

# GLOBAL DYNAMICS OF THE HOŘAVA-LIFSHITZ COSMOLOGY WITH NON-ZERO CURVATURE

FABAO GAO<sup>1,2</sup>, AND JAUME LLIBRE<sup>2</sup>

ABSTRACT. The global dynamics of a cosmological model based on Hořava-Lifshitz gravity in the presence of curvature is described by using the qualitative theory of differential equations.

## 1. INTRODUCTION

In recent years Hořava [1] proposed a spacetime asymmetric gravitational theory similar to Lifshitz’s scalar field theory, also known as Hořava-Lifshitz gravity. This theory has inspired a great deal of research for its applications in cosmology and black hole physics (see [2]-[17] or the review articles [18], [19] and the references therein).

Here we will investigate the global dynamics of the Hořava-Lifshitz scalar field cosmology under the Friedmann-Lemaître-Robertson-Walker background spacetime in the presence of curvature and no cosmological constant term. The corresponding dimensionless field equations admit the following form

$$(1) \quad \begin{aligned} \frac{dx}{dt} &= x(3x^2 - 2z^2 - 3) + \sqrt{6}s(1 - x^2 + z^2), \\ \frac{dz}{dt} &= z(3x^2 - 2z^2 - 2), \\ \frac{ds}{dt} &= -2\sqrt{6}xf(s), \end{aligned}$$

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where the power law potential  $f(s) = -s^2/(2n)$  with a natural number  $n$ . See equations (113)-(115) of [8] or (44)-(46) of [14] for more details.

The Hořava-Lifshitz cosmological model with zero curvature and without cosmological constant term was studied in [5]. Furthermore, in the references [3, 7, 8, 9, 14], the authors either studied the global dynamics of the planar case of system (1) considering only the variables  $x$  and  $z$  by using the two-dimensional Poincaré compactification, or discussed only local dynamics of system (1) without investigate the dynamics close to infinity. Our study fully describes the global dynamics of system (1) in the region  $G = \{(x, z, s) \in \mathbb{R}^3 : x^2 - z^2 \leq 1\}$  of physical interest.

## 2. PHASE PORTRAITS ON THE INVARIANT PLANES AND SURFACE

In order to study the local phase portraits of the finite and infinite equilibrium points, and the global phase portraits of system (1) in the region  $G$ , which is the meaningful region for cosmology, see again [8] or [14]. We start discussing the phase portraits on its invariant planes and surface

$$z = 0, \quad s = 0, \quad x^2 - z^2 = 1.$$

**2.1. The invariant plane  $z = 0$ .** On this plane system (1) becomes

$$(2) \quad \begin{aligned} \frac{dx}{dt} &= (x^2 - 1)(3x - \sqrt{6}s), \\ \frac{ds}{dt} &= \frac{\sqrt{6}}{n} xs^2. \end{aligned}$$

The phase portraits of system (2) in the strip  $z = 0$  and  $x^2 - z^2 \leq 1$ , i.e. in  $z = 0$ ,  $-1 \leq x \leq 1$ , it has been studied in [5], and the phase

portraits is shown in Figure 1, where the hyperbolic equilibrium point  $e_0 = (0, 0)$  is a saddle, both the semi-hyperbolic equilibrium points  $e_1 = (1, 0)$  and  $e_2 = (-1, 0)$  are saddle-nodes.

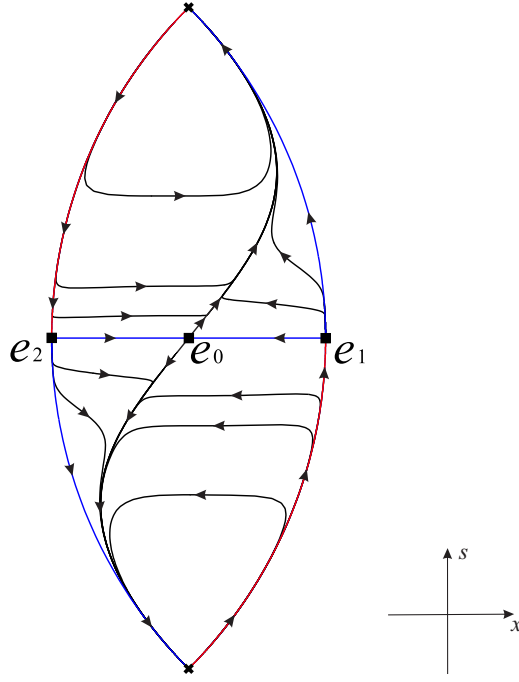


FIGURE 1. The phase portraits of the invariant plane  $z = 0$  in the region  $-1 \leq x \leq 1$ .

**2.2. The invariant plane  $s = 0$ .** On this plane system (1) becomes

$$(3) \quad \frac{dx}{dt} = x(3x^2 - 2z^2 - 3), \quad \frac{dz}{dt} = z(3x^2 - 2z^2 - 2),$$

which has three equilibrium points  $e_0 = (0, 0)$ ,  $e_1 = (1, 0)$  and  $e_2 = (-1, 0)$ . Here  $e_0$  is a hyperbolic stable node with eigenvalues -3 and -2, both  $e_1$  and  $e_2$  are unstable hyperbolic nodes with eigenvalues 6 and 1.

On the local chart  $U_1$  (see Chapter 5 of [20] for more details on the Poincaré compactification) system (3) becomes

$$(4) \quad \frac{du}{dt} = uv^2, \quad \frac{dv}{dt} = v(-3 + 2u^2 + 3v^2).$$

Since this system vanishes at  $v = 0$ , all the points at infinity are equilibrium points. Taking the transformation with respect to time  $d\tau = vdt$  yields

$$(5) \quad \frac{du}{d\tau} = uv, \quad \frac{dv}{d\tau} = -3 + 2u^2 + 3v^2.$$

This system has two hyperbolic points at infinity,  $e_3 = (-\sqrt{6}/2, 0)$  and  $e_4 = (\sqrt{6}/2, 0)$ , both of them are unstable hyperbolic saddle points with eigenvalues  $\pm\sqrt{6}$ .

On the local chart  $U_2$  system (3) writes

$$(6) \quad \frac{du}{dt} = -uv^2, \quad \frac{dv}{dt} = v(2 - 3u^2 + 2v^2).$$

Rescaling the time of system (6) by letting  $d\tau = vdt$  we obtain

$$(7) \quad \frac{du}{d\tau} = -uv, \quad \frac{dv}{d\tau} = 2 - 3u^2 + 2v^2.$$

In view of  $(0, 0)$  is not an equilibrium point of system (7), we will not continue to study other infinite equilibrium points of system (7) because they have been studied in local chart  $U_1$ .

Therefore the global phase portraits of system (3) can be found in Figure 2.

**2.3. The invariant surface  $x^2 - z^2 = 1$ .** First we prove that the surface  $x^2 - z^2 = 1$  is invariant under the flow of system (1). If  $l = l(x, z, s) = x^2 - z^2 - 1$ , then in order that the surface  $x^2 - z^2 = 1$  be

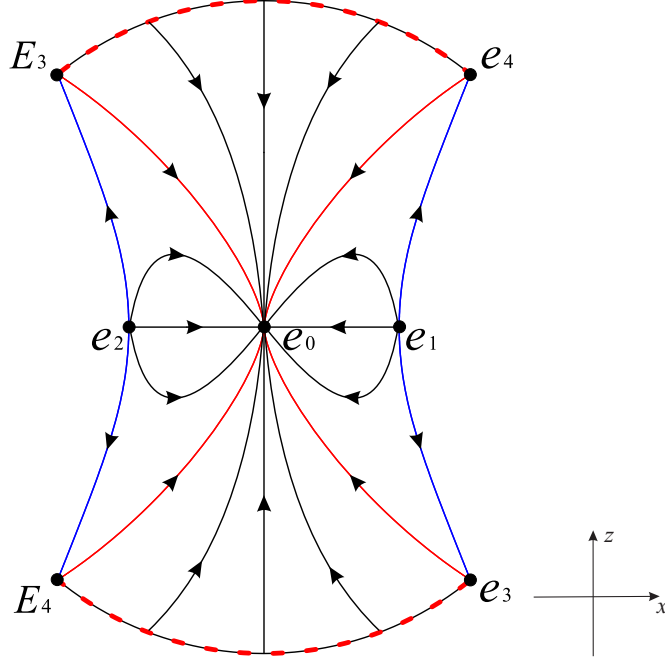


FIGURE 2. The phase portrait on the invariant plane  $s = 0$  restricted to the region  $x^2 - z^2 \leq 1$ .

invariant we must have

$$\frac{\partial l}{\partial x} \dot{x} + \frac{\partial l}{\partial z} \dot{z} + \frac{\partial l}{\partial s} \dot{s} = Kl,$$

for some polynomial  $K$ , and this is the case with  $K = 2(3x^2 - 2z^2 - \sqrt{6}xs)$ .

On the surface  $x^2 - z^2 = 1$  system (1) can be written as

$$(8) \quad \frac{dx}{dt} = x(x^2 - 1), \quad \frac{ds}{dt} = \frac{\sqrt{6}}{n} xs^2.$$

Then except for all the points on  $x = 0$  which are equilibrium points, system (8) also admits two finite equilibrium points  $e_1 = (1, 0)$  and  $e_2 = (-1, 0)$ . By using Theorem 2.19 of [20], we can find that both  $e_1$  and  $e_2$  are semi-hyperbolic saddle-nodes.

On the local chart  $U_1$  system (8) becomes

$$(9) \quad \frac{du}{dt} = u \left( \frac{\sqrt{6}}{n} u + v^2 - 1 \right), \quad \frac{dv}{dt} = v(v^2 - 1).$$

It has two infinite equilibrium points  $e_5 = (0, 0)$  and  $e_6 = (\sqrt{6}n/6, 0)$ , where  $e_5$  is a hyperbolic stable node with eigenvalues  $-1$  of multiplicity two, and  $e_6$  is a hyperbolic unstable saddle point with eigenvalues  $\pm 1$ .

On the local chart  $U_2$  system (8) writes

$$(10) \quad \frac{du}{dt} = u \left( -\frac{\sqrt{6}}{n} u + u^2 - v^2 \right), \quad \frac{dv}{dt} = -\frac{\sqrt{6}}{n} uv.$$

Let  $d\tau = udt$  we obtain

$$(11) \quad \frac{du}{d\tau} = -\frac{\sqrt{6}}{n} u + u^2 - v^2, \quad \frac{dv}{d\tau} = -\frac{\sqrt{6}}{n} v.$$

The origin  $e_7 = (0, 0)$  on the local chart  $U_2$  is a hyperbolic stable node with eigenvalues  $-\sqrt{6}/n$  of multiplicity two.

In short the global phase portraits of system (8) is shown in Figure 3.

**2.4. The finite equilibrium points.** It is noted that system (1) admits three finite equilibrium points  $p_0 = (0, 0, 0)$  with eigenvalues  $(-3, -2, 0)$ ,  $p_1 = (1, 0, 0)$  and  $p_2 = (-1, 0, 0)$  with the same eigenvalues  $(6, 1, 0)$ . Here  $p_1$  and  $p_2$  are located at the intersection of the invariant planes  $z = 0$ ,  $s = 0$  and the invariant surface  $x^2 - z^2 = 1$ , corresponding to the equilibrium points  $e_1$  and  $e_2$  in subsections 2.1-2.3, respectively. The origin  $p_0$  of system (1) lies at the intersection of the invariant planes  $z = 0$  and  $s = 0$ , which is the same point as the equilibrium point  $e_0$  studied in subsections 2.1 and 2.2.

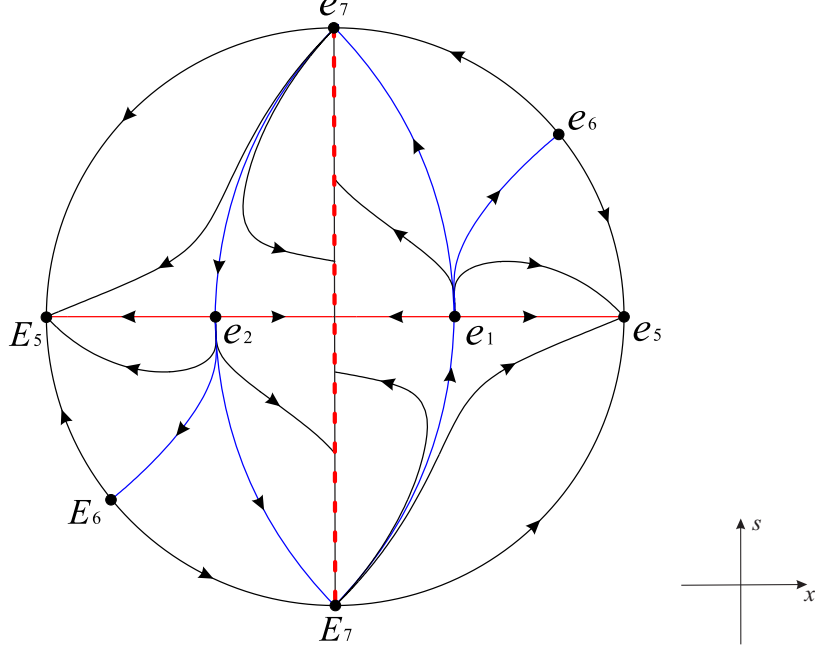


FIGURE 3. The phase portrait of the invariant surface  $x^2 - z^2 = 1$ .

### 3. PHASE PORTRAITS ON THE POINCARÉ SPHERE AT INFINITY

In order to describe the dynamics of system (1) at infinity. We use the method of the three-dimensional Poincaré compactification (see [21] for more details)  $x = 1/z_3$ ,  $z = z_1/z_3$ ,  $s = z_2/z_3$ , then the analytical vector field of system (1) on the local chart  $U_1$  becomes

$$\begin{aligned}
 \frac{dz_1}{dt} &= z_1 [z_3^2 - \sqrt{6}z_2(-1 + z_1^2 + z_3^2)], \\
 \frac{dz_2}{dt} &= z_2 \left[ -3 + 2z_1^2 + 3z_3^2 + \sqrt{6}z_2 \left( \frac{1}{n} + 1 - z_1^2 - z_3^2 \right) \right], \\
 \frac{dz_3}{dt} &= -z_3 [z_1^2(-2 + \sqrt{6}z_2) + (-3 + \sqrt{6}z_2)(-1 + z_3^2)].
 \end{aligned}
 \tag{12}$$

In the different local charts of the Poincaré sphere, the infinity of  $\mathbb{R}^3$  corresponds to  $z_3 = 0$ , then system (12) has the equilibrium points listed in Table 1. Except that the equilibrium point  $u_{31}$  denotes the

origin of the local chart  $U_3$ , the rest equilibrium points lie in the local chart  $U_1$ . Moreover the straight line  $s = 0$  of the local chart  $U_1$  is filled with the equilibrium points  $u_{a0}$  for all  $a \in \mathbb{R}$ .

TABLE 1. Equilibrium points on the different local charts of the Poincaré sphere at the infinity of  $\mathbb{R}^3$ .

Equilibrium points	Eigenvalues
$u_{11} = (0, 0, 0)$	$(-3, -3, 0)$
$u_{12} = \left(-\frac{\sqrt{6}}{2}, 0, 0\right)$	$(0, 0, 0)$
$u_{13} = \left(\frac{\sqrt{6}}{2}, 0, 0\right)$	$(0, 0, 0)$
$u_{14} = \left(-1, \frac{\sqrt{6}}{6}n, 0\right)$	$(-1, 1, -2n)$
$u_{15} = \left(1, \frac{\sqrt{6}}{6}n, 0\right)$	$(-1, 1, -2n)$
$u_{16} = \left(0, \frac{\sqrt{6}}{2} \frac{n}{1+n}, 0\right)$	$\left(-\frac{3}{1+n}, \frac{3n}{1+n}, 3\right)$
$u_{a0} = (a, 0, 0)$	$(0, -3 + 2a^2, -3 + 2a^2)$
$u_{31} = (0, 0, 0)$	$(0, 0, 0)$

For the case  $z_3 = 0$  system (12) is reduced to

$$(13) \quad \begin{aligned} \frac{dz_1}{dt} &= -\sqrt{6}z_1z_2(-1 + z_1^2), \\ \frac{dz_2}{dt} &= z_2 \left[ -3 + 2z_1^2 + \sqrt{6}z_2 \left( \frac{1}{n} + 1 - z_1^2 \right) \right]. \end{aligned}$$

After changing the of time  $d\tau = z_2dt$  system (13) becomes

$$(14) \quad \begin{aligned} \frac{dz_1}{d\tau} &= -\sqrt{6}z_1(-1 + z_1^2), \\ \frac{dz_2}{d\tau} &= -3 + 2z_1^2 + \sqrt{6}z_2 \left( \frac{1}{n} + 1 - z_1^2 \right). \end{aligned}$$



Then this system has three equilibrium points  $e_{i,1}$ ,  $e_{i,2}$  and  $e_{i,3}$  with coordinates  $(-1, \sqrt{6}n/6)$ ,  $(1, \sqrt{6}n/6)$  and  $(0, \sqrt{6}n/(2(1+n)))$ , respectively. Here both  $e_{i,1}$  and  $e_{i,2}$  are unstable saddle points with eigenvalues  $\sqrt{6}/n$  and  $-2\sqrt{6}$ .  $e_{i,3}$  is an unstable node with eigenvalues  $\sqrt{6}$  and  $\sqrt{6}(1+n)/n$ . The phase portrait on the Poincaré sphere at infinity on local chart  $U_1$  is shown in Figure 4.

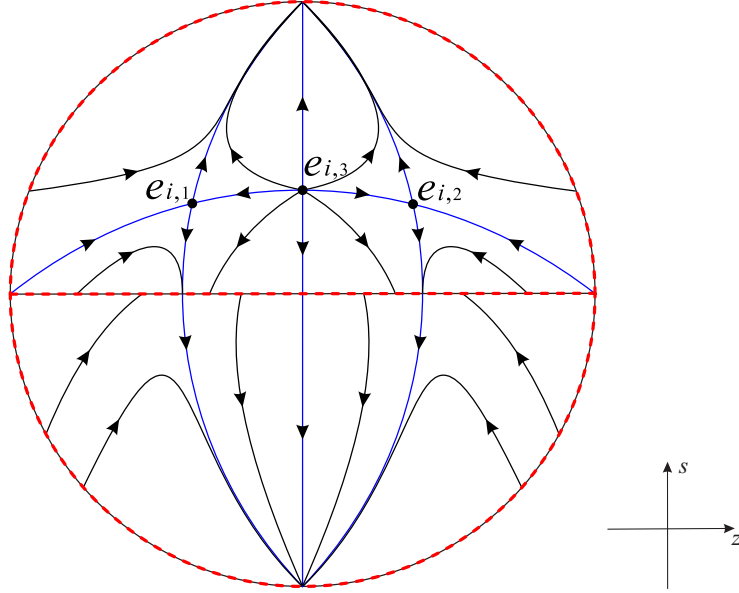


FIGURE 4. The phase portrait of system (1) at infinity on local chart  $U_1$ .

On the local chart  $U_2$  in view of Poincaré compactification  $x = z_1/z_3$ ,  $z = 1/z_3$ ,  $s = z_2/z_3$ , system (1) writes

$$\begin{aligned}
 \frac{dz_1}{dt} &= -z_1 z_3^2 + \sqrt{6} z_2 (1 - z_1^2 + z_3^2), \\
 \frac{dz_2}{dt} &= z_2 \left( 2 - 3z_1^2 + \frac{\sqrt{6}}{n} z_1 z_2 + 2z_3^2 \right), \\
 \frac{dz_3}{dt} &= z_3 (2 - 3z_1^2 + 2z_3^2).
 \end{aligned}
 \tag{15}$$

Since we want to study the infinity we take  $z_3 = 0$ , rescaling the time  $d\tau = z_2 dt$  system (15) is equivalent to

$$(16) \quad \begin{aligned} \frac{dz_1}{d\tau} &= \sqrt{6}(1 - z_1^2), \\ \frac{dz_2}{d\tau} &= 2 - 3z_1^2 + \frac{\sqrt{6}}{n}z_1z_2. \end{aligned}$$

For any constant  $b$ , since  $(b, 0)$  is not the equilibrium point of the system (16), we will not continue to discuss other infinite equilibrium points of the system (16). Because this has been discussed in the case  $U_1$ .

On the local chart  $U_3$  the three-dimensional Poincaré compactification is  $x = z_1/z_3$ ,  $z = z_2/z_3$ ,  $s = 1/z_3$ , then system (1) becomes

$$(17) \quad \begin{aligned} \frac{dz_1}{dt} &= z_1 \left[ -\frac{\sqrt{6}(1+n)}{n}z_1 + 3z_1^2 - 2z_2^2 - 3z_3^2 \right] + \sqrt{6}(z_2^2 + z_3^2), \\ \frac{dz_2}{dt} &= z_2 \left[ -\frac{\sqrt{6}}{n}z_1 + 3z_1^2 - 2(z_2^2 + z_3^2) \right], \\ \frac{dz_3}{dt} &= -\frac{\sqrt{6}}{n}z_1z_3. \end{aligned}$$

For the case  $z_3 = 0$  from system (17) we obtain

$$(18) \quad \begin{aligned} \frac{dz_1}{dt} &= z_1 \left[ -\frac{\sqrt{6}(1+n)}{n}z_1 + 3z_1^2 - 2z_2^2 \right] + \sqrt{6}z_2^2, \\ \frac{dz_2}{dt} &= z_2 \left( -\frac{\sqrt{6}}{n}z_1 + 3z_1^2 - 2z_2^2 \right). \end{aligned}$$

It is noted that  $(0, 0)$  is a linearly zero equilibrium point, the topological index is zero from the Poincaré-Hopf theory (see Theorem 6.30 of [20] for more details). In order to study its local phase portrait we shall use vertical blow-ups (see for instance [22]) by letting  $w = z_2/z_1$ , then

we obtain

$$(19) \quad \begin{aligned} \frac{dz_1}{dt} &= z_1^2 \left[ \sqrt{6} \left( -\frac{1}{n} - 1 + w^2 \right) + (3 - 2w^2) z_1 \right], \\ \frac{dw}{dt} &= -\sqrt{6} z_1 w (-1 + w^2). \end{aligned}$$

Eliminating the common factor  $z_1$  of system (19) by changing the time  $d\tau = z_1 dt$  it yields

$$(20) \quad \begin{aligned} \frac{dz_1}{d\tau} &= z_1 \left[ \sqrt{6} \left( -\frac{1}{n} - 1 + w^2 \right) + (3 - 2w^2) z_1 \right], \\ \frac{dw}{d\tau} &= -\sqrt{6} w (-1 + w^2). \end{aligned}$$

System (20) admits three equilibrium points  $e_{i,4} = (0, -1)$ ,  $e_{i,5} = (0, 1)$  and  $e_{i,6} = (0, 0)$  on  $z_1 = 0$ , where  $e_{i,4}$  and  $e_{i,5}$  are two hyperbolic stable nodes with eigenvalues  $-\sqrt{6}/n$  and  $-2\sqrt{6}$ ,  $e_{i,6}$  is a hyperbolic unstable saddle point with eigenvalues  $\sqrt{6}$  and  $-\sqrt{6}(1+n)/n$ . The local phase portraits around  $e_{i,4}$ ,  $e_{i,5}$  and  $e_{i,6}$  are shown in Figure 5(a). Note that there is a time rescaling  $d\tau = z_1 dt$  between systems (19) and (20), so the direction of the trajectories in the local phase portraits of system (19) is opposite to that of Figure 5(a) when  $z_1 < 0$ , see Figure 5(b) for more details. In addition, all points on the  $w$  axis, i.e.  $z_1 = 0$ , are singularities of system (19). Thus the local phase portraits at  $(0, 0)$  of system (18) is shown in Figure 5(c). Then the phase portrait in the local chart  $U_3$  is shown in Figure 6.

In summary joining the previous information we obtain the global phase portraits at infinity in the Poincaré sphere in Figure 7.

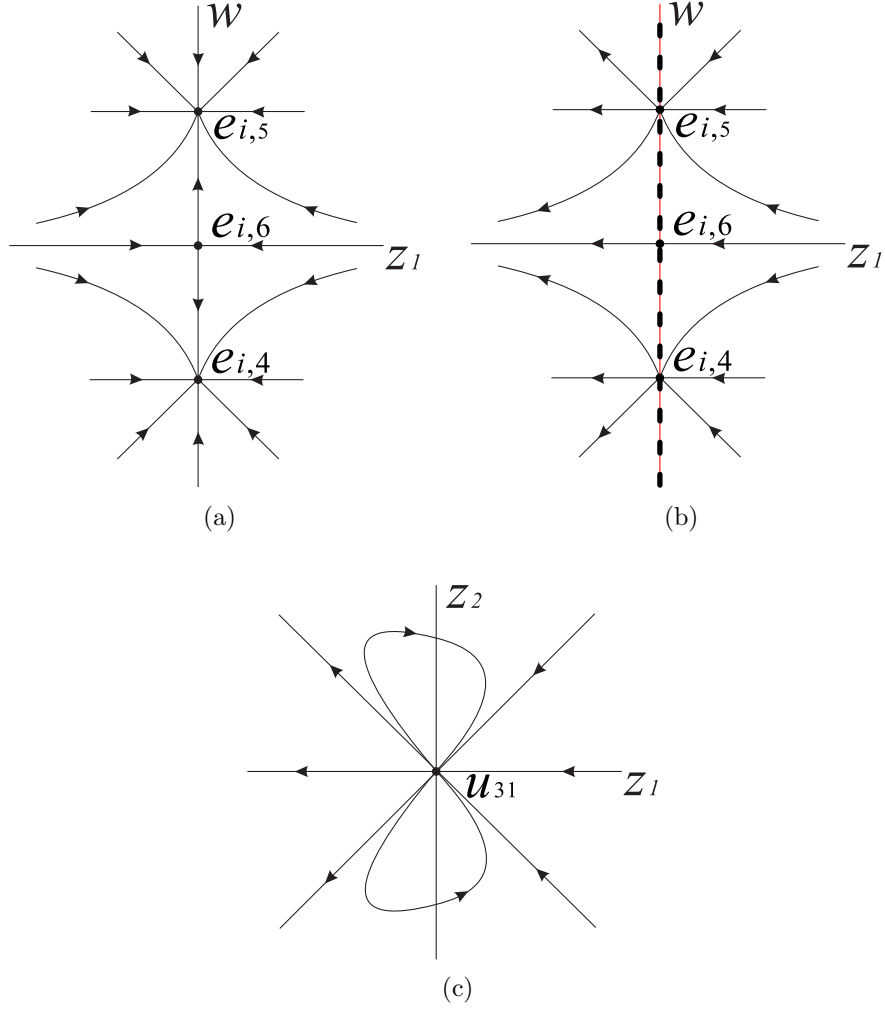


FIGURE 5. In (a), (b) and (c) there are the local phase portraits of the equilibrium points in systems (20), (19) and (18), respectively.

#### 4. PHASE PORTRAITS INSIDE THE POINCARÉ BALL RESTRICTED TO

$$x^2 - z^2 \leq 1$$

It is noted that system (1) is invariant under the symmetries  $(x, z, s) \mapsto (-x, -z, -s)$  and  $(x, z, s) \mapsto (-x, z, -s)$ , so it is invariant under the

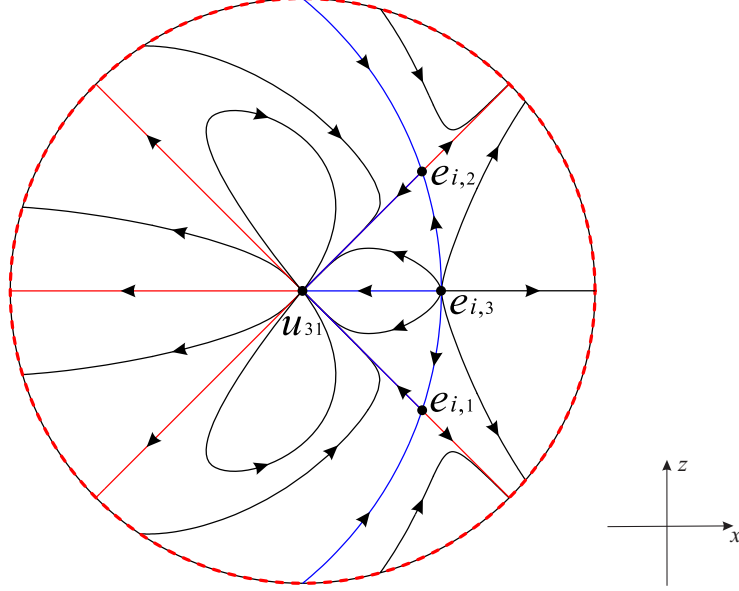


FIGURE 6. The phase portrait of system (18), i.e. the phase portrait at infinity in the local chart  $U_3$  of system (1).

symmetry with respect to the origin and to the  $z$ -axis. Now we divide the Poincaré ball restricted to  $x^2 - z^2 \leq 1$  into the following four regions:

$$\begin{aligned} R_1 : z \leq 0, s \geq 0. \quad R_2 : z \leq 0, s \leq 0. \\ R_3 : z \geq 0, s \geq 0. \quad R_4 : z \geq 0, s \leq 0. \end{aligned}$$

Then due to the symmetries we only need to study the phase portraits of system (1) in the region  $R_1$ .

Combining the phase portraits of the invariant surface  $x^2 - z^2 = 1$  with the phase portraits of the planes  $z = 0$ , and  $s = 0$ , together with the phase portrait at infinity, we get the phase portraits on the boundary of the region  $R_1$  as shown in Figures 8-10. Here we explain the definition of the three-dimensional cartesian coordinate system in this paper as follows: we regard the  $xz$ -plane as the horizontal plane

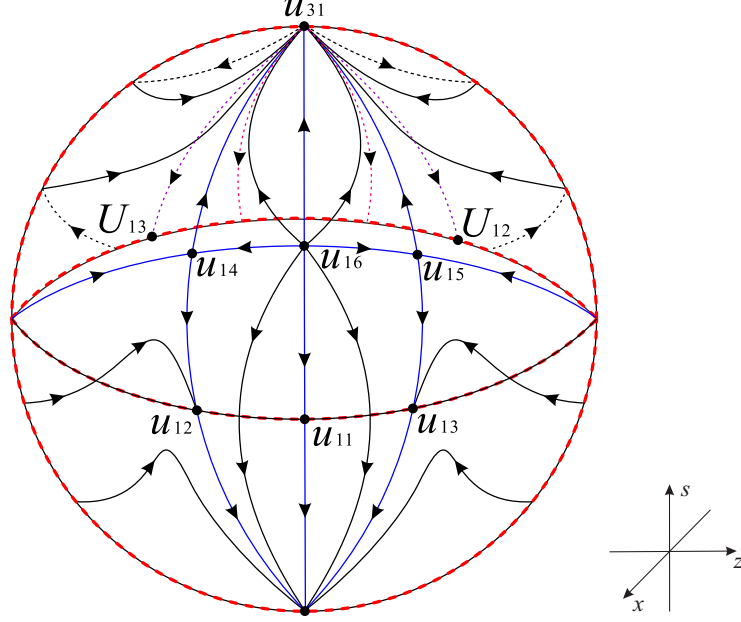


FIGURE 7. The global phase portrait at infinity in the Poincaré sphere. The North Pole of the Poincaré ball is the equilibrium point  $u_{31}$ . The symmetric points of  $u_{12}$  and  $u_{13}$  with respect to the center of the sphere are  $U_{12}$  and  $U_{13}$ , respectively.

in  $\mathbb{R}^3$ , where the direction of the  $z$ -axis is horizontal towards the right. If the  $z$ -axis axis rotates 90 degrees clockwise we get the  $x$ -axis. The  $s$ -axis is vertical upward, and then  $xzs$  constitutes a three-dimensional right-handed cartesian coordinate system.

In order to show the phase portraits more clearly, the boundary of the region  $R_1$  is divided into six surfaces according to the orientation towards us, the back to us and the bottom area. See Figure 11 for more details. It is noted that the North Pole  $u_{31}$  of the Poincaré ball on the front boundary surfaces  $F_1$  and  $F_2$  is stable, and there is an elliptic sector and a stable parabolic sector segment of  $U_3$  on the boundary  $F_3$ , but it is unstable on the back boundary surfaces  $B_1$  and  $B_2$ .

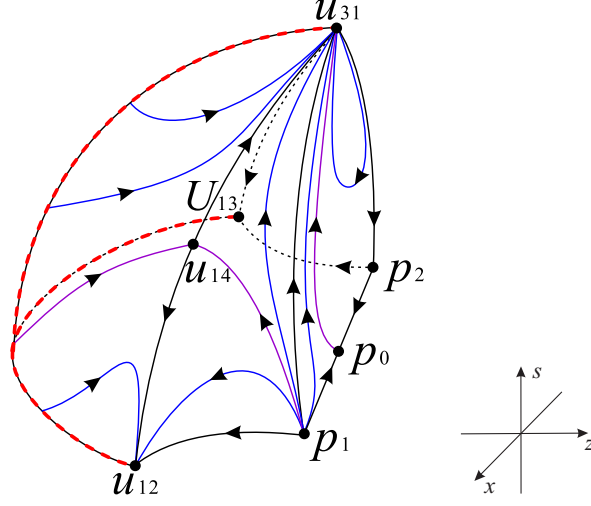


FIGURE 8. Phase portrait in the front boundary of the region  $R_1$ .

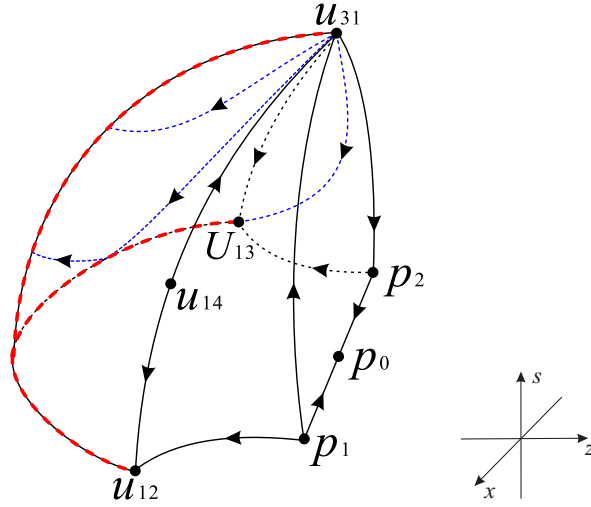


FIGURE 9. Phase portrait in the back boundary of the region  $R_1$ .

## 5. DYNAMICS IN THE INTERIOR OF THE REGION $R_1$

Note that the original system (1) admits the three finite equilibrium points  $p_0$ ,  $p_1$  and  $p_2$  in the three-dimensional cartesian coordinate

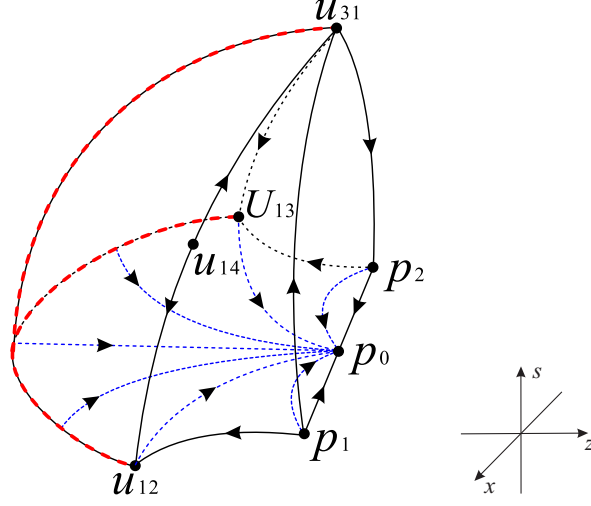


FIGURE 10. Phase portrait in the bottom of the region  $R_1$ .

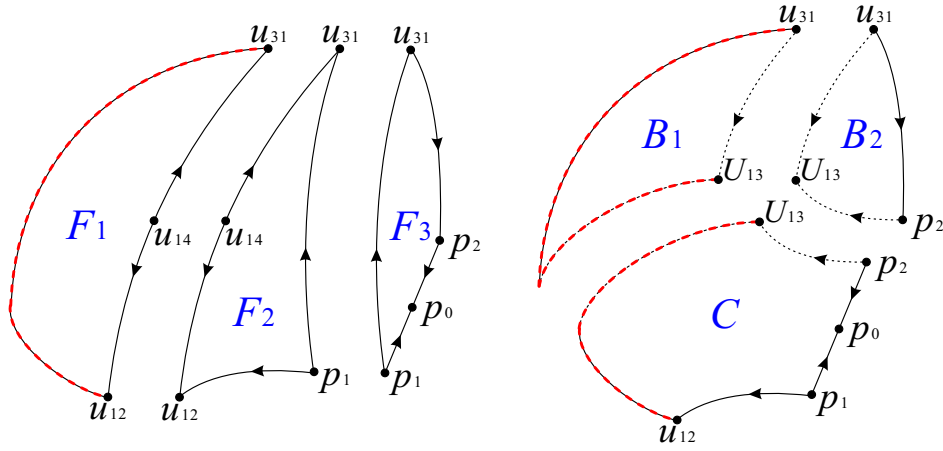


FIGURE 11. The six boundary surfaces of the region  $R_1$ .

system. The dynamical behavior of the system inside the region  $R_1$  depends on the behavior of the flow in the following surfaces and planes

$$h(x, z, s) = 0, \quad g(x, z) = 0, \quad x = 0, \quad z = 0, \quad s = 0,$$



where

$$\begin{aligned} h(x, z, s) &= x(3x^2 - 2z^2 - 3) + \sqrt{6}s(1 - x^2 + z^2), \\ g(x, z) &= 3x^2 - 2z^2 - 2. \end{aligned}$$

These surfaces and planes divide the region  $R_1$  into seven different subregions  $R_{1i}$ ,  $i = (1, 2, \dots, 7)$ , see Figures 12-19 for more details. It should be noted that  $R_{11}$  and  $R_{13}$  represent parallel tunnels inside  $R_1$ , respectively. It is easy to verify that  $h > 0$  in the subregions  $R_{12}$ ,  $R_{14}$ ,  $R_{15}$  and  $R_{16}$ , and  $h < 0$  in the subregions  $R_{11}$ ,  $R_{13}$  and  $R_{17}$ . Similarly, we can find that  $g > 0$  in the subregions  $R_{11}$ ,  $R_{12}$ ,  $R_{16}$  and  $R_{17}$ , and  $g < 0$  in the subregions  $R_{13}$ ,  $R_{14}$  and  $R_{15}$ . It should be noted that the dotted and solid lines in Figures 13-19 are consistent with those in Figure 12.

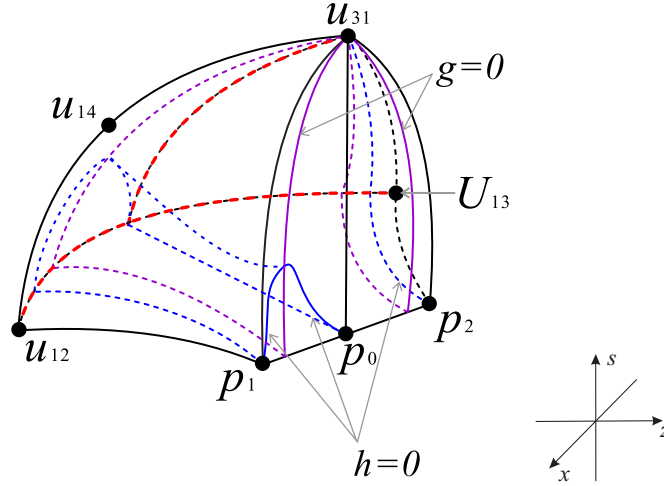
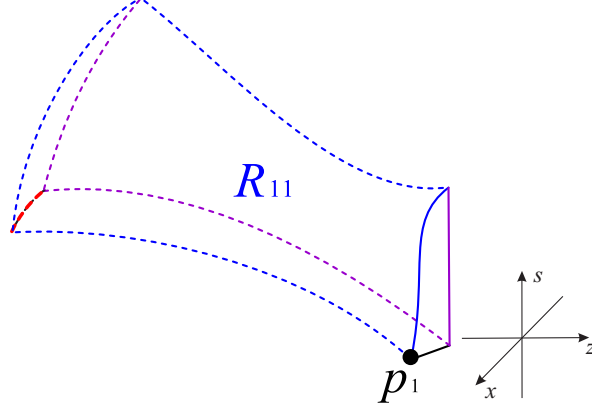
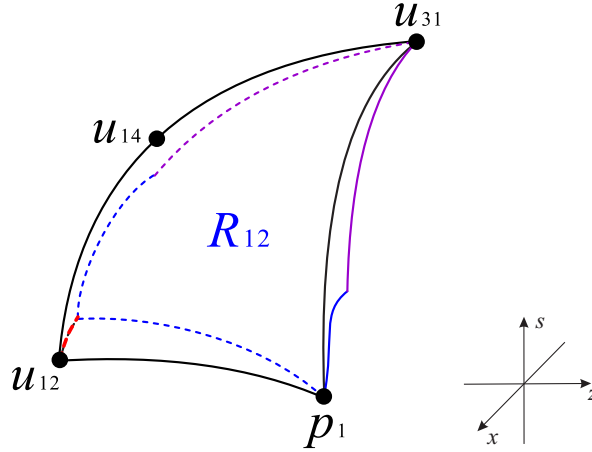


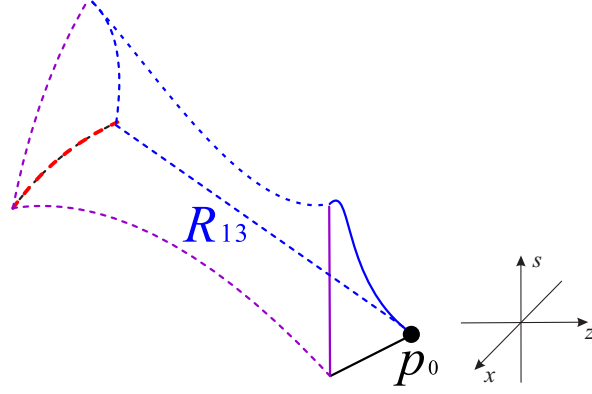
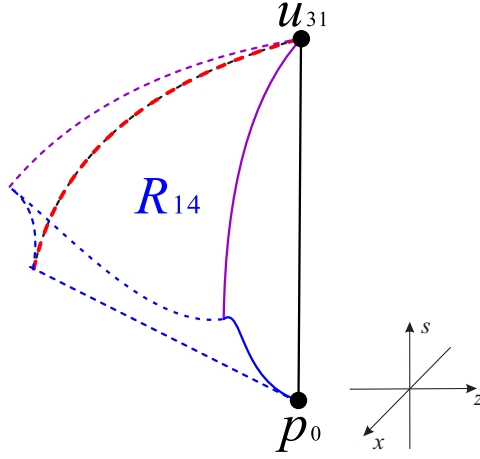
FIGURE 12. There are seven subregions inside the region  $R_1$  of the Poincaré ball.

As shown in the subregion  $R_{11}$  (see Figure 13) the front surface consists of three dashed lines and one solid line contained in the surface

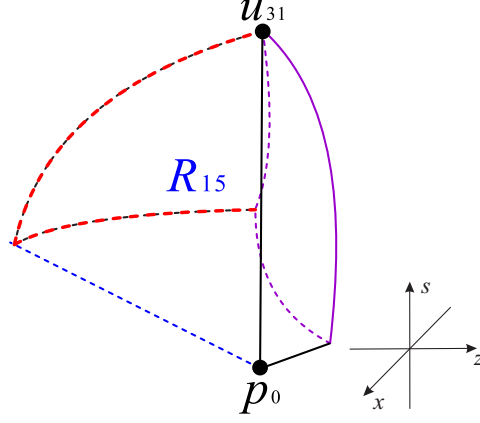
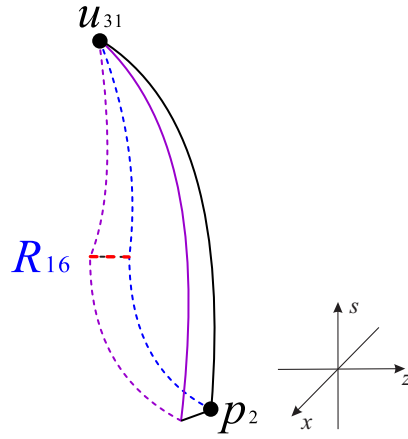
FIGURE 13. The subregion  $R_{11}$ .FIGURE 14. The subregion  $R_{12}$ .

$h = 0$ , and the surface on the back side (opposite to the above front surface) contained in the surface  $g = 0$ .

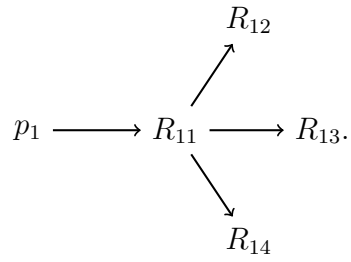
In Table 2 we describe the behavior of  $\dot{x}$ ,  $\dot{z}$  and  $\dot{s}$  in the seven subregions  $R_{11}, \dots, R_{17}$ . From this table we obtain that the variables  $x$  and  $z$  in the subregion  $R_{11}$  decrease monotonically, and the variable  $s$  increases monotonically, so an orbit in the subregion  $R_{11}$  either crosses the boundary  $h = 0$  and enters into the subregion  $R_{12}$ , or crosses the

FIGURE 15. The subregion  $R_{13}$ .FIGURE 16. The subregion  $R_{14}$ .

boundary  $g = 0$  and goes to the subregion  $R_{13}$ , or crosses the intersection curve of the subregions  $R_{11}$ ,  $R_{12}$  and  $R_{13}$  into the subregion  $R_{14}$  with very low probability, i.e. an orbit in the subregion  $R_{11}$  will not stay in the future in this region, but will exit through its boundaries into other subregions, and these orbits in backwards time come from the equilibrium point  $p_1$  in the subregion  $R_{11}$ . This process can be

FIGURE 17. The subregion  $R_{15}$ .FIGURE 18. The subregion  $R_{16}$ .

simply summarized as follows



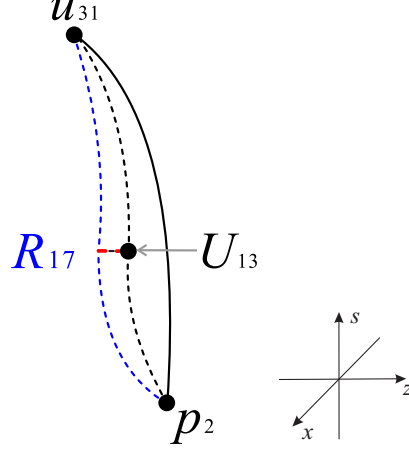
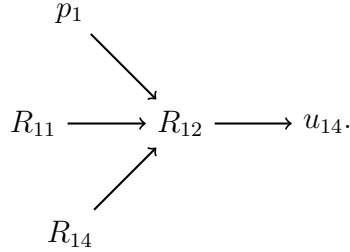
FIGURE 19. The subregion  $R_{17}$ .

TABLE 2. Dynamical behavior in seven different subregions

Subregions	Corresponding Region	Increase or decrease
$R_{11}$	$h < 0, g > 0, x > 0, z < 0, s > 0$	$\dot{x} < 0, \dot{z} < 0, \dot{s} > 0$
$R_{12}$	$h > 0, g > 0, x > 0, z < 0, s > 0$	$\dot{x} > 0, \dot{z} < 0, \dot{s} > 0$
$R_{13}$	$h < 0, g < 0, x > 0, z < 0, s > 0$	$\dot{x} < 0, \dot{z} > 0, \dot{s} > 0$
$R_{14}$	$h > 0, g < 0, x > 0, z < 0, s > 0$	$\dot{x} > 0, \dot{z} > 0, \dot{s} > 0$
$R_{15}$	$h > 0, g < 0, x > 0, z < 0, s > 0$	$\dot{x} > 0, \dot{z} > 0, \dot{s} < 0$
$R_{16}$	$h > 0, g > 0, x > 0, z < 0, s > 0$	$\dot{x} > 0, \dot{z} < 0, \dot{s} < 0$
$R_{17}$	$h < 0, g > 0, x > 0, z < 0, s > 0$	$\dot{x} < 0, \dot{z} < 0, \dot{s} < 0$

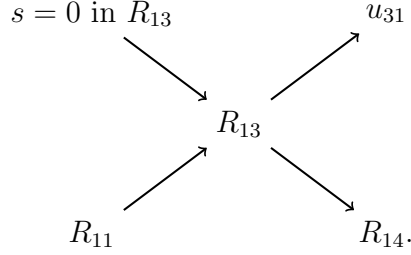
The front boundary of  $R_{12}$  (see Figure 14), i.e. the surface containing the equilibrium points  $u_{12}$ ,  $u_{14}$ ,  $u_{31}$  and  $p_1$ , is contained in the invariant surface  $x^2 - z^2 = 1$  ( $x > 0$ ). The surface on the back side of  $R_{12}$  that contains the equilibrium points  $p_1$  and  $u_{31}$  consists of two parts: the upper part surface containing the infinite equilibrium point  $u_{31}$  is the intersection of the subregions  $R_{12}$  and  $R_{14}$ , which is contained in the

surface  $g = 0$ , and the lower part surface containing the finite equilibrium point  $p_1$ , is the intersection of subregions  $R_{11}$  and  $R_{12}$  that is contained in the surface  $h = 0$ . The equilibrium points in the subregion  $R_{12}$  are the finite equilibrium point  $p_1$ , the infinite equilibrium points on the Poincaré sphere at  $s = 0$  and the equilibrium point  $u_{14}$  at infinity. However an orbit in the subregion  $R_{12}$  does not start from these infinite equilibrium points at  $s = 0$  or return to these points, but it comes from the finite equilibrium point  $p_1$  or from the subregions  $R_{11}$  and  $R_{14}$  which have common boundaries with the subregion  $R_{12}$ . This is due to the fact that the variable  $z$  is decreasing monotonically and the variables  $x$  and  $s$  are increasing monotonically inside the subregion  $R_{12}$  according to Table 2. Moreover a trajectory in the subregion  $R_{12}$  will tend to the equilibrium point  $u_{14}$  at infinity. In summary, this dynamic behavior process can be represented as

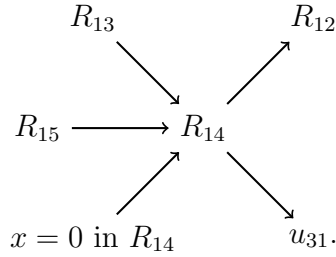


In the subregion  $R_{13}$  (see Figure 15) the front surface composed of three dashed lines and one solid line is contained in the surface  $g = 0$ , and its opposite surface is contained in the surface  $h = 0$ . From Table 2 an orbit in the subregion  $R_{13}$  may only comes from the equilibrium points locate at the infinity on  $s = 0$  in  $R_{13}$ , or it comes from the subregions  $R_{11}$ . Then the orbit passes through the intersection of subregions  $R_{13}$  and  $R_{14}$  (contained in the surface  $h = 0$ ) and enters

the subregion  $R_{14}$ , or tends to the infinite equilibrium point  $u_{31}$ , i.e. lies in the North Pole of the Poincaré sphere. So we obtain the following dynamics

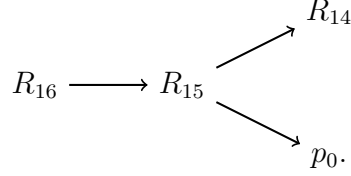


In subregion  $R_{14}$  (see Figure 16) the front triangular surface is contained in  $g = 0$ , and its opposite triangular surface is on the back side, which is included in the plane  $x = 0$ . According to Table 2 an orbit of the subregion  $R_{14}$  may come from the infinite equilibrium points on  $x = 0$ , or come from the subregion  $R_{13}$  through the surface  $h = 0$ , or from subregion  $R_{15}$  traversing plane  $x = 0$ , then go through the surface  $g = 0$  into the subregion  $R_{12}$ , or directly goes to the infinite equilibrium point  $u_{31}$  at the North Pole of the Poincaré sphere. This can be represented as



It can be noted from Figure 17 that the surface on the left of subregion  $R_{15}$  (consisting of two dashed lines and one solid line) is contained in plane  $x = 0$ , and the opposite surface to it is included in the surface  $g = 0$ . According to Table 2 an orbit in the subregion  $R_{15}$  can only come from the adjacent subregion  $R_{16}$ , and then either enter into the

subregion  $R_{14}$  adjacent to it, or directly tend to the finite equilibrium point  $p_0$  located at the center of the Poincaré ball. This dynamics can be denoted as



We observe that the curved surfaces of the left and right sides of the subregion  $R_{16}$  (see Figure 18) are contained in the surfaces  $g = 0$  and  $h = 0$ , respectively. It is known from Table 2 that an orbit in the subregion  $R_{16}$  can only come from the subregion  $R_{17}$  and crossing the surface  $g = 0$ , it enters into the subregion  $R_{15}$ . Thus we get that

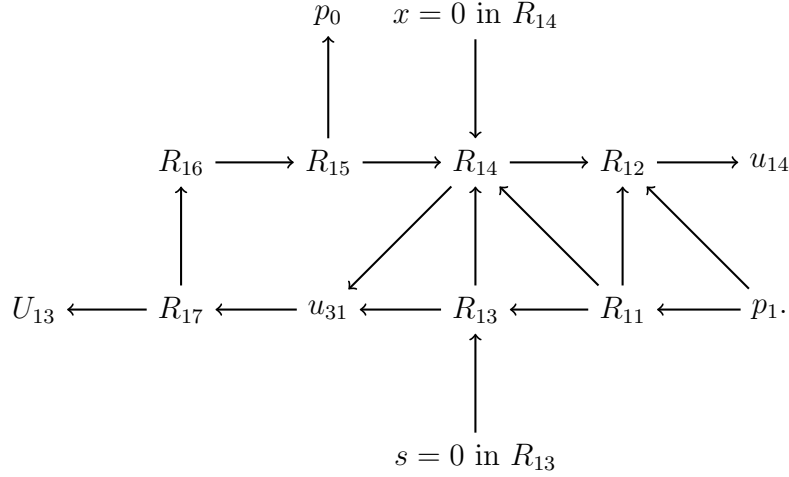
$$R_{17} \longrightarrow R_{16} \longrightarrow R_{15}.$$

The left and right surfaces of subregion  $R_{17}$  (see Figure 19) are included in the surface  $h = 0$  and in the invariant surface  $x^2 - z^2 = 1$  ( $x < 0$ ), respectively. According to Table 2, it is known that all three variables in subregion  $R_{17}$  are decreasing monotonically, so an orbit in this subregion must come from the infinite equilibrium point  $u_{31}$  located at the North Pole of Poincaré sphere and eventually tends to the infinite equilibrium point  $U_{13}$  in the invariant plane  $s = 0$ . Then we have that

$$u_{31} \longrightarrow R_{17} \longrightarrow U_{13}.$$



The dynamic behavior of the orbits inside the seven subregions of  $R_1$  discussed above can be summarized as



The above flow chart shows that the orbits of system (1) contained in the interior of the region  $R_1$  have  $\alpha$ -limit at the finite equilibrium point  $p_1$ , and the some orbits on the boundary of the region  $R_1$  have  $\alpha$ -limit at the  $x = 0$  in  $R_{14}$ , or  $s = 0$  in  $R_{13}$ . Moreover the orbits have  $\omega$ -limit either at the finite equilibrium point  $p_0$ , or at the infinite equilibrium points  $u_{14}$  and  $U_{13}$ , where  $u_{14}$  is located on the intersection curve of the Poincaré sphere and the invariant surface  $x^2 - z^2 = 1$  ( $x > 0$ ) at infinity, and  $U_{13}$  is located on the intersection of the Poincaré sphere and the invariant surfaces  $s = 0$ ,  $x^2 - z^2 = 1$  ( $x < 0$ ) at infinity (see Figure 12). Furthermore the orbits of system (1) on the boundary of  $R_1$  also have  $\alpha$ -limit and  $\omega$ -limit at the infinite equilibrium point  $u_{31}$ , i.e. the North Pole of the Poincaré sphere.

In this way the qualitative global dynamic behavior of system (1) is described.

## 6. DISCUSSION AND CONCLUSIONS

The physical meaning of the variables  $x$ ,  $z$  and  $s$  in this paper can be found in references [3, 7, 8, 9, 14]. So we have

$$(21) \quad x = \frac{\dot{\phi}}{2\sqrt{6}H}, \quad z = \frac{\mu}{4(3\lambda - 1)a^2H} \quad s = -\frac{V'(\phi)}{V(\phi)},$$

where  $\lambda$  is a dimensionless constant,  $V(\phi)$  is the potential,  $\mu$  is a constant, and  $H = \dot{a}/a$  is the Hubble parameter, and  $a$  is the dimensionless scale factor for the expanding universe.

Near infinity we note that the phase portrait described in Figure 2 has been obtained in [9], where the phase portrait of system (1) is limited to  $s = 0$ . Now we have completely described the phase portrait of system (1) with all  $s$  values in the region  $G$  as shown in Figures 8-10 and 12. By using the fact that system (1) remains invariant under two aforementioned symmetries, we provide the complete global phase portrait of system (1) in  $G$ . The phase portrait shows that the final evolution of the orbital direction of system (1) within  $G$  tends to the center of the Poincaré ball as well as the infinite equilibrium points  $u_{14}$  and  $U_{13}$  located at the waist and equator of the Poincaré sphere (except for the initial conditions  $s = 0$ ), respectively. Based on Hořava-Lifshitz gravity in a Friedmann-Lemaître-Robertson-Walker space-time with non-zero curvature and without the cosmological constant term, equations (21) imply that the value of Hubble parameter  $H$  is tending to be zero in the cosmological model as time goes on.

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<sup>1</sup>SCHOOL OF MATHEMATICAL SCIENCE, YANGZHOU UNIVERSITY, YANGZHOU 225002, CHINA

E-mail: gaofabao@sina.com (Fabao Gao, ORCID 0000-0003-2933-1017)

<sup>2</sup> DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, BELLATERRA 08193, BARCELONA, CATALONIA, SPAIN

E-mail: jllibre@mat.uab.cat (Jaume Llibre, ORCID 0000-0002-9511-5999)