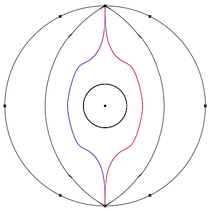
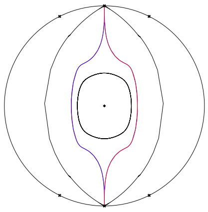


FIGURE 16



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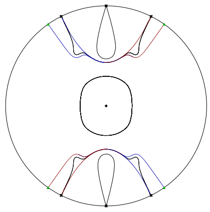

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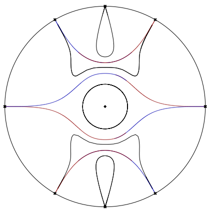


FIGURE 19

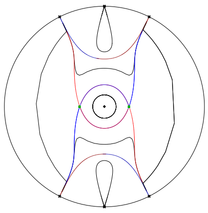


FIGURE 20

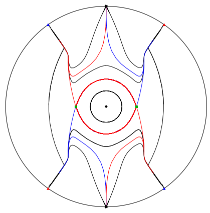


FIGURE 21

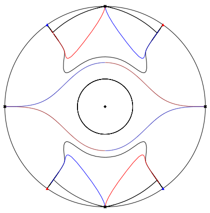
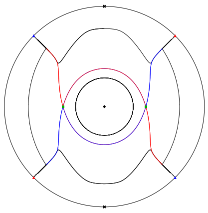


FIGURE 22



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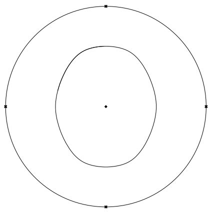

FIGURE 24

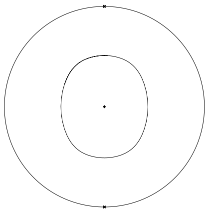

FIGURE 25

polynomial differential systems

$$
\begin{aligned}
\dot{x} &= -y + \sum_{s=1}^{6} \varepsilon^s \sum_{0\le i+j\le 7} a_{ij}^s x^i y^j, \\
\dot{y} &= x + axy^6 + bx^3y^4 + cx^5y^2 + dx^7 + \sum_{s=1}^{6} \varepsilon^s \sum_{0\le i+j\le 7} b_{ij}^s x^i y^j,
\end{aligned} \tag{3}
$$

where a_{ij}^s and b_{ij}^s are real parameters, for $0 \le i, j \le 7$ and $1 \le s \le 6$.

Theorem 2. *For $|\varepsilon| \neq 0$ sufficiently small the maximum number of small amplitude limit cycles of the differential system* (3) *bifurcating from the periodic solutions of the center* (2) *is*

(a) 0 *if the first order average function f_1 is non-zero,*
(b) 0 *if $f_1 = 0$ and the second order average function f_2 is non-zero,*
(c) 1 *if $f_1 = f_2 = 0$ and the third order average function f_3 is non-zero,*
(d) 1 *if $f_1 = f_2 = f_3 = 0$ and the fourth order average function f_4 is non-zero,*
(e) 2 *if $f_1 = f_2 = f_3 = f_4 = 0$ and the fifth order average function f_5 is non-zero,*
(f) 2 *if $f_1 = f_2 = f_3 = f_4 = f_5 = 0$ and the sixth order average function f_6 is non-zero.*

Moreover, assume that $f_j = 0$ for $j = 1, \ldots, k-1$ and $f_k \neq 0$. Then if $\overline{r}$ is a simple zero of f_k, the small amplitude limit cycle $(x(t, \varepsilon), y(t, \varepsilon))$ associated to this zero is of the form $(x(t, \varepsilon), y(t, \varepsilon)) = \varepsilon(\overline{r} \cos t, \overline{r} \sin t) + O(\varepsilon^2)$.

Theorem 2 is proved in section 4.

2. Preliminaries and basic results

In this section we present some basic results and notations which are necessary for proving our results.

2.1. **Poincaré compactification.** Let $X = (-y, x+axy^6+bx^3y^4+cx^5y^2+dy^7)$ be the planar polynomial vector field associated to system (2). We define the *Poincaré compactified vector field* $p(X)$ associated to X as follows (see all the details for instance [11] or Chapter 5 of [4]).

The *Poincaré sphere* is defined as $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ and its tangent space at the point $y \in \mathbb{S}^2$ is denoted by $T_y\mathbb{S}^2$. We identify the plane $\mathbb{R}^2$ where we have our vector field X with the plane $T_{(0,0,1)}\mathbb{S}^2$. We define the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ as follows: to each point $q \in T_{(0,0,1)}\mathbb{S}^2$ the central projection associates the two intersection points of the straight line which connects the points q and $(0, 0, 0)$ with the sphere $\mathbb{S}^2$. This central projection gives two copies of X in $\mathbb{S}^2$, one in each hemisphere. Let X' be the vector field $Df \circ \mathcal{X}$, which is defined in $\mathbb{S}^2$ minus its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. The equator $\mathbb{S}^1$ can be identified with the infinity of $\mathbb{R}^2$. We extend the vector field X' on $\mathbb{S}^2 \setminus \mathbb{S}^1$ to a vector field $p(X)$ on $\mathbb{S}^2$ as follows: $p(X)$ is the unique analytic extension of $y_3^7\mathcal{X}'$ to $\mathbb{S}^2$. In summary, we have two symmetric copies of X on $\mathbb{S}^2 \setminus \mathbb{S}^1$, and studying the dynamics of $p(X)$ near $\mathbb{S}^1$, we have the dynamics of X at infinity. The *Poincaré disc*, denoted by $\mathbb{D}^2$, is the closed northern hemisphere of $\{y \in \mathbb{S}^2 : y_3 \geq 0\}$ projected on $y_3 = 0$ under the projection $(y_1, y_2, y_3) \mapsto (y_1, y_2)$.

The infinity $\mathbb{S}^1$ is invariant under the flow of the Poincaré compactifcation $p(X)$. Here two polynomial vector fields X and Y associated to systems (1) are *topologically equivalent* if there is a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow of $p(X)$ into orbits of the flow of $p(Y)$, either reversing or preserving the sense of all orbits.

For computing the analytic expression of $p(X)$ we use the fact that $\mathbb{S}^2$ is a differentiable manifold. Thus we take the six local charts $U_i = \{y_2 \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y_2 \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1, 2, 3$; and the associated diffeomorphisms $F_i : U_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1, 2, 3$ are respectively the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$; $(-1, 0, 0)$; $(0, 1, 0)$; $(0, -1, 0)$; $(0, 0, 1)$ and $(0, 0, -1)$. The value of $F_i(y)$ or $G_i(y)$ for some $i = 1, 2, 3$ is denoted by $z = (z_1, z_2)$, consequently according to the local charts under consideration the same letter z represents different coordinates.

After a rescaling in the independent variable in the local chart (U_1, F_1) the expression for $p(X)$ is

$$\dot{u} = v^n \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right], \qquad \dot{v} = -v^{n+1}P\left(\frac{1}{v}, \frac{u}{v}\right);$$

in the local chart (U_2, F_2) the expression for $p(X)$ is

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right] \qquad \dot{v} = -v^{n+1}Q\left(\frac{u}{v}, \frac{1}{v}\right);$$

and for the local chart (U_3, F_3) the expression for $p(X)$ is

$$\dot{u} = P(u, v), \qquad \dot{v} = Q(u, v).$$

In the chart (V_i, G_i) the expression for $p(X)$ is the same than in the chart (U_i, F_i) multiplied by $(-1)^6$ for $i = 1, 2, 3$. We note that the points at the infinity $\mathbb{S}^1$ in any chart have coordinates $(u, v) = (u, 0)$.

The equilibrium points of $p(X)$ which come from the equilibrium points of X are called *finite equilibrium points* of X, and the equilibrium points of $p(X)$ which are in $\mathbb{S}^1$ are called *infinite equilibrium points* of X.

We observe that the unique infinite equilibrium points which cannot be contained in the charts $U_1 \cup V_1$ are the origins of the local charts U_2 and V_2. Therefore when we study the infinite equilibrium points on the charts $U_2 \cup V_2$, we only need to verify if the origin of these charts are equilibrium points.

2.2. **Topological equivalence.** Two polynomial vector fields X and Y on $\mathbb{R}^2$ are *topologically equivalent* if there is a homeomorphism on the Poincaré sphere $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying trajectories of the flow of $p(X)$ into trajectories of the flow of $p(Y)$, either preserving or reversing the sense of all trajectories.

Here a *separatrix* of the Poincaré compactification $p(X)$ is a trajectory which is either an equilibrium point, or a limit cycle, or a trajectory which belongs to the boundary of a hyperbolic sector at an equilibrium point, finite or infinity, or any trajectory contained at the infinity $\mathbb{S}^1$. We denote by $S(p(X))$ the set formed by all separatrices of $p(X)$. It is known that the set $S(p(X))$ is closed, see for instance Neumann [14].

A *canonical region* of $p(X)$ is an open connected component of $\mathbb{S}^2 \setminus S(p(X))$. The union of $S(p(X))$ plus one trajectory chosen from each canonical region is the *separatrix configuration* of $p(X)$. Two separatrix configurations $S(p(X))$ and $S(p(Y))$ are *equivalent* if there is a homeomorphism in $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying trajectories of $S(p(X))$ into trajectories of $S(p(Y))$, either preserving or reversing the sense of all orbits.

Markus [12], Neumann [14] and Peixoto [15] characterized the topologically equivalence between two Poincaré compactified vector fields as follows.

Theorem 3. *Assume that two Poincaré compactified polynomial vector fields $p(X)$ and $p(Y)$ have finitely many separatrices. Then their separatrix configurations are equivalent if and only if $S(p(X))$ and $S(p(Y))$ are topologically equivalent.*

Theorem 3 says that essentially we need to determine the α– and the ω–limit sets of all the separatrices of $p(X)$ for obtaining the phase portrait of a Poincaré compactified polynomial vector field $p(X)$ with finitely many separatrices.

2.3. **The averaging theory up to order 6.** In this subsection we present some results on the averaging theory that we shall need for studying the limit cycles which bifurcate from the center localized at the origin of coordinates of the systems (2), when they are perturbed inside the class of all polynomial differential systems of degree 7.

We deal with a non–autonomous differential system

$$\dot{x}(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{4}$$

where the functions $F_i : \mathbb{R}\times D \to \mathbb{R}$ for $i = 0, 1, \cdots, k$ and $R : \mathbb{R}\times D\times(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ are T-periodic in the first variable and continuous functions, D is an open interval of $\mathbb{R}$, and ε a small parameter. Following the results of [8] we define the functions $y_j(t,z)$ for $j = 1, 2, 3, 4, 5$ related to system (13) as

$$\begin{aligned}
y_1(t,z) = & \int_0^t F_1(s,z)ds,\\
y_2(t,z) = & \int_0^t \Big(2F_2(s,z) + 2\partial F_1(s,z)y_1(s,z)\Big)ds,\\
y_3(t,z) = & \int_0^t \Big(6F_3(s,z) + 6\partial F_2(s,z)y_1(t,z)\\
& +3\partial^2 F_1(s,z)y_1(s,z)^2 + 3\partial F_1(s,z)\, y_2(s,z)\Big)ds,\\
y_4(t,z) = & \int_0^t \Big(24F_4(s,z) + 24\partial F_3(s,z)y_1(s,z)\\
& +12\partial^2 F_2(s,z)y_1(s,z)^2 + 12\partial F_2(s,z)y_2(s,z)\\
& +12\partial^2 F_1(s,z)y_1(s,z)y_2(s,z)\\
& +4\partial^3 F_1(s,z)y_1(s,z)^3 + 4\partial F_1(s,z)y_3(s,z)\Big)ds,\\
y_5(t,z) = & \int_0^t \Big(120F_5(s,z) + 120\partial F_4(s,z)y_1(s,z)\\
& +60\partial^2 F_3(s,z)y_1(s,z)^2 + 60\partial F_3(s,z)y_2(s,z)\\
& +60\partial^2 F_2(s,z)y_1(s,z)y_2(s,z) + 20\partial^3 F_2(s,z)y_1(s,z)^3\\
& +20\partial F_2(s,z)y_3(s,z) + 20\partial^2 F_1(s,z)y_1(s,z)y_3(s,z)\\
& +15\partial^2 F_1(s,z)y_2(s,z)^2 + 30\partial^3 F_1(s,z)y_1(s,z)^2 y_2(s,z)\\
& +5\partial^4 F_1(s,z)y_1(s,z)^4 + 5\partial F_1(s,z)y_4(s,z)\Big)ds,
\end{aligned}$$

where the k–th partial derivative of the function $F_\ell(s,z)$ with respect to the variable z has been denoted by $\partial^k F_\ell(s,z)$. From [8] the functions averaged functions $f_j(z)$ for $j=1,\ldots,6$ are

$$
\begin{aligned}
f_1(z)= &\int_0^T F_1(t,z)dt,\\
f_2(z)= &\int_0^T \Big(F_2(t,z)+\partial F_1(t,z)y_1(t,z)\Big)dt,\\
f_3(z)= &\int_0^T \Big(F_3(t,z)+\partial F_2(t,z)y_1(t,z)\\
&+\frac{1}{2}\partial^2 F_1(t,z)y_1(t,z)^2+\frac{1}{2}\partial F_1(t,z)y_2(t,z)\Big)dt,\\
f_4(z)= &\int_0^T \Big(F_4(t,z)+\partial F_3(t,z)y_1(t,z)\\
&+\frac{1}{2}\partial^2 F_2(t,z)y_1(t,z)^2+\frac{1}{2}\partial F_2(t,z)y_2(t,z)\\
&+\frac{1}{2}\partial^2 F_1(t,z)y_1(t,z)y_2(t,z)dt+\frac{1}{6}\partial^3 F_1(t,z)y_1(t,z)^3\\
&+\frac{1}{6}\partial F_1(t,z)y_3(t,z)\Big)dt,\\
f_5(z)= &\int_0^T \Big(F_5(t,z)+\partial F_4(t,z)y_1(t,z)+\frac{1}{2}\partial^2 F_3(t,z)y_1(t,z)^2\\
&+\frac{1}{2}\partial F_3(t,z)y_2(t,z)+\frac{1}{2}\partial^2 F_2(t,z)y_1(t,z)y_2(t,z)\\
&+\frac{1}{6}\partial^3 F_2(t,z)y_1(t,z)^3+\frac{1}{6}\partial F_2(t,z)y_3(t,z)\\
&+\frac{1}{6}\partial^2 F_1(t,z)y_1(t,z)y_3(t,z)+\frac{1}{8}\partial^2 F_1(t,z)y_2(t,z)^2\\
&+\frac{1}{4}\partial^3 F_1(t,z)y_1(t,z)^2y_2(t,z)+\frac{1}{24}\partial^4 F_1(t,z)y_1(t,z)^4\\
&+\frac{1}{24}\partial F_1(t,z)y_4(t,z)\Big)dt,\\
f_6(z)= &\int_0^T \Big(F_6(t,z)+\partial F_5(t,z)y_1(t,z)+\frac{1}{2}\partial F_4(t,z)y_2(t,z)\\
&+\frac{1}{2}\partial^2 F_4(t,z)y_1(t,z)^2+\frac{1}{6}\partial F_3(t,z)y_3(t,z)\\
&+\frac{1}{2}\partial^2 F_3(t,z)y_1(t,z)y_2(t,z)+\frac{1}{6}\partial^3 F_3(t,z)y_1(t,z)^3\\
&+\frac{1}{24}\partial F_2(t,z)y_4(t,z)+\frac{1}{6}\partial^2 F_2(t,z)y_1(t,z)y_3(t,z)\\
&+\frac{1}{4}\partial^3 F_2(t,z)y_1(t,z)^2y_2(t,z)+\frac{1}{8}\partial^2 F_2(t,z)y_2(t,z)^2\\
&+\frac{1}{24}\partial^4 F_2(t,z)y_1(t,z)^4+\frac{1}{120}\partial F_1(t,z)y_5(t,z)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{24}\partial^2 F_1(t,z)y_1(t,z)y_4(t,z)+\frac{1}{12}\partial^2 F_1(t,z)y_2(t,z)y_3(t,z)\\
&+\frac{1}{12}\partial^3 F_1(t,z)y_1(t,z)^2y_3(t,z)+\frac{1}{12}\partial^4 F_2(t,z)y_1(t,z)^3y_2(t,z)\\
&+\frac{1}{8}\partial^3 F_1(t,z)y_1(t,z)y_2(t,z)^2+\frac{1}{120}\partial^5 F_1(t,z)y_1(t,z)^5\Big)dt.
\end{aligned}
$$

The averaging theory for studying the periodic solutions of a non–autonomous differential system (13) works as follows, see [8] for more details. Suppose that the average functions $f_j(z)=0$ for $j=1,\ldots,k-1$ and $f_k(z)\neq 0$ for some $k\geq 1$, we assume that $f_0(z)=0$. By [8] if $\overline{z}$ is a simple zero of $f_k(z)$, then there is a limit cycle $r(\theta,\varepsilon)$ of system (13) such that $r(0,\varepsilon)=\overline{z}+O(\varepsilon)$.

3. Phase portraits in the Poincaré disc

Now we shall study the phase portraits of the Poincaré compactified polynomial differential systems (2) with $(a,b,c,d)\neq(0,0,0,0)$.

Remark 4. *The polynomial differential systems* (2) *are reversible because they remain the same under the transformations* $(x,y,t)\to(x,-y,-t)$ *and* $(x,y,t)\to(-x,y,-t)$. *Therefore their phase portraits are symmetric with respect to the x–axis and y–axis.*

We shall study the phase portrait of a polynomial differential system (2) in the Poincaré disc as follows. First we shall determine the local phase portrait at all its finite and infinite equilibrium points. After with the help of the symmetries of its trajectories with respect to both coordinate axes, we shall determine its phase portrait in the Poincaré disc.

Here we classify an equilibrium point p as *hyperbolic* when the eigenvalues of the linear part of system (2) at p have nonzero real part, as *semi–hyperbolic* when only one of these two eigenvalues is zero, as *nilpotent* when both eigenvalues are zero but the linear part of system (2) at p is not identically zero, and finally as *linearly zero* when the linear part of system (2) at p is identically zero.

The local phase portraits of the hyperbolic, semi–hyperbolic and nilpotent equilibrium points can be determined using, for instance the Theorems 2.15, 2.19 and 3.5 of the book [4]. In order to determine the local phase portrait of a nilpotent equilibrium point at infinity it is not sufficient the mentioned Theorem 3.5, and we must studied it doing the changes of variables called blow–ups. These changes of variables are also necessary for analyze the local phase portraits of the linearly zero equilibrium points. For more information about the blow ups see Chapter 3 of [4] or [1].

3.1. **Finite equilibrium points.** For a planar polynomial differential system (2) its finite equilibrium points are characterized in the following result.

Proposition 5. *Always the origin of the polynomial differential system* (2) *is a center. Furthermore if $d<0$ then there are two additional equilibrium points, namely $(\pm|d|^{-1/6},0)$, which are hyperbolic saddles.*

Proof. Since the eigenvalues of the linear part of system (2) at the origin are $\pm i$ such equilibrium point is either a focus or a center, but due to the fact that the system is symmetric with respect to both coordinate axes, it is a center.

Clearly when $d < 0$ the system has the two equilibria $(\pm|d|^{-1/6}, 0)$, and the eigenvalues of the linear part at these equilibria are $\pm\sqrt{6}$. So these equilibria are hyperbolic, and by Theorem 2.15 of [4] they are saddles. □

3.2. **Infinite equilibrium points.** We shall use the notations and definitions given in subsection 2.2, for determining the local phase portraits at the infinite equilibrium points in the Poincaré disc.

System (2) **in the local chart** U_1**.** The differential system (2) in the local chart U_1 is

$$\dot{u} = d + cu^2 + bu^4 + au^6 + v^6 + u^2v^6, \qquad \dot{v} = uv^7. \tag{5}$$

An infinite equilibrium point of system (5) is a point $(u_0, 0)$ such that u_0 is a real root of the polynomial $d + cu^2 + bu^4 + au^6$. So the infinite equilibria of system (2) are $(\pm\sqrt{r_j}, 0)$ when $r_j > 0$, where the r_j's are the ones which appear in the polynomials of the cases (1) to (62) described in section 1.

The Jacobian matrix of system (5) evaluated at $(\pm\sqrt{r_j}, 0)$ is

$$\begin{pmatrix} \pm\big(2cr_1^{1/2} - 4br_1^{3/2} + 6ar_1^{5/2}\big) & 0 \\ 0 & 0 \end{pmatrix}.$$

Then all infinite equilibria of differential system (5) are semi–hyperbolic or linearly zero.

In the next proposition we only provide the local phase portrait of the infinite equilibrium point $(\sqrt{r_j}, 0)$ with $r_j \geq 0$, because due to the symmetry $(x, y, t) \to (x, -y, -t)$ of system (2) with respect to x–axis the local phase portrait at the infinite equilibrium point $(-\sqrt{r_j}, 0)$ is the same than at the equilibrium $(\sqrt{r_j}, 0)$ after reversing the sense of the trajectories.

Proposition 6. *The local phase portraits at the infinite equilibrium points* $(\sqrt{r_j}, 0)$ *with* $r_j \geq 0$ *of the local chart* U_1 *ordered from the smallest value of* r_j *to the biggest one are formed by*

(a) *a semi–hyperbolic unstable node, a semi–hyperbolic saddle and a semi–hyperbolic unstable node in case* (1)*;*
(b) *a linearly zero at the origin of coordinates with two hyperbolic sectors, a semi–hyperbolic saddle and a semi–hyperbolic unstable node in case* (2)*;*
(c) *a semi–hyperbolic saddle and a semi–hyperbolic unstable node in cases* (3) *and* (37)*;*
(d) *a linearly zero with six hyperbolic sectors (the infinity line separates them in two groups of three sectors) at the origin of coordinates and a semi–hyperbolic unstable node in cases* (4) *and* (38)*;*
(e) *a semi–hyperbolic unstable node in cases* (5)*,* (39)*,* (47) *and* (55)*;*
(f) *a linearly zero at the origin of coordinates with two hyperbolic sectors in cases* (6)*,* (18)*,* (26)*,* (32)*,* (40)*,* (48) *and* (56)*;*

(g) *no infinite equilibria in the local chart* U_1 *in cases* (7)*,* (14)*,* (24)*,* (27)*,* (30)*,* (33)*,* (36)*,* (41)*,* (46)*,* (49)*,* (52)*–*(54)*,* (57)*,* (60)*–*(62)*;*

(h) *a semi–hyperbolic saddle, a semi–hyperbolic unstable node and a semi–hyperbolic saddle in case* (8)*;*

(i) *a linearly zero with six hyperbolic sectors (the infinity line separates them in two groups of three sectors) at the origin of coordinates, a semi–hyperbolic unstable node and a semi–hyperbolic saddle in case* (9)*;*

(j) *a semi–hyperbolic unstable node and a semi–hyperbolic saddle in cases* (10) *and* (42)*;*

(k) *a linearly zero at the origin of coordinates with two hyperbolic sectors and a semi–hyperbolic saddle in cases* (11) *and* (43)*;*

(l) *a semi–hyperbolic saddle in cases* (12)*,* (28)*,* (34)*,* (44)*,* (50) *and* (58)*;*

(m) *a linearly zero at the origin of coordinates with six hyperbolic sectors (the infinity line separates them in two groups of three sectors) in cases* (13)*,* (23)*,* (29)*,* (35)*,* (45)*,* (51) *and* (59)*;*

(n) *a semi–hyperbolic unstable node and a linearly zero with two hyperbolic sectors in case* (15)*;*

(o) *two linearly zero with two hyperbolic sectors, one of this equilibria is located at the origin of coordinates, in case* (16)*;*

(p) *a linearly zero with two hyperbolic sectors in case* (17)*;*

(q) *a semi–hyperbolic saddle and a linearly zero with two hyperbolic sectors and one parabolic sector, the straight line at infinity separates the two hyperbolic sectors and divides the parabolic one in case* (20)*;*

(r) *a linearly zero with six hyperbolic sectors (the infinity line separates them in two groups of three sectors) at the origin of coordinates, and a linearly zero with two adjacent hyperbolic sectors separated and with a parabolic sector, the infinite line separates the two hyperbolic sectors and divides the parabolic one in case* (21)*;*

(s) *a linearly zero saddle–node, the infinite line separates the two hyperbolic sectors and divides the parabolic one in case* (22)*;*

(t) *a linearly zero unstable node in cases* (25) *and* (31)*;*

Proof. As we have mention near the beginning of this section the phase portraits of the semi–hyperbolic equilibrium can be determined using the Theorem 2.19 of [4], and the phase portraits of the linearly zero equilibrium points doing the blow–up changes of variables. Here we shall prove with all details the statements (a) and (b), the other statements are proved in a similar way.

In statements (a) and (b) system (1) writes as

$$\dot{x} = -y; \qquad \dot{y} = x + ax(y^2 - r_1x^2)(y^2 - r_2x^2)(y^2 - r_3x^2), \tag{6}$$

with $a > 0$. This system in the local chart U_1 becomes

$$\begin{aligned} \dot{u} = &\; -ar_1r_2r_3 + a(r_1r_2 + r_1r_3 + r_2r_3)u^2 \\ &\; -a(r_1 + r_2 + r_3)u^4 + au^6 + v^6 + u^2v^6, \\ \dot{v} = &\; uv^7. \end{aligned} \tag{7}$$

Assume $a > 0$ and $0 < r_1 < r_2 < r_3$. Then the eigenvalues at the infinite equilibrium point $(\sqrt{r_j}, 0)$ for $j = 1, 2, 3$ are 0 and $a\big(2r_j^{1/2}(r_1r_2 + r_1r_3 + r_2r_3) -$

$4r_j^{3/2}(r_1+r_2+r_3)+6r_j^{5/2}) = h'(\sqrt{r_j}) \neq 0$, because the six roots $\pm\sqrt{r_j}$ for $j = 1,2,3$ of the polynomial $h(u) = \dot{u}|_{v=0} = -ar_1r_2r_3 + a(r_1r_2 + r_1r_3 + r_2r_3)u^2 - a(r_1 + r_2 + r_3)u^4 + au^6$ are simple. Moreover $h'(\sqrt{r_1}) > 0$, $h'(\sqrt{r_2}) < 0$ and $h'(\sqrt{r_3}) > 0$. Hence the points $(\sqrt{r_j}, 0)$ for $j = 1,2,3$ are semi–hyperbolic equilibria.

In what follows we shall apply Theorem 2.19 of [4] for determining the local phase portraits of the infinite equilibria $(\sqrt{r_j}, 0)$ for $j = 1,2,3$, and we shall use the notation of that theorem. First we translate the equilibrium point $(\sqrt{r_j}, 0)$ to the origin of coordinates doing the change $(u, v) = (Y + \sqrt{r_j}, X)$. Thus we obtain the differential system

$$
\begin{aligned}
\dot{X} &= A(X,Y) = \operatorname{sign}(h'(\sqrt{r_j}))\big(\sqrt{r_j}X^7 + X^7Y\big),\\
\dot{Y} &= \operatorname{sign}(h'(\sqrt{r_j}))\big(h'(\sqrt{r_j})Y + B(X,Y)\big),
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
B(X,Y) = \;& a(r_1r_2 + r_1r_3 + r_2r_3 - 6r_1r_j - 6r_2r_j - 6r_3r_j + 15r_j^2)Y^2\\
& -4a(r_1 + r_2 + r_3 - 5r_j)\sqrt{r_j}Y^3 - a(r_1 + r_2 + r_3 - 15r_j)Y^4\\
& +6a\sqrt{r_j}Y^5 + aY^6 + (1 + r_j)X^6 + 2\sqrt{r_j}YX^6 + Y^2X^6.
\end{aligned}
$$

If $h'(\sqrt{r_j}) < 0$ we have changed the sign of the independent variable in the differential system (8) in order that the coefficient of Y in the expression of $\dot{Y}$ be positive as it is necessary in order to apply Theorem 2.19. Now the functions $f(X)$ and $g(X)$ of that theorem are

$$
\begin{aligned}
f(X) &= -\frac{1 + r_j}{h'(\sqrt{r_j})}X^6 + \text{h.o.t.},\\
g(X) &= \operatorname{sign}(h'(\sqrt{r_j}))\sqrt{r_j}X^7 + \text{h.o.t.},
\end{aligned}
$$

where h.o.t. denotes higher order terms. So using the notation of Theorem 2.19 we have that $a_m = \operatorname{sign}(h'(\sqrt{r_j}))\sqrt{r_j}$ and $m = 7$. Consequently, by Theorem 2.19 the equilibria $(\sqrt{r_j}, 0)$ for $j = 1,3$ are unstable nodes, and the equilibrium $(\sqrt{r_2}, 0)$ is a saddle. This completes the proof of statement (a).

Assume $a > 0$ and $r_1 = 0 < r_2 < r_3$. The proof that the equilibria $(\sqrt{r_2}, 0)$ and $(\sqrt{r_3}, 0)$ are a semi–hyperbolic saddle and a semi–hyperbolic unstable node respectively, is identical to the proof of statement (a). So in order to complete the proof of statement (b) we shall show that the local phase portrait of the equilibrium $(\sqrt{r_1}, 0) = (0,0)$ is formed by two hyperbolic sectors.

We do the blow–up change of variables $v = wu$ to system (7) and it becomes

$$
\begin{aligned}
\dot{u} = \;& ar_2r_3u^2 - a(r_2 + r_3)u^4 + au^6 + u^6w^6 + u^8w^6,\\
\dot{w} = \;& -ar_2r_3uw + ar_2u^3w + ar_3u^3w - au^5w - u^5w^7.
\end{aligned} \tag{9}
$$

Now we remove the common factor u of $\dot{u}$ and $\dot{w}$ doing to system (9) the rescaling $d\tau = udt$ in the independent variable, we get the differential system

$$
\begin{aligned}
u' = \;& ar_2r_3u - a(r_2 + r_3)u^3 + au^5 + u^5w^6 + u^7w^6,\\
w' = \;& -ar_2r_3w + ar_2u^2w + ar_3u^2w - au^4w - u^4w^7,
\end{aligned} \tag{10}
$$

where the prime denotes derivative with respect to the new independent variable τ. The unique equilibrium on the w–axis of system (10) is the origin of coordinates, which is a saddle whose four separatrices are contained in the two axes. Going back through the two changes of variables, first the change $d\tau = udt$ and after the change $v = wu$, and taking into account that in system (7) we have that $\dot{u}|_{u=0} = v^6$, we obtain that the local phase portrait at the origin of system (7) is formed by two hyperbolic sectors. This completes the proof of statement (b). □

System (2) **in the local chart** U_2**.** The differential system (2) in the local chart U_1 is

$$\begin{aligned}\dot{u} &= -au^2 - bu^4 - cu^6 - du^8 - v^6 - u^2v^6,\\ \dot{v} &= -auv - bu^3v - cu^5v - du^7v - uv^7.\end{aligned}$$

Therefore the origin $(0,0)$ always is an infinite equilibrium point, which is linearly zero. Recall that from subsection 2.1 in this local chart the unique infinite equilibrium point that we must study is its origin when it is an equilibrium point, as in the present case.

Proposition 7. *The local phase portrait at the origin of the local chart* U_2 *is formed by a linearly zero equilibrium point with*

(a) *two hyperbolic sectors separated by two parabolic sectors and the line of the infinity is contained into the two parabolic sectors in cases* (1)–(7), (15)–(19), (25)–(27), (31)–(33), (37)–(41) *and* (53)*;*
(b) *two elliptic sectors separated by the infinity in cases* (8)–(13), (20)–(23), (28)–(29), (34)–(35), (42)–(45), (50), (51), (58) *and* (59)*;*
(c) *two elliptic sectors separated by two parabolic sectors and the line of the infinity is contained into the two parabolic sectors in cases* (14), (24), (30), (36), (46), (52), (54), (60) *and* (62).
(d) *two hyperbolic sectors in cases* (47)–(49), (55)–(57) *and* (61).

Proof. We shall prove statement (a) in the case (1). Statement (a) in the other cases, and the other statements are proved in a similar way.

In the case (1) system (1) becomes system (6). This last system in the local chart U_2 writes as

$$\begin{aligned}\dot{u} = &\ -au^2 + a(r_1+r_2+r_3)u^4 - a(r_1r_2+r_1r_3+r_2r_3)u^6\\ &+ar_1r_2r_3u^8 - v^6 - u^2v^6,\\ \dot{v} = &\ -auv + a(r_1+r_2+r_3)u^3v - a(r_1r_2+r_1r_3+r_2r_3)u^5v\\ &+ar_1r_2r_3u^7v - uv^7.\end{aligned} \tag{11}$$

This system doing the blow–up change of variables $v = wu$ becomes

$$\begin{aligned}\dot{u} = &\ -au^2 + a(r_1+r_2+r_3)u^4 - a(r_1r_2+r_1r_3+r_2r_3)u^6\\ &+ar_1r_2r_3u^8 - u^6w^6 - u^8w^6,\\ \dot{w} = &\ u^5w^7.\end{aligned} \tag{12}$$

We eliminate the common factor u^2 of $\dot{u}$ and $\dot{w}$ in system (12) doing the rescaling of the independent variable $d\tau = u^2dt$, and we obtain the differential system

$$\begin{aligned}\dot{u} = &\ -a + a(r_1+r_2+r_3)u^2 - a(r_1r_2+r_1r_3+r_2r_3)u^4\\ &+ar_1r_2r_3u^6 - u^4w^6 - u^6w^6,\\ \dot{w} = &\ u^3w^7.\end{aligned}$$

This last system has no equilibria on the w–axis and on it we have that $\dot{u}|_{u=0} = -a < 0$. Then going back through the two changes of variables, first $d\tau = u^2 dt$ and second $v = wu$, and taking into account that in system (11) we have $\dot{u}|_{u=0} = -v^6$, we obtain that the local phase portrait at the origin of system (11) is formed by two hyperbolic sectors whose separatrices are tangent to the w–axis, one in $v > 0$ and the other in $v < 0$, separated by two parabolic sectors which contain locally the invariant u–axis, i.e. the infinite line. This completes the proof of case (1) of statement (a). $\square$

We note that in the proof of statement (b) doing blow–ups changes of variables we only obtain that the local phase portrait is formed by two elliptic sectors separated by two parabolic sectors and the line of the infinity is contained into the two parabolic sectors, but that when we shall consider the global phase portrait in the Poincaré disc these two local parabolic sectors are contained in two global elliptic sectors, for this reason in statement (b) we only said two elliptic sectors instead of two elliptic sectors separated by two parabolic sectors and the line of the infinity is contained into the two parabolic sectors.

Proof of Theorem 1. This proof follows taking into account the local phase portraits of the finite and infinite equilibria described in Propositions 5, 6 and 7, and taking into account first Theorem 3 (i.e. that for obtaining the global phase portrait in the Poincaré disc we essentially must determine the $\alpha-$ and ω–limit of all the separatrices of the system) and after that the global phase portrait of a system (2) is symmetric with respect both coordinate axes. $\square$

4. Proof of Theorem 2

For studying the limit cycles which bifurcate in a Hopf bifurcation from the center of the differential system (2) when it is perturbed inside the class of all polynomial differential systems of degree 7, see (3), we work as follows. First doing the scaling $x = \varepsilon X$, $y = \varepsilon Y$ we introduce a small parameter ε. Thus we obtain the differential system $(\dot{X}, \dot{Y})$. Now performing the polar change of coordinates $X = r\cos\theta$, $Y = r\sin\theta$, the differential system $(\dot{X}, \dot{Y})$ written in polar coordinates becomes a differential system $(\dot{r}, \dot{\theta})$. Taking as independent variable the angle θ the differential system $(\dot{r}, \dot{\theta})$ produces the differential equation $dr/d\theta$. Finally doing a Taylor expansion in the variable r at $r = 0$ and truncating at 6–th order in ε we obtain the differential equation

$$r' = \frac{dr}{d\theta} = \sum_{i=0}^{6} \varepsilon^i F_i(\theta, r) + O(\varepsilon^7). \tag{13}$$

The functions $F_i(\theta, r)$ $i = 1, \ldots, 6$ of the differential system (13) are analytic, and since the independent variable θ appears through the sinus and cosinus of θ, they are 2π–periodic. Hence the assumptions for applying the averaging theory described in subsection 2.3 are satisfied.

Now we shall study the limit cycles bifurcating from the center of system (2) when it is perturbed as in (3) following the steps described in subsection 2.3. We give only the expressions of functions $F_1(r, \theta)$ and $F_2(r, \theta)$. The explicit expressions of $F_i(r,\theta)$ for $i=3,\dots,6$ are very long, therefore we shall omit them here. Thus we have

$$\begin{aligned} F_1(r,\theta) = \quad & a^2_{00}\cos\theta + b^2_{00}\sin\theta + \frac{1}{2}r(b^1_{01}+a^1_{10}-b^1_{01}\cos 2\theta + a^1_{10}\cos 2\theta \\ & +b^1_{10}\sin 2\theta + a^1_{01}\sin 2\theta). \end{aligned}$$

and

$$\begin{aligned} F_2(r,\theta) = \quad & \frac{1}{r}(a^2_{00}\cos\theta + a^1_{10}r\cos^2\theta + b^2_{00}\sin\theta + b^1_{10}r\cos\theta\sin\theta + a^1_{01}r\cos\theta\sin\theta \\ & +b^1_{01}r\sin^2\theta)(-b^2_{00}\cos\theta - b^1_{10}r\cos^2\theta + a^2_{00}\sin\theta - b^1_{01}r\cos\theta\sin\theta \\ & +a^1_{01}r\sin^2\theta) + (a^3_{00}\cos\theta + a^2_{10}r\cos^2\theta + a^1_{20}r^2\cos^3\theta + b^3_{00}\sin\theta \\ & +a^2_{01}r\cos\theta\sin\theta + b^1_{20}r^2\cos^2\theta\sin\theta + a^1_{11}r^2\cos^2\theta\sin\theta + b^2_{01}r\sin^2\theta \\ & +b^1_{11}r^2\cos\theta\sin^2\theta + a^1_{02}r^2\cos\theta\sin^2\theta + b^1_{02}r^2\sin^3\theta + a^1_{10}r\cos\theta\sin\theta \\ & +b^2_{10}r\cos\theta\sin\theta). \end{aligned}$$

Using the formulas given in subsection 2.3 the averaged function of first order is

$$f_1(r) = (b^1_{01}+a^1_{10})r.$$

Since the polynomial $f_1(r)=0$ has no positive roots, the first average function does not give any information on the limit cycles that bifurcate from the center when we perturb it as in system (3). This proves statement (a).

Taking $b^1_{01}=-a^1_{10}$, we obtain $f_1(r)\equiv 0$. Therefore we can apply the averaging theory of second order. Thus the averaged function of second order is

$$f_2(r) = (b^2_{01}+a^2_{10})r.$$

Again the second averaged function does not provide information on the bifurcating limit cycles. This completes the proof of statement (b).

Now taking $b^2_{01}=-a^2_{10}$ we get $f_2(r)\equiv 0$, and applying the averaging theory of third order, we obtain the third averaged function

$$f_3(r) = -(b^1_{11}b^2_{00}-b^3_{01}+2b^2_{00}a^1_{20}-2b^1_{02}a^2_{00}-a^1_{11}a^2_{00}-a^3_{10})r+\frac{1}{4}(3b^1_{03}+b^1_{21}+a^1_{12}+3a^1_{30})r^3.$$

Since the polynomial $f_3(r)$ can have at most one positive real root, statement (c) of the theorem is proved.

For applying the averaging theory of fourth order we must have $f_3(r)\equiv 0$, so we take

$$a^3_{10} = b^1_{11}b^2_{00} - b^3_{01} + 2b^2_{00}a^1_{20} - 2b^1_{02}a^2_{00} - a^1_{11}a^2_{00}, \qquad a^1_{12} = -3b^1_{03} - b^1_{21} - 3a^1_{30}.$$

The averaged function of fourth order is

$$
\begin{aligned}
f_4(r) = \quad & -\frac{1}{4}r^3(b_{02}^1 b_{11}^1 + b_{11}^1 b_{20}^1 + b_{10}^1 b_{21}^1 - 3b_{03}^2 - b_{21}^2 + b_{21}^1 a_{01}^1 - 2b_{02}^1 a_{02}^1 \\
& -2b_{12}^1 a_{10}^1 - a_{02}^1 a_{11}^1 + 2b_{20}^1 a_{20}^1 - a_{11}^1 a_{20}^1 - 2a_{10}^1 a_{21}^1 + 3b_{10}^1 a_{30}^1 \\
& +3a_{01}^1 a_{30}^1 - a_{12}^2 - 3a_{30}^2) - r(-b_{10}^1 b_{11}^1 b_{00}^2 + b_{00}^2 b_{11}^2 + b_{11}^1 b_{00}^3 \\
& -b_{01}^4 + 2b_{02}^1 b_{00}^2 a_{10}^1 + b_{00}^2 a_{10}^1 a_{11}^1 - 2b_{10}^1 b_{00}^2 a_{20}^1 + 2b_{00}^3 a_{20}^1 \\
& -2b_{02}^1 a_{01}^1 a_{00}^2 - b_{11}^1 a_{10}^1 a_{00}^2 - a_{01}^1 a_{11}^1 a_{00}^2 - 2a_{10}^1 a_{20}^1 a_{00}^2 - a_{00}^2 a_{11}^2 \\
& -2b_{02}^2 a_{00}^2 + 2b_{00}^2 a_{20}^2 - 2b_{02}^1 a_{00}^3 - a_{11}^1 a_{00}^3 - a_{10}^4).
\end{aligned}
$$

From the expression of the polynomial $f_4(r)$ we get that it has at most one positive real root. Hence statement (d) of the theorem is proved. Doing

$$
\begin{aligned}
a_{10}^4 = \quad & -b_{10}^1 b_{11}^1 b_{00}^2 + b_{00}^2 b_{11}^2 + b_{11}^1 b_{00}^3 - b_{01}^4 + 2b_{02}^1 b_{00}^2 a_{10}^1 + b_{00}^2 a_{10}^1 a_{11}^1 \\
& +2b_{00}^3 a_{20}^1 - 2b_{02}^2 a_{00}^2 - 2b_{02}^1 a_{01}^1 a_{00}^2 - b_{11}^1 a_{10}^1 a_{00}^2 - a_{01}^1 a_{11}^1 a_{00}^2 \\
& -a_{00}^2 a_{11}^2 + 2b_{00}^2 a_{20}^2 - 2b_{02}^1 a_{00}^3 - a_{11}^1 a_{00}^3 - 2b_{10}^1 b_{00}^2 a_{20}^1 - 2a_{10}^1 a_{20}^1 a_{00}^2. \\
a_{12}^2 = \quad & b_{02}^1 b_{11}^1 + b_{11}^1 b_{20}^1 + b_{10}^1 b_{21}^1 - 3b_{03}^2 - b_{21}^2 + b_{21}^1 a_{01}^1 - 2b_{02}^1 a_{02}^1 - 3a_{30}^2 \\
& -2b_{12}^1 a_{10}^1 - a_{02}^1 a_{11}^1 + 2b_{20}^1 a_{20}^1 - a_{11}^1 a_{20}^1 - 2a_{10}^1 a_{21}^1 + 3a_{01}^1 a_{30}^1 \\
& +3b_{10}^1 a_{30}^1,
\end{aligned}
$$

we have that $f_4(r) \equiv 0$, and we can apply the averaging theory of order five, and the fifth averaged function is $f_5(r) = K_3 r^5 + K_2 r^3 + K_1 r$, where

$$
\begin{aligned}
K_1 = \quad & -((b_{10}^1)^2 b_{11}^1 b_{00}^2 - b_{21}^1 (b_{00}^2)^2 - b_{11}^1 b_{00}^2 b_{10}^2 - b_{10}^1 b_{00}^2 b_{11}^2 - b_{10}^1 b_{11}^1 b_{00}^3 + b_{11}^2 b_{00}^3 \\
& +b_{00}^2 b_{11}^3 + b_{11}^1 b_{00}^4 - 2b_{02}^1 b_{10}^1 b_{00}^2 a_{10}^1 + 2b_{00}^2 b_{02}^2 a_{10}^1 + 2b_{02}^1 b_{00}^3 a_{10}^1 \\
& +2b_{02}^1 b_{00}^2 a_{01}^1 a_{10}^1 + b_{11}^1 b_{00}^2 (a_{10}^1)^2 - b_{10}^1 b_{00}^2 a_{10}^1 a_{11}^1 + b_{00}^3 a_{10}^1 a_{11}^1 \\
& +b_{00}^2 a_{01}^1 a_{10}^1 a_{11}^1 + 2(b_{10}^1)^2 b_{00}^2 a_{20}^1 - 2b_{00}^2 b_{10}^2 a_{20}^1 - 2b_{10}^1 b_{00}^3 a_{20}^1 + 2b_{00}^4 a_{20}^1 \\
& +2b_{00}^2 (a_{10}^1)^2 a_{20}^1 - 3(b_{00}^2)^2 a_{30}^1 + 2b_{12}^1 b_{00}^2 a_{00}^2 - 2b_{02}^3 a_{00}^2 - 2b_{02}^2 a_{01}^1 a_{00}^2 \\
& -2b_{02}^1 (a_{01}^1)^2 a_{00}^2 + b_{10}^1 b_{11}^1 a_{10}^1 a_{00}^2 - b_{11}^2 a_{10}^1 a_{00}^2 - b_{11}^1 a_{01}^1 a_{10}^1 a_{00}^2 - a_{11}^2 a_{00}^3 \\
& -(a_{01}^1)^2 a_{11}^1 a_{00}^2 - (a_{10}^1)^2 a_{11}^1 a_{00}^2 + 2b_{10}^1 a_{10}^1 a_{20}^1 a_{00}^2 - 2a_{01}^1 a_{10}^1 a_{20}^1 a_{00}^2 \\
& +b_{21}^1 (a_{00}^2)^2 + 3a_{30}^1 (a_{00}^2)^2 - 2b_{02}^1 a_{00}^2 a_{01}^2 - a_{11}^1 a_{00}^2 a_{01}^2 + 2b_{02}^1 b_{00}^2 a_{10}^2 \\
& -b_{11}^1 a_{00}^2 a_{10}^2 - 2a_{20}^1 a_{00}^2 a_{10}^2 + b_{00}^2 a_{10}^1 a_{11}^2 - a_{01}^1 a_{00}^2 a_{11}^2 - 2b_{10}^1 b_{00}^2 a_{20}^2 \\
& +2b_{00}^3 a_{20}^2 - 2a_{10}^1 a_{00}^2 a_{20}^2 - 2b_{02}^2 a_{00}^3 - 2b_{02}^1 a_{01}^1 a_{00}^3 - b_{11}^1 a_{10}^1 a_{00}^3 - a_{01}^1 a_{11}^1 a_{00}^3 \\
& -a_{00}^2 a_{11}^3 + 2b_{00}^2 a_{20}^3 - a_{10}^5 - 2b_{02}^1 a_{00}^4 - a_{11}^1 a_{00}^4 - b_{01}^5 + b_{00}^2 a_{11}^1 a_{10}^2 \\
& -2b_{02}^1 (a_{10}^1)^2 a_{00}^2 - 2a_{10}^1 a_{20}^1 a_{00}^3 + 2b_{00}^2 a_{21}^1 a_{00}^2), \\
K_2 = \quad & \frac{-1}{4}(-b_{02}^1 b_{10}^1 b_{11}^1 - 2b_{10}^1 b_{11}^1 b_{20}^1 - (b_{10}^1)^2 b_{21}^1 + b_{11}^1 b_{02}^2 + b_{21}^1 b_{10}^2 + b_{02}^1 b_{11}^2 \\
& +b_{20}^1 b_{11}^2 + b_{11}^1 b_{20}^2 + b_{10}^1 b_{21}^2 - 3b_{03}^3 - b_{21}^3 - b_{11}^1 b_{20}^1 a_{01}^1 - b_{10}^1 b_{21}^1 a_{01}^1 \\
& -2b_{02}^2 a_{02}^1 - 2b_{02}^1 a_{01}^1 a_{02}^1 + 2(b_{02}^1)^2 a_{10}^1 + (b_{11}^1)^2 a_{10}^1 + 2b_{10}^1 b_{12}^1 a_{10}^1 \\
& -2b_{12}^2 a_{10}^1 - b_{11}^1 a_{02}^1 a_{10}^1 - a_{01}^1 a_{02}^1 a_{11}^1 - b_{02}^1 a_{10}^1 a_{11}^1 + b_{20}^1 a_{10}^1 a_{11}^1 \\
& -4b_{10}^1 b_{20}^1 a_{20}^1 + 2b_{20}^2 a_{20}^1 - 2b_{20}^1 a_{01}^1 a_{20}^1 + b_{11}^1 a_{10}^1 a_{20}^1 - 2a_{02}^1 a_{10}^1 a_{20}^1 \\
& +b_{10}^1 a_{11}^1 a_{20}^1 - 2a_{10}^1 (a_{20}^1)^2 + 2b_{10}^1 a_{10}^1 a_{21}^1 + 2b_{00}^2 a_{22}^1 - 3(b_{10}^1)^2 a_{30}^1 \\
& +3b_{10}^2 a_{30}^1 - 3b_{10}^1 a_{01}^1 a_{30}^1 + 12b_{00}^2 a_{40}^1 - 12b_{04}^1 a_{00}^2 - 2b_{22}^1 a_{00}^2 + b_{21}^1 a_{01}^2 \\
& 3a_{30}^1 a_{01}^2 - 2b_{02}^1 a_{02}^2 - a_{11}^1 a_{02}^2 - 2b_{12}^1 a_{10}^2 - 2a_{21}^1 a_{10}^2 - a_{02}^1 a_{11}^2 \\
& -a_{20}^1 a_{11}^2 + 2b_{20}^1 a_{20}^2 - a_{11}^1 a_{20}^2 - 2a_{10}^1 a_{21}^2 + 3b_{10}^1 a_{30}^2 - a_{10}^1 (a_{11}^1)^2 \\
& 3a_{01}^1 a_{30}^2 - a_{12}^3 - 3a_{30}^3 + b_{21}^2 a_{01}^1 + 2b_{02}^1 b_{20}^1 a_{10}^1), \\
K_3 = \quad & \frac{1}{8}(5b_{05}^1 + b_{23}^1 + a_{32}^1 + 5a_{50}^1).
\end{aligned}
$$

The rank of the Jacobian matrix of the function $\mathcal{K} = (K_1, K_2, K_3)$ with respect to the coefficient a_{ij}^s and b_{ij}^s which appear in their expressions is maximal, i.e. it is 3. Then the coefficients K_i for $i = 1, 2, 3$ which appear in th expression of $f_5(r)$ are linearly independent. By the roots of a quadratic polynomial in the variable r^2 it follows that $f_5(r)$ can have at most two positive real roots. Therefore statement (e) of the theorem is proved.

Imposing that $K_1 = 0$, $K_2 = 0$ and $K_3 = 0$ we obtain that $f_5(r) \equiv 0$, and applying the averaging theory of order six we obtain the sixth average function

$f_6(r) = r(C_1 + C_2 r^2 + C_3 r^4)$, where

$$
\begin{aligned}
C_1 = \; & \frac{1}{2}(2(b^1_{10})^3 b^1_{11} b^2_{00} - 4b^1_{02} b^1_{11}(b^2_{00})^2 - 2b^1_{11} b^1_{20}(b^2_{00})^2 - 4b^1_{10} b^1_{21}(b^2_{00})^2 - 4b^1_{10} b^1_{11} b^2_{00} b^2_{10} \\
& -2(b^1_{10})^2 b^2_{00} b^2_{11} + 2b^2_{00} b^2_{10} b^2_{11} + 2(b^2_{00})^2 b^2_{21} - 2(b^1_{10})^2 b^1_{11} b^3_{00} + 4b^1_{21} b^2_{00} b^3_{00} \\
& +2b^1_{10} b^2_{11} b^3_{00} + 4b^1_{02} b^2_{00} b^3_{01} + 2b^1_{11} b^2_{00} b^3_{10} + 2b^1_{10} b^2_{00} b^3_{11} - 2b^3_{00} b^3_{11} + 2b^1_{10} b^1_{11} b^4_{00} \\
& -2b^2_{00} e11 + 2b^6_{01} - 4b^1_{02}(b^1_{10})^2 b^2_{00} a^1_{10} + 4(b^1_{12})^2 a^1_{10} + 4b^1_{10} b^2_{00} b^2_{02} a^1_{10} + 2b^1_{11} b^2_{10} b^3_{00} \\
& +4b^1_{02} b^2_{00} b^2_{10} a^1_{10} + 4b^1_{02} b^1_{10} b^3_{00} a^1_{10} - 4b^2_{02} b^3_{00} a^1_{10} - 4b^2_{00} b^3_{02} a^1_{10} - 4b^1_{02} b^4_{00} a^1_{10} \\
& +4b^1_{02} b^1_{10} b^2_{00} a^1_{01} a^1_{10} - 4b^2_{00} b^2_{02} a^1_{01} a^1_{10} - 4b^1_{02} b^3_{00} a^1_{01} a^1_{10} - 4b^1_{02} b^2_{00}(a^1_{01})^2 a^1_{10} \\
& +4b^1_{10} b^1_{11} b^2_{00}(a^1_{10})^2 - 2b^2_{00} b^2_{11}(a^1_{10})^2 - 2b^1_{11} b^3_{00}(a^1_{10})^2 - 2b^1_{11} b^2_{00} a^1_{01}(a^1_{10})^2 - 2b^2_{11} b^4_{00} \\
& -4b^1_{02} b^2_{00}(a^1_{10})^3 - 2b^1_{11}(b^2_{00})^2 a^1_{11} + 2b^2_{00} b^3_{01} a^1_{11} - 2(b^1_{10})^2 b^2_{00} a^1_{10} a^1_{11} + 2b^2_{00} b^2_{10} a^1_{10} a^1_{11} \\
& +2b^1_{10} b^3_{00} a^1_{10} a^1_{11} - 2b^4_{00} a^1_{10} a^1_{11} + 2b^1_{10} b^2_{00} a^1_{01} a^1_{10} a^1_{11} - 2b^3_{00} a^1_{01} a^1_{10} a^1_{11} - 4b^1_{20}(b^2_{00})^2 a^1_{20} \\
& -2b^2_{00}(a^1_{01})^2 a^1_{10} a^1_{11} - 2b^2_{00}(a^1_{10})^3 a^1_{11} + 4(b^1_{10})^3 b^2_{00} a^1_{20} - 4b^1_{02}(b^2_{00})^2 a^1_{20} + 4(b^2_{00})^2 a^1_{10} a^1_{21} \\
& -8b^1_{10} b^2_{00} b^2_{10} a^1_{20} - 4(b^1_{10})^2 b^3_{00} a^1_{20} + 4b^2_{10} b^3_{00} a^1_{20} + 4b^2_{00} b^3_{10} a^1_{20} + 4b^1_{10} b^4_{00} a^1_{20} \\
& +8b^1_{10} b^2_{00}(a^1_{10})^2 a^1_{20} - 4b^3_{00}(a^1_{10})^2 a^1_{20} - 4b^2_{00} a^1_{01}(a^1_{10})^2 a^1_{20} - 2(b^2_{00})^2 a^1_{11} a^1_{20} \\
& -12b^1_{10}(b^2_{00})^2 a^1_{30} + 12b^2_{00} b^3_{00} a^1_{30} + 8(b^1_{02})^2 b^2_{00} a^2_{00} + 2(b^1_{11})^2 b^2_{00} a^2_{00} + 4b^1_{10} b^1_{12} b^2_{00} a^2_{00} \\
& -4b^2_{00} b^2_{12} a^2_{00} - 4b^1_{12} b^3_{00} a^2_{00} - 2b^1_{11} b^3_{01} a^2_{00} + 4b^4_{02} a^2_{00} - 4b^1_{12} b^2_{00} a^1_{01} a^2_{00} + 4b^3_{02} a^1_{01} a^2_{00} \\
& +4b^2_{02}(a^1_{01})^2 a^2_{00} + 4b^1_{02}(a^1_{01})^3 a^2_{00} + 2(b^1_{10})^2 b^1_{11} a^1_{10} a^2_{00} - 2b^1_{11} b^2_{10} a^1_{10} a^2_{00} \\
& -2b^1_{10} b^2_{11} a^1_{10} a^2_{00} + 2b^3_{11} a^1_{10} a^2_{00} - 2b^1_{10} b^1_{11} a^1_{01} a^1_{10} a^2_{00} + 2b^2_{11} a^1_{01} a^1_{10} a^2_{00} \\
& +2b^1_{11}(a^1_{01})^2 a^1_{10} a^2_{00} - 4b^1_{02} b^1_{10}(a^1_{10})^2 a^2_{00} + 4b^2_{02}(a^1_{10})^2 a^2_{00} + 8b^1_{02} a^1_{01}(a^1_{10})^2 a^2_{00} \\
& +2b^1_{11}(a^1_{10})^3 a^2_{00} + 4b^1_{02} b^2_{00} a^1_{11} a^2_{00} + 2(a^1_{01})^3 a^1_{11} a^2_{00} - 2b^1_{10}(a^1_{10})^2 a^1_{11} a^2_{00} \\
& +4b^1_{11} b^2_{00} a^1_{20} a^2_{00} - 4b^3_{01} a^1_{20} a^2_{00} + 4(b^1_{10})^2 a^1_{10} a^1_{20} a^2_{00} - 4b^2_{10} a^1_{10} a^1_{20} a^2_{00} \\
& -4b^1_{10} a^1_{01} a^1_{10} a^1_{20} a^2_{00} + 4(a^1_{01})^2 a^1_{10} a^1_{20} a^2_{00} + 4(a^1_{10})^3 a^1_{20} a^2_{00} + 4b^1_{10} b^2_{00} a^1_{21} a^2_{00} \\
& -4b^3_{00} a^1_{21} a^2_{00} - 4b^2_{00} a^1_{01} a^1_{21} a^2_{00} + 2b^1_{11}(b^1_{20} a^2_{00})^2 + 2b^1_{10} b^1_{21}(a^2_{00})^2 - 2b^2_{21}(a^2_{00})^2 \\
& -2b^1_{21} a^1_{01}(a^2_{00})^2 - 4b^1_{02} a^1_{20}(a^2_{00})^2 + 4b^1_{20} a^1_{20}(a^2_{00})^2 - 2a^1_{11} a^1_{20}(a^2_{00})^2 + 6b^1_{10} a^1_{30}(a^2_{00})^2 \\
& -6a^1_{01} a^1_{30}(a^2_{00})^2 - 4b^1_{02} b^2_{00} a^1_{10} a^2_{01} - 2b^2_{00} a^1_{10} a^1_{11} a^2_{01} + 4b^2_{02} a^2_{00} a^2_{01} + 8b^1_{02} a^1_{01} a^2_{00} a^2_{01} \\
& +2b^1_{11} a^1_{10} a^2_{00} a^2_{01} + 4a^1_{01} a^1_{11} a^2_{00} a^2_{01} + 4a^1_{10} a^1_{20} a^2_{00} a^2_{01} + 4b^1_{02} b^1_{10} b^2_{00} a^2_{10} + 4b^1_{10} b^3_{00} a^2_{20} \\
& -4b^2_{00} b^2_{02} a^2_{10} - 4b^1_{02} b^3_{00} a^2_{10} - 4b^1_{02} b^2_{00} a^1_{01} a^2_{10} - 4b^1_{11} b^2_{00} a^1_{10} a^2_{10} + 4b^1_{02}(a^1_{01})^2 a^3_{00} \\
& +2b^1_{10} b^2_{00} a^1_{11} a^2_{10} - 2b^3_{00} a^1_{11} a^2_{10} - 2b^2_{00} a^1_{01} a^1_{11} a^2_{10} - 8b^2_{00} a^1_{10} a^1_{20} a^2_{10} - 2b^1_{10} b^1_{11} a^2_{00} a^2_{10} \\
& +2b^2_{11} a^2_{00} a^2_{10} + 2b^1_{11} a^1_{01} a^2_{00} a^2_{10} + 8b^1_{02} a^1_{10} a^2_{00} a^2_{10} + 4a^1_{10} a^1_{11} a^2_{00} a^2_{10} - 4b^1_{10} a^1_{20} a^2_{00} a^2_{10} \\
& +4a^1_{01} a^1_{20} a^2_{00} a^2_{10} + 2b^1_{10} b^2_{00} a^1_{10} a^2_{11} - 2b^3_{00} a^1_{10} a^2_{11} - 2b^2_{00} a^1_{01} a^1_{10} a^2_{11} + 2(a^1_{01})^2 a^2_{00} a^2_{11} \\
& +2(a^1_{10})^2 a^2_{00} a^2_{11} + 2a^2_{00} a^2_{01} a^2_{11} - 2b^2_{00} a^2_{10} a^2_{11} - 4(b^1_{10})^2 b^2_{00} a^2_{20} + 4b^2_{00} b^2_{10} a^2_{20} \\
& -4b^4_{00} a^2_{20} - 4b^2_{00}(a^1_{10})^2 a^2_{20} - 4b^1_{10} a^1_{10} a^2_{00} a^2_{20} + 4a^1_{01} a^1_{10} a^2_{00} a^2_{20} + 4a^2_{00} a^2_{10} a^2_{20} \\
& -4b^2_{00} a^2_{00} a^2_{21} + 6(b^2_{00})^2 a^2_{30} - 6(a^2_{00})^2 a^2_{30} - 4b^1_{12} b^2_{00} a^3_{00} + 4b^3_{02} a^3_{00} + 4b^2_{02} a^1_{01} a^3_{00} \\
& -2b^1_{10} b^1_{11} a^1_{10} a^3_{00} + 2b^2_{11} a^1_{10} a^3_{00} + 2b^1_{11} a^1_{01} a^1_{10} a^3_{00} + 4b^1_{02}(a^1_{10})^2 a^3_{00} + 2(a^1_{01})^2 a^1_{11} a^3_{00}
\end{aligned}
$$

$$
\begin{aligned}
&+2(a^1_{10})^2a^1_{11}a^3_{00}-4b^1_{10}a^1_{10}a^1_{20}a^3_{00}+4a^1_{01}a^1_{10}a^1_{20}a^3_{00}-4b^2_{00}a^1_{21}a^3_{00}-4b^1_{21}a^2_{00}a^3_{00}\\
&-12a^1_{30}a^2_{00}a^3_{00}+4b^1_{02}a^2_{01}a^3_{00}+2a^1_{11}a^2_{01}a^3_{00}+2b^1_{11}a^2_{10}a^3_{00}+4a^1_{20}a^2_{10}a^3_{00}\\
&+2a^1_{01}a^2_{11}a^3_{00}+4a^1_{10}a^2_{20}a^3_{00}+4b^1_{02}a^2_{00}a^3_{01}+2a^1_{11}a^2_{00}a^3_{01}-2b^2_{00}a^1_{10}a^3_{11}+2a^1_{11}a^5_{00}\\
&+2a^1_{01}a^2_{00}a^3_{11}+2a^3_{00}a^3_{11}+4b^1_{10}b^2_{00}a^3_{20}-4b^3_{00}a^3_{20}+4a^1_{10}a^2_{00}a^3_{20}+4b^1_{02}a^5_{00}\\
&+2a^6_{10}+4b^2_{00}2a^4_{00}+4b^1_{02}a^1_{01}a^4_{00}+2b^1_{11}a^1_{10}a^4_{00}+2a^1_{01}a^1_{11}a^4_{00}+4a^1_{10}a^1_{20}a^4_{00}\\
&+2a^2_{00}v11-4b^2_{00}v20-2b^1_{11}b^5_{00}-4a^1_{20}b^5_{00}+4a^1_{01}(a^1_{10})^2a^1_{11}a^2_{00}+2a^2_{11}a^4_{00}).
\end{aligned}
$$

$$
\begin{aligned}
C_2 = \ &\frac{1}{8}(-2b^1_{02}(b^1_{10})^2b^1_{11}-6(b^1_{10})^2b^1_{11}b^1_{20}-2(b^1_{10})^3b^1_{21}+12b^1_{02}b^1_{03}b^2_{00}+6b^1_{11}b^1_{12}b^2_{00}\\
&+8b^1_{02}b^1_{21}b^2_{00}+8b^1_{20}b^1_{21}b^2_{00}+6b^1_{11}b^1_{30}b^2_{00}+2b^1_{10}b^1_{11}b^2_{00}2+2b^1_{02}b^1_{11}b^2_{10}\\
&+4b^1_{11}b^1_{20}b^2_{10}+4b^1_{10}b^1_{21}b^2_{10}+2b^1_{02}b^1_{10}b^2_{11}+4b^1_{10}b^1_{20}b^2_{11}-2b^2_{00}2b^2_{11}\\
&+4b^1_{10}b^1_{11}b^2_{20}-2b^2_{11}b^2_{20}+2(b^1_{10})^2b^2_{21}-2b^2_{10}b^2_{21}-4b^1_{12}b^3_{01}-2b^1_{11}b^3_{02}\\
&-2b^1_{21}b^3_{10}-2b^1_{02}b^3_{11}-2b^1_{20}b^3_{11}-2b^1_{11}b^3_{20}-2b^1_{10}b^3_{21}+6b^4_{03}+2b^4_{21}\\
&-4b^1_{10}b^1_{11}b^1_{20}a^1_{01}-2(b^1_{10})^2b^1_{21}a^1_{01}+2b^1_{21}b^2_{10}a^1_{01}+2b^1_{20}b^2_{11}a^1_{01}+4b^3_{02}a^1_{02}\\
&+4b^1_{02}a^1_{01}a^1_{02}+4b^1_{02}(a^1_{01})^2a^1_{02}+4(b^1_{02})^2b^1_{10}a^1_{10}+4b^1_{10}(b^1_{11})^2a^1_{10}+2a^4_{12}\\
&+4(b^1_{10})^2b^1_{12}a^1_{10}+8b^1_{02}b^1_{10}b^1_{20}a^1_{10}-24b^1_{04}b^2_{00}a^1_{10}-12b^1_{22}b^2_{00}a^1_{10}-2b^3_{21}a^1_{01}\\
&-8b^1_{02}b^1_{02}a^1_{10}-4b^1_{20}b^1_{02}a^1_{10}-4b^1_{12}b^2_{10}a^1_{10}-4b^1_{11}b^2_{11}a^1_{10}-4b^1_{10}b^2_{12}a^1_{10}\\
&-4b^1_{02}b^2_{20}a^1_{10}+4b^3_{12}a^1_{10}-4(b^1_{02})^2a^1_{01}a^1_{10}-2b^1_{10}b^1_{11}a^1_{02}a^1_{10}+2b^1_{11}b^2_{20}a^1_{01}\\
&+2b^2_{11}a^1_{02}a^1_{10}+2b^1_{11}a^1_{01}a^1_{02}a^1_{10}-6b^1_{02}b^1_{11}(a^1_{10})^2-2b^1_{11}b^1_{20}(a^1_{10})^2-4b^1_{12}b^2_{00}a^1_{02}\\
&+4b^1_{02}a^1_{02}(a^1_{10})^2+6b^1_{03}b^2_{00}a^1_{11}+4b^1_{21}b^2_{00}a^1_{11}+2(a^1_{01})^2a^1_{02}a^1_{11}+2b^1_{10}b^2_{21}a^1_{01}\\
&-2b^1_{02}b^1_{10}a^1_{10}a^1_{11}+4b^1_{10}b^1_{20}a^1_{10}a^1_{11}+2b^1_{02}a^1_{10}a^1_{11}-2b^2_{20}a^1_{10}a^1_{11}+2b^1_{02}a^1_{01}a^1_{10}a^1_{11}\\
&+2a^1_{02}(a^1_{10})^2a^1_{11}-2b^1_{10}a^1_{10}(a^1_{11})^2+2a^1_{01}a^1_{10}(a^1_{11})^2-12(b^1_{10})^2b^1_{20}a^1_{20}\\
&+12b^1_{30}b^2_{00}a^1_{20}+8b^1_{20}b^2_{10}a^1_{20}+8b^1_{10}b^2_{20}a^1_{20}-4b^3_{20}a^1_{20}-8b^1_{10}b^1_{20}a^1_{01}a^1_{20}\\
&+4b^2_{20}a^1_{01}a^1_{20}+4b^1_{10}b^1_{11}a^1_{10}a^1_{20}-2b^2_{11}a^1_{10}a^1_{20}-4b^1_{10}a^1_{02}a^1_{10}a^1_{20}-4b^1_{10}b^1_{22}a^2_{00}\\
&+4a^1_{01}a^1_{02}a^1_{10}a^1_{20}-4b^1_{20}(a^1_{10})^2a^1_{20}+2(b^1_{10})^2a^1_{11}a^1_{20}-2b^2_{10}a^1_{11}a^1_{20}\\
&+6(a^1_{10})^2a^1_{11}a^1_{20}-8b^1_{10}a^1_{10}(a^1_{20})^2+4b^1_{11}b^2_{00}a^1_{21}-4b^3_{01}a^1_{21}-4b^2_{00}a^1_{02}a^1_{21}\\
&+4(b^1_{10})^2a^1_{10}a^1_{21}-4b^2_{10}a^1_{10}a^1_{21}-4b^2_{00}a^1_{20}a^1_{21}+4b^1_{10}b^2_{00}a^1_{22}-4b^3_{00}a^1_{22}\\
&-6(b^1_{10})^3a^1_{30}+12b^1_{02}b^2_{00}a^1_{30}+24b^1_{20}b^2_{00}a^1_{30}+12b^1_{10}b^2_{10}a^1_{30}-6b^3_{10}a^1_{30}\\
&-6(b^1_{10})^2a^1_{01}a^1_{30}+6b^2_{10}a^1_{01}a^1_{30}+6b^2_{00}a^1_{11}a^1_{30}+48b^1_{10}b^2_{00}a^1_{40}-24b^3_{00}a^1_{40}\\
&+24b^2_{00}a^1_{01}a^1_{40}-6b^1_{03}b^1_{11}a^2_{00}-12b^1_{02}b^1_{12}a^2_{00}-4b^1_{12}b^1_{20}a^2_{00}-4b^1_{11}b^1_{21}a^2_{00}\\
&+24b^2_{00}4a^2_{00}+4c22a^2_{00}+24b^1_{04}a^1_{01}a^2_{00}-8b^1_{21}a^1_{02}a^2_{00}+12b^1_{02}a^1_{03}a^2_{00}\\
&+6a^1_{03}a^1_{11}a^2_{00}-12b^1_{03}a^1_{20}a^2_{00}-8b^1_{21}a^1_{20}a^2_{00}-8b^1_{02}a^1_{21}a^2_{00}-4b^1_{20}a^1_{21}a^2_{00}\\
&+2a^1_{11}a^1_{21}a^2_{00}+12a^1_{10}a^1_{22}a^2_{00}-6b^1_{11}a^1_{30}a^2_{00}-24a^1_{02}a^1_{30}a^2_{00}-12a^1_{20}a^1_{30}a^2_{00}\\
&+24a^1_{10}a^1_{40}a^2_{00}+2b^1_{11}b^1_{20}a^2_{01}+2b^1_{10}b^1_{21}a^2_{01}-2b^2_{21}a^2_{01}+4b^1_{02}a^1_{02}a^2_{01}\\
&+2a^1_{02}a^1_{11}a^2_{01}+4b^1_{20}a^1_{20}a^2_{01}+6b^1_{10}a^1_{30}a^2_{01}+4b^2_{00}2a^2_{02}+4b^1_{02}a^1_{01}a^2_{02}\\
&+2b^1_{11}a^1_{10}a^2_{02}+2a^1_{01}a^1_{11}a^2_{02}+4a^1_{10}a^1_{20}a^2_{02}-4(b^1_{02})^2a^2_{10}-2(b^1_{11})^2a^2_{10}
\end{aligned}
$$

$$
\begin{aligned}
& -4b_{10}^1b_{12}^1a_{10}^2 - 4b_{02}^1b_{20}^1a_{10}^2 + 4b_{12}^2a_{10}^2 + 2b_{11}^1a_{02}^1a_{10}^2 + 2b_{02}^1a_{11}^1a_{10}^2 - 6b_{10}^1a_{30}^3 \\
& -2b_{20}^1a_{11}^1a_{10}^2 + 2(a_{11}^1)^2a_{10}^2 - 2b_{11}^1a_{20}^1a_{10}^2 + 4a_{02}^1a_{20}^1a_{10}^2 + 4(a_{20}^1)^2a_{10}^2 \\
& -4b_{10}^1a_{21}^1a_{10}^2 + 2a_{01}^1a_{02}^1a_{11}^2 + 2b_{02}^1a_{10}^1a_{11}^2 - 2b_{20}^1a_{10}^1a_{11}^2 + 4a_{10}^1a_{11}^1a_{11}^2 \\
& -2b_{10}^1a_{20}^1a_{11}^2 + 2a_{02}^2a_{11}^2 + 8b_{10}^1b_{20}^1a_{20}^2 - 4b_{20}^2a_{20}^2 + 4b_{20}^1a_{01}^1a_{20}^2 - 6a_{01}^1a_{30}^3 \\
& -2b_{11}^1a_{10}^1a_{20}^2 + 4a_{02}^1a_{10}^1a_{20}^2 - 2b_{10}^1a_{11}^1a_{20}^2 + 8a_{10}^1a_{20}^1a_{20}^2 + 2a_{11}^2a_{20}^2 + 2a_{11}^1a_{20}^3 \\
& -4b_{10}^1a_{10}^1a_{21}^2 + 4a_{10}^2a_{21}^2 - 4b_{00}^2a_{22}^2 + 6(b_{10}^1)^2a_{30}^2 - 6b_{10}^2a_{30}^2 + 6a_{30}^4 \\
& +6b_{10}^1a_{01}^1a_{30}^2 - 6a_{01}^2a_{30}^2 - 24b_{00}^2a_{40}^2 + 24b_{04}^1a_{00}^3 + 4b_{22}^1a_{00}^3 - 2b_{21}^1a_{01}^3 \\
& -6a_{30}^1a_{01}^3 + 4b_{02}^1a_{02}^3 + 2a_{11}^1a_{02}^3 + 2a_{02}^1a_{11}^3 + 2a_{20}^1a_{11}^3 - 4b_{20}^1a_{20}^3 + 4a_{10}^1a_{21}^3).
\end{aligned}
$$

$$
\begin{aligned}
C_3 = \quad & \frac{1}{24}(-6b_{03}^1b_{12}^1 - 3b_{12}^1b_{21}^1 - b_{11}^1b_{22}^1 - 3b_{21}^1b_{30}^1 - 3b_{11}^1b_{40}^1 + 15b_{05}^1 - 15b_{10}^1a_{50}^1 \\
& +3b_{23}^2 + 15b_{05}^1(b_{10}^1 + a_{01}^1) + 2b_{22}^1a_{02}^1 - 3b_{21}^1a_{03}^1 + 15a_{50}^2 - 5a_{11}^1a_{40}^1 \\
& +6b_{02}^1a_{04}^1 + 6b_{32}^1a_{10}^1 + 3a_{04}^1a_{11}^1 + 4b_{22}^1a_{20}^1 - 6b_{40}^1a_{20}^1 + 3a_{32}^2 \\
& +b_{04}^1(5b_{11}^1 + 20a_{02}^1 + 22a_{20}^1) - 6b_{03}^1a_{21}^1 - 3b_{21}^1a_{21}^1 - 4b_{02}^1a_{22}^1 \\
& -2b_{20}^1a_{22}^1 + a_{11}^1a_{22}^1 + 6a_{10}^1a_{23}^1 - 3b_{12}^1a_{30}^1 - 9b_{30}^1a_{30}^1 - 15a_{01}^1a_{50}^1 \\
& -9a_{03}^1a_{30}^1 - 3a_{21}^1a_{30}^1 - 22b_{02}^1a_{40}^1 - 20b_{20}^1a_{40}^1).
\end{aligned}
$$

Since the rank of the Jacobian matrix of the function $\mathcal{C} = (C_1, C_2, C_3)$ with respect to the coefficients a_{ij}^s and b_{ij}^s which appear in their expressions is 3, the three coefficients C_i for $i = 1, 2, 3$ of the polynomial $f_6(r)$ are linearly independent. Therefore il follows that the polynomial $f_6(r)$ can have at most 2 positive real roots. Consequently, by subsection 2.3 for $|\varepsilon| > 0$ sufficiently small the differential equation $(\dot r, \dot\theta)$ has at most 2 limit cycles surrounding the origin, going back through the changes of variables we obtain that there are at most 2 limit cycles bifurcating from the center of system (2) when we perturb it as in system (3) if $f_k = 0$ for $k = 1, 2, 3, 4, 5$ and $f_6 \neq 0$. So statement (e) of the theorem is proved for the averaged function of order 6.

Acknowledgments

The second author is partially supported by a FEDER-MINECO grant MTM2016-77278-P, a MINECO grant MTM2013-40998-P, and an AGAUR grant number 2014SGR-568.

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