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INTEGRABILITY AND ZERO-HOPF BIFURCATION IN THE SPROTT A SYSTEM

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Abstract. The first objective of this paper is to study the Darboux integrability of the polynomial differential system

$$\dot{x} = y, \quad \dot{y} = -x - yz, \quad \dot{z} = y^2 - a$$

and the second one is to show that for $a > 0$ sufficiently small this model exhibits one small amplitude periodic solution that bifurcates from the origin of coordinates when $a = 0$. This model was introduced by Hoover as the first example of a differential equation with a hidden attractor and it was used by Sprott to illustrate a differential equation having a chaotic behavior without equilibrium points, and now this system is known as the Sprott A system.

1. Introduction and statement of the main results

In differential equations it is very important to know whether a differential system presents chaos. One might think that it is not possible to have chaotic behavior in a system without equilibrium points, but now we know that this is not the case. In this paper we study one of these differential systems. More precisely we consider the well-known example of differential equation with a hidden chaotic attractor, which is a special case of the Nosé–Hoover oscillator [10] and as showed in [18] describes many natural phenomena. The Nosé–Hoover thermostat is a differential method used in molecular dynamics to keep the temperature around an average. It was first introduced by Nosé in 1985 [17] and developed further by Hoover [9]. In this paper we consider the Nosé–Hoover equation in the form

$$\dot{x} = y, \quad \dot{y} = -x - yz, \quad \dot{z} = y^2 - a, \tag{1}$$

where $a \in \mathbb{R}$ is a parameter and the dot denotes derivative with respect to the time t. In [21] Sprott listed this system as one of the differential systems without equilibrium points having chaotic behavior (when $a = 1$), and from then system (1) is usually called *Sprott A system*, see for instance [16]. When $a = 0$ the system has a continuum of equilibria, i.e. $(0, 0, z)$ for

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all $z \in \mathbb{R}$. For $a \neq 0$ it has no equilibria and for $a = 1$ it generates a chaotic attractor, see for more details [21].

A number of dynamical aspects of system (1) have been analysed such as the formation of hidden chaotic attractors as well as the formation of nested invariant tori, and the dynamics at infinity (see [16] and the references therein). Here our first main objective is to further contribute to the understanding the complexity of the system, or more precisely the dynamics of system (1) by studying its integrability. We note that the existence of a first integral allows to reduce its study in one dimension. Moreover, inside the class of first integrals, the simpler ones are the so called Darboux first integrals, for more information about these kind of first integrals see for instance [5, 11, 14, 15] and subsection 2.1.

The second objective of this paper is to study the zero-Hopf bifurcation which exhibits the polynomial differential system (1) when $a = 0$. The main tool up to now for studying a zero-Hopf bifurcation is to pass the system to the normal form of a zero-Hopf bifurcation, later on this introduction we provide the references about this. Our analysis of the zero-Hopf bifurcation is different, we study them directly using the averaging theory, see subsection 2.2 for a summary of the results of this theory that we need in this paper. We note that this paper is one of the few papers where we study a zero-Hopf bifurcation that is not isolated.

Let U be an open and dense subset of $\mathbb{R}^3$, we say that a non-locally constant C^1 function $H\colon U \to \mathbb{R}$ is a *first integral* of system (1) on U if $H(x(t), y(t), z(t))$ is constant for all of the values of t such that for which $(x(t), y(t), z(t))$ is a solution of system (1) contained in U.

The first main result of this paper is the following. See section 2 for the definition of Darboux polynomial and Darboux first integral.

Theorem 1. *The next statements hold for system* (1)*:*

(i) *If $a \neq 0$ it has no Darboux polynomials.*
(ii) *If $a = 0$ the unique Darboux polynomials have cofactor zero and are polynomials in the variable $H = x^2 + y^2 + z^2$.*
(iii) *If $a \neq 0$ it has no Darboux first integrals.*
(iv) *If $a = 0$ the unique Darboux first integrals are Darboux functions in the variable H.*

The second main objective of this paper is to show that system (1) exhibits one small amplitude periodic solution for $a > 0$ sufficiently small that bifurcates from the non-isolated zero-Hopf equilibrium $(0, 0, 0)$ localized at the z-axis filled of equilibria when $a = 0$.

We recall that a *zero-Hopf equilibrium* of a 3-dimensional autonomous differential system is an equilibrium such that it has a zero real eigenvalue and a pair of purely imaginary eigenvalues. We know that generically an isolated zero-Hopf equilibrium exhibits a zero-Hopf bifurcation, which is a two-parameter unfolding of a 3-dimensional autonomous differential system with a zero-Hopf equilibria. The unfolding can exhibit different topological type of dynamics in a small neighborhood of this non-isolated equilibria and in some cases the zero-Hopf bifurcation can imply a local birth of chaos (see for instance [3, 7, 8, 12, 20]). This previous unfolding of a zero-Hopf bifurcation cannot be applied to our differential system (1) because first our system has only one parameter and second because the zero-Hopf equilibrium of system (1) is not isolated. For studying the zero-Hopf bifurcation of system (1), we shall use the averaging theory in a similar way at it was used in [2].

The second main result of this paper characterizes the zero-Hopf bifurcation of system (1).

Theorem 2. *The differential system* (1) *has a zero-Hopf bifurcation at the non isolated equilibrium point* $(0,0,0)$ *when* $a=0$ *producing one small periodic solution for* $a>0$ *sufficiently small with shape*

$$x(t)=\sqrt{2a}\cos t+O(a),\quad y(t)=-\sqrt{2a}\sin t+O(a),\quad z(t)=O(a).$$

From the numerical work [16] on the differential system (1) we see that the periodic orbit found in Theorem 2 is surrounded by invariant tori.

The paper is organized as follows. In section 2 we present the basic definitions and necessary results to prove Theorems 1 and 2. In section 3 we prove Theorem 1 and in section 4 we prove Theorem 2.

2. Preliminaries

2.1. **Darboux theory of integrability.** We denote by $\mathbb{C}[x,y,z]$ the ring of polynomials in the variables x,y,z and coefficients in $\mathbb{C}$. Given $f\in\mathbb{C}[x,y,z]\setminus\mathbb{C}$ we say that the polynomial $f(x,y,z)$ is a *Darboux polynomial* of system (1) if there exists $K\in\mathbb{C}[x,y,z]$ called the *cofactor* such that

$$y\frac{\partial f}{\partial x}-(x+yz)\frac{\partial f}{\partial y}+(y^2-a)\frac{\partial f}{\partial z}=Kf. \tag{2}$$

The degree of K is at most 1. If $K=0$ then f is a *polynomial first integral.* When a real polynomial system has a complex Darboux polynomial then it has also its conjugate. It is important to consider the complex Darboux polynoials of the real polynomial differential systems because sometimes they force the real integrability of the system. Note that we can write f as

$$Xf=Kf,\quad X=y\frac{\partial}{\partial x}-(x+yz)\frac{\partial}{\partial y}+(y^2-a)\frac{\partial}{\partial z}. \tag{3}$$

Let $f,g\in\mathbb{C}[x,y,z]$ and assume that f and g are relatively prime in the ring $\mathbb{C}[x,y,z]$, or that $g=1$. Then the function $\exp(f/g)$ is called an *exponential factor* of system (1) if for some polynomial $L \in \mathbb{C}[x, y, z]$ of degree one, we have

$$y\frac{\partial \exp(f/g)}{\partial x} - (x + yz)\frac{\partial \exp(f/g)}{\partial y} + (y^2 - a)\frac{\partial \exp(f/g)}{\partial z} = L \exp(f/g).$$

As before, we call L the *cofactor* of $\exp(f/g)$. We note that in the definition of exponential factor if $f, g \in \mathbb{C}[x, y, z]$ then the exponential factor is a complex function. Again, when a real polynomial differential system has a complex exponential factor, then it has also its conjugate, and both are important for the existence of real first integrals of the system. The exponential factors are related with the multiplicity of the Darboux polynomials, see for more details [4], Chapter 8 of [6] and [14, 15].

A first integral is called a *Darboux first integral* if it is a first integral of the form

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}, \tag{4}$$

where f_i are Darboux polynomials of system (1) for $i = 1, \dots, p$ and F_j are exponential factors of system (1) for $j = 1, \dots, q$ and $\lambda_i, \mu_j \in \mathbb{C}$. The next result proved in [6] indicates how to find Darboux first integrals.

Proposition 3. *Assume that a polynomial differential system of degree m admits p Darboux polynomials f_i with cofactors k_i for $i = 1, \dots, p$ and q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$. Then there exist λ_i and $\mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$$

if and only if the function (4) *is a Darboux first integral of that polynomial system.*

The following result is proved in [6].

Proposition 4. *Let f be a Darboux polynomial and $f = \prod_{j=1}^{s} f_j^{\alpha_j}$ its decomposition into irreducible factors in $\mathbb{C}[x, y, z]$. Then f us a Darboux polynomial if and only if all f_j are Darboux polynomials. Moreover, if K and K_j are the cofactors of f and f_j, then $K = \sum_{j=1}^{s} \alpha_j K_j$.*

The following proposition is proved in [14, 15].

Proposition 5. *Let $f, g \in \mathbb{C}[x, y, z]$. The following statements hold.*

1. *If $\exp(f/g)$ is an exponential factor for the polynomial differential system* (1) *and g is not a constant function, then g is a Darboux polynomial of* (1).
2. *The existence of an exponential factor of the form $\exp(f)$ is due to the multiplicity of the invariant plane at infinity.*

The following proposition is well known and its proof is similar to the proof of Proposition 4.

Proposition 6. *Let $F_j = \exp(f_j/g_j)$ for $j = 1, \dots, s$ be exponential factors of system* (1) *with cofactors L_j. Then*

$$F = \prod_{j=1}^{s} F_j = \exp\left(\sum_{j=1}^{s} \frac{f_j}{g_j}\right)$$

is an exponential factor of system (1) *with cofactor $L = \sum_{j=1}^{s} L_j$.*

2.2. **Averaging theory.** We present a result from the averaging theory that we shall need for proving Theorem 2, for a general introduction to the averaging theory of integrability see [13, 19].

We consider the initial value problems

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x,\varepsilon), \quad x(0) = x_0, \tag{5}$$

and

$$\dot{y} = \varepsilon g(y), \quad y(0) = x_0, \tag{6}$$

with x, y and x_0 in some open subset Ω of $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in (0, \varepsilon_0]$. We assume that F_1 and F_2 are periodic of period T in the variable t, and we set

$$g(y) = \frac{1}{T}\int_0^T F_1(t,y)\,dt. \tag{7}$$

We will also use the notation $D_x g$ for all the first derivatives of g, and $D_{xx} g$ for all the second derivatives of g. For a proof of the next result, see [22].

Theorem 7. *Assume that $F_1, D_x F_1, D_{xx} F_1$ and $D_x F_2$ are continuous and bounded by a constant independent of ε in $[0,\infty) \times \Omega \times (0, \varepsilon_0]$ and that $y(t) \in \Omega$ for $t \in [0, 1/\varepsilon]$. Then the following statements hold:*

(1) *For $t \in [0, 1/\varepsilon]$, we have $x(t) - y(t) = O(\varepsilon)$ as $\varepsilon \to 0$;*
(2) *If $p \neq 0$ is a singular point of system* (6) *such that*

$$\det D_y g(p) \neq 0, \tag{8}$$

then there exists a periodic solution $x(t,\varepsilon)$ of period T os system (5) *which is close to p and such that $x(0,\varepsilon) - p = O(\varepsilon)$ as $\varepsilon \to 0$;*
(3) *The stability of the periodic solution $x(t,\varepsilon)$ is given by the stability of the singular point p.*

3. Proof of Theorem 1

To prove Theorem 1 we state and prove several auxiliary results. Let $\mathbb{N}$ be the set of positive integers and $\tau\colon \mathbb{C}[x,y,z] \to \mathbb{C}[x,y,z]$ be the automorphism

$$\tau(x) = -x, \quad \tau(y) = -y, \quad \tau(z) = z.$$

Lemma 8. *If f is a Darboux polynomial for system* (1) *with cofactor K, then $\tau(f)$ is a Darboux polynomial of system* (1) *with cofactor $\tau(K)$.*

Proof. Since f is a Darboux polynomial for system (1) with cofactor K in view of (3) we can write it as $Xf = Kf$. Since (1) is invariant by τ we have that $\tau_*(X) = X$, where τ_* is the push-forward associated to automorphism τ. Therefore

$$\tau(Xf)(x, y, z) = \tau(Kf)(x, y, z).$$

The left-hand side is

$$\begin{aligned}\tau(Xf)(x, y, z) &= \tau_*(X)\tau(f)(x, y, z) = Xf(\tau^{-1}(x, y, z)) = Xf(\tau(x, y, z))\\ &= Xf(-x, -y, z) = X\tau(f)(x, y, z),\end{aligned}$$

where we have used that $\tau^{-1} = \tau$. The right-hand side is

$$\begin{aligned}\tau(Kf)(x, y, z) &= ((Kf) \circ \tau^{-1})(x, y, z) = ((Kf) \circ \tau)(x, y, z) = (Kf)(-x, -y, z)\\ &= K(-x, -y, z)f(-x, -y, z) = \tau(K)\tau(f)(x, y, z).\end{aligned}$$

From these two previous equalities we get

$$X\tau(f) = \tau(K)\tau(f),$$

and the lemma follows. □

Lemma 9. *If f is an irreducible Darboux polynomial of degree $n \geq 1$ for system* (1) *with cofactor $K = k_0+k_1x+k_2y+k_3z$ with $k_i \in \mathbb{C}$ for $i = 0, 1, 2, 3$, then $g = f \cdot \tau(f)$ is a Darboux polynomial invariant by τ of degree $2n$ with cofactor $2(k_0 + k_3z)$. If the cofactor of f is zero, then the cofactor of g is also zero.*

Proof. It follows from Lemma 8 that $\tau(f)$ is a Darboux polynomial of system (1) with cofactor $\tau(K)$. Therefore, by Proposition 4 the cofactor of $g = f\cdot\tau(f)$ is $K+\tau(K) = 2(k_0+k_3z)$. This concludes the proof of the lemma. □

Lemma 10. *Let g be a Darboux polynomial of system* (1) *of degree n. Then the polynomial $f = g\cdot\tau(g)$ is a Darboux polynomial invariant by τ with degree $2n$ and with cofactor $K = 2k_0 - 2mz$ for $k_0 \in \mathbb{C}$ and $m \in \mathbb{N}\cup\{0\}$. Moreover,*

$$f_{2n}(x, y, z) = y^{2m} \sum_{l=0}^{n-m} a_l (x^2)^{n-m-l}(y^2 + z^2)^l, \quad a_l \in \mathbb{C}. \tag{9}$$

Proof. We write f in sum of its homogeneous parts as

$$f = \sum_{i=0}^{2n} f_i, \quad f_i \text{ is a homogeneous polynomial of degree } i.$$

Without loss of generality we can assume that $f_{2n} \neq 0$ and $n \geq 1$. Computing the terms of degree $2n + 1$ in (2) we get

$$-yz\frac{\partial f_{2n}}{\partial y} + y^2\frac{\partial f_{2n}}{\partial z} = 2k_3zf_{2n}.$$

Solving this linear partial differential equation we get

$$f_{2n} = C_n(x, y^2 + z^2)y^{-2k_3},$$

where C_n is a polynomial in the variables $x, y^2 + z^2$. Since f_{2n} is a polynomial we must have $k_3 = -m$ with $m \in \mathbb{N} \cup \{0\}$. Moreover, since f_{2n} is a polynomial of degree $2n$ it can be written as in (9) (note that f_{2n} is invariant by τ). This concludes the proof of the lemma. □

Lemma 11. *Let g be a Darboux polynomial of system* (1) *of degree n. Then the polynomial $f = g \cdot \tau(g)$ is a Darboux polynomial invariant by τ with degree $2n$ and with cofactor $K = -2mz$ for $m \in \mathbb{N} \cup \{0\}$, $f_{2n} = \alpha_0 y^{2m}(x^2 + y^2 + z^2)^{n-m}$ for some $\alpha_0 \in \mathbb{C} \setminus \{0\}$ and*

$$f_{2n-1} = 2\alpha_0 m z y^{2m-1} \sum_{l=0}^{n-m} \binom{n-m}{l} x^{2n-2m-2l+1}(y^2 + z^2)^{l-1}.$$

Proof. We compute the terms of degree $2n-1$ using f_{2n} given in (9). Taking the terms of degree $2n$ from (2) we get

$$\mathcal{L}[f_{2n-1}] = -2mz f_{2n-1} - y\frac{\partial f_{2n}}{\partial x} + x\frac{\partial f_{2n}}{\partial y} + 2k_0 f_{2n}, \tag{10}$$

where

$$\mathcal{L} = -yz\frac{\partial}{\partial y} + y^2\frac{\partial}{\partial z}. \tag{11}$$

Note that in view of the definition of f_{2n} we have

$$\begin{aligned} R(x, y, z) &= -y\frac{\partial f_{2n}}{\partial x} + x\frac{\partial f_{2n}}{\partial y} + 2k_0 f_{2n} \\ &= -2xy^{2m+1} \sum_{l=0}^{n-m-1} a_l(n-m-l)(x^2)^{n-m-l-1}(y^2 + z^2)^l \\ &\quad + 2mxy^{2m-1} \sum_{l=0}^{n-m} a_l(x^2)^{n-m-l}(y^2 + z^2)^l \\ &\quad + 2xy^{2m+1} \sum_{l=1}^{n-m} a_l l(x^2)^{n-m-l}(y^2 + z^2)^{l-1} \\ &\quad + 2k_0 y^{2m} \sum_{l=0}^{n-m} a_l(x^2)^{n-m-l}(y^2 + z^2)^l \end{aligned}$$

$$
\begin{aligned}
&= 2xy^{2m+1}\sum_{l=0}^{n-m-1} b_l(x^2)^{n-m-l-1}(y^2+z^2)^l\\
&+ 2mxy^{2m-1}\sum_{l=0}^{n-m} a_l(x^2)^{n-m-l}(y^2+z^2)^l\\
&+ 2k_0y^{2m}\sum_{l=0}^{n-m} a_l(x^2)^{n-m-l}(y^2+z^2)^l,
\end{aligned}
$$

where

$$b_l = -(n-m-l)a_l + (l+1)a_{l+1}.$$

We apply now the method of characteristic curves to solve (10). For the method of characteristic curves for solving linear partial differential equations we refer the reader, for instance, to Bleecker and Csordas [1].

The characteristic equations associated with the first linear partial differential equation of system

$$\dot{y} = -yz, \quad \dot{z} = y^2$$

are of the form

$$\frac{dy}{dz} = -\frac{z}{y}.$$

This system has the general solution $y^2+z^2=\kappa$, where κ is a constant.

According with the method of characteristics we make the change of variables

$$u = x, \quad v = z, \quad w = y^2+z^2. \tag{12}$$

Its inverse transformation is

$$x = u, \quad y = \pm\sqrt{w-v^2}, \quad z = v. \tag{13}$$

In the following for simplicity we only consider the case $y = +\sqrt{w-z^2}$. In what follows we always write $\overline{\theta}$ to denote a function $\theta = \theta(x,y,z)$ written in the (u,v,w) variables, that is, $\overline{\theta} = \overline{\theta}(u,v,w)$. Under changes (12) and (13), equation in (10) becomes the following ordinary differential equation (for fixed u,w)

$$(w-v^2)\frac{d\overline{f}_{2n-1}}{dv} = -2mv\overline{f}_{2n-1} + \overline{R}(u,w,v) \tag{14}$$

where

$$\overline{R}(u,w,v) = 2u\sqrt{w-v^2}^{2m+1}\sum_{l=0}^{n-m-1} b_l(u^2)^{n-m-l-1}w^l$$
$$+ 2mu\sqrt{w-v^2}^{2m-1}\sum_{l=0}^{n-m} a_l(u^2)^{n-m-l}w^l$$
$$+ 2k_0\sqrt{w-v^2}^{2m}\sum_{l=0}^{n-m} a_l(u^2)^{n-m-l}w^l.$$

Equation (14) has the general solution

$$\overline{f}_{2n-1} = \overline{F}_{2n-1}(u,w)\sqrt{w-v^2}^{2m} + \sqrt{w-v^2}^{2m}\int \frac{\overline{R}(u,w,v)}{\sqrt{w-v^2}^{2m+2}}\,dv$$
$$= \overline{F}_{n-1}(u,w)\sqrt{w-v^2}^{2m}$$
$$+ 2\sqrt{w-v^2}^{2m}u\sum_{l=0}^{n-m-1} b_l(u^2)^{n-m_2-l-1}w^l\int\frac{1}{\sqrt{w-v^2}}\,dv$$
$$+ 2m\sqrt{w-v^2}^{2m}u\sum_{l=0}^{n-m} a_l(u^2)^{n-m-l}w^l\int\frac{1}{\sqrt{w-v^2}^3}\,dv$$
$$+ 2k_0\sqrt{w-v^2}^{2m}\sum_{l=0}^{n-m} a_l(u^2)^{n-m-l}w^l\int\frac{1}{w-v^2}\,dv,$$

where F_{2n-1} is a smooth function in the variables u, w. Using that

$$\int\frac{1}{\sqrt{w-v^2}}\,dv = \arctan\Big(\frac{v}{\sqrt{w-v^2}}\Big),$$
$$\int\frac{1}{(w-v^2)^{3/2}}\,dv = \frac{v}{w\sqrt{w-v^2}},$$
$$\int\frac{1}{w-v^2}\,dv = \frac{1}{\sqrt{w}}\arctan\Big(\frac{v}{\sqrt{w}}\Big),$$

and that f_{2n-1} must be a homogeneous polynomial of degree $2n-1$ we must have

$$k_0 = 0 \quad \text{and} \quad b_l = 0, \quad l = 0,\dots,n-m-1.$$

Therefore, we have

$$a_l = \frac{n-m-l+1}{l}a_{l-1} \quad \text{that is} \quad a_l = \binom{n-m}{l}\alpha_0, \quad \alpha_0 \in \mathbb{C}.$$

This implies that

$$f_{2n} = y^{2m}\sum_{l=0}^{n-m}\binom{n-m}{l}\alpha_0(x^2)^{n-m-l}(y^2+z^2)^l$$
$$= \alpha_0 y^{2m}(x^2+y^2+z^2)^{n-m}. \tag{15}$$

Note that

$$f_{2n-1} = y^{2m} F_{n-1}(x, y^2+z^2) + 2mzy^{2m-1} \sum_{l=0}^{n-m} a_l x^{2n-2m-2l+1} (y^2+z^2)^{l-1}.$$

Since f_{2n-1} is a homogeneous polynomial of degree $2n-1$ and must be invariant by τ we must have that F_{2n-1} is even in the variable x, and so F_{2n-1} must be zero. Then

$$(16)\qquad \begin{aligned} f_{2n-1} &= 2mzy^{2m-1} \sum_{l=0}^{n-m} a_l x^{2n-2m-2l+1} (y^2+z^2)^{l-1} \\ &= 2\alpha_0 mzy^{2m-1} \sum_{l=0}^{n-m} \binom{n-m}{l} x^{2n-2m-2l+1} (y^2+z^2)^{l-1}. \end{aligned}$$

This completes the proof of the lemma. □

Theorem 12. *The following holds for system* (1).

- (i) *If $a \neq 0$ it has no Darboux polynomials invariant by τ.*
- (ii) *If $a = 0$ the unique Darboux polynomials invariant by τ have cofactor zero and are polynomials in the variable $H = x^2 + y^2 + z^2$.*

Proof. Assume that system (1) has a Darboux polynomial under the assumptions of Lemma 10. Using the notation and the computations in the proof of Lemma 10 we shall compute the terms of degree $2n-2$ of f using the terms of degree $2n-1$ of (2). Thus we have

$$\mathcal{L}[f_{2n-2}] = -2mzf_{2n-2} - y\frac{\partial f_{2n-1}}{\partial x} + x\frac{\partial f_{2n-1}}{\partial y} + a\frac{\partial f_{2n}}{\partial z}$$

where $\mathcal{L}$ is given in (11). So from (15) and (16) we have

$$\begin{aligned} S(x,y,z) &= -y\frac{\partial f_{2n-1}}{\partial x} + x\frac{\partial f_{2n-1}}{\partial y} + a\frac{\partial f_{2n}}{\partial z} \\ &= -2\alpha_0 mzy^{2m} \sum_{l=0}^{n-m} \binom{n-m}{l} (2n-2m-2l+1) x^{2n-2m-2l} (y^2+z^2)^{l-1} \\ &\quad + 2\alpha_0 mz(2m-1)y^{2m-2} \sum_{l=0}^{n-m} \binom{n-m}{l} x^{2n-2m-2l+2} (y^2+z^2)^{l-1} \\ &\quad + 4\alpha_0 mzy^{2m} \sum_{l=0}^{n-m} \binom{n-m}{l} x^{2n-2m-2l+2} (l-1)(y^2+z^2)^{l-2} \\ &\quad + 2a\alpha_0 (n-m) zy^{2m} (x^2+y^2+z^2)^{n-m-1}. \end{aligned}$$

Changing the variables from (x,y,z) to (u,v,w) we obtain

$$(w-v^2)\frac{d\overline{f}_{2n-2}}{dv} = -2mv\overline{f}_{2n-2} + \overline{S}(u,v,w).$$

Solving this linear differential equation we get

$$\begin{aligned}
\overline{f}_{2n-2} &= \overline{F}_{2n-2}(u,w)\sqrt{w-v^2}^{2m} \\
&- 2\alpha_0 m\sqrt{w-v^2}^{2m}\sum_{l=0}^{n-m}\binom{n-m}{l}(2n-2m-2l+1)u^{2n-2m-2l}w^{l-1}\int \frac{v}{w-v^2}\,dv \\
&+ 2\alpha_0 m(2m-1)\sqrt{w-v^2}^{2m}\sum_{l=0}^{n-m}\binom{n-m}{l}u^{2n-2m-2l+2}w^{l-1}\int \frac{v}{(w-v^2)^2}\,dv \\
&+ 4\alpha_0 m\sqrt{w-v^2}^{2m}\sum_{l=0}^{n-m}\binom{n-m}{l}u^{2n-2m-2l+2}(l-1)w^{l-2}\int \frac{v}{w-v^2}\,dv \\
&+ 2a\alpha_0(n-m)\sqrt{w-v^2}^{2m}(u^2+w)^{n-m-1}\int \frac{v}{w-v^2}\,dv,
\end{aligned}$$

where F_{2n-2} is a smooth function in the variables u, w. Using that

$$\int \frac{v}{w-v^2}\,dv = -\frac{1}{2}\log(w-v^2),$$
$$\int \frac{v}{(w-v^2)^2}\,dv = \frac{1}{2(w-v^2)}.$$

From the fact that f_{2n-2} must be a polynomial we conclude that

$$\alpha_0 m = 0 \quad \text{and} \quad a\alpha_0(n-m) = 0.$$

Taking into account that $\alpha_0 n \neq 0$ we must have $m = 0$ and $a = 0$.

Thus if $a \neq 0$ there are no Darboux polynomials invariant by τ for system (1). This proves statement (i) of Theorem 12.

If $a = 0$ then $m_2 = 0$, $f_{2n} = \alpha_0(x^2+y^2+z^2)^n$, $f_{2n-1} = 0$ and f_{2n-2} as in (9) with n replaced by $n-1$. Proceeding as we did for f_{2n} we get that

$$f_{2n-2k} = \alpha_k(x^2+y^2+z^2)^{n-k}, \quad \text{for } k = 0,\ldots,n$$

and

$$f_{2n-2k-1} = 0, \quad \text{for } k = 0,\ldots,n-1.$$

So if $a = 0$ there are no Darboux polynomials invariant by τ with nonzero cofactor for system (1) and the polynomial first integrals are polynomials in the variable $H = x^2+y^2+z^2$. This completes the proof of Theorem 12. □

Proof of Theorem 1. Let f be an irreducible Darboux polynomial of degree n. In view of Lemma 10, the polynomial $g = f\cdot\tau(f)$ is a Darboux polynomial of system (1) with degree $2n$ invariant by τ. By Theorem 12 (i) if $a \neq 0$ this is not possible and so such f cannot exists proving statement (i) of Theorem 1. If $a = 0$ then in view of Theorem 12 (ii), the cofactor must be zero and $g = g(x^2+y^2+z^2)$. Since $x^2+y^2+z^2$ is irreducible, we get that both f and $\tau(f)$ must be polynomials in the variable $x^2+y^2+z^2$ proving statement (ii) of Theorem 1.

To prove statement (iii) we proceed by contradiction. Assume that G is a first integral of Darboux type of system (1). Then in view of its definition in (4) and taking into account statement (i) of Theorem 1 as well as Propositions 5 and 6, we conclude that G must be of the form

$$G = \exp(f), \quad \text{where } f \in \mathbb{C}[x,y,z] \setminus \mathbb{C}.$$

Since G is a first integral, after simplifying by $\exp(f)$, it must satisfy

$$y\frac{\partial f}{\partial x} - (x+yz)\frac{\partial f}{\partial y} + (y^2-a)\frac{\partial f}{\partial z} = 0.$$

So f must be a polynomial first integral, which in view of statement (i) it is not possible proving statement (iii).

Now we prove statement (iv). Let $a = 0$ and G be a first integral of Darboux type of system (1). Then in view of its definition in (4) and taking into account statement (ii) of Theorem 1 as well as Propositions 5 and 6, we conclude that G must be of the form

$$G = \exp(f/g(x^2+y^2+z^2)),$$

with $f \in \mathbb{C}[x,y,z]$, g a polynomial coprime with f. Since G and $g(x^2+y^2+z^2)$ are first integrals, after simplifying by $\exp(f/g(x^2+y^2+z^2))$, we have

$$y\frac{\partial f}{\partial x} - (x+yz)\frac{\partial f}{\partial y} + (y^2-a)\frac{\partial f}{\partial z} = 0.$$

So f must be a polynomial first integral, which in view of statement (ii) of Theorem 1 must be a polynomial in the variable $H = x^2+y^2+z^2$. This concludes the proof of statement (iv) and completes the proof of Theorem 1. ☐

4. Proof of Theorem 2

We introduce the scaling

$$X = \frac{x}{\sqrt{a}}, \quad Y = \frac{y}{\sqrt{a}}, \quad Z = \frac{z}{\sqrt{a}},$$

where $a > 0$. In these new variables system (1) becomes

$$\dot{X} = Y, \quad \dot{Y} = -X - \sqrt{a}YZ, \quad \dot{Z} = \sqrt{a}(Y^2-1). \tag{17}$$

Doing the cylindrical change of variables

$$X = r\cos\theta, \quad Y = r\sin\theta, \quad Z = Z \tag{18}$$

in the region $r > 0$, system (17) becomes

$$\dot{r} = -\sqrt{a}rZ\sin^2\theta, \quad \dot{\theta} = -1 - \sqrt{a}Z\cos\theta\sin\theta, \quad \dot{Z} = \sqrt{a}(-1+r^2\sin^2\theta). \tag{19}$$

It is easy to show that in a suitable neighborhood of $(r, Z) = (0, 0)$ with $r > 0$ we have $\dot{\theta} \neq 0$. Then choosing θ as the new independent variable, system (19) in a neighborhood of $(r, Z) = (0, 0)$ with $r > 0$ becomes

$$\frac{dr}{d\theta} = \frac{\sqrt{a} r Z \sin^2\theta}{1 + \sqrt{a} Z \cos\theta \sin\theta},$$
$$\frac{dZ}{d\theta} = \frac{\sqrt{a}(1 - r^2 \sin^2\theta)}{1 + \sqrt{a} Z \cos\theta \sin\theta},$$

Note that this system is periodic or period 2π in θ and system (17) becomes

$$\begin{aligned} \frac{dr}{d\theta} &= \sqrt{a} F_{11}(\theta, r, Z) + O(a), \\ \frac{dZ}{d\theta} &= \sqrt{a} F_{12}(\theta, r, Z) + O(a), \end{aligned} \tag{20}$$

where

$$F_{11}(\theta, r, Z) = rZ \sin^2\theta, \quad F_{21}(\theta, r, Z) = 1 - r^2 \sin^2\theta.$$

Using the notation of the averaging theory in Theorem 7, we have that if we take $t = \theta$, $T = 2\pi$, $\varepsilon = \sqrt{a}$, $x = (r, Z)^T$ and

$$F_1(t, x) = F_1(\theta, r, Z) = \begin{pmatrix} F_{11}(\theta, r, Z) \\ F_{12}(\theta, r, Z) \end{pmatrix},$$
$$\varepsilon^2 F_2(t, x) = O(a),$$

it is immediate to check that the differential system (20) is written in the normal form (5) for applying the averaging theory and that it satisfies the assumptions of Theorem 7. Now we compute the integral in (7) with $y = (r, Z)^T$, and denoting

$$g(y) = g(r, Z) = \begin{pmatrix} g_{11}(r, Z) \\ g_{12}(r, Z) \end{pmatrix},$$

we obtain

$$g_{11}(r, Z) = \frac{rZ}{2},$$
$$g_{12}(r, Z) = \frac{1}{2}(2 - r^2).$$

System $g_{11}(r, z) = g_{12}(r, Z) = 0$ has a unique solution with $r > 0$, namely $Z = 0$, $r = \sqrt{2}$. The Jacobian (8) is

$$\begin{vmatrix} Z/2 & r/2 \\ -r & 0 \end{vmatrix} = \frac{r^2}{2}$$

and evaluated on the solution $(r, Z) = (\sqrt{2}, 0)$ takes the value 1. Then by Theorem 7 it follows that for any $a > 0$ sufficiently small, system (20) has a periodic solution $(r(\theta, a), Z(\theta, a))$ such that $(r(0, a), Z(0, a))$ tends to $(\sqrt{2}, 0)$ when a tends to zero.

Going back to the differential system (19) we get that this system for $a > 0$ sufficiently small has one periodic solution of period approximately 2π of the form

$$r(t) = \sqrt{2} + O(\sqrt{a}), \quad \theta(t) = -t + O(\sqrt{a}), \quad Z(t) = O(\sqrt{a}).$$

This periodic solution become for the differential system (18) into one periodic solution of period also close to 2π of the form

$$X(t) = \sqrt{2}\cos t + O(\sqrt{a}), \quad Y(t) = -\sqrt{2}\sin t + O(\sqrt{a}), \quad Z(t) = O(\sqrt{a})$$

for $a > 0$ sufficiently small. Finally, we get for the differential system (1) the periodic solution

$$x(t) = \sqrt{2a}\cos t + O(a), \quad y(t) = -\sqrt{2a}\sin t + O(a), \quad z(t) = O(a)$$

of period near 2π when $a > 0$ sufficiently small. Clearly this periodic orbit tends to the equilibrium point $(0, 0, 0)$ when a tends to zero. This concludes the proof of the theorem.

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