TOPOLOGICAL ENTROPY OF CONTINUOUS SELF-MAPS
ON CLOSED SURFACES

JUAN LUIS GARCÍA GUIRAO\textsuperscript{1} AND JAUME LLIBRE\textsuperscript{2} AND WEI GAO\textsuperscript{1,3}

Abstract. The objective of the present work is to present sufficient conditions for having positive topological entropy for continuous self-maps defined on a closed surface by using the action of this map on the homological groups of the closed surface.

1. Introduction

Along this work by a closed surface we denote a connected compact surface with or without boundary, orientable or not. More precisely, an orientable connected compact surface without boundary of genus \( g \geq 0 \), \( M_g \), is homeomorphic to the sphere if \( g = 0 \), to the torus if \( g = 1 \), or to the connected sum of \( g \) copies of the torus if \( g \geq 2 \). An orientable connected compact surface with boundary of genus \( g \geq 0 \), \( M_{g,b} \), is homeomorphic to \( M_g \) minus a finite number \( b > 0 \) of open discs having pairwise disjoint closure. In what follows \( M_{g,0} = M_g \).

A non-orientable connected compact surface without boundary of genus \( g \geq 1 \), \( N_g \), is homeomorphic to the real projective plane if \( g = 1 \), or to the connected sum of \( g \) copies of the real projective plane if \( g > 1 \). A non-orientable connected compact surface with boundary of genus \( g \geq 1 \), \( N_{g,b} \), is homeomorphic to \( N_g \) minus a finite number \( b > 0 \) of open discs having pairwise disjoint closure. In what follows \( N_{g,0} = N_g \).

Let \( f : X \to X \) be a continuous map on a closed surface \( X \). A point \( x \in X \) is periodic of period \( n \) if \( f^n(x) = x \) and \( f^k(x) \neq x \) for \( k = 1, \ldots, n - 1 \).

The topological entropy of a continuous map \( f : X \to X \) denoted by \( h(f) \) is a non-negative real number (possibly infinite) which measures how much \( f \) mixes up the phase space of \( X \). When \( h(f) \) is positive the dynamics of the system is said to be complicated and the positivity of \( h(f) \) is used as a measure of the so called topological chaos.

Here we introduce the topological entropy using the definition of Bowen [4].

\textsuperscript{1}Key words and phrases. Closed surface, continuous self-map, Lefschetz fixed point theory, periodic point, set of periods.
\textsuperscript{2}2010 Mathematics Subject Classification: 58F20, 37C05, 37C25, 37C30.
Since it is possible to embedded any surface orientable or not in $\mathbb{R}^4$ by the Whitney immersion theorem see [11], we consider the distance between two points of $X$ as the distance of these two points in $\mathbb{R}^4$. Now, we define the distance $d_n$ on $G$ by

$$d_n(x, y) = \max_{0 \leq i \leq n} d(f_i(x), f_i(y)), \quad \forall x, y \in G.$$ 

A finite set $S$ is called $(n, \varepsilon)$-separated with respect to $f$ if for different points $x, y \in S$ we have $d_n(x, y) > \varepsilon$. We denote by $S_n$ the maximal cardinality of an $(n, \varepsilon)$-separated set. Define

$$h(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log S_n.$$ 

Then

$$h(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon)$$

is the topological entropy of $f$.

We have chosen the definition by Bowen because, probably it is the shorter one. The classical definition was due to Adler, Konheim and McAndrew [1]. See for instance the book of Hasselblatt and Katok [7] and [3] for other equivalent definitions and properties of the topological entropy. See [1, 2, 12] for more details on the topological entropy.

Our main results are the following.

**Theorem 1.** Let $\mathcal{M}_g$ be an orientable connected compact surface without boundary of genus $g$. Then the following statements hold.

(a) If the degree $d \notin \{-1, 0, 1\}$, then the topological entropy of $f$ is positive.

(b) If the degree $d \in \{-1, 0, 1\}$ and the number of roots of the characteristic polynomial of $f_{x_1}$ equal to $\pm 1$ or $0$ taking into account their multiplicities is not even, then the topological entropy of $f$ is positive.

**Theorem 2.** Let $\mathcal{M}_{g,b}$, $b > 0$, be an orientable connected compact surface with boundary of genus $g$. Then the following statements hold.

(a) If $2g + b - 1$ is even and the number of roots of the characteristic polynomial of $f_{x_1}$ equal to $\pm 1$ or $0$ taking into account their multiplicities is not even, then the topological entropy of $f$ is positive.

(b) If $2g + b - 1$ is odd and the number of roots of the characteristic polynomial of $f_{x_1}$ equal to $\pm 1$ or $0$ taking into account their multiplicities is not odd, then the topological entropy of $f$ is positive.

**Theorem 3.** Let $\mathcal{N}_{g,b}$, $b \geq 0$, be a non-orientable connected compact surface with boundary of genus $g$. Then the following statements hold.

(a) If $g + b - 1$ is even and the number of roots of the characteristic polynomial of $f_{x_1}$ equal to $\pm 1$ or $0$ taking into account their multiplicities is not even, then the topological entropy of $f$ is positive.
(b) If \( g+b-1 \) is odd and the number of roots of the characteristic polynomial of \( f_x \) equal to \( \pm 1 \) or \( 0 \) taking into account their multiplicities is not odd, then the topological entropy of \( f \) is positive.

2. Lefschetz zeta functions for surfaces

Let \( f \) be a continuous self-map defined on \( \mathbb{M}_{g,b} \) or \( \mathbb{N}_{g,b} \), respectively. For a closed surface the homological groups with coefficients in \( \mathbb{Q} \) are linear vector spaces over \( \mathbb{Q} \). We recall the homological spaces of \( \mathbb{M}_{g,b} \) with coefficients in \( \mathbb{Q} \), i.e.

\[
H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}^{n_1} \oplus \mathbb{Q},
\]

where \( n_0 = 1, n_1 = 2g \) if \( b = 0 \), \( n_1 = 2g + b - 1 \) if \( b > 0 \), \( n_2 = 1 \) if \( b = 0 \), and \( n_2 = 0 \) if \( b > 0 \); and the induced linear maps \( f_{sk} : H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \rightarrow H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \)

by \( f \) on the homological group \( H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \) are \( f_{s0} = (1), f_{s2} = (d) \) where \( d \) is the degree of the map \( f \) if \( b = 0 \), \( f_{s2} = (0) \) if \( b > 0 \), and \( f_{s1} = A \) where \( A \) is an \( n_1 \times n_1 \) integral matrix (see for additional details [10, 13]).

We recall that the homological groups of \( \mathbb{N}_{g,b} \) with coefficients in \( \mathbb{Q} \), i.e.

\[
H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}^{n_1} \oplus \mathbb{Q},
\]

where \( n_0 = 1, n_1 = g + b - 1 \) and \( n_2 = 0 \); and the induced linear maps are \( f_{s0} = (1) \) and \( f_{s1} = A \) where \( A \) is an \( n_1 \times n_1 \) integral matrix (see again for additional details [10, 13]).

Let \( f : X \rightarrow X \) be a continuous map and let \( X \) be either \( \mathbb{M}_{g,b} \) or \( \mathbb{N}_{g,b} \). Then the Lefschetz number of \( f \) is defined by

\[
L(f) = \text{trace}(f_{s0}) - \text{trace}(f_{s1}) + \text{trace}(f_{s2}).
\]

We shall use the Lefschetz numbers of the iterates of \( f \), i.e. \( L(f^n) \). In order to study the whole sequence \( \{L(f^n)\}_{n \geq 1} \) it is defined the formal Lefschetz zeta function of \( f \) as

\[
Z_f(t) = \exp \left( \sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n \right).
\]

The Lefschetz zeta function is in fact a generating function for the sequence of the Lefschetz numbers \( L(f^n) \).

From the work of Franks in [6] we have for a continuous self-map of a closed surface that its Lefschetz zeta function is the rational function

\[
Z_f(t) = \frac{\det(I - tf_{s1})}{\det(I - tf_{s0})\det(I - tf_{s2})},
\]

(1)
where in \( I - tf_{sk} \) the \( I \) denotes the \( n_k \times n_k \) identity matrix, and \( \det(I - tf_2) = 1 \) if \( f_2 = (0) \). Then for a continuous map \( f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b} \) we have

\[
Z_f(t) = \begin{cases} 
\frac{\det(I - tA)}{(1-t)(1 - dt)} & \text{if } b = 0, \\
\frac{\det(I - tA)}{1-t} & \text{if } b > 0,
\end{cases}
\]

and for a continuous map \( f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b} \) we have

\[
Z_f(t) = \frac{\det(I - tA)}{1-t}.
\]

3. Basic results

In this section we present the main result stated in Theorem 7 for proving Theorems 1, 2 and 3. Since its proof is short and important for this work we provide it here.

For a polynomial \( H(t) \) we define \( H^*(t) \) by

\[
H(t) = (1 - t)^\alpha (1 + t)^\beta t^\gamma H^*(t),
\]

where \( \alpha, \beta \) and \( \gamma \) are non-negative integers such that \( 1 - t, 1 + t \) and \( t \) do not divide \( H^*(t) \).

The spectral radii of the maps \( f_{sk} \) are denoted \( sp(f_{sk}) \), and they are equal to the largest modulus of all the eigenvalues of the linear map \( f_{sk} \). The spectral radius of \( f_s \) is

\[
sp(f_s) = \max_{k=0,...,m} sp(f_{sk}).
\]

The next result is due to Manning [9].

**Theorem 4.** Let \( f : X \to X \) be a continuous map on a closed surface \( X \). Then \( \log \max\{1, \sp(f_{s1})\} \leq h(f) \).

**Lemma 5.** Let \( f : \mathbb{X} \to \mathbb{X} \) be a continuous map and let \( \mathbb{X} \) be a closed surface. If the topological entropy of \( f \) is zero, then all the eigenvalues of the induced homomorphism \( f_{s1} \) are zero or root of unity.

**Proof.** Since the topological entropy is zero, by Theorem 4 we have \( \sp(f_{s1}) = 1 \). So, all the eigenvalues of \( f_{s1} \) have modulus in the interval \([0,1]\) and at least one of them is 1. Then the characteristic polynomial of \( f_{s1} \) is of the form \( t^m p(t) \), where \( m \) is a non-negative integer, positive if the zero is an eigenvalue. And \( p(t) \) is a polynomial with integer coefficients and whose independent term \( a_0 \) is non-zero. Since the product of all non-zeros eigenvalues of \( f_{s1} \) is the integer \( a_0 \) and, these eigenvalues have modulus in \((0,1)\), we have that any of these eigenvalues can have modulus smaller than one, otherwise we are in contradiction with the fact \( a_0 \) is an integer. In short, all the non-zero eigenvalues have modulus one, and consequently \( a_0 = 1 \).
Since if a polynomial has integer coefficients, constant term 1 and all of whose roots have modulus 1, then all of its roots are roots of unity, see [14], the lemma follows.

The $n$-th cyclotomic polynomial is defined by

$$c_n(t) = \prod_k (w_k - t),$$

being $w_k = e^{2\pi ik/n}$ a primitive $n$-th root of unity and where $k$ runs over all the relative primes $\leq n$. See [8] for the properties of these polynomials.

For a positive integer $n$ the Euler function is $\varphi(n) = n \prod_{p|n, p\text{ prime}} (1 - \frac{1}{p})$.

It is known that the degree of the polynomial $c_n(t)$ is $\varphi(n)$. Note that $\varphi(n)$ is even for $n > 2$.

A proof of the next result can be found in [8].

**Proposition 6.** Let $\xi$ be a primitive $n$-th root of the unity and $P(t)$ a polynomial with rational coefficients. If $P(\xi) = 0$ then $c_n(t)|P(t)$.

The proofs of our results are strongly based in the next theorem.

**Theorem 7** (Theorem 3.2 of [5]). Let $\mathbb{X}$ be a closed surface, $f : \mathbb{X} \to \mathbb{X}$ be a continuous self-map, and let $Z_f(t) = \frac{P(t)}{Q(t)}$ be its Lefschetz zeta function. If $P^*(t)$ or $Q^*(t)$ has odd degree, then the topological entropy of $f$ is positive.

**Proof.** From the definitions of a polynomial $H^*$ and of the Lefschetz zeta function we have

$$Z_f(t) = \frac{P(t)}{Q(t)} = (1 - t)^a(1 + t)^b H^* \frac{P^*(t)}{Q^*(t)},$$

where $a, b$ and $c$ are integers.

Assume now that the topological entropy $h(f) = 0$. Then by Lemma 5 all the eigenvalues of the induced homomorphisms $f_{*1}$'s are zero or roots of unity. Therefore, by (1) all the roots of the polynomials $P^*(t)$ and $Q^*(t)$ are roots of the unity different from $\pm 1$ and zero. Hence, by Proposition 6 the polynomials $P^*(t)$ and $Q^*(t)$ are product of cyclotomic polynomials different from $c_1(t) = 1 - t$ and $c_2(t) = 1 + t$. Consequently $P^*(t)$ and $Q^*(t)$ have even degree because all the cyclotomic polynomials which appear in them have even degree due to the fact that the Euler function $\varphi(n)$ for $n > 2$ only takes even values. But this is a contradiction with the assumption that $P^*(t)$ or $Q^*(t)$ has odd degree. \qed
4. Proof of Theorems 1, 2 and 3

Proof of Theorem 1. Since \( \mathbb{M}_g \) is an orientable connected compact surface without boundary of genus \( g \), then the Lefschetz zeta function of \( f \) is equal to

\[
Z_f(t) = \frac{\det(I - tA)}{(1 - t)(1 - dt)},
\]

where \( d \) is the degree of \( f \) and \( 2g \) is the dimension of the characteristic polynomial \( \det(I - tA) \) of \( f_{*1} = A \). Note here that if \( d \notin \{-1, 0, 1\} \), then \( Q^*(t) = 1 - dt \) and therefore by Theorem 7 statement (a) of Theorem 1 is proved.

Assume now that \( d \in \{-1, 0, 1\} \). Note that in this case \( Q(t) = (1 - t)(1 - dt) \) and \( Q^*(t) = 1 \). So, by Theorem 7 the main role will be play by the \( 2g \) degree polynomial \( P(t) = \det(I - tA) \) where \( f_{*1} = A \). Since \( 2g \) is even and the number of roots of the characteristic polynomial of \( f_{*1} \) equal to \( \pm 1 \) or 0 taking into account their multiplicities is not even, then \( P^*(t) \) has odd degree. Therefore, statement (b) of Theorem 1 follows by the application of Theorem 7.

\[\square\]

Proof of Theorem 2. Note now, since \( \mathbb{M}_{g,b} \) is an orientable connected compact surface with boundary \( (b > 0) \) of genus \( g \), then the Lefschetz zeta function of \( f \) is equal to

\[
Z_f(t) = \frac{\det(I - tA)}{1 - t}
\]

being \( 2g + b - 1 \) the degree of the characteristic polynomial \( \det(I - tA) \) of \( f_{*1} = A \). Now the proof is similar to the statements (b) and (c) of Theorem 1.

\[\square\]

Proof of Theorem 3. For a non–orientable connected compact surface with or without boundary \( (b \geq 0) \) of genus \( g \geq 1 \), the Lefschetz zeta function of \( f \) is equal to

\[
Z_f(t) = \frac{\det(I - tA)}{1 - t}
\]

being \( g + b - 1 \) the degree of the characteristic polynomial \( \det(I - tA) \) of \( f_{*1} = A \). Then the proof if this theorems follows in a similar way to the proof of statements (b) and (c) of Theorem 1.

\[\square\]

Acknowledgements

The second author is partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grants MTM-2016-77278-P (FEDER) and MDM-2014-0445, the Agència de Gestió d’Àgits Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.
REFERENCES


Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Hospital de Marina, 30203-Cartagena, Región de Murcia, Spain.

E-mail address: juan.garcia@upct.es

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, 08193-Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

School of Information Science and Technology, Yunnan Normal University, Kunming 650500, China

E-mail address: gaowei@ynnu.edu.cn